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Stability and conservation properties of Hermite-based approximations of the Vlasov-Poisson system

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Abstract

Spectral approximation based on Hermite-Fourier expansion of the Vlasov-Poisson model for a collisionless plasma in the electrostatic limit is provided by adding high-order artificial collision operators of Lenard-Bernstein type. These differential operators are suitably designed in order to preserve the physically-meaningful invariants (number of particles, momentum, energy). In view of time-discretization, stability results in appropriate norms are presented. In this study, necessary conditions link the magnitude of the artificial collision term, the number of spectral modes of the discretization, as well as the time-step. The analysis, carried out in full for the Hermite discretization of a simple linear problem in one-dimension, is then partly extended to cover the complete nonlinear Vlasov-Poisson model.

Key words: Vlasov equation, spectral methods, conservation laws, Hermite polyomials
1991 MSC: 65N35, 35Q83

1. Introduction

The numerical approximation of physical systems described by kinetic equations is a formidable challenge [34]. These equations are, indeed, highly dimensional, strongly non-linear, and describe phenomena that are extremely multi-scale, as the behavior of the physical system at macroscopic scales is influenced by the microscopic particle dynamics. In plasma physics, scale separation occurs at the kinetic level because of the difference in mass between electrons and ions [18]. Other important applications that may be worth mentioning can be found in fluid dynamics, atmospheric and climate research [1], and multidimensional radiative transfer problems [25]. In all these fields, performing macroscale simulations that accurately include effects from the underlying microscale particle dynamics is still an open challenge.

In this work, we focus on the numerical approximation of the kinetics equation describing the behavior of electrically charged particle in a noncollisional plasmas, also known as *the Vlasov equation*. Such an equation governs the time evolution of the distribution function of plasma particles, through the action of an electromagnetic field generated by the charge and current densities of the same moving particles. The resulting coupling through Maxwell's equations (or Poisson's equation, in the electrostatic limit) is highly nonlinear, since the electromagnetic sources in such equations, i.e. charge and current densities, depend on the distribution functions themselves [15].

In his historical and pioneering paper [16], H. Grad proposed to expand the velocity distribution function of a noncollisional plasma at equilibrium using Hermite functions. Hermite functions are Hermite polynomials multiplied by the Gaussian exponential function $w(v) = \exp(-v^2)$, where v is the velocity in $1D$ of the plasma particles. Such a weight w is indeed the velocity distribution of a plasma at equilibrium and is a steady state solution of the Vlasov equation. Since a plasma at equilibrium is actually described by w, we expect that only a few modes may be needed to describe a plasma in a perturbed state still close to equilibrium. Moreover, it has to be pointed out that, when the solution of the Vlasov equation is expanded by Hermite basis functions, the equations for the first three coefficients correspond to the conservation laws for the number of particles, momentum and energy. These three quantities characterize the macroscopic (i.e., fluid-like) behavior of a plasma. The successive terms of the Hermite expansion introduce kinetics effects in a very straightforward manner, thus providing a strategy to realize the coupling between micro- and macro-physics. The micro/macro coupling is an intrinsic and specific feature of the Hermite approach, which cannot be replicated if we choose a different set of approximating functions. For the above reasons, Hermite functions constitute an "ideal" basis for solving numerically Vlasov-based models of noncollisional plasmas.

Since late sixties throughout the last five decades, Grad's idea has extensively been applied to the development of plasma simulators [2, 14, 21, 19, 37, 6, 35], where the Hermite basis for velocity is coupled with the Fourier basis in space. A renewed interest has been manifested in very recent years towards these approximation methods [4, 5, 9, 10], as the excellent properties mentioned above make them the natural numerical framework of high resolution and computationally efficient solvers [42, 36]. Moreover, the accuracy of Hermite's approximations can be improved by order of magnitudes by introducing a *translation factor*, u, and a *scaling factor*, α, in the so-called *generalized weight* [40]: $w(v) = exp\big(-((v-u)/\alpha)^2\big)$. Empirical evidence that a convenient choice of the scaling factor α can improve the accuracy in Hermite discretizations of the Vlasov equation was shown in [37]. Generalized basis functions of Hermite type have been investigated for solving time-dependent parabolic problems in [26] and, more recently, in [11] for the approximation of the Vlasov phase space. An adaptive strategy is currently under investigation [33], where both u and α may change depending on how the plasma evolves in time during a numerical simulation. Such an adaptive strategy is sought to improve the computational efficiency by using only a few spectral modes where a macroscopic description of the system is appropriate, and adding more modes where the microscopic physics is important [41]. This aspect offers the possibility of selecting the most meaningful number of spectral modes for a given resolution in phase space.

The strong point in favour of spectral schemes is that they can be extremely accurate because of the excellent convergence rate [8, 7, 3, 13, 12, 38]. Stability of spectral techniques for Vlasov-based systems can be ensured in different ways. For example, if we assume that the velocity domain remains bounded during simulation, we can use Legendre polynomials and enforce stability through a penalty technique acting on boundary terms [30, 31]. A common way to enforce numerical stability is by adding a suitable artificial dissipation to the right-hand side. Such a modification must not destroy the conservation properties of the original method, and this is a major concern in the case of Vlasov equation. Discrete schemes preserving basic quantities are available in spectral discretizations in combination to Fourier expansions [19, 37, 5, 9], or the discontinuous Galerkin method [29, 28, 32, 22, 23].

As specified above, in the spectral discretization of the Vlasov equation using Hermite basis functions, the conservation of number of particles, momentum and energy, is strictly related to the lowest-order modes. These can be heavily modified during the evolution by adding numerical dissipation in a straightforward fashion. A possible way to maintain a perfect preservation of low modes is to design the dissipation terms through Lenard-Bernstein-like operators of order $2k$, with $k \geq 1$ integer [24]. In the new formulation, the $1D-1D$ Vlasov-Poisson system takes the form

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - E\frac{\partial f}{\partial v} = -(-1)^k \nu \widetilde{L}^{(k)} L^{(k)} f \qquad \text{in } \Omega \times [0,T], \tag{1}$$

$$\frac{\partial E}{\partial x} = 1 - \int_{\Omega_v} f\,dv \qquad \text{in } \Omega \times [0,T], \tag{2}$$

where $f = f(x,v,t)$ is the distribution function, $E = E(x,t)$ the electric field, $\widetilde{L}^{(k)}$ and $L^{(k)}$ are the Lenard-Bernstein-like operators only acting onto the velocity variable v. The positive parameter ν is a sort of *artificial viscosity* used to tune the action of the differential operator $\widetilde{L}^{(k)} L^{(k)}$ on f. This kind of dissipation was proposed in previous works to control the filamentation process based on an empirical argument [4, 5, 9, 10, 30, 31].

There are of course other techniques to add viscosity. The *Spectral Vanishing Viscosity* (SVV) method was introduced at the end of the '80 and further investigated in the '90 [39, 27, 20, 17], to solve scalar conservation laws and hyperbolic problems using spectral methods. In [39] SVV is considered in the Fourier spectral discretization of a periodic conservation law, whereas the non-periodic case is treated in [27] through Legendre expansion. This approach makes it possible to preserve spectral accuracy and, at the same time, guarantee stability even in situations developing shocks. The trick is to introduce artificial viscosity only at the highest frequencies, according to special rules. For a given N, a way to represent the diffusion operator is to construct some coefficients Q_k such that

$$\begin{cases} Q_k = 0 & \text{if } k \le m_N \\ 0 < Q_k \le 1 & \text{if } m_N < k \le N, \end{cases}$$

where $m_N = \sqrt{N}$ is the spectral viscosity activation mode. If the function ϕ admits an expansion of the type $\phi(x) = \sum_{k=0}^{\infty} C_k P_k(x)$, where P_k is the k-th Legendre polynomial, the SVV operator Q^N is defined in the following way

$$\big(Q^N \phi\big)(x) = \sum_{k=0}^{N} Q_k C_k \, P_k(x).$$

Correspondingly, in the framework of collocation techniques, the conservation law and its polynomial approximation assume respectively the forms

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \qquad \frac{\partial u^N}{\partial t} + \frac{\partial I^N\big(f(u^N)\big)}{\partial x} = \epsilon^N \frac{\partial}{\partial x}\left[Q^N\left(\frac{\partial u^N}{\partial x}\right)\right],$$

where I^N is a suitable interpolation operator and the coefficient $\epsilon_N = \mathcal{O}(1/N)$ is the spectral viscosity amplitude. As we said, a strict peculiarity of the AW Hermite approximations is the possibility to preserve, perfectly unchanged, a certain number of moments associated with the lower modes. Unfortunately, discretization techniques based on Hermite functions are intrinsically unstable, so that the schemes require a mandatory mechanism of dissipation that should leave the basic modes untouched. The SVV approach could be a viable alternative to the combined Lenard-Bernstein operators. Contrary to what happens for the formulation (1), that has been successfully experimented in applications [4, 5, 9, 10, 30, 31], as far as we know, there are no examples of implementation of the SVV method in the framework of Hermite expansions for the Vlasov equation. The new analysis would be certainly interesting, but, at the moments, out of the scopes of this paper. The topic could be the subject of possible future work.

Commonly, there are two different choices of Hermite functions (i.e., Hermite polynomials multiplied by a suitable Gaussian function). The classical polynomial orthogonality weighted by $w(v) = e^{-v^2}$ leads to the so called *Asymmetrically Weighted* (AW) case, whereas the orthogonality of Hermite functions, each one weighted by $w(v) = e^{-v^2/2}$, leads to the *Symmetrically Weighted* (SW) case. This terminology will be better clarified in the coming sections. Accordingly, we have two different definitions of the Lenard-Bernstein differential operators $\widetilde{L}^{(k)}$ and $L^{(k)}$. In both cases, the basis elements are eigenfunctions of the combined operator. The crucial point is that the eigenvalues corresponding to the first $k-1$ modes are zero. This says that the action of the operators does not modify such modes, or, in other words, that $\widetilde{L}^{(k)} L^{(k)}$ induces dissipation only for the modes grater or equal to k. Despite these common properties, the two discrete formulations resulting from using AW and SW Hermite functions are substantially different. In fact, concerning time-discretization, it turns out that the SW formulation can easily be proven to be algebraically stable with or without the diffusive term [19, 37], whereas for the AW formulation the issue is far more delicate. More precisely, the stability result in the $L^2(\Omega)$ norm that we are interested to investigate reads as

$$\frac{d}{dt}\big|\big|f(\cdot,\cdot,t)\big|\big|^2_{L^2(\Omega)} \le 0. \tag{3}$$

The main criticism to the SW formulation is that, although stable, it does not preserve the lowest modes during time evolution, even in absence of artificial dissipation. On the contrary, the AW formulation perfectly conserves all the basic invariants, but its stability requires artificial dissipation. What we are able to prove in our work is an $L^2(\Omega)$ stability result when ν is sufficiently large. The result is achieved thanks to a suitable extension of the Poincarè inequality in weighted norms defined on the real line. Stability then follows by classical estimates for bilinear forms in Sobolev spaces. When instead ν is small, the result is certainly not true in the continuous case, but still holds in the framework of numerical time discretizations, by suitably linking ν to the parameter Δt, the final time T, and the maximum integer N used for the Hermite representation in the variable v. Preliminary, we show how to get these relations for a simple linear advection-diffusion model problem, and successively we partly extend our arguments to equation (1). By the way, from the practical viewpoint the use of the viscous term should not just be interpreted as a way to improve the time-stability of the schemes for the Vlasov-Poisson system, but has an important role in the reduction of the negative phenomenon known as *filamentation* [4], which shows up as a polluting effect on the computed solutions, due to the nonlinearity of the problem in conjunction with the truncation of the high modes.

We would like to remark that a stability result for the Hermite approximation of $1D-1V$ Vlasov-Poisson model was provided in [14], where L^2 boundedness has been proven with respect to the parameter N. However, that paper fails in proving absolute stability with respect to time, since the estimate there provided contains an exponential growth in T on the right-hand side of the inequality. The major result of our work is in achieving a stability estimate where boundedness in time is uniformly guaranteed.

The outline of the paper is as follows. In Section 2, we introduce the discretization framework and basic definitions concerning with the spectral Hermite method. In Sections 3 and 4 we present the Lenard-Bernstein operators for the AW and SW Hermite functions, respectively. We prove their dissipative nature and that they preserve unchanged the first modes of the spectral expansion. In Sections 5 and 6 we study the absolute stability in time of the modified advection problem

$$\frac{\partial f}{\partial t} - \frac{\partial f}{\partial v} = -(-1)^k \nu \widetilde{L}^{(k)} L^{(k)} f, \tag{4}$$

for the unknown scalar field $f(v,t)$, with the initial condition $f(v,0) = f_0(v)$, and study the impact of the stabilizing operator $\widetilde{L}^{(k)} L^{(k)}$. In Section 7, we apply an implicit time discretization scheme to the system of coefficients resulting from the Hermite expansion, and investigate its absolute stability in suitable weighted norms. In Section 8, we extend our approach to the full spectral discretization of the 1D-1V Vlasov-Poisson equations, and derive several sufficient conditions to guarantee the stability of the method. In Section 9 we provide our final remarks and conclusions. The appendix reports a couple of original theorems concerning with the Poincaré inequality involving Hermite functions. These results are crucial for the analysis carried out in the paper.

2. Hermite polynomials and their properties

We start by pointing out some well-known relations concerning Hermite polynomials that are, as usual, denoted by $H_n(v)$, and considered as functions of the independent variable $v \in \mathbb{R}$. The integer number n is the polynomial degree. First of all, we have the recursion formula

$$H_0 = 1, \quad H_1 = 2v, \qquad H_{n+1} = 2vH_n - 2nH_{n-1} \quad \text{for } n \geq 1, \tag{5}$$

and the differential equation

$$H_n'' - 2vH_n' + 2nH_n = 0, \tag{6}$$

which holds for any $n \in \mathbb{N}$. Here the primes denote differentiation with respect to v. The next formulas link Hermite polynomials of different degrees

$$H_n' = 2vH_n - H_{n+1}, \tag{7}$$

$$H_0' = 0 \quad \text{and} \quad H_n' = 2nH_{n-1}, \qquad \forall n \geq 1. \tag{8}$$

The last relation can recursively be generalized as follows

$$H_n^{(m)} = \frac{d^m H_n}{dv^m} = \begin{cases} 0 & n < m, \\ 2^m \dfrac{n!}{(n-m)!} H_{n-m} & n \geq m. \end{cases} \tag{9}$$

We recall that Hermite polynomials are orthogonal with respect to the weight function e^{-v^2} and are normalized in such a way that

$$\int_{\mathbb{R}} H_n^2 e^{-v^2} \, dv = \sqrt{\pi}\, 2^n \, n!. \tag{10}$$

For $n > m$, we recursively find that

$$\int_{\mathbb{R}} \big(H_n^{(m)}\big)^2 e^{-v^2}\, dv = 2^m \frac{n!}{(n-m)!} \int_{\mathbb{R}} H_n^2 e^{-v^2}\, dv. \tag{11}$$

In fact, by examining relation (8), it turns out that the derivatives of the Hermite polynomials are also orthogonal with respect to the weight e^{-v^2}. Using (8) and (10) for $n \geq 1$, we can obtain

$$\begin{aligned}\int_{\mathbb{R}} \big(H_n'\big)^2 e^{-v^2}\, dv &= 4n^2 \int_{\mathbb{R}} \big(H_{n-1}\big)^2 e^{-v^2}\, dv = 4n^2 \sqrt{\pi}\, 2^{n-1}\, (n-1)! \\ &= 2n\sqrt{\pi}\, 2^n\, n! = 2n \int_{\mathbb{R}} H_n^2 e^{-v^2}\, dv. \end{aligned} \tag{12}$$

The above relation is trivially satisfied also for $n = 0$. Equation (11) follows by applying (10) recursively. We then consider the generic function φ that, in appropriate circumstances, can be formally expanded as a series of Hermite polynomials $\varphi = \sum_{n=0}^{\infty} C_n H_n$. The coefficients C_n of φ are obtained as usual, i.e.

$$C_n = \frac{1}{\sqrt{\pi}\, 2^n\, n!} \int_{\mathbb{R}} \varphi H_n e^{-v^2}\, dv. \tag{13}$$

Of course, φ has to be such that all the above integrals are finite. From the orthogonality of Hermite polynomials and their derivatives, it follows that

$$\int_{\mathbb{R}} \varphi^2 e^{-v^2}\, dv = \sum_{n=0}^{\infty} C_n^2 \int_{\mathbb{R}} H_n^2 e^{-v^2}\, dv, \qquad \int_{\mathbb{R}} (\varphi')^2 e^{-v^2}\, dv = \sum_{n=0}^{\infty} C_n^2 \int_{\mathbb{R}} (H_n')^2 e^{-v^2}\, dv.$$

The last summation can also start from $n = 1$ since $H_0' = 0$.

Some important inequalities that will be used later in this paper are collected in the appendix. They are not only crucial in proving the stability of the schemes proposed, but they constitute an interesting new result in the theory of Hermite expansions.

We end this preliminary section by introducing a few definitions concerning *Hermite functions*, i.e., those functions that can be written as a linear combination (finite or infinite) of the elements of the *Hermite basis functions* $\{\psi_n\}$. Following the current literature, we will adopt a suitable notation in order to distinguish the so-called *Symmetrically-Weighted* (SW) case, from the *Asymmetrically-Weighted* (AW) one. The reason of this setting will be made clear as we proceed with the exposition. We consider the following definition

$$\psi_n(v) = \begin{cases} \gamma_n^{SW} H_n(v) e^{-v^2/2} & \text{symmetrically-weighted case,} \\ \gamma_n^{AW} H_n(v) e^{-v^2} & \text{asymmetrically-weighted case,} \end{cases} \tag{14}$$

for suitable choices of the real coefficients γ_n^{SW} and γ_n^{AW} (see below). Besides, we introduce the *dual basis* functions defined by

$$\psi^n(v) = \begin{cases} \widetilde{\gamma}_n^{SW} H_n(v) e^{-v^2/2} & \text{symmetrically-weighted case,} \\ \widetilde{\gamma}_n^{AW} H_n(v) & \text{asymmetrically-weighted case.} \end{cases} \tag{15}$$

We have

$$\gamma_n^{SW} = \widetilde{\gamma}_n^{SW} = (\sqrt{\pi} 2^n\, n!)^{-\frac{1}{2}}, \tag{16}$$

and

$$\gamma_n^{AW} = (\pi 2^n\, n!)^{-\frac{1}{2}}, \qquad \widetilde{\gamma}_n^{AW} = (2^n\, n!)^{-\frac{1}{2}}. \tag{17}$$

Such choices are compatible with the orthogonality relation

$$\int_{\mathbb{R}} \psi_n \psi^m e^{-v^2}\, dv = \delta_{n,m}. \tag{18}$$

3. Lenard-Bernstein diffusive operators in the AW case

Throughout the paper we will use indifferently the notation $\partial f/\partial v$ and f' to denote the partial derivative of functions like $f(v)$ or $f(t,v)$, regardless of their possible dependence on time. We study the differential operator ($k \geq 1$) that appears in the modified Vlasov equation (1) and in the simplified model equation (4). This section will be devoted to the AW case. In this framework, the operator is the result of the functional product of the two first-order operators

$$L = \frac{1}{2}\frac{\partial}{\partial v} + v\mathcal{I}, \qquad \widetilde{L} = \frac{\partial}{\partial v}, \tag{19}$$

where $\mathcal{I}$ is the identity. The second operator $\widetilde{L}$ is trivially the first derivative with respect to the variable v. The combination of L and $\widetilde{L}$ provides the so called second-order *Lenard-Bernstein-like operator* [24]. We investigate the action of $\widetilde{L}L$ on Hermite functions written in the form $f(v) = h(v)e^{-v^2}$, where h is a polynomial. Concerning the operator L, we have

$$Lf = \left(\frac{1}{2}\frac{\partial}{\partial v} + v\mathcal{I}\right) f = \frac{1}{2}h' e^{-v^2} - vhe^{-v^2} + vhe^{-v^2} = \frac{1}{2}h' e^{-v^2}. \tag{20}$$

Clearly, Lf is identically zero if h is a constant. Therefore, correspondingly to $h = 1$, we find that $L(e^{-v^2}) = 0$. Similarly, for $k = 2$ we have

$$L^2 f = L(Lf) = L\left(\frac{1}{2}h' e^{-v^2}\right) = \frac{1}{4}h'' e^{-v^2} - \frac{1}{2}vh' e^{-v^2} + \frac{1}{2}vh' e^{-v^2} = \frac{1}{4}h'' e^{-v^2}, \tag{21}$$

and, in general, for $k \geq 2$

$$L^k f = L(L^{k-1} f) = \frac{1}{2^k}h^{(k)} e^{-v^2}. \tag{22}$$

Equation (22) can be proved recursively starting from (20), by assuming successively that $L^{k-1} = (1/2^{k-1})h^{(k-1)}e^{-v^2}$ and applying the definition of L given in (19).

We are ready to prove our first result, aimed to show that the combined Lenard-Bernstein operator $-(-1)^k \widetilde{L}^k L^k f$, for $k \geq 1$, is dissipative.

Theorem 3.1 *Let $f(v,t) = h(v,t)e^{-v^2}$ be the solution of*

$$\frac{\partial f}{\partial t} + (-1)^k \widetilde{L}^k L^k f = 0 \qquad (k \geq 1). \tag{23}$$

We then have

$$\frac{d}{dt}\int_{\mathbb{R}} h^2 e^{-v^2}\, dv \leq 0. \tag{24}$$

Proof. We start from the case $k = 1$, that corresponds to

$$\widetilde{L}Lf = \widetilde{L}\left(\frac{1}{2}\frac{\partial}{\partial v} + v\mathcal{I}\right) f = \widetilde{L}\left(\frac{1}{2}h' e^{-v^2}\right) = \frac{1}{2}h'' e^{-v^2} - h' v e^{-v^2}. \tag{25}$$

Within the space of polynomials, $\widetilde{L}Lf$ is zero if and only if h is constant. To prove that $\widetilde{L}L$ is a dissipative operator, we rewrite (23) as

$$\frac{\partial f}{\partial t} - \widetilde{L}Lf = \frac{\partial f}{\partial t} - \frac{\partial Lf}{\partial v} = 0, \tag{26}$$

We multiply (26) by h, integrate over $\mathbb{R}$, and use integration by parts for the second term. The boundary terms are zero since they can be expressed as a polynomial multiplied by e^{-v^2}, which tends to zero for $|v| \to \infty$. Considering (20), we obtain

$$\begin{aligned} 0 &= \int_{\mathbb{R}} \left(\frac{\partial f}{\partial t} - \widetilde{L}Lf\right) h\, dv = \int_{\mathbb{R}} \left(\frac{\partial f}{\partial t} - \frac{\partial Lf}{\partial v}\right) h\, dv = \int_{\mathbb{R}} \frac{\partial f}{\partial t} h + \int_{\mathbb{R}} \left(Lf\right) h'\, dv - \big[(Lf)h\big]_{-\infty}^{+\infty} \\ &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} h^2 e^{-v^2}\, dv + \frac{1}{2}\int_{\mathbb{R}} \left(h'\right)^2 e^{-v^2}\, dv, \end{aligned} \tag{27}$$

which is equivalent to

$$\frac{d}{dt}\int_{\mathbb{R}} h^2 e^{-v^2}\,dv = -\int_{\mathbb{R}} \big(h'\big)^2 e^{-v^2}\,dv \le 0, \tag{28}$$

so that $\widetilde{L}Lf$ can be considered a dissipative operator for the weighted $L^2(\mathbb{R})$ norm. We repeat the same analysis for the fourth-order operator ($k=2$). We have the time dependent problem

$$\frac{\partial f}{\partial t} + \widetilde{L}^2 L^2 f = \frac{\partial f}{\partial t} + \frac{\partial^2 L^2 f}{\partial v^2} = 0 \tag{29}$$

(note the change of sign with respect to equation (26)). As before, we multiply (29) by h and integrate over $\mathbb{R}$. Using integration by parts (twice), we note that all the boundary terms are zero since they always consist of a polynomial multiplied by the Gaussian function e^{-v^2}, which tends to zero for $|v|\to\infty$. Thus, omitting the boundary terms and using (21) in the next calculation, we obtain

$$\begin{aligned} 0 &= \int_{\mathbb{R}} \left(\frac{\partial f}{\partial t} + \frac{\partial^2 L^2 f}{\partial v^2}\right) h\,dv = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} h^2 e^{-v^2}\,dv + \int_{\mathbb{R}} \big(L^2 f\big)\, h''\,dv \\ &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} h^2 e^{-v^2}\,dv + \frac{1}{4}\int_{\mathbb{R}} \big(h''\big)^2 e^{-v^2}\,dv. \end{aligned} \tag{30}$$

This implies that $-\widetilde{L}^2 L^2 f$ plays the role of a diffusive term, since

$$\frac{d}{dt}\int_{\mathbb{R}} h^2 e^{-v^2}\,dv = -\frac{1}{2}\int_{\mathbb{R}} \big(h''\big)^2 e^{-v^2}\,dv \le 0. \tag{31}$$

The general case can be handled in a very similar way. We write the time-dependent problem with the $2k$-th order operator as follows:

$$\frac{\partial f}{\partial t} + (-1)^k \widetilde{L}^k L^k f = \frac{\partial f}{\partial t} + (-1)^k \frac{\partial^k L^k f}{\partial v^k} = 0 \qquad (k \ge 1). \tag{32}$$

Going through the same steps it follows that $-(-1)^k \widetilde{L}^k L^k f$ is a diffusive operator. Indeed, integrating by parts k times and recalling (22), yields

$$\begin{aligned} -(-1)^k \int_{\mathbb{R}} \big(\widetilde{L}^k L^k f\big) h\,dv &= -(-1)^k \int_{\mathbb{R}} \frac{\partial^k L^k f}{\partial v^k}\, h\,dv \\ &= -(-1)^k(-1)^k \int_{\mathbb{R}} \big(L^k f\big)\, h^{(k)}\,dv = -\frac{1}{2^k}\int_{\mathbb{R}} \big(h^{(k)}\big)^2 e^{-v^2}\,dv \le 0, \end{aligned}$$

where $h^{(k)}$ denotes the k-th derivative of h with respect to v. □

We note that in the AW case, the weighted norm of h has no relation with the $L^2(\mathbb{R})$ norm of f. From the previous theorem we deduce that the operator of order $2k$ so far examined is not strictly negative definite, since its kernel is not empty. In particular, we have the following result (recall the definition (14) in the AW case).

Theorem 3.2 *For $n \ge 0$, the AW Hermite function ψ_n is an eigenfunction of the $2k$-th operator $\widetilde{L}^k L^k$ ($k \ge 1$), since we have*

$$\widetilde{L}^k L^k \psi_n = (-1)^k\, n(n-1)\dots(n-(k-1))\psi_n = (-1)^k \frac{n!}{(n-k)!}\psi_n. \tag{33}$$

The eigenvalue is zero for $0 \le n \le k-1$.

Proof. We begin with considering the case $k=1$. From (25) a direct calculation yields

$$\begin{aligned} \widetilde{L}L\psi_n &= \widetilde{L}L\big(\gamma_n^{AW} H_n e^{-v^2}\big) = \frac{\gamma_n^{AW}}{2}\big(H_n'' - 2vH_n'\big)e^{-v^2} = \frac{\gamma_n^{AW}}{2}\big(-2nH_n e^{-v^2}\big) \\ &= -n\gamma_n^{AW} H_n e^{-v^2} = -n\psi_n, \end{aligned} \tag{34}$$

where we used the differential equation (6). In other words, ψ_n is an eigenfunction of the differential operator $\widetilde{L}L$ with eigenvalue $-n$. As such eigenvalue is zero for $n = 0$, it follows that $\widetilde{L}L$ acts on Hermite functions without altering the first coefficient C_0. The minus sign in (34) is a further confirmation of the diffusive nature of the operator.

Similar relations hold for $\widetilde{L}^2L^2$ and the more general operator $\widetilde{L}^kL^k$. In the case $k = 2$, we use (21) with $h = H_n$, so to obtain

$$\widetilde{L}^2L^2\psi_n = \widetilde{L}^2L^2\big(\gamma_n^{AW}H_ne^{-v^2}\big) = \widetilde{L}^2\Big(\gamma_n^{AW}\frac{1}{4}H_n''e^{-v^2}\Big) = \frac{\gamma_n^{AW}}{4}\Big(H_n''e^{-v^2}\Big)''. \tag{35}$$

To compute the last term in the above equation we proceed in two steps, starting from the first derivative of $H''e^{-v^2}$. Using (6), we have

$$\begin{aligned}\Big(H_n''e^{-v^2}\Big)' &= \Big(\big(2vH_n' - 2nH_n\big)e^{-v^2}\Big)' = \left(2H_n' + 2vH_n'' - 2nH_n'\right)e^{-v^2} - 2v\left(2vH_n' - 2nH_n\right)e^{-v^2}\\ &= \left(2H_n' + 2vH_n'' - 2nH_n'\right)e^{-v^2} - 2vH_n''e^{-v^2} = 2(1-n)H_n'e^{-v^2}.\end{aligned} \tag{36}$$

Using again (6), we end up with

$$\Big(H_n'e^{-v^2}\Big)' = H_n''e^{-v^2} - 2vH_n'e^{-v^2} = \left(H_n'' - 2vH_n'\right)e^{-v^2} = -2nH_ne^{-v^2}. \tag{37}$$

Hence, the second derivative of $H''e^{-v^2}$ with respect to v is readily given by collecting the relations in (36) and (37). The result is

$$\Big(H_n''e^{-v^2}\Big)'' = \Big(\Big(H_n''e^{-v^2}\Big)'\Big)' = \Big(2(1-n)H_n'e^{-v^2}\Big)' = 4n(n-1)H_ne^{-v^2}. \tag{38}$$

Replacing (38) in (35), finally yields

$$\widetilde{L}^2L^2\psi_n = \frac{\gamma_n^{AW}}{4}\,4n(n-1)H_ne^{-v^2} = n(n-1)\,\gamma_n^{AW}H_ne^{-v^2} = n(n-1)\psi_n, \tag{39}$$

which shows that ψ_n is an eigenfunction of $\widetilde{L}^2L^2$ corresponding to the eigenvalue $n(n-1)$. Note that such eigenvalue is zero for $n = 0$ and $n = 1$, which means that the fourth-order operator $\widetilde{L}^2L^2$ does not modify the first two modes of the expansion of f. The above arguments can be easily repeated for a general integer $k \geq 1$. For the sake of brevity we omit the details. We conclude that every element ψ_n $(n \geq 0)$ of the basis is an eigenfunction of the $2k$-th operator $\widetilde{L}^kL^k$ with eigenvalue equal to $(-1)^k\,n!/(n-k)!$ for $n \geq k$ and zero for $0 \leq n \leq k-1$. □

We proceed our analysis by investigating the action of the operators on Hermite functions expressed as linear combinations of the basis functions ψ_n, as well as the implications on some conservation properties. Similar topics were analyzed in the more specific context of Vlasov-based models [9, 4].

We consider again the expansion of $f = he^{-v^2}$, where, based on (13), the polynomial function is given by

$$h = \sum_{n=0}^{\infty} C_nH_n. \tag{40}$$

By introducing the normalization factor γ_n^{AW} and recalling the definitions (14)-(15), we discover that

$$f = he^{-v^2} = \left(\sum_{n=0}^{\infty} C_nH_n\right)e^{-v^2} = \sum_{n=0}^{\infty}\frac{C_n}{\gamma_n^{AW}}\left(\gamma_n^{AW}H_ne^{-v^2}\right) = \sum_{n=0}^{\infty}C_n^{\star}\psi_n, \tag{41}$$

where $C_n^{\star} = C_n/\gamma_n^{AW}$. Since, according to the previous theorem, ψ_n is an eigenfunction of the generalized Lenard-Bernstein operators, we obtain the relations

$$\widetilde{L}Lf = \sum_{n=0}^{\infty} C_n^\star \widetilde{L}L\psi_n = \sum_{n=0}^{\infty}(-n)C_n^\star\psi_n, \tag{42}$$

$$\widetilde{L}^2L^2f^2 = \sum_{n=0}^{\infty} C_n^\star \widetilde{L}^2L^2\psi_n = \sum_{n=0}^{\infty} n(n-1)C_n^\star\psi_n, \tag{43}$$

$$\dots$$

$$\widetilde{L}^kL^kf = \sum_{n=0}^{\infty} C_n^\star \widetilde{L}^kL^k\psi_n = \sum_{n=0}^{\infty}(-1)^k\frac{n!}{(n-k)!}C_n^\star\psi_n. \tag{44}$$

from which it follows immediately that

$$\widetilde{L}Lf = \sum_{n=0}^{\infty} D_n^{(1)}\psi_n \qquad \text{with } D_n^{(1)} = -nC_n^\star, \tag{45}$$

$$\widetilde{L}^2L^2f = \sum_{n=0}^{\infty} D_n^{(2)}\psi_n \qquad \text{with } D_n^{(2)} = n(n-1)C_n^\star, \tag{46}$$

$$\dots$$

$$\widetilde{L}^kL^kf = \sum_{n=0}^{\infty} D_n^{(k)}\psi_n \qquad \text{with } D_n^{(k)} = (-1)^k\frac{n!}{(n-k)!}C_n^\star. \tag{47}$$

By definition, it holds that $D_0^{(k)} = D_1^{(k)} = \ldots = D_{k-1}^{(k)} = 0$, for any $k \geq 1$. The case $k = 3$ corresponds to the operator used in [5, 9]. The relations established so far allow us to deduce some conservation properties as stated by the next theorem.

Theorem 3.3 (Conservation of the m-th moment of f) *Let $f(t,v) = h(t,v)e^{-v^2}$, where h is a polynomial, be the solution of the partial differential equation* (23)*. Then, the m-th moment of f, for $0 \leq m < k$, is preserved with respect to time, i.e.*

$$\frac{d}{dt}\int_{\mathbb{R}} v^m f\,dv = 0. \tag{48}$$

Proof. For $m = 0$, f satisfies the relation

$$\frac{d}{dt}\int_{\mathbb{R}} f\,dv = 0, \tag{49}$$

known as *mass conservation*. To prove the relation for $k = 1$, we integrate (23) (see (26)) on $\mathbb{R}$, apply the fundamental theorem of calculus and substitute the expression of Lf in (20) to obtain

$$\frac{d}{dt}\int_{\mathbb{R}} f\,dv = \int_{\mathbb{R}} \widetilde{L}Lf\,dv = \int_{\mathbb{R}} \frac{\partial Lf}{\partial v}\,dv = [Lf]_{-\infty}^{\infty} = \frac{1}{2}\left[h'e^{-v^2}\right]_{-\infty}^{\infty} = 0, \tag{50}$$

since e^{-v^2} times a polynomial tends to zero for $v \to \pm\infty$.

For $k = 2$, we integrate again (23) (see (29)) on $\mathbb{R}$, and apply the fundamental theorem of calculus to obtain

$$\frac{d}{dt}\int_{\mathbb{R}} f\,dv = -\int_{\mathbb{R}} \widetilde{L}^2L^2f\,dv = -\int_{\mathbb{R}} \frac{\partial}{\partial v}(\widetilde{L}L^2f)\,dv = -\left[\widetilde{L}L^2f\right]_{-\infty}^{\infty}. \tag{51}$$

Furthermore, by using (21), we arrive at

$$\widetilde{L}L^2f = \frac{\partial L^2f}{\partial v} = \frac{1}{4}\frac{\partial}{\partial v}\left(h''e^{-v^2}\right) = \frac{1}{4}\big(h''' - 2vh''\big)e^{-v^2}. \tag{52}$$

Therefore, the last term above provides zero in (51), since the Gaussian function e^{-v^2} multiplied by a polynomial tends to zero for $v \to \pm\infty$. Finally, to obtain the general result for $k > 2$, we integrate (23) on $\mathbb{R}$ and apply the fundamental theorem of calculus. In this way, we get

$$\frac{d}{dt}\int_{\mathbb{R}} f\,dv = -(-1)^k \int_{\mathbb{R}} \widetilde{L}^k L^k f\,dv = -(-1)^k \int_{\mathbb{R}} \frac{\partial}{\partial v}(\widetilde{L}^{k-1} L^k f)\,dv = -(-1)^k \left[\widetilde{L}^{k-1} L^k f\right]_{-\infty}^{\infty} = 0, \tag{53}$$

since we can prove recursively that $\widetilde{L}^{k-1} L^k f$ is equal to a polynomial times e^{-v^2}, which tends to zero for $v \to \pm\infty$.

We now move to the case $m = 1$. Here, f must satisfy the relation

$$\frac{d}{dt}\int_{\mathbb{R}} v f\,dv = 0, \tag{54}$$

which is known as *momentum conservation*. We observe that there is no momentum conservation for the operator $\widetilde{L}L$ ($m = k = 1$). We then consider the two other cases in which f is either solution of (29) (using $-\widetilde{L}^2 L^2 f$), or (23) (using $-(-1)^k \widetilde{L}^k L^k f$). We just study the first situation, since the general case can be approached in a similar way. We start from

$$\frac{d}{dt}\int_{\mathbb{R}} v f\,dv = -\int_{\mathbb{R}} v \frac{\partial^2 L^2 f}{\partial v^2}\,dv. \tag{55}$$

Then, we integrate by parts the right-hand side in order to get

$$\frac{d}{dt}\int_{\mathbb{R}} v f\,dv = \int_{\mathbb{R}} \frac{\partial L^2 f}{\partial v}\,dv - \left[v\frac{\partial L^2 f}{\partial v}\right]_{-\infty}^{\infty} = \left[L^2 f\right]_{-\infty}^{\infty} - \left[v\frac{\partial L^2 f}{\partial v}\right]_{-\infty}^{\infty} = 0. \tag{56}$$

Again, the arguments in the square brackets are of the form of a polynomial multiplied by the Gaussian function e^{-v^2}.

Through very similar steps, that we omit for brevity, we can study the case $m > 1$, so arriving at equation (48), for $k > m$, which is our thesis. □

The conservation of momenta, for the distribution function f in the Vlasov equation, implies the conservation of physical quantities such as the total energy. We will discuss this topic at the end of this paper.

4. Lenard-Bernstein diffusive operators in the SW case

In this section we adapt the results already obtained for the AW case to the SW context. The generalized Lenard-Bernstein operators are now obtained by composing the first order operators

$$L = \frac{\partial}{\partial v} + v\mathcal{I}, \qquad \widetilde{L} = \frac{\partial}{\partial v} - v\mathcal{I}. \tag{57}$$

We remind that the weighted L^2 inner product is now

$$\big(f, g\big) = \int_{\mathbb{R}} f g\,dv = \int_{\mathbb{R}} h_f h_g e^{-v^2}\,dv, \tag{58}$$

where $f = h_f e^{-v^2/2}$ and $g = h_g e^{-v^2/2}$, and h_f and h_g are polynomials. This somehow justifies the adoption of the term "symmetric". The results are going to be analogous to those presented in the previous section. We briefly review the main points, beginning with a theorem.

Theorem 4.1 *Let $f(v,t) = h(v,t)e^{-v^2/2}$ be the solution of*

$$\frac{\partial f}{\partial t} + (-1)^k \widetilde{L}^k L^k f = 0 \qquad (k \geq 1). \tag{59}$$

We then have

$$\frac{d}{dt}\int_{\mathbb{R}} f^2\,dv = \frac{d}{dt}\int_{\mathbb{R}} h^2 e^{-v^2}\,dv \leq 0. \tag{60}$$

Proof. Again, we start from the case $k = 1$. We multiply equation (59) by f and integrate over $\mathbb{R}$, ending up with the equality

$$\int_{\mathbb{R}} \left(\frac{\partial f}{\partial t} - \widetilde{L} L f \right) f \, dv = 0. \tag{61}$$

Using the definition of $\widetilde{L}$ given in (57), we find that

$$\frac{1}{2} \int_{\mathbb{R}} \frac{\partial}{\partial t} \big(h^2 e^{-v^2} \big) \, dv - \int_{\mathbb{R}} \big((Lf)' - vLf \big) f \, dv = 0, \tag{62}$$

where again we denoted the derivative with respect to v of Lf by $(Lf)'$. We integrate by parts the second integral of (62) and note that the boundary terms for $v \to \pm\infty$ are zero. This leads us to

$$\begin{aligned} 0 &= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} h^2 e^{-v^2} \, dv + \int_{\mathbb{R}} \big(Lf \big) \, f' \, dv - [(Lf) f]_{-\infty}^{\infty} + \int_{\mathbb{R}} v \big(Lf \big) \, f \, dv \\ &= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} h^2 e^{-v^2} \, dv + \int_{\mathbb{R}} \big(Lf \big) \big(f' + vf \big) \, dv = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} h^2 e^{-v^2} \, dv + \int_{\mathbb{R}} \big(Lf \big)^2 \, dv. \end{aligned} \tag{63}$$

The last relation shows that the operator $\widetilde{L} L$ introduces dissipation.

The same result holds for the fourth-order operator and the related time dependent problem

$$\frac{\partial f}{\partial t} + \widetilde{L}^2 L^2 f = 0. \tag{64}$$

Here, the proof is a bit more involved, but still elementary. We first note that for a generic function g we have $\widetilde{L}^2 g = \widetilde{L}(\widetilde{L} g) = \widetilde{L}(g' - vg)$, from which it follows that

$$\widetilde{L}^2 g = (g' - vg)' - v(g' - vg) = g'' - 2vg' + (v^2 - 1) g. \tag{65}$$

Moreover, we have

$$L^2 f = f'' + 2vf' + (v^2 + 1) f = f'' + 2\big(vf \big)' + (v^2 - 1) f. \tag{66}$$

By multiplying equation (64) by f and integrating over $\mathbb{R}$, we find that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} h^2 e^{-v^2} \, dv + \int_{\mathbb{R}} \big(\widetilde{L}^2 L^2 f \big) f \, dv = 0. \tag{67}$$

From straightforward calculations using integration by parts and noting again that the boundary terms for $|v| \to \infty$ are zero, and then using formulas (65) with $g = L^2 f$ and (66), we are able to prove the following relation

$$\begin{aligned} \int_{\mathbb{R}} \big(\widetilde{L}^2 L^2 f \big) f \, dv &= \int_{\mathbb{R}} \Big(\big(L^2 f \big)'' - 2v \big(L^2 f \big)' + (v^2 - 1) L^2 f \Big) \, f \, dv \\ &= \int_{\mathbb{R}} \big(L^2 f \big) \, f'' \, dv + 2 \int_{\mathbb{R}} \big(L^2 f \big) \, \big(vf \big)' \, dv + \int_{\mathbb{R}} (v^2 - 1) \big(L^2 f \big) \, f \, dv \\ &= \int_{\mathbb{R}} \big(L^2 f \big) \Big(f'' + 2 \big(vf \big)' + (v^2 - 1) f \Big) \, dv = \int_{\mathbb{R}} \big(L^2 f \big)^2 \, dv. \end{aligned} \tag{68}$$

Therefore, also (64) is dissipative in the $L^2(\mathbb{R})$ weighted norm.

For a general k, with the same considerations as above, we find the relation

$$\frac{d}{dt} \int_{\mathbb{R}} h^2 e^{-v^2} \, dv = - \int_{\mathbb{R}} \big(L^k f \big)^2 \, dv \leq 0, \tag{69}$$

which easily brings us to the thesis. □

It is important to remark that, differently from the AW case, the weighted norm of h is actually the $L^2(\mathbb{R})$ norm of f. A result similar to theorem 3.2 holds. To this regard, we remind the definition given in (14) for the SW case.

Theorem 4.2 *For any $n \geq 0$, the SW Hermite function ψ_n is an eigenfunction of the $2k$-th operator $\widetilde{L}^k L^k$ ($k \geq 1$), since we have*

$$\widetilde{L}^k L^k \psi_n = (-1)^k\, 2^k\, n(n-1)\dots(n-(k-1))\psi_n = (-1)^k\, 2^k \frac{n!}{(n-k)!}\psi_n \qquad n \geq k. \tag{70}$$

The eigenvalue is zero for $0 \leq n \leq k-1$.

Proof. We recall that $L = \partial/\partial v + v\mathcal{I}$ and $\widetilde{L} = \partial/\partial v - v\mathcal{I}$. Let us take $\psi_n = \gamma_n^{SW} H_n e^{-v^2/2}$. A direct calculation yields

$$L\psi_n = \gamma_n^{SW}\left(\frac{\partial}{\partial v} + v\mathcal{I}\right) H_n e^{-v^2/2} = \gamma_n^{SW}\left(H_n' e^{-v^2/2} - vH_n e^{-v^2/2} + vH_n e^{-v^2/2}\right) = \gamma_n^{SW} H_n' e^{-v^2/2}. \tag{71}$$

Using the result above we obtain

$$\begin{aligned} L^2\psi_n = L\big(L\psi_n\big) = L\left(\gamma_n^{SW} H_n' e^{-v^2/2}\right) &= \gamma_n^{SW}\left(\frac{\partial}{\partial v} + v\mathcal{I}\right) H_n' e^{-v^2/2} \\ &= \gamma_n^{SW}\left(H_n'' e^{-v^2/2} - vH_n' e^{-v^2/2} + vH_n' e^{-v^2/2}\right) = \gamma_n^{SW} H_n'' e^{-v^2/2}. \end{aligned} \tag{72}$$

A simple recursive argument allows us to prove the formula for a generic k, i.e.

$$L^k\psi_n = \gamma_n^{SW} H_n^{(k)} e^{-v^2/2}, \tag{73}$$

where we recall that $H_n^{(k)} = d^k H_n/dv^k$. Indeed, we have already proved that (73) is true for $k=1$ and $k=2$. Since $L^{k-1}\psi_n = \gamma_n^{SW} H_n^{(k-1)} e^{-v^2/2}$, we end up with

$$\begin{aligned} L^k\psi_n = L\big(L^{(k-1)}\psi_n\big) = L\left(\gamma_n^{SW} H_n^{(k-1)} e^{-v^2/2}\right) &= \gamma_n^{SW}\left(\frac{\partial}{\partial v} + v\mathcal{I}\right)(H_n^{(k-1)} e^{-v^2/2}) \\ &= \gamma_n^{SW}\left(H_n^{(k)} e^{-v^2/2} - vH_n^{(k-1)} e^{-v^2/2} + vH_n^{(k-1)} e^{-v^2/2}\right) = \gamma_n^{SW} H_n^{(k)} e^{-v^2/2}. \end{aligned} \tag{74}$$

We then compute the action of $\widetilde{L}$, $\widetilde{L}^2$, and $\widetilde{L}^k$ on $L\psi_n$, $L^2\psi_n$, and $L^k\psi_n$, respectively. In the first case, by using (6) we recover the relation

$$\begin{aligned} \widetilde{L}L\psi_n = \gamma_n^{SW}\left(\frac{\partial}{\partial v} - v\mathcal{I}\right)(H_n' e^{-v^2/2}) &= \gamma_n^{SW}\left(H_n'' e^{-v^2/2} - vH_n' e^{-v^2/2} - vH_n' e^{-v^2/2}\right) \\ &= \gamma_n^{SW}\left(H_n'' - 2vH_n'\right) e^{-v^2/2} = \gamma_n^{SW}(-2n)H_n e^{-v^2/2} = -2n\psi_n, \qquad n \geq 1. \end{aligned} \tag{75}$$

In the second case, we first obtain

$$\begin{aligned} \widetilde{L}L^2\psi_n = \gamma_n^{SW}\left(\frac{\partial}{\partial v} - v\mathcal{I}\right)(H_n'' e^{-v^2/2}) &= \gamma_n^{SW}\left(\big(H_n''\big)' e^{-v^2/2} - vH_n'' e^{-v^2/2} - vH_n'' e^{-v^2/2}\right) \\ &= \gamma_n^{SW}\left(\big(H_n''\big)' - 2vH_n''\right) e^{-v^2/2} = \gamma_n^{SW}\left(\big(2vH_n' - 2nH_n\big)' - 2vH_n''\right) e^{-v^2/2} \\ &= \gamma_n^{SW}\left(2H_n' + 2vH_n'' - 2nH_n' - 2vH_n''\right) e^{-v^2/2} = \gamma_n^{SW}\, 2(1-n)\, H_n' e^{-v^2/2}, \quad n \geq 2, \end{aligned} \tag{76}$$

and then

$$\begin{aligned} \widetilde{L}^2 L^2\psi_n &= \widetilde{L}(\widetilde{L}L^2\psi_n) = \widetilde{L}\big(\gamma_n^{SW}\, 2(1-n)H_n' e^{-v^2/2}\big) \\ &= \gamma_n^{SW}\, 2(1-n)\left(\frac{\partial}{\partial v} - v\mathcal{I}\right)(H_n' e^{-v^2/2}) = \gamma_n^{SW}\, 2(1-n)\left(H_n'' - vH_n' - vH_n'\right) e^{-v^2/2} \\ &= \gamma_n^{SW}\, 2(1-n)\left(H_n'' - 2vH_n'\right) e^{-v^2/2} = \gamma_n^{SW}\, 2(1-n)(-2n)H_n e^{-v^2/2} = 4n(n-1)\psi_n. \end{aligned} \tag{77}$$

The final case, for a generic k, follows from a recursive argument, allowing us to prove that

$$\widetilde{L}^k L^k\psi_n = (-1)^k\, 2^k \frac{n!}{(n-k)!}\psi_n, \qquad n \geq k. \tag{78}$$

Therefore, we conclude that every ψ_n is an eigenfunction of the combined $2k$-th operator $\widetilde{L}^k L^k$ with eigenvalue $(-1)^k 2^k\, n!/(n-k)!$, for $n \geq k$ and that such operator does not modify the first k modes of the expansion of f. Except for the factor 2^k, the eigenvalue is the same as that in (33). $\square$

We end this section by investigating the action of the generalized Lenard-Bernstein operators on Hermite functions expressed as linear combinations of the basis functions. To this purpose, we consider the expansion

$$f = he^{-v^2/2} = \left[\sum_{n=0}^{\infty} C_n H_n\right] e^{-v^2/2} = \sum_{n=0}^{\infty} \frac{C_n}{\gamma_n^{SW}} \left[\gamma_n^{SW} H_n e^{-v^2/2}\right] = \sum_{n=0}^{\infty} C_n^\star \psi_n, \tag{79}$$

where $C_n^\star = C_n/\gamma_n^{SW}$. Since ψ_n is an eigenfunction, we readily find the following relations

$$\widetilde{L}Lf = \sum_{n=0}^{\infty} C_n^\star \widetilde{L}L\psi_n = \sum_{n=0}^{\infty} (-2n) C_n^\star \psi_n, \tag{80}$$

$$\widetilde{L}^2 L^2 f = \sum_{n=0}^{\infty} C_n^\star \widetilde{L}^2 L^2 \psi_n = \sum_{n=0}^{\infty} 4n(n-1) C_n^\star \psi_n, \tag{81}$$

$$\ldots$$

$$\widetilde{L}^k L^k f = \sum_{n=0}^{\infty} C_n^\star \widetilde{L}^k L^k \psi_n = \sum_{n=0}^{\infty} (-1)^k 2^k \frac{n!}{(n-k)!} C_n^\star \psi_n, \tag{82}$$

from which we deduce that

$$\widetilde{L}Lf = \sum_{n=0}^{\infty} D_n^{(1)} \psi_n \qquad \text{with } D_n^{(1)} = -2nC_n^\star, \tag{83}$$

$$\widetilde{L}^2 L^2 f = \sum_{n=0}^{\infty} D_n^{(2)} \psi_n \qquad \text{with } D_n^{(2)} = 4n(n-1)C_n^\star, \tag{84}$$

$$\ldots$$

$$\widetilde{L}^k L^k f = \sum_{n=0}^{\infty} D_n^{(k)} \psi_n \qquad \text{with } D_n^{(k)} = (-1)^k 2^k \frac{n!}{(n-k)!} C_n^\star. \tag{85}$$

By definition, it holds that $D_0^{(k)} = D_1^{(k)} = \ldots = D_{k-1}^{(k)} = 0$, for any $k \geq 1$.

As far as mass and momentum conservations are concerned, we do not have the same results of the AW case. Indeed, we can check that equations (49) and (48) do not hold anymore in the symmetric case. Instead, we can prove the conservation of the weighted integrals

$$\int_{\mathbb{R}} f(v,t) e^{-v^2/2}\, dv \quad \text{and} \quad \int_{\mathbb{R}} v f(v,t) e^{-v^2/2}\, dv,$$

which however are not associated with any physical quantity of interest in the continuous setting.

5. Hermite approximation of a pure advection equation

We take into account the following time-dependent problem for the unknown scalar field $f = f(v,t)$

$$\frac{\partial f}{\partial t} - \frac{\partial f}{\partial v} = 0, \tag{86}$$

supplemented with the initial condition

$$f(v, 0) = f_0(v). \tag{87}$$

Our aim is to study the absolute stability in time either in the SW case or the AW case. For convenience, we formally write our differential problem in variational form. Nevertheless, an analysis of the well-posedness in suitable functional spaces, and the convergence properties of a Galerkin type approximation, are not the main issues here, so we will skip such technical details to simplify the presentation.

Regarding the SW case, we present the following result.

Theorem 5.1 *The $L^2(\mathbb{R})$ norm of the function $f(v,t) = h(v,t)e^{-v^2/2}$ (where h is a polynomial in v) solving equation* (86) *in weak form, is conserved, i.e.*

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} f^2\, dv = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} h^2 e^{-v^2}\, dv = 0. \tag{88}$$

Proof. To prove the statement, we multiply (86) by the test function f and integrate over $\mathbb{R}$, so obtaining

$$0 = \int_{\mathbb{R}} \left(\frac{\partial f}{\partial t} - \frac{\partial f}{\partial v}\right) f\, dv = \int_{\mathbb{R}} \left(\frac{\partial}{\partial t}\left(\frac{f^2}{2}\right) - \frac{\partial}{\partial v}\left(\frac{f^2}{2}\right)\right) dv = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} h^2 e^{-v^2}\, dv. \tag{89}$$

The relation is true since the integral of $\partial f^2/\partial v$ is zero because $f(v,t) \to 0$ exponentially for $v \to \pm\infty$. □

This stability result is encouraging but, unfortunately, the same is not going to be true for the AW case. In fact, we may try to study the stability with the same approach followed in the proof before. This time we set $f(v,t) = h(v,t)e^{-v^2}$ (where h is a polynomial in v). We then take h as test function and integrate over $\mathbb{R}$. In this way, we obtain

$$0 = \int_{\mathbb{R}} \left(\frac{\partial f}{\partial t} - \frac{\partial f}{\partial v}\right) h\, dv = \int_{\mathbb{R}} h\frac{\partial h}{\partial t} e^{-v^2}\, dv - \int_{\mathbb{R}} h\frac{\partial f}{\partial v}\, dv. \tag{90}$$

Successively, we integrate by parts the last term, substitute $f = he^{-v^2}$ and integrate by parts again. All the boundary terms are zero since they involve a polynomial in v multiplied by a decaying exponential and are omitted. This procedure yields

$$-\int_{\mathbb{R}} h\frac{\partial f}{\partial v}\, dv = \int_{\mathbb{R}} f\frac{\partial h}{\partial v}\, dv = \int_{\mathbb{R}} \frac{\partial h}{\partial v} he^{-v^2}\, dv = \int_{\mathbb{R}} \frac{\partial}{\partial v}\left(\frac{h^2}{2}\right) e^{-v^2}\, dv = \int_{\mathbb{R}} h^2\, v\, e^{-v^2}\, dv. \tag{91}$$

Finally, we find out that

$$0 = \frac{d}{dt}\int_{\mathbb{R}} \frac{h^2}{2} e^{-v^2}\, dv + \int_{\mathbb{R}} v\, h^2 e^{-v^2}\, dv. \tag{92}$$

Since $v \in \mathbb{R}$ can assume positive or negative values, the sign of the second integral is undetermined, and therefore, the AW Hermite variational formulation is not absolutely stable in the weighted $L^2(\mathbb{R})$ norm. Note, however, that such weighted norm has no physical meaning.

The conclusion is that both the continuous case and SW case, preserve the quantity $\int_{\mathbb{R}} f^2 dv$. Unfortunately, this quantity is not preserved in the AW case. Indeed, we are in the sad situation in which neither the weighted L^2-norm nor the unweighted one are conserved. For this reason, it is necessary to make use of the dissipation operator as it will be discussed in section 6.

We proceed now by deriving the recursive equation for the coefficients of the Hermite expansion in both AW and SW cases. In order to simplify the notation, in the expressions below, we set $\gamma_n = \gamma_n^{SW}$ when we deal with the SW case or $\gamma_n = \gamma_n^{AW}$ when we deal with the AW case (we recall that these coefficients are defined in (16) and (17)). Also, we use the notation $C_n^\star = C_n/\gamma_n$ to denote the coefficients of the expansion in the basis functions ψ_n. We start by writing

$$f(v,t) = \sum_{n=0}^{\infty} C_n(t) H_n(v) e^{-v^2} = \sum_{n=0}^{\infty} C_n^\star(t)\psi_n(v). \tag{93}$$

Accordingly, the initial condition is set through the relation

$$\sum_{n=0}^{\infty} C_{n,0} H_n(v) e^{-v^2} = \sum_{n=0}^{\infty} C_{n,0}^\star \psi_n(v) = f_0(v). \tag{94}$$

To derive the system of equations for the coefficients $C_n^\star$, we multiply (86) by ψ^m (see (15)) and integrate in v over $\mathbb{R}$. All integrals can easily be computed using the orthogonality properties (see (18)). In view of the expansion (93), we have that

$$
\begin{aligned}
0 &= \int_{\mathbb{R}} \left(\frac{\partial f}{\partial t} - \frac{\partial f}{\partial v} \right) \psi^m \, dv = \sum_{n=0}^{\infty} \dot{C}_n^\star(t) \int_{\mathbb{R}} \psi_n \psi^m \, dv - \sum_{n=0}^{\infty} C_n^\star(t) \int_{\mathbb{R}} \frac{d\psi_n}{dv} \psi^m \, dv \\
&= \dot{C}_m^\star(t) - \sum_{n=0}^{\infty} C_n^\star(t) \int_{\mathbb{R}} \frac{d\psi_n}{dv} \psi^m \, dv,
\end{aligned} \tag{95}
$$

where the upper dot indicates the derivative with respect to t. The equation for each coefficient $C_n^\star(t)$ can be recovered by reformulating $d\psi_n/dv$ in terms of the basis functions ψ_n and using the orthogonality against ψ^m. We distinguish the AW and the SW cases in the following subsections.

5.1. *Symmetrically-weighted case*

To ease the notation in the developments of this section, we continue using the symbol γ_n instead of γ_n^{SW}, which is defined in (16). For $n \geq 1$, using (7)-(8), we compute $d\psi_n/dv$ as follows

$$
\begin{aligned}
\frac{d\psi_n}{dv} &= \frac{d}{dv} \left(\gamma_n H_n e^{-v^2/2} \right) = \gamma_n \left(H_n' - vH_n \right) e^{-v^2/2} = \gamma_n \left(\frac{1}{2} H_n' + \frac{1}{2} H_n' - vH_n \right) e^{-v^2/2} \\
&= \gamma_n \left(nH_{n-1} - \frac{1}{2} H_{n+1} \right) e^{-v^2/2} = \frac{n\gamma_n}{\gamma_{n-1}} \psi_{n-1} - \frac{1}{2} \frac{\gamma_n}{\gamma_{n+1}} \psi_{n+1}.
\end{aligned} \tag{96}
$$

Thus, equation (95) implies that

$$
\dot{C}_n^\star(t) = \frac{(n+1)\gamma_{n+1}}{\gamma_n} C_{n+1}^\star(t) - \frac{1}{2} \frac{\gamma_{n-1}}{\gamma_n} C_{n-1}^\star(t), \tag{97}
$$

that we supplement with the initial condition $C_n^\star(0) = C_{n,0}^\star$. Equivalently, by getting rid of the normalizing factors, one has for $n \geq 1$

$$
\dot{C}_n(t) = (n+1) C_{n+1}(t) - \frac{1}{2} C_{n-1}(t), \tag{98}
$$

with the (obvious) initial condition $C_n(0) = C_{n,0}$. The case $n = 0$ can be treated separately, by observing that

$$
\begin{aligned}
\frac{d\psi_0}{dv} &= \frac{d}{dv} \left(\gamma_0 H_0 e^{-v^2/2} \right) = -\gamma_0 v e^{-v^2/2} = -\frac{\gamma_0}{2\gamma_1} \left(\gamma_1 \, 2v e^{-v^2/2} \right) = -\frac{\gamma_0}{2\gamma_1} \left(\gamma_1 H_1 e^{-v^2/2} \right) \\
&= -\frac{\gamma_0}{2\gamma_1} \psi_1 = -\frac{1}{\sqrt{2}} \psi_1 \quad \Rightarrow \quad \int_{\mathbb{R}} \frac{d\psi_0}{dv} \psi^0 \, dv = 0,
\end{aligned}
$$

since $H_0(v) = 1$, $H_1(v) = 2v$, and $\gamma_0/\gamma_1 = \sqrt{2}$. Recalling (95) we finally get

$$
\dot{C}_0(t) = 0 \quad \Rightarrow \quad C_0(t) = C_{0,0} \quad \forall t. \tag{99}
$$

5.2. *Asymmetrically-weighted case*

As in the previous section, we adopt the symbol γ_n instead of γ_n^{AW}, which is defined in (17). In this case, using (7) and the definition of ψ_{n+1}, we have

$$
\frac{d\psi_n}{dv} = \frac{d}{dv} \left[\gamma_n H_n e^{-v^2} \right] = \gamma_n \left(H_n' - 2vH_n \right) e^{-v^2} = -\gamma_n H_{n+1} e^{-v^2} = -\frac{\gamma_n}{\gamma_{n+1}} \psi_{n+1}, \tag{100}
$$

which now provides the differential equation for $n \geq 1$

$$
\dot{C}_n^\star(t) = -\frac{\gamma_{n-1}}{\gamma_n} C_{n-1}^\star(t), \tag{101}
$$

supplemented with the initial condition $C^\star_n(0) = C^\star_{n,0}$. This is equivalent to

$$\dot{C}_n(t) = -C_{n-1}(t). \tag{102}$$

Moreover we have the initial conditions $C_n(0) = C_{n,0}$, hence, $C_0(t) = C_{0,0}$ for every $t \geq 0$. For $n = 0$ we have the same situation as in (99).

We continue by providing the explicit solution to the above system of equations. For instance, when $n = 1$, we need to solve

$$\dot{C}_1(t) = -C_0(t) \quad \Rightarrow \quad C_1(t) = C_{1,0} - C_{0,0}t. \tag{103}$$

Clearly, this coefficient grows in magnitude with t. By successive integrations, one can prove that the n-th coefficient behaves as t^n. In practice, it is possible to find numbers $\alpha_\ell^{(n)}$ in such a way that

$$C_n(t) = \gamma_n C^\star_n(t) = \sum_{\ell=0}^{n} \alpha_\ell^{(n)} t^\ell, \tag{104}$$

which is clearly unbounded for t tending to infinity. We already proved that the advection problem in the AW case is not absolutely stable in the $L^2(\mathbb{R})$-weighted norm. The result in (104) confirms this statement. A way to stabilize the approximation scheme is to introduce some numerical dissipation, though this may not be the only option. We will study this approach in the next section.

6. The advection equation with the stabilization term in the AW case

Since we know that the Lenard-Bernstein combined operator has a dissipative nature, we consider a modified advection equation with such an operator at the right-hand side. We aim at investigating how this modification may impact in the stability of the AW case (remind that the SW case is already stable with no need of dissipation). We start with $k = 1$, so that adding the second-order operator $\nu\widetilde{L}L$ to the right-hand side of the advection equation yields

$$\frac{\partial f}{\partial t} - \frac{\partial f}{\partial v} = \nu\widetilde{L}Lf. \tag{105}$$

The stability of the AW Hermite discretization is stated by the next theorem.

Theorem 6.1 (Stability for the AW case with dissipation) *Let $f(v,t) = h(v,t)e^{-v^2}$, with h polynomial in v, be the solution of the differential problem* (105) *with the initial condition* (87). *Then, it holds that*

$$\int_{\mathbb{R}} h^2 e^{-v^2}\, dv \leq \int_{\mathbb{R}} h_0^2\, dv = \int_{\mathbb{R}} h(v,0)^2\, dv, \tag{106}$$

provided $\nu > \sqrt{2}$.

Proof. We will prove that the term on the right-hand side of (105) acts like a stabilization term. To this end, we set $f = he^{-v^2}$, take h as the test function, and integrate (105) over $\mathbb{R}$. For the term $\nu\widetilde{L}Lf$ we argue as done in (27). Thus, we get

$$\int_{\mathbb{R}} \left(\frac{\partial f}{\partial t} - \frac{\partial f}{\partial v}\right) h\, dv = \nu \int_{\mathbb{R}} \left(\widetilde{L}Lf\right) h\, dv = -\frac{\nu}{2}\int_{\mathbb{R}} \left(h'\right)^2 e^{-v^2}\, dv. \tag{107}$$

After integration by parts, we apply the Young inequality (with constant σ) to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} h^2 e^{-v^2}\, dv &= -\int_{\mathbb{R}} h'he^{-v^2}\, dv - \frac{\nu}{2}\int_{\mathbb{R}} \left(h'\right)^2 e^{-v^2}\, dv \leq \left|\int_{\mathbb{R}} h'he^{-v^2}\, dv\right| - \frac{\nu}{2}\int_{\mathbb{R}} \left(h'\right)^2 e^{-v^2}\, dv \\ &\leq \frac{1}{2\sigma}\int_{\mathbb{R}} h^2 e^{-v^2}\, dv + \frac{1}{2}(\sigma - \nu)\int_{\mathbb{R}} \left(h'\right)^2 e^{-v^2}\, dv,\end{aligned} \tag{108}$$

where we used the fact that the boundary contributions from the integration by parts are zero. From the Poincarè inequality (A.1) (take $\varphi = h$) we have

$$-\frac{1}{2}\int_{\mathbb{R}} \left(h'\right)^2 e^{-v^2}\, dv \leq -\int_{\mathbb{R}} h^2 e^{-v^2}\, dv + \sqrt{\pi}C_0^2. \tag{109}$$

Assuming $\nu > \sigma$ in (108), using (109) we find that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} h^2 e^{-v^2}\, dv \leq \left(\frac{1}{2\sigma} - (\nu-\sigma)\right)\int_{\mathbb{R}} h^2 e^{-v^2}\, dv + (\nu-\sigma)\sqrt{\pi}C_0^2. \tag{110}$$

The coefficient $\big(1/(2\sigma) - (\nu-\sigma)\big)$ is negative if $\nu > \sigma + 1/(2\sigma)$. Since the minimum of $\sigma + 1/(2\sigma)$ for $\sigma > 0$ is $\sqrt{2}$ we can assume that $\nu > \sqrt{2}$. Using this choice, we find that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} h^2 e^{-v^2}\, dv \leq -\left(\nu - \sqrt{2}\right)\int_{\mathbb{R}} h^2 e^{-v^2}\, dv + (\nu-1)\sqrt{\pi}C_0^2. \tag{111}$$

Now, we consider $C_0(t) = C_0(0) = C_{0,0}$ and introduce the quantities

$$K = \frac{\nu-1}{\nu-\sqrt{2}}\sqrt{\pi}C_{0,0}^2 \qquad \text{and} \qquad Y(t) = \int_{\mathbb{R}} h^2 e^{-v^2}\, dv - K, \tag{112}$$

so we can rewrite (111) as

$$\frac{1}{2}\frac{d}{dt}Y(t) \leq -\left(\nu-\sqrt{2}\right)Y(t), \tag{113}$$

since K is constant. Note that for $t = 0$ we have

$$Y(0) = \int_{\mathbb{R}} h_0^2\, e^{-v^2}\, dv - K, \tag{114}$$

where $h_0 = h(v, 0)$, which is recovered from the expansion of the initial solution f_0. Finally, an application of the Gronwall's inequality yields

$$Y(t) \leq Y(0)\exp\left(-2\left(\nu-\sqrt{2}\right)t\right) \leq Y(0), \tag{115}$$

since the argument of the exponential is negative. Using the expression of $Y(t)$ and $Y(0)$, respectively given in (112) and (114), the condition $Y(t) \leq Y(0)$ implies

$$\int_{\mathbb{R}} h^2 e^{-v^2}\, dv \leq \int_{\mathbb{R}} h_0^2\, e^{-v^2}\, dv = \int_{\mathbb{R}} h(v,0)^2\, e^{-v^2}\, dv, \tag{116}$$

which is the stability result we were looking for. Note that $\nu > \sqrt{2}$ is a sufficient but not necessary conditions for stability. □

Concerning the case $k > 1$, a result of stability for ν sufficiently large, can be given following the same steps of the case $k = 1$. We just provide here a sketch of the proof. Thanks to (27), formula (107) can be rewritten as

$$\int_{\mathbb{R}}\left(\frac{\partial f}{\partial t} - \frac{\partial f}{\partial v}\right) h\, dv = -(-1)^k \nu \int_{\mathbb{R}}\big(\widetilde{L}^{(k)} L^{(k)} f\big)\, h\, dv = -\frac{\nu}{2^k}\int_{\mathbb{R}}\big(h^{(k)}\big)^2 e^{-v^2}\, dv. \tag{117}$$

As in (108) we apply the Schwarz and Young inequalities. Successively, we estimate the right-hand side of (117) by using (A.7) with $p = 1$ and $m = k$. Through (109), we arrive at

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} h^2 e^{-v^2}\, dv \leq \Phi_1 \int_{\mathbb{R}} h^2 e^{-v^2}\, dv + \Phi_2, \tag{118}$$

where

$$\Phi_1 = \frac{1}{2\sigma} - \nu(k-1)! + \sigma \quad \text{and} \quad \Phi_2 = \big(\nu(k-1)! - \sigma\big)\sqrt{\pi}C_0^2 + \nu(k-1)!\sqrt{\pi}\sum_{\ell=1}^{k-1} 2^\ell \frac{(\ell!)^2}{(\ell-1)!}C_\ell^2. \tag{119}$$

Now, we define

$$K = \frac{\big(\nu(k-1)! - \sigma\big)}{\nu(k-1)! - \sigma - \frac{1}{2\sigma}}\sqrt{\pi}C_0^2 \qquad \text{and} \qquad Y(t) = \int_{\mathbb{R}} h^2 e^{-v^2}\, dv - K, \tag{120}$$

so that equation (108) can be rewritten as:

$$\frac{1}{2}\frac{d}{dt}Y(t) \le \Phi_1 Y(t) + \Psi_1(t), \qquad \text{where} \qquad \Psi_1(t) = \nu(k-1)!\sqrt{\pi}\sum_{\ell=1}^{k-1} 2^\ell \frac{(\ell!)^2}{(\ell-1)!} C_\ell^2, \tag{121}$$

where we used that $C_0 = C_{0,0}$ is independent of t. The application of the Gronwall's lemma leads to the final result

$$Y(t) \le Y(0)e^{-2\Phi_1 t} + \int_0^t \Psi_1(\tau)d\tau. \tag{122}$$

Choosing, for example, $\sigma = 1$ and by taking $\nu(k-1)! > 3/2$, it is easy to get a stability estimate that generalizes (115) to any $k \ge 1$. We also note that the diffusion parameter ν is now multiplied by $(k-1)!$.

We confirm the stability result for $k = 1$ by deriving the explicit recursive formula for the expansion coefficients. To this end, we consider the right-most sum in (93) and repeat the calculation of section 5.2. We now include the stabilization term $\nu\widetilde{L}Lf$, which can be treated in the AW case with the help of Theorem 3.2 with $k = 1$, cf. (33). Thus, we get

$$\begin{aligned}
0 &= \int_{\mathbb{R}} \left(\frac{\partial f}{\partial t} - \frac{\partial f}{\partial v} - \nu\widetilde{L}Lf \right) \psi^m \, dv \\
&= \sum_{n=0}^{\infty} \dot{C}_n^\star(t) \int_{\mathbb{R}} \psi_n \psi^m \, dv - \sum_{n=0}^{\infty} C_n^\star(t) \int_{\mathbb{R}} \frac{d\psi_n}{dv}\psi^m \, dv + \nu \sum_{n=0}^{\infty} nC_n^\star(t) \int_{\mathbb{R}} \psi_n\psi^m \, dv \\
&= \dot{C}_m^\star(t) - \sum_{n=0}^{\infty} C_n^\star(t) \int_{\mathbb{R}} \frac{d\psi_n}{dv}\psi^m \, dv + \nu m C_m^\star(t).
\end{aligned} \tag{123}$$

We compute the last integral using again (100) to obtain

$$\dot{C}_n^\star(t) = -\frac{\gamma_{n-1}}{\gamma_n} C_{n-1}^\star(t) - \nu n C_n^\star(t), \tag{124}$$

which holds for $n \ge 1$, while for $n = 0$ we find that $C_0^\star(t) = C_{0,0}^\star$ is constant. We rewrite the above system of equations as follows

$$\dot{C}_n(t) = -C_{n-1}(t) - \nu n C_n(t), \tag{125}$$

which actually corresponds to (102) when $\nu = 0$. For instance, for $n = 1$, we find the ordinary differential equation

$$\dot{C}_1(t) = -C_0(t) - \nu C_1(t) = -C_{0,0} - \nu C_1(t), \tag{126}$$

the solution of which is

$$C_1(t) = \left(C_{1,0} + \frac{1}{\nu} C_{0,0} \right) e^{-\nu t} - \frac{1}{\nu} C_{0,0}. \tag{127}$$

Since ν is positive, $C_1(t)$ is clearly bounded with respect to t. It is not hard to show that, for a generic n, the expression of the n-th coefficient takes the form

$$C_n(t) = \sum_{\ell=0}^{n} \alpha_\ell^{(n)} e^{-\ell\nu t}, \tag{128}$$

where the constants $\alpha_\ell^{(n)}$ depend on n and the diffusion parameter ν. It is important to analyze such a dependence on ν. Indeed, by substituting (128) into (125) for $n \ge 1$ and $0 \le \ell \le n-1$, we deduce the recursive relation

$$\alpha_\ell^{(n)} = -\frac{1}{\nu(n-\ell)} \alpha_\ell^{(n-1)},$$

from which

$$\alpha_\ell^{(n)} = \frac{(-1)^{n-\ell}}{(n-\ell)!\,\nu^{n-\ell}} \alpha_\ell^{(\ell)}.$$

From the initial condition

$$C_{n,0} = C_n(0) = \sum_{\ell=0}^{n} \alpha_\ell^{(n)} = \alpha_n^{(n)} + \sum_{\ell=0}^{n-1} \alpha_\ell^{(n)},$$

we find the expression of $\alpha_n^{(n)}$, which is given by

$$\alpha_n^{(n)} = C_{n,0} - \sum_{\ell=0}^{n-1} \alpha_\ell^{(n)} = C_{n,0} + \sum_{\ell=0}^{n-1} \frac{(-1)^{n-\ell}}{(n-\ell)!\,\nu^{n-\ell}} \alpha_\ell^{(\ell)}.$$

For example, starting from $\alpha_0^0 = C_{0,0}$, for $n = 1$, we find that $\alpha_1^{(1)} = C_{1,0} - \alpha_0^0/\nu = C_{1,0} - C_{0,0}/\nu$. Similarly, $\alpha_2^{(2)}$ is computed from $\alpha_0^{(0)}$ and $\alpha_1^{(1)}$, and the subsequent coefficients are obtained from those already computed. One realizes that ν appears at the denominator to the n-th power. It turns out that the coefficients $C_n(t)$ in (128) are of the form $C_n(0)$ plus a dissipative term. The stronger dissipation is obtained when $\ell = 1$, which provides a contribution like $e^{-\nu t}/\nu$ (see (127)). If we do not want this dissipation to be too heavy so that the perturbation is of order ε when we integrate until the final time T, we can consider $e^{-\nu T} \approx \nu\varepsilon$ and choose $T \approx |\ln(\nu\epsilon)|/\nu$.

7. Time discretization of the 1-D problem

We study the time discretization of the system of differential equations in (125). We use an implicit conservative method in time such as the trapezoidal rule. For a time-step $\Delta t > 0$, we write for $j \geq 1$

$$\frac{C_n^j - C_n^{j-1}}{\Delta t} = -\frac{C_{n-1}^j + C_{n-1}^{j-1}}{2} - \nu n \frac{C_n^j + C_n^{j-1}}{2}, \tag{129}$$

with the initial condition $C_n^0 = C_{n,0}$, for $n \geq 1$. Instead, for $n = 0$ we have $C_0^j = C_{0,0}, \forall j \geq 0$. As an example we can make the formula explicit for $n = 1$

$$C_1^j \left(1 + \frac{\nu}{2}\Delta t\right) = C_1^{j-1} \left(1 - \frac{\nu}{2}\Delta t\right) - \Delta t C_{0,0}. \tag{130}$$

After defining $\chi_n = (1 - \frac{1}{2}\nu n \Delta t)/(1 + \frac{1}{2}\nu n \Delta t), n \geq 1$, we trivially get $|\chi_n| < 1$. By recursive arguments, one can show that the expression for C_1^j has the form of a linear combination of powers of χ_1, i.e.

$$C_1^j = \sum_{\ell=0}^{j} (\chi_1)^\ell \alpha_\ell, \tag{131}$$

where the numbers α_ℓ depend on ν and Δt. This expression is inserted in (129) in order to compute the sequence $C_2^j, \forall j \geq 0$, and so on.

We may assume that the approximate solution of (105) belongs to the space Hermite functions where the polynomials degree is less or equal to N. When n reaches the value N, the expression of the corresponding coefficients $C_N^j, \forall j \geq 0$, is a combination of all the powers $(\chi_n)^\ell$ with $1 \leq n \leq N$ and $0 \leq \ell \leq j$. Since $|\chi_N| < 1$, the discretization method is always unconditionally stable. However, a wise relation between the parameters N, ν and Δt should be set up in order to avoid unpleasant numerical effects due to the *stiffness* of the originating differential system (125) for N large. A rule of thumb is to require that the product $\nu N \Delta t$ is of the order of unity. Actually, if we analyze (128) when $n = N$, the most significant term is that given by the exponential $e^{-N\nu t}$, displaying a very steep tangent for $t = 0$. Although there are in principle no restrictions on Δt for the trapezoidal scheme, such quick variations in time are well resolved only if the time-step is maintained suitably small.

The above arguments indicate that absolute stability for the continuous problem may be expected for any $\nu > 0$, whereas in (106) the proof was only provided for $\nu > \sqrt{2}$.

Indeed, we conjecture that the stability in the L^2-weighted norm is not verified for values of ν less than a certain constant. However, it is possible to construct milder weighted norms where a result of stability can still be achieved for any ν. We formally state this result in the next theorem, which is also our main stability theorem for the AW discretization applied to the advection equation with the addition of the Lenard-Bernstein type operator.

Theorem 7.1 (Improvement of the stability theorem for the AW Hermite method) *Let f be the solution of* (105) *with the initial datum* (87). *Let $C_n(t)$ be the coefficients defined by the differential system* (125). *Then, there exists a sequence of positive weights $\{w_n\}_{n\geq 1}$ such that*

$$\sum_{n=1}^{\infty} w_n C_n^2(t) \leq \sum_{n=1}^{\infty} w_n C_n^2(0). \tag{132}$$

This inequality holds for any $\nu > 0$.

Proof. For any $n \geq 1$, we multiply (125) by C_n and use the Young inequality on the right-hand side to obtain

$$\frac{1}{2}\frac{d}{dt}C_n^2 = -C_nC_{n-1} - \nu n C_n^2 \leq \frac{1}{2\sigma_n}C_n^2 + \frac{\sigma_n}{2}C_{n-1}^2 - \nu n C_n^2. \tag{133}$$

The family of parameters $\sigma_n > 0$ will be decided later on. We multiply both sides of the inequality above by a weight $w_n > 0$ and sum over index n, so obtaining

$$\frac{1}{2}\frac{d}{dt}\sum_{n=1}^{\infty} w_n C_n^2 \leq \sum_{n=1}^{\infty}\frac{1}{2\sigma_n}w_n C_n^2 + \sum_{n=1}^{\infty}\frac{\sigma_n}{2}w_n C_{n-1}^2 - \sum_{n=1}^{\infty}\nu n w_n C_n^2. \tag{134}$$

By shifting the index in the sum containing C_{n-1} and collecting the corresponding terms under the same symbol of summation, we get

$$\frac{1}{2}\frac{d}{dt}\sum_{n=1}^{\infty} w_n C_n^2 \leq \sum_{n=1}^{\infty}\left[\Big(\frac{1}{2\sigma_n} - \nu n\Big)w_n + \frac{\sigma_{n+1}}{2}w_{n+1}\right]C_n^2 + \frac{\sigma_1}{2}w_1 C_0^2. \tag{135}$$

For example, we may consider to choose $\sigma_n = 1/(\nu n)$. Successively, we impose that the expression in the square brackets is equal to $-(\nu/4)w_n$, which implies the recursion formula

$$w_{n+1} = \nu^2(n+1)\left(n - \frac{1}{2}\right)w_n. \tag{136}$$

We assume that $w_1 = 1$, although the initial setting is irrelevant to our analysis. Hence, (135) becomes

$$\frac{1}{2}\frac{d}{dt}\sum_{n=1}^{\infty} w_n C_n^2 \leq -\frac{\nu}{4}\sum_{n=1}^{\infty} w_n C_n^2 + \frac{1}{2\nu}w_1 C_0^2. \tag{137}$$

Finally, by setting

$$Y(t) = \sum_{n=1}^{\infty} w_n [C_n(t)]^2 - \big(2/\nu^2\big)w_1[C_0(t)]^2, \tag{138}$$

we obtain

$$\frac{1}{2}Y'(t) \leq -\frac{\nu}{4}Y(t). \tag{139}$$

Thus, by applying the Gronwall's lemma, we conclude with the estimate

$$Y(t) \leq Y(0)e^{-(\nu/2)t} \leq Y(0), \tag{140}$$

for all $t \geq 0$, from which we can find the stability result (132), since the term $\big(2/\nu^2\big)w_1 C_0^2$ in (138) is independent of time and can be removed. □

As a final exercise, we would like to characterize the weights w_n of the recursive relation (136). Assuming that $w_1 = 1$, from a straightforward calculation, we find

$$w_n = \big(2\nu^2\big)^{n-1} n!\,\frac{(2n-3)!}{2^{n-2}\,(n-2)!} = 2\big(\nu^2\big)^{n-1} n(n-1)\,(2n-3)!\,. \tag{141}$$

By substituting (141) into (138), we are finally able to give an expression to the stability norm. Note that such a norm depends on ν. We can go ahead with our computations by noting that

$$w_n \geq 2\big(\nu^2\big)^{n-1} n(n-1)2^{n-2}\,(n-2)! = \frac{1}{2}\big(\nu^2\big)^{n-1} 2^n\, n!\,. \tag{142}$$

Therefore, if for example $\nu \geq 1$, and, hence, $\nu^{2n} \geq 1$, we discover that

$$Y(t) = \sum_{n=1}^{\infty} w_n C_n^2 + \frac{2}{\nu^2} C_0^2 \geq \frac{1}{2} \sum_{n=1}^{\infty} (\nu^2)^{n-1} 2^n \, n! \, C_n^2 + \frac{2}{\nu^2} C_0^2 \geq \frac{1}{2\nu^2} \sum_{n=1}^{\infty} \nu^{2n} 2^n \, n! C_n^2 + \frac{1}{2\nu^2} C_0^2$$
$$\geq \frac{1}{2\sqrt{\pi}\nu^2} \left(\sqrt{\pi} \sum_{n=0}^{\infty} 2^n \, n! \, C_n^2 \right). \tag{143}$$

Thus, if we can bound Y, we automatically bound the last term in parenthesis, which corresponds to the square of the classical L^2-weighted norm of the solution f expanded as in (93). This confirms that, if ν is sufficiently large, stability is ensured in the standard way. On the other hand, when $0 < \nu < 1$, we can only rely on the stability result involving the weights w_n, as stated in the last theorem.

Let us finally remark that, if we are in finite dimension ($n \leq N$), the norms are equivalent for any $\nu > 0$, but with constants heavily dependent on N. For example, for $\nu \leq 1$, which implies that $(\nu^2)^{n-1} \geq (\nu^2)^{N-1}$, we can write

$$Y(t) = \sum_{n=1}^{N} w_n C_n^2 + \frac{2}{\nu^2} C_0^2 \geq \frac{\left(\nu^2\right)^{N-1}}{2} \sum_{n=1}^{N} 2^n \, n! \, C_n^2 + \frac{2}{\nu^2} C_0^2 \geq \frac{\left(\nu^2\right)^{N-1}}{2\sqrt{\pi}} \left(\sqrt{\pi} \sum_{n=0}^{N} 2^n \, n! \, C_n^2 \right). \tag{144}$$

This shows that, when Y is bounded by a constant, the classical L^2-weighted norm of the solution f is bounded by that constant multiplied by a factor behaving as the inverse of ν^{2N}. Such a constant grows as $\mathcal{O}(\nu^{-2N})$ as ν approaches zero, and the stability control on the L^2-weighted norm of the solution f provided by inequality (144) is lost.

8. Full discretization of the Vlasov-Poisson equation

We consider the AW Hermite discretization of the $1D - 1D$ Vlasov-Poisson problem (1)-(2) for the distribution function $f(x, v, t) = h(x, v, t)e^{-v^2}$, stabilized by the Lenard-Bernstein-like operator of order $2k$ with $k \geq 1$. The system is completed by assigning a sufficiently regular initial solution $f(x, v, 0) = f_0(x, v)$. We assume that Ω is of the form $\Omega_x \times \Omega_v$, and we specialize the discussion to periodic boundary conditions in space, i.e., at the boundaries of Ω_x.

Some of the reasons for approaching the Vlasov problem by Hermite discretizations have been pointed out in the introduction. The AW context is the one that guarantees a large number of conservation laws, even with the addition of the diffusion term discussed so far. To discretize the Vlasov-Poisson equations in time, we integrate equation (1) with respect to the independent unknown t between t^{j-1} and t^j by applying the trapezoidal rule, whereas we evaluate equation (2) at t^j. To ease the exposition, we assume a constant time step $\Delta t = t^j - t^{j-1}$. For $j \geq 1$, the procedure yields

$$\frac{f^j - f^{j-1}}{\Delta t} + v \frac{\partial}{\partial x} \left(\frac{f^j + f^{j-1}}{2} \right) - \frac{E^j + E^{j-1}}{2} \frac{\partial}{\partial v} \left(\frac{f^j + f^{j-1}}{2} \right)$$
$$= -(-1)^k \nu \widetilde{L}^{(k)} L^{(k)} \left(\frac{f^j + f^{j-1}}{2} \right) \tag{145}$$

$$\frac{\partial E^j}{\partial x} = 1 - \int_{\Omega_v} f^j \, dv. \tag{146}$$

For $j = 0$ we impose f_0 as initial datum.

Following the guidelines of the previous section, a proof of the absolute stability in time of this scheme can be provided for a sufficiently large parameter ν. The situation gets more technically involved if ν is relatively small. We remind you that at the end of section 7 we distinguished between $\nu \geq 1$ and $\nu < 1$. In the latter case, stability is only achieved in a suitable norm, and the generalization of this proof to the Vlasov-Poisson system becomes rather complicated. We present here below a series of results.

8.1. *Restrictions on the parameters to guarantee absolute stability*

Here, our goal is to derive sufficient stability conditions that relate the time step Δt, the collisional factor ν and the degree of the approximating polynomial N. To this end, let us first introduce the quantity

$$\mathcal{M} = \max_{x\in\Omega_x} \left|E^j + E^{j-1}\right|. \tag{147}$$

From now on the analysis is not rigorous, since we will not take into consideration that the problem is actually nonlinear. Indeed, the value E^j has still to be computed, since it is strictly linked to f^j through the relation (146). We may assume that for Δt sufficiently small, $E^j \approx E^{j-1}$, though, as we said, we have no theorems that guarantee this fact. Thus, $\mathcal{M} \approx 2\max_{x\in\Omega_x}\left|E^{j-1}\right|$. Our first stability inequality reads as follows

$$\Delta t \le \frac{16\nu}{\mathcal{M}^2}. \tag{148}$$

We can get a proof by writing (145) in operator form by collecting all the terms involving the unknown variable f^j on the left-hand side, and putting all other terms that are recoverable from what is known from the previous time step into the right-hand side term g^{j-1}, i.e.

$$\left[\mathcal{I} + \frac{\Delta t}{2} v \frac{\partial}{\partial x} - \frac{\Delta t}{2}\left(\frac{E^j + E^{j-1}}{2}\right)\frac{\partial}{\partial v} + (-1)^k \frac{\Delta t}{2}\nu \widetilde{L}^{(k)} L^{(k)}\right] f^j = g^{j-1}. \tag{149}$$

We first set $f^j = h^j e^{-v^2}$. To simplify the exposition, we remove the label j from h^j and introduce the notation

$$\mathcal{A}(x) = \left(\int_{\Omega_v} h^2 e^{-v^2}\,dv\right)^{\frac{1}{2}}, \qquad \overline{\mathcal{A}} = \left(\int_{\Omega_x} \mathcal{A}^2\,dx\right)^{\frac{1}{2}}, \tag{150}$$

$$\mathcal{B}(x) = \left(\int_{\Omega_v} \left|\frac{\partial h}{\partial v}\right|^2 e^{-v^2}\,dv\right)^{\frac{1}{2}}, \qquad \overline{\mathcal{B}} = \left(\int_{\Omega_x} \mathcal{B}^2\,dx\right)^{\frac{1}{2}}. \tag{151}$$

Then, we rewrite problem (149) in weak form. To this end, we multiply (149) by the test function ϕ, integrate over $\Omega = \Omega_x \times \Omega_v$, and define the bilinear form

$$\begin{aligned} B(h,\phi) = &\int_\Omega h\phi e^{-v^2}\,dv\,dx + \frac{\Delta t}{2}\int_\Omega \phi v\left(\frac{\partial h}{\partial x}\right)e^{-v^2}\,dv\,dx \\ &- \frac{\Delta t}{4}\int_{\Omega_x}\left(E^j + E^{j-1}\right)\left[\int_{\Omega_v}\frac{\partial\left(he^{-v^2}\right)}{\partial v}\phi\,dv\right]dx + \frac{\nu\Delta t}{2^{k+1}}\int_\Omega \frac{\partial^k h}{\partial v^k}\frac{\partial^k \phi}{\partial v^k}e^{-v^2}\,dv\,dx, \end{aligned} \tag{152}$$

where the last term is obtained after successive integration by parts and using formula (22) for $L^{(k)}f$. Now, we consider the problem of finding $f = he^{-v^2}$ such that

$$B(h,\phi) = \int_\Omega g^{j-1}\phi\,dv\,dx, \tag{153}$$

for every ϕ. Both h and ϕ will be represented as a suitable expansion (finite or infinite) of Hermite polynomials. We skip the details concerning the formulation in the proper functional spaces, since this aspect is not relevant for the analysis we are carrying out in this paper.

We want the bilinear form B to be positive definite. First, we discuss the case $k = 1$, and note that the last integral in (152) can be transformed as follows

$$\frac{\nu\Delta t}{4}\int_\Omega \frac{\partial h}{\partial v}\frac{\partial h}{\partial v}e^{-v^2}\,dv\,dx = \frac{\nu\Delta t}{4}\overline{\mathcal{B}}^2. \tag{154}$$

In this way, we get

$$B(h,h) = \overline{\mathcal{A}}^2 + \int_{\Omega_v} \frac{\Delta t}{2} v \underbrace{\left(\frac{1}{2} \int_{\Omega_x} \frac{\partial h^2}{\partial x}\, dx \right)}_{=0} e^{-v^2}\, dv$$

$$- \frac{\Delta t}{4} \int_{\Omega_x} \left(E^j + E^{j-1} \right) \left[\int_{\Omega_v} \frac{\partial}{\partial v} \left(h e^{-v^2} \right) h\, dv \right] dx + \nu \frac{\Delta t}{4} \overline{\mathcal{B}}^2, \tag{155}$$

where we noted that the integral of $\partial h^2/\partial x$ over Ω_x is zero because we assumed periodicity in space. We successively integrate by parts the third term on the right

$$- \frac{\Delta t}{4} \int_{\Omega_x} \left(E^j + E^{j-1} \right) \left[\int_{\Omega_v} \frac{\partial}{\partial v} \left(h e^{-v^2} \right) h\, dv \right] dx = \frac{\Delta t}{4} \int_{\Omega_x} \left(E^j + E^{j-1} \right) \left[\int_{\Omega_v} h \frac{\partial h}{\partial v} e^{-v^2}\, dv \right] dx. \tag{156}$$

An evaluation of Δt is practically possible in view of our assumption that $\mathcal{M} \approx 2 \max_{x \in \Omega_x} \left| E^{j-1} \right|$, and noting that $\mathcal{A} < \overline{\mathcal{A}}$ and $\mathcal{B} < \overline{\mathcal{B}}$. As a matter of fact we estimate (156) by applying the Schwartz and Young inequalities as follows

$$- \frac{\Delta t}{4} \int_{\Omega_x} \left(E^j + E^{j-1} \right) \left[\int_{\Omega_v} h \frac{\partial h}{\partial v} e^{-v^2}\, dv \right] dx$$

$$\geq - \frac{\Delta t}{4} \int_{\Omega_x} \left| E^j + E^{j-1} \right| \left[\int_{\Omega_v} h^2 e^{-v^2}\, dv \right]^{\frac{1}{2}} \left[\int_{\Omega_v} \left(\frac{\partial h}{\partial v} \right)^2 e^{-v^2}\, dv \right]^{\frac{1}{2}} dx$$

$$\geq - \frac{\Delta t}{4} \mathcal{M} \int_{\Omega_x} \mathcal{A}\mathcal{B}\, dx \geq - \frac{\Delta t}{4} \mathcal{M} \int_{\Omega_x} \left(\frac{\sigma}{2} \mathcal{A}^2 + \frac{1}{2\sigma} \mathcal{B}^2 \right) dx = - \frac{\Delta t}{4} \mathcal{M} \left(\frac{\sigma}{2} \overline{\mathcal{A}}^2 + \frac{1}{2\sigma} \overline{\mathcal{B}}^2 \right), \tag{157}$$

where $\sigma > 0$ is an arbitrary parameter. Using this estimate in (155), we find the inequality

$$B(h,h) \geq \overline{\mathcal{A}}^2 + \nu \frac{\Delta t}{4} \overline{\mathcal{B}}^2 - \frac{\Delta t}{4} \mathcal{M} \left(\frac{\sigma}{2} \overline{\mathcal{A}}^2 + \frac{1}{2\sigma} \overline{\mathcal{B}}^2 \right). \tag{158}$$

To derive sufficient conditions for the positivity of the bilinear form B, i.e., $B(h,h) \geq 0$, we can proceed in different ways. First, we may impose that

$$\frac{\Delta t}{4} \mathcal{M} \left(\frac{\sigma}{2} \overline{\mathcal{A}}^2 + \frac{1}{2\sigma} \overline{\mathcal{B}}^2 \right) \leq \overline{\mathcal{A}}^2 + \nu \frac{\Delta t}{4} \overline{\mathcal{B}}^2. \tag{159}$$

We then suggest to set $1/\sigma^2 = \nu \Delta t/4$, or, equivalently $\sigma = 2/\sqrt{\nu \Delta t}$. With this value of σ inequality (159) becomes

$$\frac{\Delta t}{4} \mathcal{M} \left(\frac{\sigma}{2} \overline{\mathcal{A}}^2 + \frac{1}{2\sigma} \overline{\mathcal{B}}^2 \right) = \frac{\Delta t \mathcal{M}}{4\sqrt{\nu \Delta t}} \left(\overline{\mathcal{A}}^2 + \nu \frac{\Delta t}{4} \overline{\mathcal{B}}^2 \right) \leq \left(\overline{\mathcal{A}}^2 + \nu \frac{\Delta t}{4} \overline{\mathcal{B}}^2 \right), \tag{160}$$

from which we immediately have the condition

$$\frac{\Delta t \mathcal{M}}{4\sqrt{\nu \Delta t}} \leq 1, \tag{161}$$

that with little manipulation brings us to (148). The constraint (148) constitutes a sufficient condition to realize the invertibility of problem (149) for $k = 1$. Unfortunately, we are unable to provide a similar result in the case when $k > 1$. The problem is that inequality (159) becomes of the form

$$\frac{\Delta t}{4} \mathcal{M} \left(\frac{\sigma}{2} \overline{\mathcal{A}}^2 + \frac{1}{2\sigma} \overline{\mathcal{B}}^2 \right) \leq \overline{\mathcal{A}}^2 + \frac{\nu \Delta t}{2^{k+1}} \int_{\Omega} \left(\frac{\partial^k h}{\partial v^k} \right)^2 e^{-v^2}\, dv\, dx. \tag{162}$$

We could bound $\overline{\mathcal{B}}$ (that only contains first derivatives) by an expression containing higher order derivatives through a Poincaré type inequality, where an appropriate number of low modes of h is constrained to be zero. This would be certainly not in line with the finality in this paper, which is aimed to preserve the low modes without imposing any restriction to them.

To recover an alternative estimate of the time step Δt that does not involve the diffusion parameter ν, we suppose that h is a linear combination of a finite number of Hermite polynomials with degree less than or equal to N. Thus, our second stability condition reads as follows

$$\Delta t \leq \frac{4}{\mathcal{M}\sqrt{2N}}, \tag{163}$$

for any positive value of ν and any $k \geq 1$. In practice, when h is a finite sum, we can rely on the inverse type inequality

$$\overline{\mathcal{B}} \;\leq\; \sqrt{2N}\,\overline{\mathcal{A}}, \tag{164}$$

which is easily deducible from (A.9). Thus, to control the last term at the end of (158) we proceed by writing

$$-\frac{\Delta t}{4}\mathcal{M}\left(\frac{\sigma}{2}\overline{\mathcal{A}}^2 + \frac{1}{2\sigma}\overline{\mathcal{B}}^2\right) \geq -\frac{\Delta t}{4}\mathcal{M}\left(\frac{\sigma}{2} + \frac{N}{\sigma}\right)\overline{\mathcal{A}}^2 = -\frac{\Delta t}{4}\mathcal{M}\sqrt{2N}\,\overline{\mathcal{A}}^2, \tag{165}$$

where we noticed that the absolute value of the term in the middle is minimized by the choice $\sigma = \sqrt{2N}$. In this way, the positivity of the bilinear form is realized by requiring that the last term in (165) is less than $\overline{\mathcal{A}}^2 + \frac{1}{4}\nu\Delta t\overline{\mathcal{B}}^2$. This is true if we actually assume (163). Moreover, this calculation does not involve any explicit expression from the Lenard-Bernestein diffusion operators on the right-hand side of (152) since this term is just eliminated because of its positivity for $\phi = h$. This means that this time the relation between N, Δt, and $\mathcal{M}$ holds for any value of $k \geq 1$.

We can make further considerations by putting together the inequalities (148) and (163). If Δt is chosen to be consistent with both of them, we get

$$\Delta t \approx \frac{16\nu}{\mathcal{M}^2} \quad \text{and} \quad \Delta t \approx \frac{4}{\mathcal{M}\sqrt{2N}} \quad \Rightarrow \quad \nu \approx \frac{\mathcal{M}}{4\sqrt{2N}}. \tag{166}$$

Similarly,by choosing the parameters in such a way that $1/\mathcal{M} = \sqrt{2N}\Delta t/4$, we recover from the above relations

$$\Delta t \approx 16\nu\frac{1}{\mathcal{M}^2} = 16\nu\frac{2N\Delta t^2}{16} = 2N\nu\Delta t^2, \tag{167}$$

from which we derive the estimate

$$\nu N\Delta t \approx \frac{1}{2}. \tag{168}$$

The last relation agrees with the suggestion, made at the end of section 6, that the product $N\nu\Delta t$ should be of order of the unity.

We can say something more if the electric field is treated explicitly, i.e.: $(E^j + E^{j-1})/2 \approx E^{j-1}$, with $\mathcal{M} = 2\max_{x\in\Omega_x}|E^{j-1}|$. The maximum norm can be bounded through the first derivative. This is done in the following way

$$\mathcal{M}^2 \leq |\Omega_x|\int_{\Omega_x}\left(\frac{\partial E^{j-1}}{\partial x}\right)^2 dx = |\Omega_x|\int_{\Omega_x}\left(2 - \int_{\Omega_v} f^{j-1}dv\right)^2 dx, \tag{169}$$

where $|\Omega_x|$ denotes the measure of Ω_x. Next, we use a standard inequality and the Schwartz inequality to obtain

$$\begin{aligned}\frac{\mathcal{M}^2}{|\Omega_x|} &\leq \int_{\Omega_x}\left(2 - \int_{\Omega_v} f^{j-1}dv\right)^2 dx \leq 2\int_{\Omega_x}\left[4 + \left(\int_{\Omega_v} f^{j-1}dv\right)^2\right] dx \\ &\leq 8|\Omega_x| + \int_{\Omega_x}\left[\int_{\Omega_v}(f^{j-1})^2 e^{v^2}dv\int_{\Omega_v} e^{-v^2}dv\right] dx \\ &\leq 8|\Omega_x| + \sqrt{\pi}\int_{\Omega_x}\int_{\Omega_v}(h^{j-1})^2 e^{-v^2}dvdx = 8|\Omega_x| + \sqrt{\pi}\mathcal{H},\end{aligned} \tag{170}$$

where we denoted the last integral by $\mathcal{H}$. We consider again inequality (158) from which we remove the nonnegative term $\nu\Delta t\overline{\mathcal{B}}^2/4$ to obtain a sufficient condition that is independent of ν. Using (170) in the right-hand side of (165), we end up with

$$\frac{\Delta t}{4}|\Omega_x|^{1/2}\left(8|\Omega_x| + \sqrt{\pi}\mathcal{H}\right)^{1/2}\sqrt{2N}\overline{\mathcal{A}}^2 \leq \overline{\mathcal{A}}^2, \tag{171}$$

which implies

$$\Delta t \left[|\Omega_x|^{1/2} \left(8|\Omega_x| + \sqrt{\pi}\mathcal{H} \right)^{1/2} \right] \leq \frac{4}{\sqrt{2N}}, \tag{172}$$

This last condition is substantially similar to (163). However, this derivation implies that having a knowledge of either $\mathcal{M}$ or $\mathcal{H}$ at the step $j-1$, we have an idea on how to set up the new time-step for the successive iteration.

8.2. *Conservation properties for the full approximation of the Vlasov-Poisson system*

In the final part of our study, we put together what we have learned in the previous sections, and investigate the interplay between time stability and conservation properties. We consider the conservation of the mass, which is the zero-th order moment of the Vlasov distribution function f.

Theorem 8.1 *For any $k \geq 1$, the solution f of* (145)-(146) *conserves the total mass, i.e.*

$$\frac{d}{dt} \int_{\Omega_x \times \Omega_v} f(x,v,t)\, dxdv = 0. \tag{173}$$

Proof. After discretization in time, we assume that f^j is expanded as

$$f^j(x,v) = \sum_{n=0}^{\infty} C_n^{\star,j}(x)\psi_n(v). \tag{174}$$

The variational formulation for the coefficients $C_n^{\star,j}$ is obtained by substituting (174) in (145), multiplying by the test function ψ^m and integrating on $\Omega = \Omega_x \times \Omega_v$

$$\begin{aligned}
&\sum_{n=0}^{\infty} \left[\int_\Omega \frac{C_n^{\star,j} - C_n^{\star,j-1}}{\Delta t} \psi_n \psi^m dxdv \right] + \sum_{n=0}^{\infty} \left[\int_\Omega \frac{\partial}{\partial x} \left(\frac{C_n^{\star,j} + C_n^{\star,j-1}}{2} \right) v \psi_n \psi^m dxdv \right] \\
&\quad - \sum_{n=0}^{\infty} \left[\int_\Omega \frac{E^j + E^{j-1}}{2} \; \frac{C_n^{\star,j} + C_n^{\star,j-1}}{2} \frac{\partial \psi_n}{\partial v} \psi^m dxdv \right] \\
&\quad + (-1)^k \nu \sum_{n=0}^{\infty} \left[\int_\Omega \widetilde{L}^{(k)} L^{(k)} \left(\frac{C_n^{\star,j} + C_n^{\star,j-1}}{2} \right) \psi_n \psi^m dxdv \right] = 0.
\end{aligned} \tag{175}$$

We separate the integration with respect to x from that with respect to v, obtaining

$$\begin{aligned}
&\int_{\Omega_x} \frac{C_m^{\star,j} - C_m^{\star,j-1}}{\Delta t} dx + \sum_{n=0}^{\infty} \left[\int_{\Omega_x} \frac{\partial}{\partial x} \left(\frac{C_n^{\star,j} + C_n^{\star,j-1}}{2} \right) dx \int_{\Omega_v} v \psi_n \psi^m dv \right] \\
&\quad + \frac{\gamma_m}{\gamma_{m+1}} \int_{\Omega_x} \frac{E^j + E^{j-1}}{2} \; \frac{C_{m-1}^{\star,j} + C_{m-1}^{\star,j-1}}{2} dx \\
&\quad - m(m-1)\cdots(m-k+1)\nu \int_{\Omega_x} \frac{C_m^{\star,j} + C_m^{\star,j-1}}{2}\, dx = 0.
\end{aligned} \tag{176}$$

We further note that, due to the periodic boundary conditions, the integral in the variable x of the second term is zero. By adjusting the normalizing coefficients, equation (176) becomes

$$\begin{aligned}
&\int_{\Omega_x} \frac{C_n^j - C_n^{j-1}}{\Delta t} dx + \sqrt{\frac{n+1}{n}} \int_{\Omega_x} \frac{E^j + E^{j-1}}{2} \; \frac{C_{n-1}^j + C_{n-1}^{j-1}}{2} dx \\
&\quad - n(n-1)\cdots(n-k+1)\nu \int_{\Omega_x} \frac{C_n^j + C_n^{j-1}}{2}\, dx = 0.
\end{aligned} \tag{177}$$

This system is coupled with (146). As a consequence of the orthogonality, we have

$$\int_{\Omega_v} f^j \, dv = \sum_{n=0}^{\infty} \int_{\Omega_v} C_n^j H_n e^{-v^2} = \sqrt{\pi}\, C_0^j. \tag{178}$$

Thus, the discretized Poisson equation takes the form:

$$\frac{\partial E^j}{\partial x} = 1 - \sqrt{\pi}\, C_0^j. \tag{179}$$

By integrating this last relation with respect to x and using the boundary conditions for E^j, we discover that $\int_{\Omega_x} C_0^j dx$ is constant for all $j \geq 0$. This is maintained by the scheme (177), whatever is $k \geq 1$. $\square$

More in general, conservation of momenta $\int_\Omega v^m f^j \, dxdv$, $j \geq 0$, is guaranteed up to $m \leq k-1$. This corresponds to the generalization for arbitrary k of the conservation properties that were proven in [5] for $k = 3$.

9. Conclusions

We investigated the dissipative nature of combined Lenard-Bernstein operators that were previously proposed in the literature to stabilize the spectral Hermite approximation of the Vlasov equation for the numerical modeling of collisionless plasmas in the electrostatic limit. We proved that a suitable design of such operators leads to a stabilizing term that does not change the first lowest-order modes of the spectral expansion. This property makes it possible to define discrete numerical invariants that can be identified with meaningful physical quantities whose conservation is a must in numerical simulations, e.g. mass, momentum, and high order velocity moments of the distribution function. We first carried out our analysis on simplified one-dimensional models and prove the absolute stability in time for the spectral Hermite representations using basis functions of symmetric or asymmetric type. This problem, in the asymmetric case, has been unresolved for many years. We proved in addition that, even in presence of dissipation, the main conservation properties are maintained.

The application of this strategy to the Vlasov-Poisson model allowed us to derive several sufficient conditions for the stability of the asymmetric numerical formulation that relate the main parameters of the approximation, which are the number of the espansion terms, the time stepsize, and a scaling coefficient associated to the magnitude of the artificial dissipation.

Declarations

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Appendix A

We prove here some inequalities concerning Hermite expansions. In what follows, φ is supposed to be a function such that $\varphi = \sum_{n=0}^{\infty} C_n H_n$, where the coefficients are computed with the help of (13). We first present a particular version of the *Poincaré inequality*.

Theorem A.1 *If $\varphi = \sum_{n=0}^{\infty} C_n H_n$, then*

$$\int_{\mathbb{R}} \varphi^2 e^{-v^2}\,dv \leq \frac{1}{2}\int_{\mathbb{R}} \big(\varphi'\big)^2 e^{-v^2}\,dv + \sqrt{\pi}C_0^2. \tag{A.1}$$

The inequality can be generalized to derivatives of order $m > 1$

$$\int_{\mathbb{R}} \varphi^2 e^{-v^2}\,dv \leq \frac{1}{2^m\,m!}\int_{\mathbb{R}} \big(\varphi^{(m)}\big)^2 e^{-v^2}\,dv + \sqrt{\pi}\sum_{\ell=0}^{m-1} 2^\ell\,\ell!\,C_\ell^2. \tag{A.2}$$

Proof. The orthogonality of the first derivatives of the Hermite polynomials, equation (12), and the fact that $2n \geq 2$ for $n \geq 1$, imply

$$\begin{aligned}\int_{\mathbb{R}} \big(\varphi'\big)^2 e^{-v^2}\,dv &= \int_{\mathbb{R}} \big(\sum_{n=1}^{\infty} C_n H_n'\big)^2 e^{-v^2}\,dv = \sum_{n=1}^{\infty} C_n^2 \int_{\mathbb{R}} \big(H_n'\big)^2 e^{-v^2}\,dv \\ &= \sum_{n=1}^{\infty} C_n^2\,2n\int_{\mathbb{R}} H_n^2 e^{-v^2}\,dv \geq 2\sum_{n=1}^{\infty} C_n^2 \int_{\mathbb{R}} H_n^2 e^{-v^2}\,dv,\end{aligned} \tag{A.3}$$

where all summations start from $n = 1$ since $H_0 = 1$ and $H_0' = 0$. Successively, we add and subtract the weighted integral of the zeroth-order mode, i.e, $C_0^2 H_0^2$, to the last member of inequality (A.3) and use the expansion of φ, so to have

$$\begin{aligned}\int_{\mathbb{R}} \big(\varphi'\big)^2 e^{-v^2}\,dv &\geq 2\sum_{n=0}^{\infty} C_n^2 \int_{\mathbb{R}} H_n^2 e^{-v^2}\,dv - 2C_0^2\int_{\mathbb{R}} H_0^2 e^{-v^2}\,dv \\ &= 2\int_{\mathbb{R}} \varphi^2 e^{-v^2}\,dv - 2\sqrt{\pi}C_0^2.\end{aligned} \tag{A.4}$$

By reversing this inequality we find that

$$\int_{\mathbb{R}} \varphi^2 e^{-v^2}\,dv \leq \frac{1}{2}\int_{\mathbb{R}} \big(\varphi'\big)^2 e^{-v^2}\,dv + \sqrt{\pi}C_0^2,$$

which is the first inequality (A.1). We generalize the result to derivatives of order $m > 1$ as follows. Since $H_n^{(m)} = 0$ for $n < m$, using formulas (9) and (11), we find

$$\begin{aligned}\int_{\mathbb{R}} \big(\varphi^{(m)}\big)^2 e^{-v^2}\,dv &= \int_{\mathbb{R}} \big(\sum_{n=m}^{\infty} C_n H_n^{(m)}\big)^2 e^{-v^2}\,dv = \sum_{n=m}^{\infty} C_n^2 \int_{\mathbb{R}} \big(H_n^{(m)}\big)^2 e^{-v^2}\,dv \\ &= \sum_{n=m}^{\infty} C_n^2\,2^m\frac{n!}{(n-m)!}\int_{\mathbb{R}} H_n^2 e^{-v^2}\,dv \geq 2^m\,m!\sum_{n=m}^{\infty} C_n^2 \int_{\mathbb{R}} H_n^2 e^{-v^2}\,dv,\end{aligned} \tag{A.5}$$

as $n!/(n-m)! \geq m!$, when $n \geq m$. Now, we add and subtract the weighted integral of the first m modes, i.e., $\big(C_\ell H_\ell\big)^2$, $\ell = 0, \dots, m-1$, to the last member of (A.5), and use the normalization of the Hermite polynomials to find out that

$$\int_{\mathbb{R}} \big(\varphi^{(m)}\big)^2 e^{-v^2}\, dv \geq 2^m\, m! \left(\sum_{n=0}^{\infty} C_n^2 \int_{\mathbb{R}} H_n^2 e^{-v^2}\, dv - \sum_{\ell=0}^{m-1} C_\ell^2 \int_{\mathbb{R}} H_\ell^2 e^{-v^2}\, dv \right)$$

$$= 2^m\, m! \left(\int_{\mathbb{R}} \varphi^2 e^{-v^2}\, dv - \sqrt{\pi} \sum_{\ell=0}^{m-1} 2^\ell\, \ell!\, C_\ell^2 \right). \tag{A.6}$$

By reversing this inequality we easily arrive at (A.2). □

A further generalization of the previous result is provided by the following statement.

Theorem A.2 (Generalized Poincaré-type inequality) *If $\varphi = \sum_{n=0}^{\infty} C_n H_n$, then for $m > p$ one has*

$$\int_{\mathbb{R}} \big(\varphi^{(p)}\big)^2 e^{-v^2}\, dv \leq \frac{1}{2^{m-p}\,(m-p)!} \int_{\mathbb{R}} \big(\varphi^{(m)}\big)^2 e^{-v^2}\, dv + 2^p \sqrt{\pi} \sum_{\ell=p}^{m-1} 2^\ell\, \frac{(\ell!)^2}{(\ell-p)!}\, C_\ell^2. \tag{A.7}$$

Proof. By noting that $H^m = H^{(p+(m-p))} = \big(H^{(p)}\big)^{(m-p)}$, a straightforward calculation exploiting the orthogonality of the derivatives of the Hermite polynomials yields

$$\int_{\mathbb{R}} \big(\varphi^{(m)}\big)^2 e^{-v^2}\, dv = \int_{\mathbb{R}} \big(\sum_{n=m}^{\infty} C_n H_n^{(m)} \big)^2 e^{-v^2}\, dv = \int_{\mathbb{R}} \big(\sum_{n=m}^{\infty} C_n H_n^{(p+(m-p))} \big)^2 e^{-v^2}\, dv$$

$$= \sum_{n=m}^{\infty} C_n^2\, 2^{m-p} \frac{n!}{(n-(m-p))!} \int_{\mathbb{R}} \big(H_n^{(p)}\big)^2 e^{-v^2}\, dv$$

$$\geq 2^{m-p}\,(m-p)! \sum_{n=m}^{\infty} C_n^2 \int_{\mathbb{R}} \big(H_n^{(p)}\big)^2 e^{-v^2}\, dv, \tag{A.8}$$

where we also used the fact that $n!/(n-(m-p))! > (m-p)!$, for $n > 1$. We add and subtract the weighted integrals of $C_\ell^2 \big(H_\ell^{(p)}\big)^2$ for $\ell = p, \dots, p+(m-p)-1$, to the last member of (A.8) and we repeat the same argument as above until we obtain

$$\int_{\mathbb{R}} \big(\varphi^{(p)}\big)^2 e^{-v^2}\, dv \leq \frac{1}{2^{m-p}\,(m-p)!} \int_{\mathbb{R}} \big(\varphi^{(m)}\big)^2 e^{-v^2}\, dv + \sum_{\ell=p}^{m-1} C_\ell^2 \int_{\mathbb{R}} \big(H_\ell^{(p)}\big)^2 e^{-v^2}\, dv$$

$$= \frac{1}{2^{m-p}\,(m-p)!} \int_{\mathbb{R}} \big(\varphi^{(m)}\big)^2 e^{-v^2}\, dv + \sum_{\ell=p}^{m-1} C_\ell^2\, 2^p \frac{\ell!}{(\ell-p)!} \int_{\mathbb{R}} H_\ell^2 e^{-v^2}\, dv$$

$$= \frac{1}{2^{m-p}\,(m-p)!} \int_{\mathbb{R}} \big(\varphi^{(m)}\big)^2 e^{-v^2}\, dv + 2^p \sqrt{\pi} \sum_{\ell=p}^{m-1} 2^\ell\, \frac{(\ell!)^2}{(\ell-p)!}\, C_\ell^2,$$

which is our assertion. □

In particular, if φ belongs to the space of polynomials of degree at most N, we have $2n \leq 2N$, so that the relations in (A.3) can be adjusted to obtain the so called *inverse inequality*

$$\int_{\mathbb{R}} \big(\varphi'\big)^2 e^{-v^2}\, dv \leq 2N \int_{\mathbb{R}} \varphi^2 e^{-v^2}\, dv. \tag{A.9}$$

Another useful inequality is

$$\int_{\mathbb{R}} v^2 H_n^2 e^{-v^2}\, dv \leq \frac{3}{4} \int_{\mathbb{R}} \big(H_n'\big)^2 e^{-v^2}\, dv, \qquad \forall n \geq 1. \tag{A.10}$$

In order to prove it, we combine (7) and (8) to get

$$2vH_n = H_n' + H_{n+1} = 2nH_{n-1} + H_{n+1} \quad \forall n \geq 1. \tag{A.11}$$

Finally, we deduce (A.10) from the sequence of relations

$$\begin{aligned}
\int_{\mathbb{R}} v^2 H_n^2 e^{-v^2}\, dv &= \int_{\mathbb{R}} n^2 H_{n-1}^2 e^{-v^2}\, dv + \frac{1}{4}\int_{\mathbb{R}} H_{n+1}^2 e^{-v^2}\, dv \\
&= \sqrt{\pi}\left[n^2\, 2^{n-1}\,(n-1)! + \frac{1}{4}\, 2^{n+1}\,(n+1)!\right] = \sqrt{\pi}\left[2^{n-1}\, n\, n! + 2^{n-1}(n+1)n!\right] \\
&= \sqrt{\pi} 2^{n-1}(2n+1)n! \ \leq \ \sqrt{\pi}\,\frac{3}{4}\, 2^{n+1}\, n\, n! = \frac{3}{4}\int_{\mathbb{R}} \left(H_n'\right)^2 e^{-v^2}\, dv, \qquad \forall n \geq 1,
\end{aligned} \tag{A.12}$$

where we noted that $2n + 1 \leq 2n + n = 3n$. The last equality follows from (12).

The inequality (A.10) is the starting point to show that

$$\int_{\mathbb{R}} v^2 \varphi^2 e^{-v^2}\, dv \leq \frac{3}{4} N \int_{\mathbb{R}} \left(\varphi'\right)^2 e^{-v^2}\, dv, \tag{A.13}$$

which holds for every polynomial $\varphi = \sum_{n=1}^{N} C_n H_n$ with degree less or equal to N and $C_0 = 0$. We argue as follows. For a given set of values α_n, the relation here below is a consequence of the Schwartz inequality

$$\left(\sum_{n=1}^{N} \alpha_n\right)^2 = \left(\sum_{n=1}^{N} 1 \cdot \alpha_n\right)^2 \leq \sum_{n=1}^{N} 1^2 \ \sum_{n=1}^{N} \alpha_n^2 = N \sum_{n=1}^{N} \alpha_n^2. \tag{A.14}$$

With the help of the above inequality, the orthogonality of the Hermite polynomials implies

$$\begin{aligned}
\int_{\mathbb{R}} v^2 \varphi^2 e^{-v^2}\, dv &= \int_{\mathbb{R}} v^2 \Big(\sum_{n=1}^{N} C_n H_n\Big)^2 e^{-v^2}\, dv \leq N \sum_{n=1}^{N} C_n^2 \int_{\mathbb{R}} v^2 H_n^2 e^{-v^2}\, dv \\
&\leq \frac{3}{4} N \sum_{n=1}^{N} C_n^2 \int_{\mathbb{R}} \left(H_n'\right)^2 e^{-v^2}\, dv = \frac{3}{4} N \int_{\mathbb{R}} \left(\varphi'\right)^2 e^{-v^2}\, dv,
\end{aligned}$$

that actually corresponds to (A.13).