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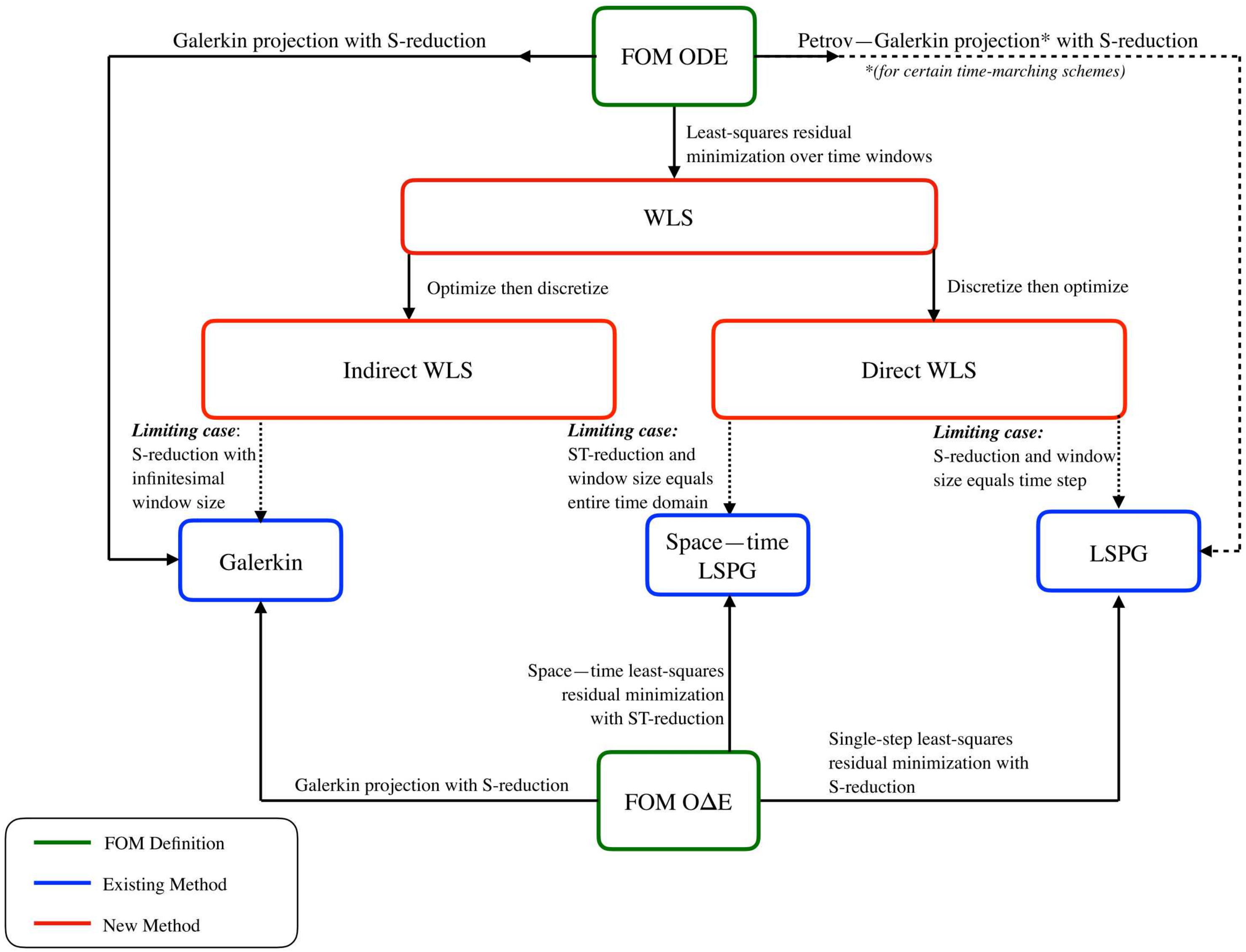


*https://arxiv.org/abs/1910.11388*

# Model Reduction via Windowed Least-Squares

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*Seattle, WA*

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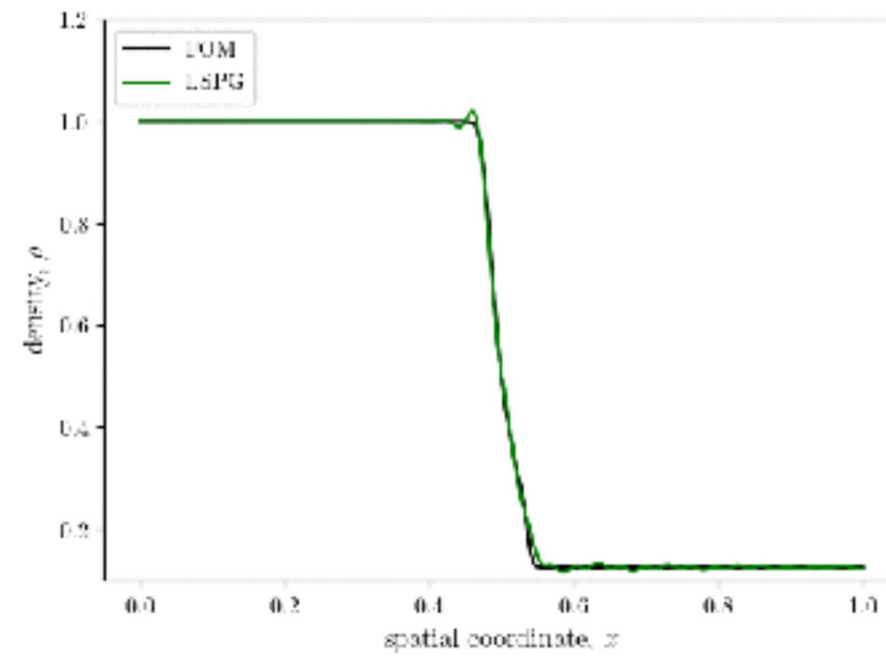
# Motivation

- **Projection-based reduced-order models:** Promising approach for time-critical and many-query problems

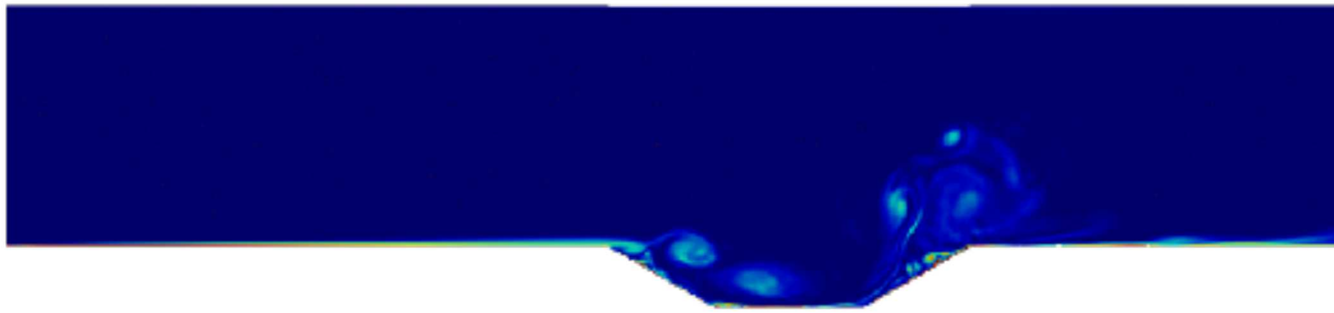
- **Projection-based ROMs:** Compute solutions to the governing equations in a low-dimensional subspace

- **Motivation for work:** current ROMs have limitations




*LSPG ROM of the Sod shock tube*

- **Require better ROM techniques**
  - Introduce the **windowed least-squares** approach

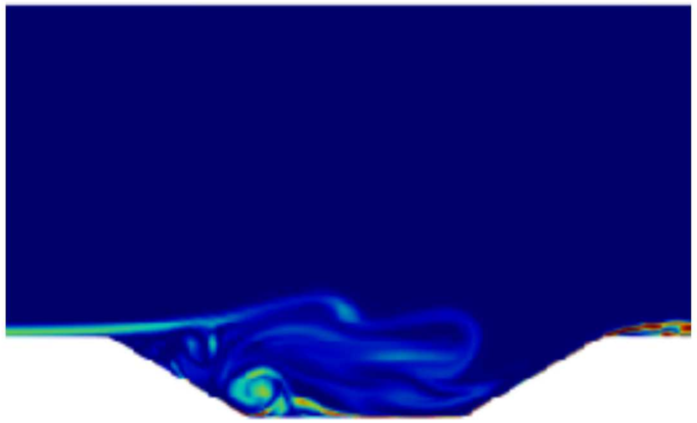


*Collocated LSPG ROM of a cavity flow*

# *Mathematical setting*

- We focus on model-reduction of the dynamical system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$$
$$\boldsymbol{x} : [0,T] \rightarrow \mathbb{R}^N$$



- Solving this system can be ***computationally expensive***
  - ***Motivates model-order reduction***

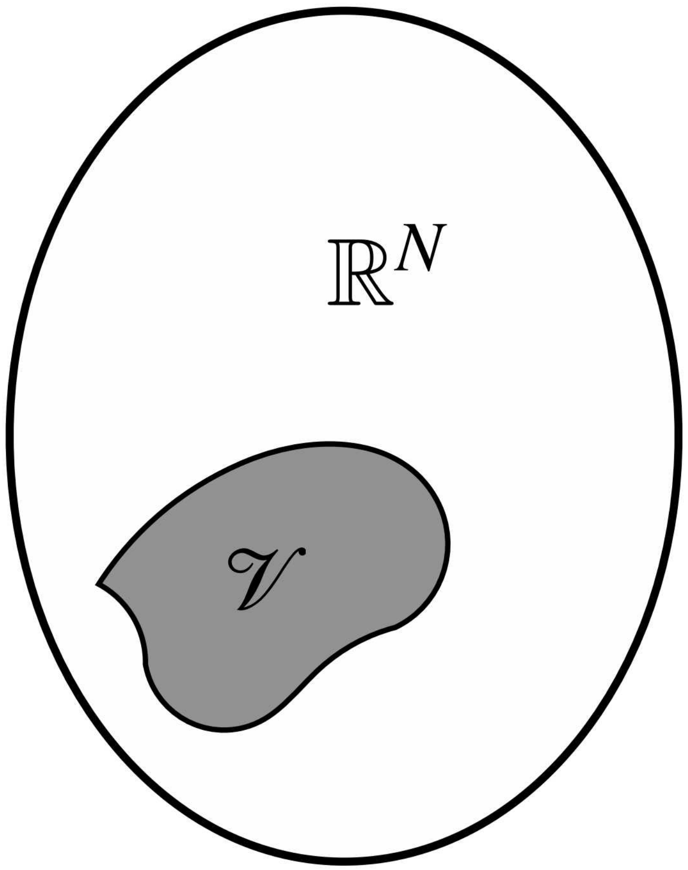
- Model-order reduction: **approximate state in a low-dimensional trial subspace and solve the system**
  - We focus on POD ROMs

# Trial subspaces



- Approximate state in low dimensional trial subspace

$$\boldsymbol{x}(t) \approx \tilde{\boldsymbol{x}}(t) \in \mathscr{V} \subseteq \mathbb{R}^N$$
$$dim(\mathscr{V}) = K, \ (K \ll N)$$

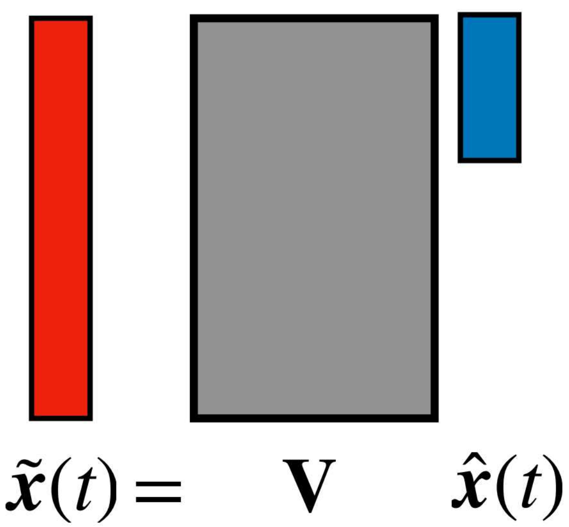



- Subspace described by orthonormal trial basis:

$$\mathbf{V} \in \mathbb{R}^{N \times K}, \mathbf{V}^T\mathbf{V} = \mathbf{I}$$
$$Range(\mathbf{V}) = \mathscr{V}$$

- Approximation can be written as

$$\tilde{\boldsymbol{x}}(t) = \mathbf{V}\hat{\boldsymbol{x}}(t)$$
$$\hat{\boldsymbol{x}} : [0, T] \rightarrow \mathbb{R}^K$$




- Dynamical system can be solved via, e.g., Galerkin, LSPG

# *Galerkin and LSPG*

- **Galerkin:** Enforces the residual to be orthogonal to the trial space

$$\dot{\hat{\boldsymbol{x}}}_G = \mathbf{V}^T \boldsymbol{f}(\mathbf{V}\hat{\boldsymbol{x}}_G)$$

- Can alternatively be cast as a minimization problem:

$$\dot{\hat{\boldsymbol{x}}}_G(t) = \underset{\hat{\mathbf{y}} \in \mathbb{R}^K}{arg\ min}\ \|\mathbf{V}\hat{\mathbf{y}} - \boldsymbol{f}(\mathbf{V}\hat{\boldsymbol{x}}_G(t))\|_2^2$$

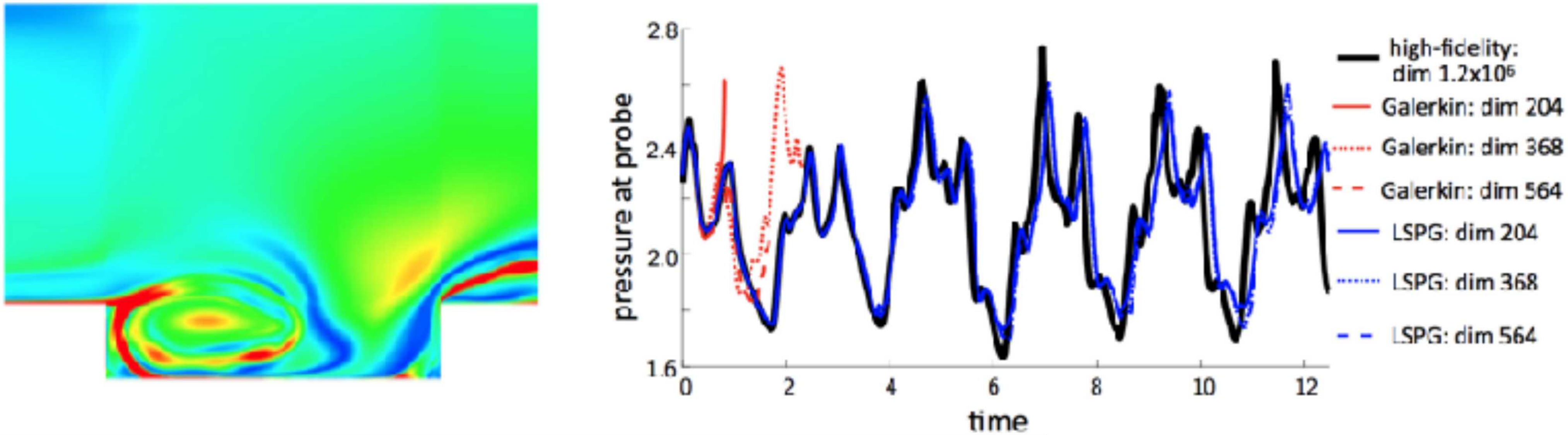
- **Galerkin minimizes the time-instantaneous ODE residual**

- **LSPG:** minimizes the time-discrete residual

$$\tilde{\mathbf{x}}^n_{LSPG} = \underset{\mathbf{y} \in \mathscr{V}}{arg\ min}\ \left\|\frac{\mathbf{y} - \mathbf{x}^{n-1}_{LSPG}}{\Delta t} - \boldsymbol{f}(\mathbf{y})\right\|_2^2$$

- **LSPG minimizes the ODE residual over a finite time window**

- LSPG can perform better than Galerkin




*K. Carlberg, M. Barone, and H. Antil, Galerkin v. least-squares Petrov-Galerkin projection in nonlinear model reduction, Journal of Computational Physics, 330 (2017), pp. 693–734.*

- LSPG is observed to be **more stable** than Galerkin
- LSPG is often **more accurate** than Galerkin
  - Minimizes the residual over a **finite time window**

- ***Problem solved?***

# *LSPG*

*K. Carlberg, M. Barone, and H. Antil, Galerkin v. least-squares Petrov-Galerkin projection in nonlinear model reduction, Journal of Computational Physics, 330 (2017), pp. 693–734.*

- LSPG is subject to several nuances:

1. Sensitivity to the time step
   - In the limit $\Delta t \rightarrow 0$, LSPG recovers Galerkin

2. Sensitivity to the time scheme
   - LSPG recovers Galerkin for explicit schemes

3. Lacks *a priori* guarantees of accuracy, stability, etc

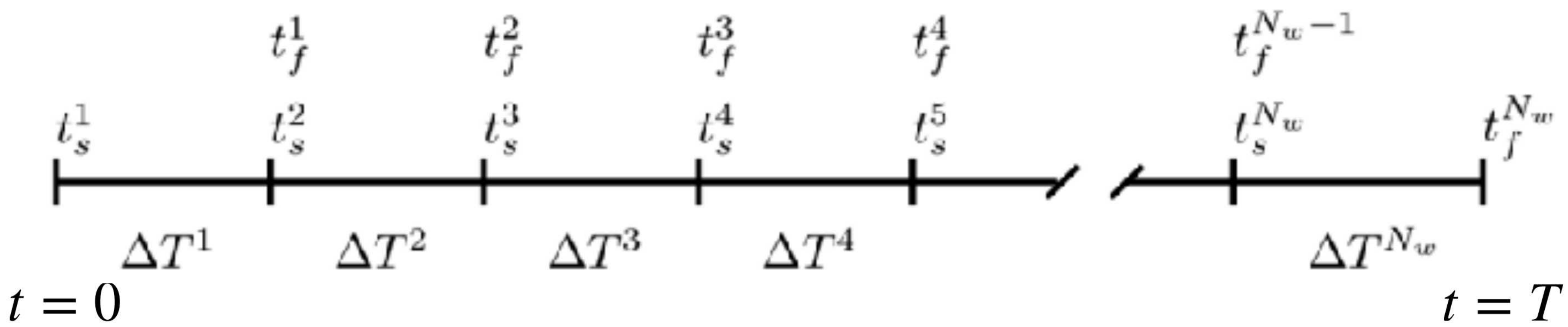
- **Motivates the need for more robust ROM techniques**

# *Windowed least-squares (WLS)*

- **Underpinning thought**: LSPG is *thought* to outperform Galerkin because it minimizes the residual over a **finite time window**
  - LSPG doesn’t directly do this
  - Byproduct of the fully discrete formulation
- **Common sense:** Formulate a model-order reduction approach that directly minimizes the time-continuous residual over a finite time window
- **Result of this is the windowed least-squares (WLS) approach**

# *WLS formulation*

- Start by partitioning $[0,T]$ into $N_w$ ***windows***




$$[t_s^n, t_f^n] \subseteq [0,T]$$
$$\Delta T^n = t_f^n - t_s^n$$
$$t_s^1 = 0,\ t_f^{N_w} = T$$

- Sequentially solve minimization problems over each window

$$\tilde{\boldsymbol{x}}^n = \underset{\boldsymbol{y} \in \mathcal{S}\mathcal{T}^n}{arg\ min} \int_{t_s^n}^{t_f^n} \|\dot{\boldsymbol{y}}(t) - \boldsymbol{f}(\boldsymbol{y}(t))\|_2^2 dt$$

- WLS computes a **function** that minimizes a **functional**
  - Minimization statement comprises a problem from the calculus of variations

# *Solving WLS*

- Two types of solution techniques are possible

  - **Indirect** (*optimize then discretize*)

    - Solve the stationary conditions (Euler—Lagrange equations)

  - **Direct** (*discretize then optimize*)

    - Discretize the objective functional and solve with, e.g., Gauss —Newton

# *Indirect approach:*
# *Euler—Lagrange equations*

- Solutions to Euler—Lagrange equations are the stationary conditions
- Euler—Lagrange gives the forward—backward problem over each window

$$\dot{\hat{\boldsymbol{x}}}^n - \mathbf{V}^T \boldsymbol{f}(\mathbf{V}\hat{\boldsymbol{x}}^n) = \boldsymbol{\lambda}^n$$

$$\hat{\boldsymbol{x}}^n(t_s^n) = \mathbf{x}_0^n$$

$$\dot{\boldsymbol{\lambda}}^n + \mathbf{V}^T \Big[\frac{\partial f}{\partial \mathbf{y}}(\mathbf{V}\hat{\boldsymbol{x}}^n)\Big]^T \mathbf{V}\boldsymbol{\lambda}^n = -\mathbf{V}^T \Big[\frac{\partial f}{\partial \mathbf{y}}(\mathbf{V}\hat{\boldsymbol{x}}^n)\Big]^T \Big(\mathbf{I} - \mathbf{V}\mathbf{V}^T\Big)\Big(\mathbf{V}\dot{\hat{\boldsymbol{x}}}^n - \boldsymbol{f}(\mathbf{V}\hat{\boldsymbol{x}}^n)\Big)$$

$$\boldsymbol{\lambda}^n(t_f^n) = \mathbf{0}$$

- Forward system is a Galerkin ROM with forcing
- Backwards system is an adjoint equation forced by the residual
- **Recovers Galerkin for infinitesimal window size**

# *Direct approach: (implicit Euler)*

- Step 1: discretize the state into $N_s$ instances
  $\boldsymbol{x}^n \rightarrow \mathbf{x}^{n,1}, \ldots, \mathbf{x}^{n,N_s}$
- Step 2: discretize the objective functional

$$\int_{t_s^n}^{t_f^n} \|\dot{\boldsymbol{x}}^n(t) - \boldsymbol{f}(\boldsymbol{x}^n(t))\|_2^2 dt \longrightarrow \sum_{i=1}^{N_s} \Delta t \| \frac{\mathbf{x}^{n,i} - \mathbf{x}^{n,i-1}}{\Delta t} - \boldsymbol{f}(\mathbf{x}^{n,i})\|_2^2$$

- Step 3: solve the optimization problem

$$\hat{\mathbf{x}}^{n,1}, \ldots, \hat{\mathbf{x}}^{n,N_s} = \underset{\hat{\mathbf{y}}^1, \ldots, \hat{\mathbf{y}}^{N_s} \in \mathbb{R}^K}{\arg\min} \sum_{i=1}^{N_s} \Delta t \| \mathbf{V} \frac{\hat{\mathbf{y}}^i - \hat{\mathbf{y}}^{i-1}}{\Delta t} - \boldsymbol{f}(\mathbf{V}\hat{\mathbf{y}}^i) \|_2^2$$

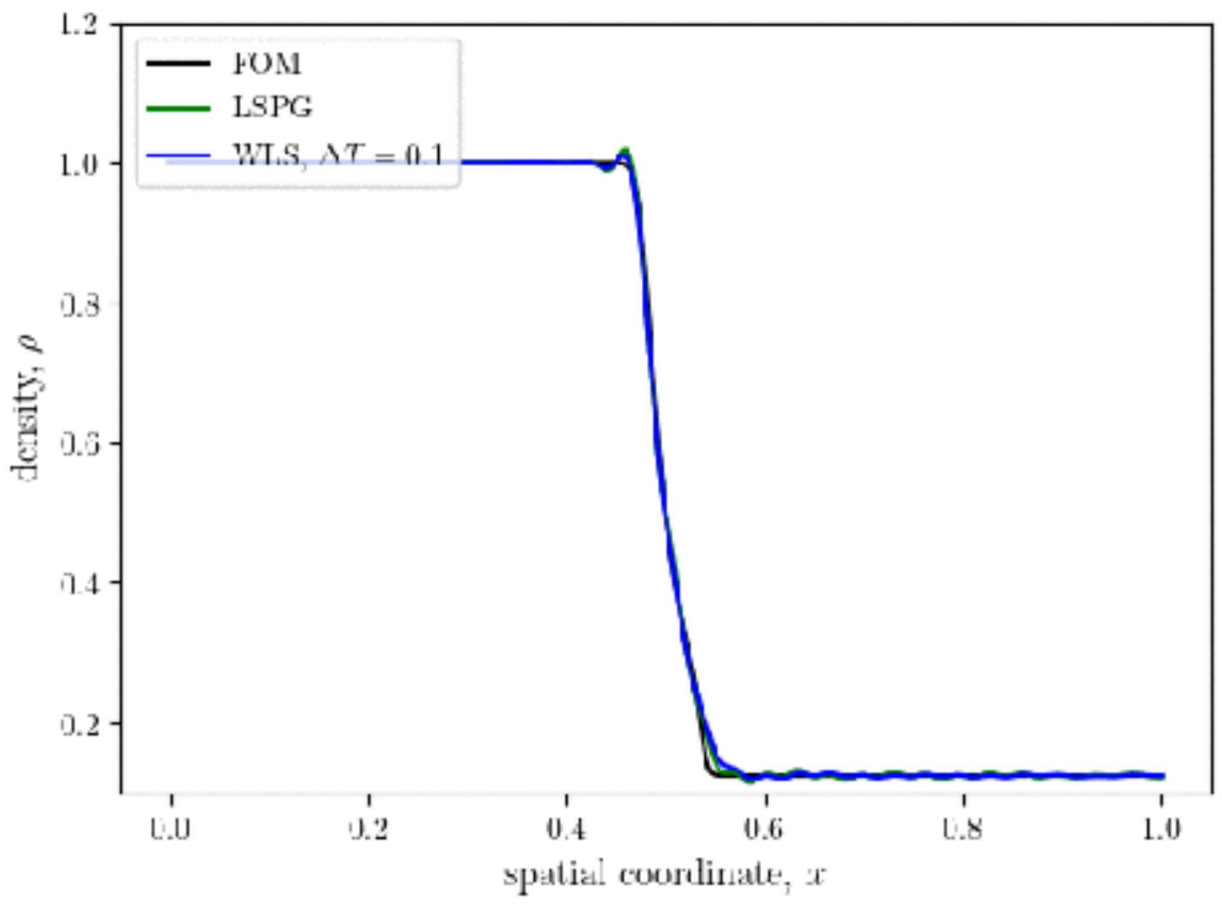
- **Recovers LSPG for window size = step size**

# *Numerical Experiments*

# *Sod shock tube*

- Reduced-order models of the SOD shock tube
  - FOM: $\Delta t = 0.002$
  - LSPG: $\Delta t = 0.002$
  - WLS: $\Delta t = 0.002, \Delta T = 0.1$ (10 windows)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$
$$\frac{\partial \rho u}{\partial t} + \frac{\partial}{\partial x}(\rho u^2 + p) = 0$$
$$\frac{\partial \rho E}{\partial t} + \frac{\partial}{\partial x}(u(\rho E + p) = 0$$
$$t \in [0,1]$$




***ROM details***

*Type*: reconstructive

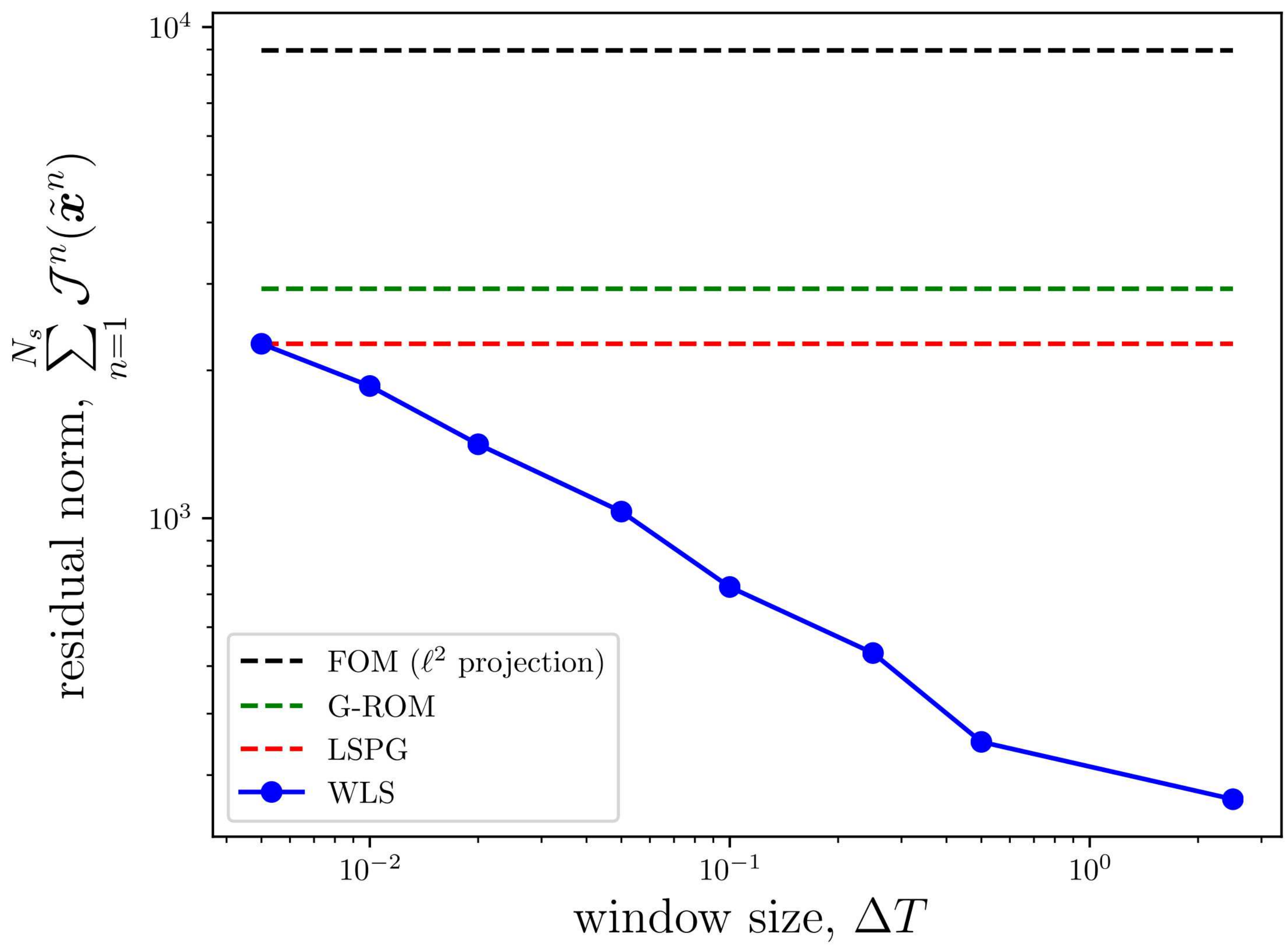
*Energy criterion*:

99.99%

*POD Modes:* 46

# *Numerical results: errors vs window size*

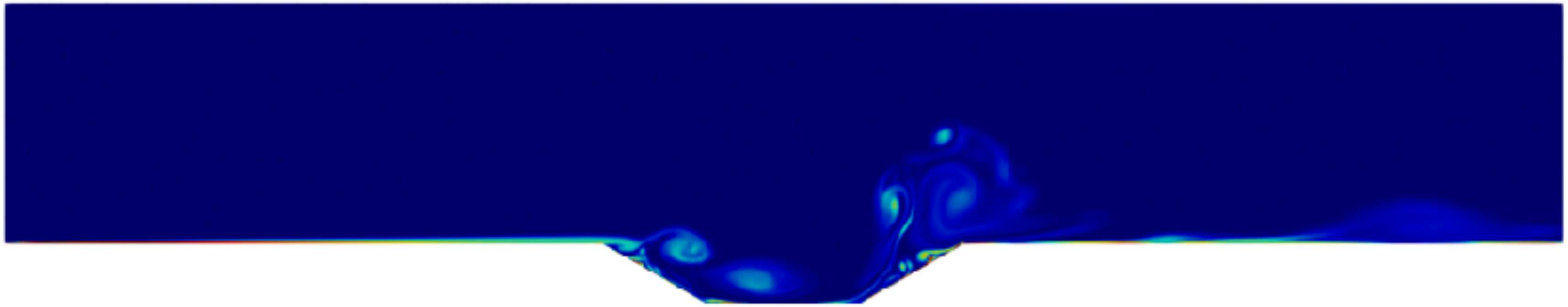
- Quantitative results:




***Residual norm***

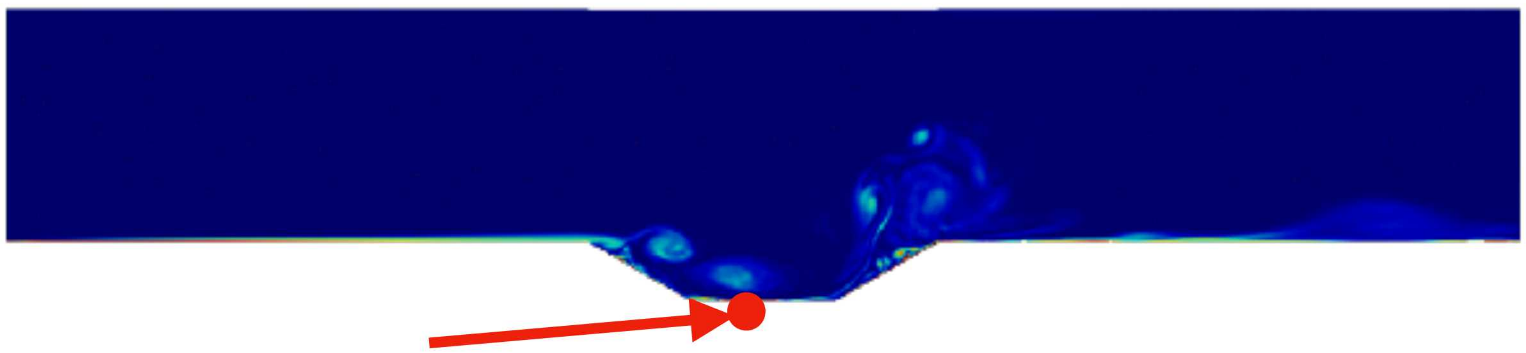
# *Cavity flow: full-order model*

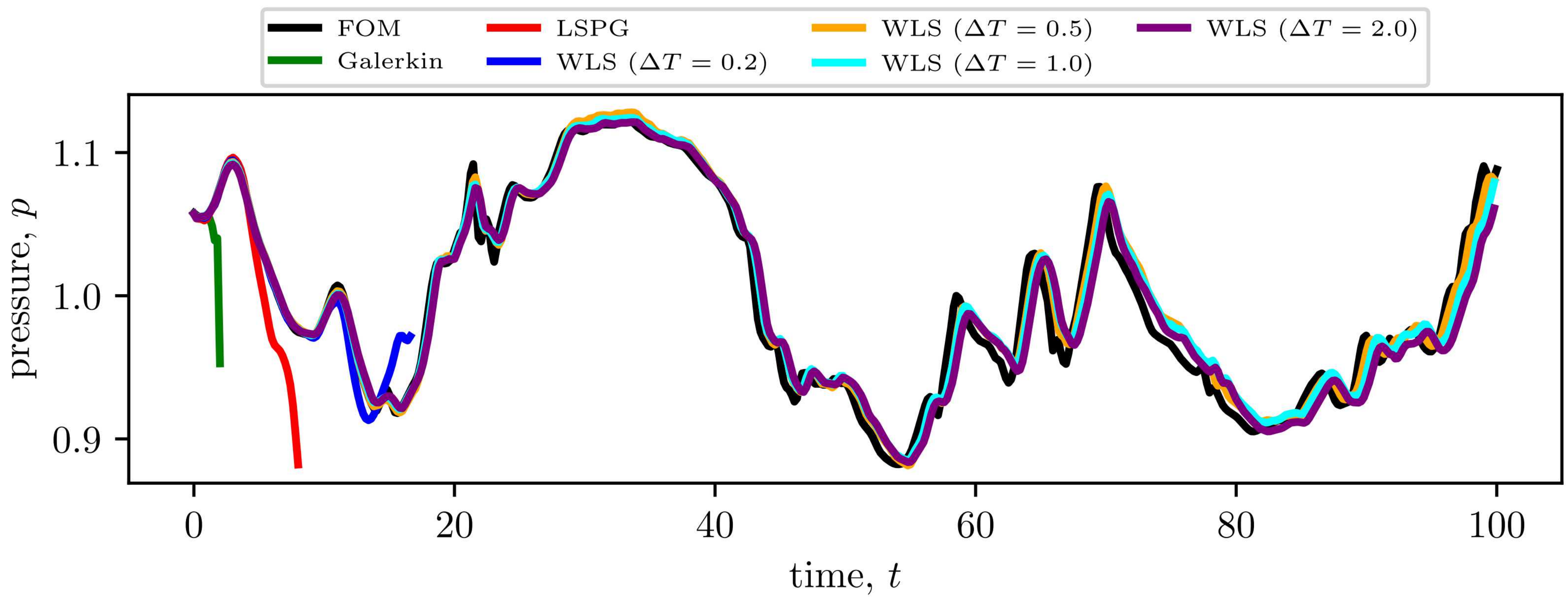
- Consider reduced-order models of a cavity flow
- Governed by the compressible Navier-Stokes equations
  - 2D (no turbulence model)
  - Re=10,000, M = 0.5
- FOM: Third-order discontinuous Galerkin discretization with 150k DOF
- Time-stepping: explicit SSP RK3



# *Cavity flow: results*

- Pressure response



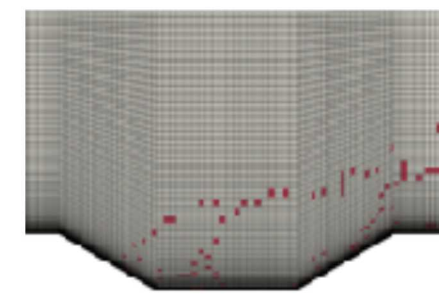



***ROM details***

*Type*: reconstructive

*Energy criterion*: 97%

*POD Modes*: 193



- **WLS accurately captures response where LSPG/Galerkin fails**

# Conclusions

- Outlined the windowed least-squares approach
  - Minimizes the time-continuous residual over windows
  - Comprises a problem from the calculus of variations
  - WLS comprises a generalization of existing approaches
  - Numerical experiments showed that WLS outperforms Galerkin and LSPG

# *Questions?*

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## Preprint

*Parish, E.J., and Carlberg, K.T., "**Windowed least-squares model-reduction for dynamical systems**" https://arxiv.org/abs/1910.11388*

## Thanks for your time!

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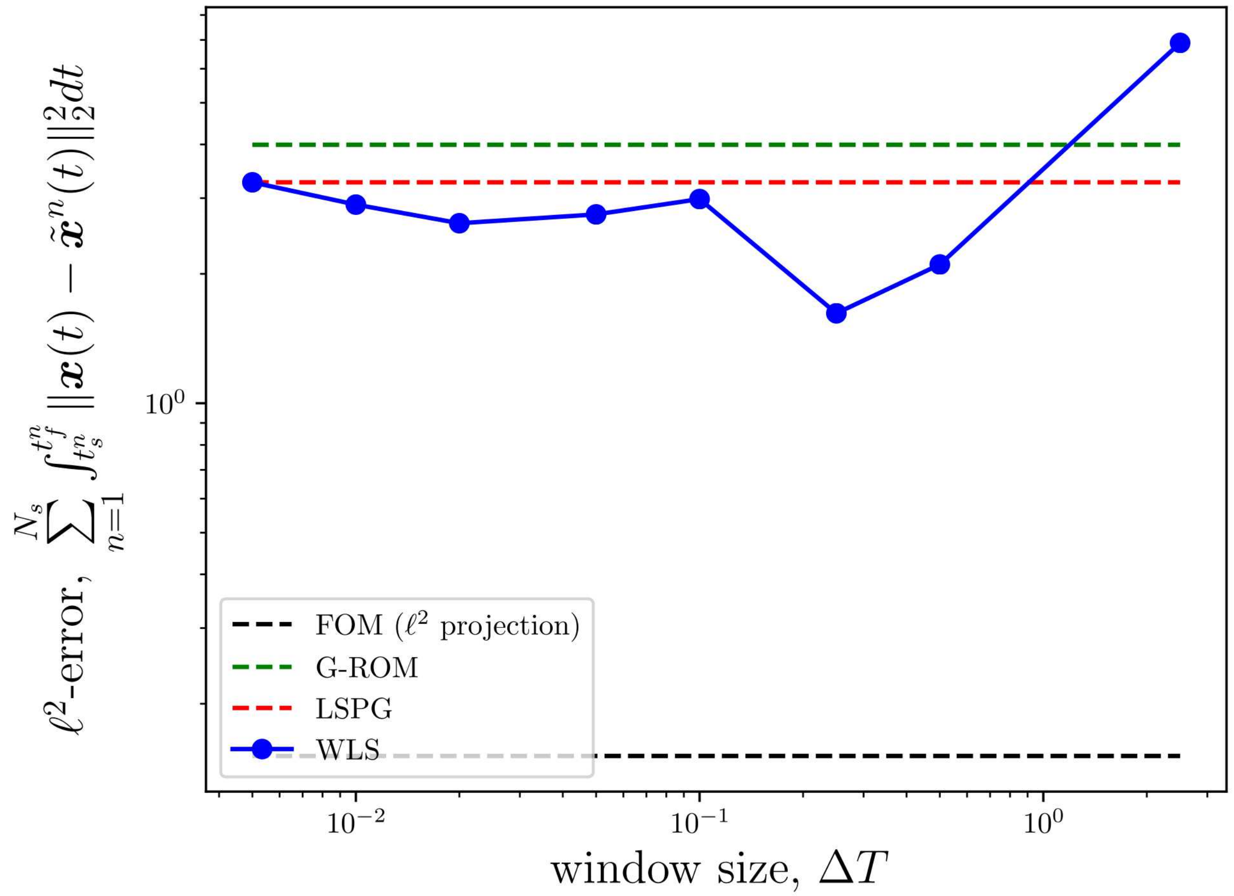
# *Numerical results: errors vs window size*

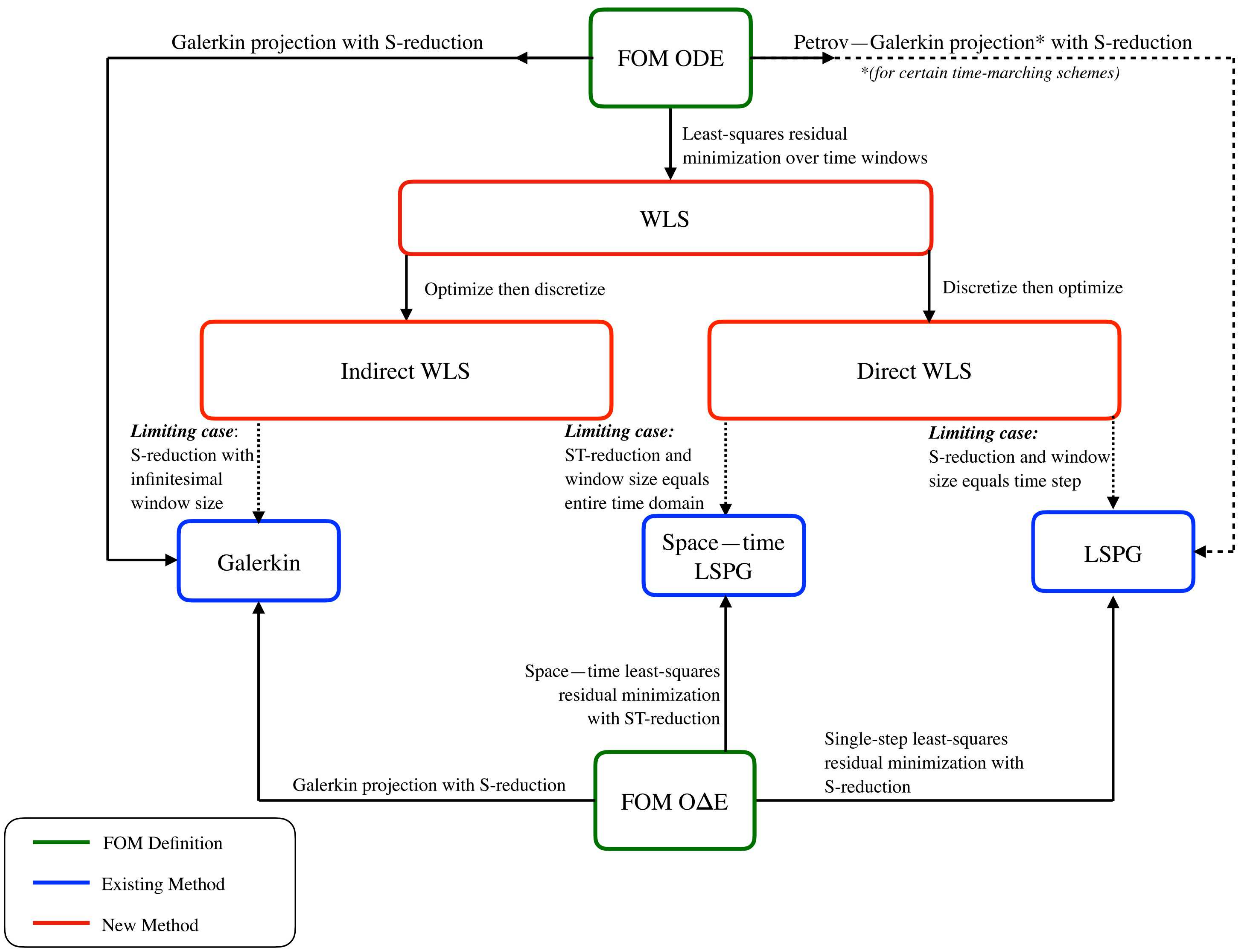
- Quantitative results:




***Error norm***

- **Lower residuals, but not necessarily lower errors!**

# Questions?



FOM ODE
Galerkin projection with S-reduction
Petrov—Galerkin projection* with S-reduction
*(for certain time-marching schemes)
Least-squares residual
minimization over time windows
WLS
Optimize then discretize
Discretize then optimize
Indirect WLS
Direct WLS
Limiting case:
S-reduction with
infinitesimal
window size
Limiting case:
ST-reduction and
window size equals
entire time domain
Limiting case:
S-reduction and window
size equals time step
Galerkin
Space—time
LSPG
LSPG
Space—time least-squares
residual minimization
with ST-reduction
Single-step least-squares
residual minimization with
S-reduction
Galerkin projection with S-reduction
FOM OΔE
FOM Definition
Existing Method
New Method