The Stabilizer Rank of the T-Gate Magic State

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SQuInT 2020

SAND2020-0959C

Stabilizer states

- given a state |Ψ>, the stabilizer group is the set of tensor products of Pauli operators that stabilizes |Ψ>
e.g. I|0> = |0> and Z|0> = |0>.

- stabilizer states $\{|\phi_i>\}_i$ on n d-dimensional qudits are the set of states stabilized by dxn commuting stabilizer group elements

- we define the Clifford gateset to be closed for stabilizer states
  - gateset is not universal

- projection onto Paulis also takes stabilizer states to themselves:
Clifford subtheory = Pauli measurement + Clifford gateset
+ (convex combos of) stabilizer states

Gottesman-Knill theorem: there is a polynomial-time classical algorithm to simulate the Clifford subtheory [1].

- stabilizer states $\{|\phi_i>\}_i$ form an overcomplete basis.
  - therefore, any state |Ψ> can be expressed as $|\Psi> = \Sigma^{\chi}_i\, c_i\, |\phi_i>$

- the ***stabilizer rank χ(Ψ)*** of a pure state |Ψ> is the minimal number χ of states required in a stabilizer state decomposition of |Ψ>.

Trivial tensor bound property: Let χ(n) be the stabilizer rank of $|\Psi>^n$. Since the tensor product of two stabilizer states is a stabilizer state, it follows that $\chi(m+n) \leqslant \chi(m)\chi(n)$.

T-Gate Magic State

$T> = 2^{-1/2}(|0> + e^{\pi i/4}\,|1>)$.
The T-gate magic state extends the Clifford subtheory to universality in the limit of t->inty, where t denotes the power of tensoring the $|T>^t$
- T-gate magic state equivalent to a T-gate by teleportation protocol
- imperfect T-gates can be "distilled" to increase purity by magic state distillation [2]

It is postulated that χ(t) grows slowest with increasing number of qubits.
- for t = 1 qubit T gate magic state, χ(1) = 2.
- for t = 2 qubit T gate magic states, χ(2) = 2.
- for t = 3 qubit T gate magic states, χ(3) = 3.
- for t = 6 qubit T gate magic states, χ(6) ⩽ 7.

It is conjectured that this bound is tight [3].

Using the tensor bound property, we can obtain upper bounds on the T-gate stabilizer rank as t→∞ (see Figure 1):
- $\chi(1)^t = 2^t$.
- $\chi(2)^{t/2} = 2^{0.5t}$ where t is even.
- $\chi(3)^{t/3} = 3^{t/3} \approx 2^{0.53t}$ where t is a multiple of 3.
- $\chi(6)^{t/6} = 7^{t/3} \approx 2^{0.47t}$ where t is a multiple of 6.

The last bound provides the most favorable asymptotic scaling, and so the outcome of this procedure is often reported as a scaling of $O(2^{0.468t})$ for the qubit T gate magic state.

Stabilizer Rank as a Computational Cost Metric for Pauli-Based Computation

- the inner product of two stabilizer states $<\phi_i|\phi_j>$ is governed by Gaussian elimination and therefore scales as $O(n^3)$.

- in a Pauli-based computation, given a projector $E = \Pi^t_{i=1}(I+\sigma_i\, P_a)/2$ where P_a is a Pauli operator, it follows that $E_{ij}{}^{\chi(t)}=c_i x c_j <\phi_i|\Pi|\phi_j>$

- since the number of terms is $\chi(t)^2$, we want to use the lowest χ(t) that we can to simulate this classically

- therefore, stabilizer state decompositions are a good way to measure the cost of strong simulation (i.e. classical computation of the probability outcome of a measurement)
  - the stabilizer rank of states (like the T-gate magic state) that extend the Clifford subtheory to a universal set necessarily grow exponentially with tensoring
  - the exponential factor of this scaling determines how large of a universal quantum computer can be simulated by today's classical computers

Current Technique for Finding Stabilizer Rank Not Adequate

The fastest method for finding the stabilizer rank of the T-gate magic state is numerical and based on Monte Carlo (Glauber dynamics) sampling of the full stabilizer state space [3]
- for greater than t>7, such an approach ceases to converge

To find a better asymptotic tensor scaling requires reaching larger t, which therefore requires relying on a new method.
- a non-numerical method would be especially attractive

Alternative Method: Odd Prime-Dimensional Wigner-Weyl-Moyal (WWM) Formalism

Instead of considering our magic state in terms of vectors in Hilbert space, we can use a kernel (or quasiprobability) representation instead.

Given a set of operators R(x) indexed by x, that are Hilbert-Schmidt orthogonal, any operator A can be represented:

$$A=d^{-1}\sum_x \mathrm{Tr}(R(x)A)R(x)\equiv\sum_x A_x(x)R(x).$$

We choose R(x) to be Hermitian, self-inverse and unitary, and so $A_x(x)$ are real.
In particular, we consider the odd-prime-d Wigner "reflection" operators [4].

Corresponence between Quadratic Gauss Sums in WWM and Orthogonal Clifford-Separable Pointer Stabilizer Rank (t<3)

This representation of finite odd-dimensional quantum states is especially simple for the Clifford subtheory:
- stabilizer states $\rho_x(x)$ are non-negative
- Clifford gates $U_x(x)$ are symplectic positive maps.

General quantum states |Ψ> can be expressed in terms of quadratic Gauss sums that correspond to ***orthogonal Clifford-separable pointer stabilizer states*** [5]:

an equiprobable linear combination of orthogonal stabilizer states, which can be written after some Clifford transformation U_C, as products of orthogonal single-qudit stabilizer states $\{|\phi_{ij}>\}_j$ and single-qudit stabilizer states $\{|\psi_{ik}>\}_k$:

$$U_C|\Psi> = \Sigma_i c_i \Pi_j|\phi_{ij}>\Pi_k|\psi_{ik}>$$

- this is not a numerical approach!
  - to find the minimal number of quadratic Gauss sums you search for the number of non-quadratic phase space (x) variables after Clifford (unit Jacobian) transformations

  - for the T-gate magic state with t=1, and t=2, such a decomposition equals the stabilizer rank
    - therefore, the minimal number of quadratic Gauss sums provably equals the stabilizer rank
  - however, for t>3, the T-gate magic state stabilizer rank is generally non-orthogonal

Pushing to find Correspondence with (Non-Orthogonal) Stabilizer Rank (t>3)

For qutrits (d=3), the minimal number of quadratic Gauss sums equals the stabilizer rank for t=1 and t=2 (see Figure 2)
- qualitatively, the qubit (d=2) and qutrit (d=3) correspond approximately $2^{\alpha t} \leftrightarrow 3^{\alpha t}$
- however, qutrit stabilizer rank for t>3 is not known (search space too large for Monte Carlo)

- despite the T-gate magic state's optimal stabilizer decompositions not being orthogonal for t>3, the number of quadratic Gauss sums continues to qualitatively correspond to the stabilizer rank for t=3 and t=6 (see Figure 2).

New Optimal Exponential Coefficient Found for the Asymptotic Stabilizer Rank of the T-Gate Magic State?

Can this correspondence be used to push the search for the T-gate magic state stabilizer rank beyond six qudits and find a better asymptotic scaling?

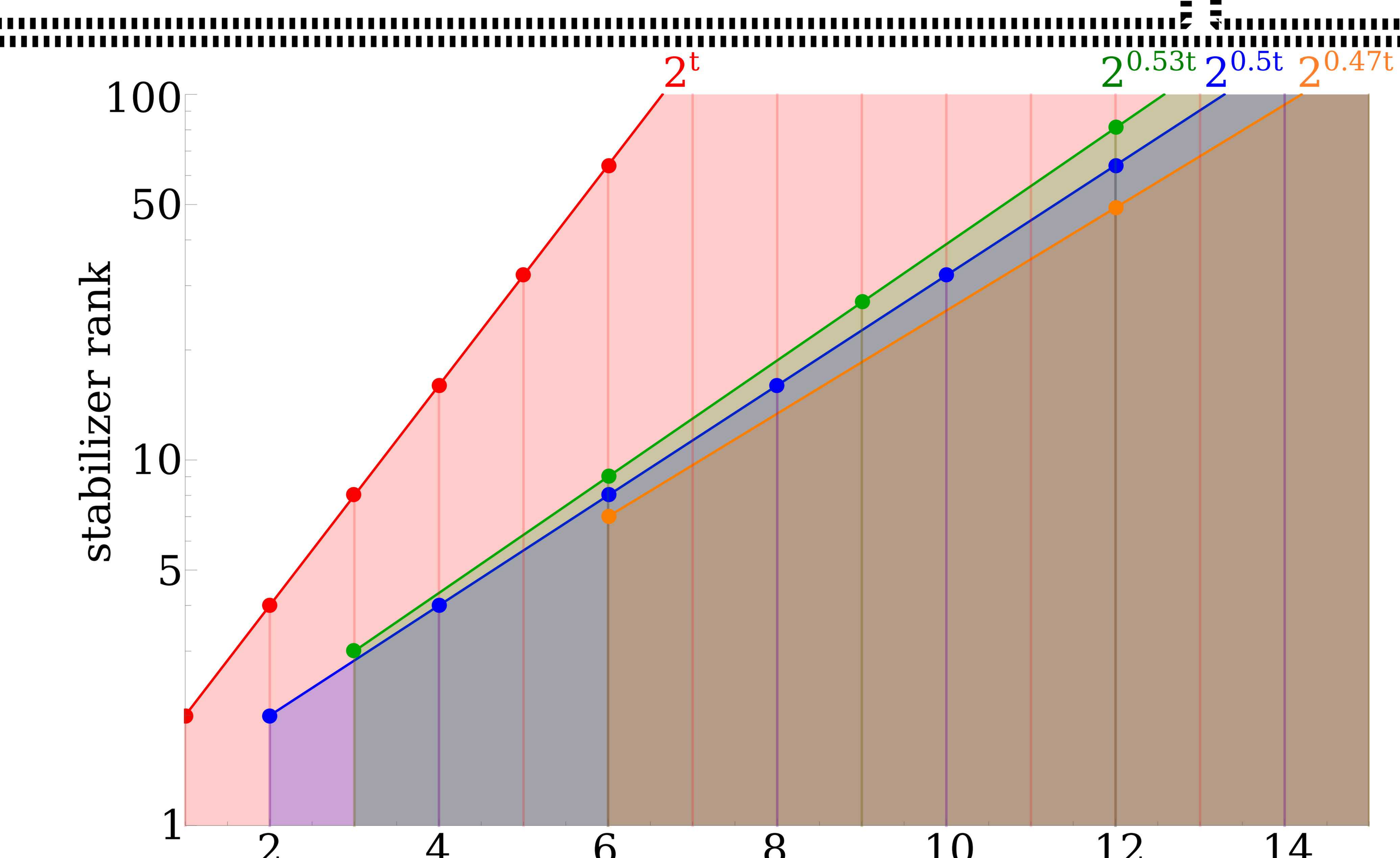

Figure 1: Qubit (d=2) stabilizer rank upper bound w.r.t t from the trivial tensor bound property using the stabilizer rank found for 1, 2, 3, and 6 tensored T-gate magic states

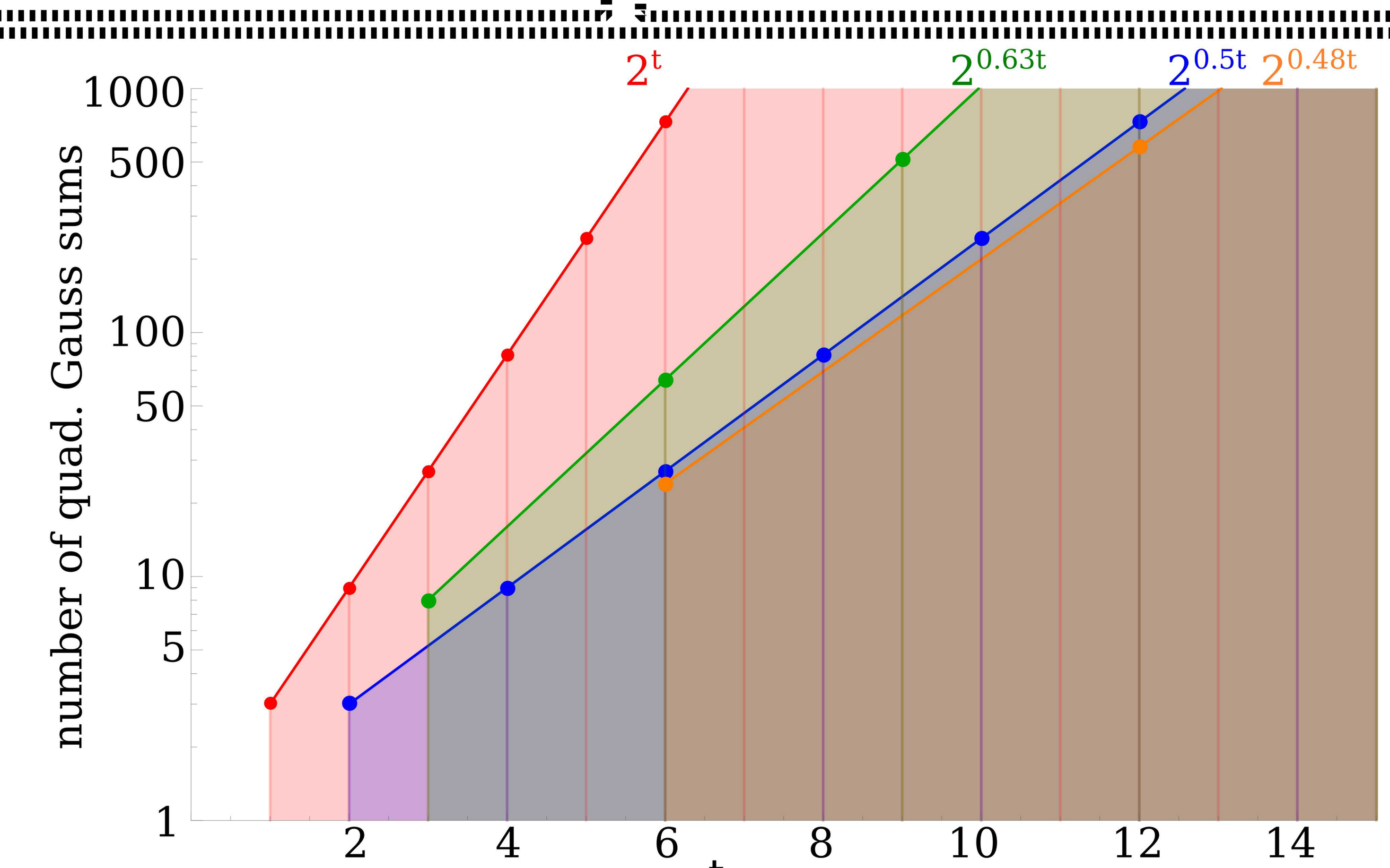

Figure 2: Qutrit (d=3) quadratic Gauss sum upper bound w.r.t t from the trivial tensor bound property using the minimal quadratic Gauss sum found for 1, 2, 3, and 6 tensored T-gate magic states

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