SAND2019-9645R

A meshfree mimetic divergence operator

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(joint work with Nathaniel Trask and Mauro Perego)

Mimetic methods [3] discretize divergence by restricting the Gauss theorem to mesh cells. Because point clouds lack such geometric entities, construction of a compatible meshfree divergence is a challenge. In this work, we define an abstract Meshfree Mimetic Divergence (MMD) operator on point clouds by contraction of *field* and *virtual face* moments. This MMD satisfies a discrete divergence theorem, provides a discrete local conservation principle, and is first-order accurate.

A mimetic divergence operator on a primal-dual mesh [4] motivates our construction. For homogeneous boundary conditions such an operator is given by

$$(DIV\boldsymbol{u}^h)_i := \frac{1}{\mu_i} \sum_{\boldsymbol{f}_{ij} \in \partial\omega_i} u_{ij}\mu_{ij} \qquad \forall \omega_i \in C. \tag{1}$$

where ω_i is a dual cell corresponding to a primal vertex $\boldsymbol{v}_i$, $\mu_i = |\omega_i|$, $\boldsymbol{f}_{ij} \in \partial\omega_i$ is a dual face with oriented measure $\mu_{ij} = \int_{\boldsymbol{f}_{ij}} dS$, and u_{ij} is a normal component of the vector field $\boldsymbol{u}^h$.

Let $X = \{\boldsymbol{x}_i\}_{i=1}^N$ denote a point cloud with a fill distance h_X defined on a domain Ω. We approximate vector fields $\boldsymbol{u}$ by their point samples $\boldsymbol{u}^h$ on X. Using (1) as a template we define the following abstract MMD operator

$$(DIV\boldsymbol{u}^h)_i := \frac{1}{\mu_i} \sum_{\boldsymbol{f}_{ij} \in \widetilde{\partial}\omega_i} \boldsymbol{t}_{ij}(\boldsymbol{u}^h) \cdot \boldsymbol{\mu}_{ij} \qquad \forall \omega_i \in \widetilde{C}. \tag{2}$$

Here $\widetilde{C}$ is collection of virtual dual cells such that every $\omega_i \in \widetilde{C}$ corresponds to a point $\boldsymbol{x}_i \in X$, $\widetilde{\partial}$ is a virtual boundary operator mapping virtual cells to virtual faces $\boldsymbol{f}_{ij} \in \widetilde{F}$, and $\boldsymbol{t}_{ij}$ is an operator mapping point samples to *field moments* on the virtual faces. Finally, μ_i and $\boldsymbol{\mu}_{ij}$ are *metric moments* providing information about the measures of the virtual cells and faces, respectively.

To ensure that (2) has the same mimetic properties as its mesh-based parent (1) we require that

- **T.1:** The virtual face volumes satisfy $\mu_i > 0$, $\mu_i = O(h_X^d)$, and $\sum_i \mu_i = \mu(\Omega)$.
- **T.2:** The virtual face moments $\{\boldsymbol{\mu}_{ij}\}$ are antisymmetric: $\boldsymbol{\mu}_{ij} = -\boldsymbol{\mu}_{ji}$.
- **T.3:** The operator $\boldsymbol{t}_{ij}$ is symmetric: $\boldsymbol{t}_{ij}(\boldsymbol{u}^h) = \boldsymbol{t}_{ji}(\boldsymbol{u}^h)$.

Assuming that **T.1**-**T.3** hold one can prove [1] that the abstract MMD operator (2) is locally conservative with respect to the virtual dual cells. Under some additional conditions on the metric and the field data it is also possible to show that (2) is first order accurate [1], i.e.,

$$\left\|\nabla \cdot \boldsymbol{u} - (DIV\boldsymbol{u}^h)\right\|_{\ell^\infty, X} \le Ch\|\boldsymbol{u}\|_{C^2(\Omega)}. \tag{3}$$

We consider two instantiations of (2). The first one assumes a background primal-dual mesh complex and uses generalized moving least squares (GMLS) [2] to obtain the necessary field and face moments. This MMD instance is appropriate

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for settings where a mesh is available but its quality is insufficient for a robust and accurate mesh-based discretization. The MMD with a background mesh is given by

$$(\overset{\text{GMLS}}{DIV}\boldsymbol{u}^h)_i = \frac{1}{\mu_i} \sum_{\boldsymbol{f}_{ij} \in \partial\omega_i} \boldsymbol{c}_{ij}(\boldsymbol{u}^h) \cdot \boldsymbol{\mu}_{ij} \qquad \forall \omega_i \in C. \tag{4}$$

In this definition $\mu_i = |\omega_i|$,

$$\boldsymbol{c}_{ij}(\boldsymbol{u}^h) = \underset{\boldsymbol{b} \in \mathbb{R}^n}{\operatorname{argmin}} \frac{1}{2} \left| B\boldsymbol{b} - \boldsymbol{u}^h \right|^2_{W(\boldsymbol{f}_{ij})} \quad \text{and} \quad \boldsymbol{\mu}_{ij} = \int_{\boldsymbol{f}_{ij}} \boldsymbol{p} \cdot \boldsymbol{n}_{\boldsymbol{f}} dS \,,$$

where $\boldsymbol{p}$ is basis of the GMLS *reproduction* space, the matrix B contains samples of the basis $\boldsymbol{p}$, and $W(\boldsymbol{f}_{ij})$ is a diagonal weight matrix; see [2]. If $\boldsymbol{u}^h$ is a sample of a vector field $\boldsymbol{v}$ then the product $\boldsymbol{c}_{ij}(\boldsymbol{u}^h) \cdot \boldsymbol{\mu}_{ij}$ is the GMLS approximation of the flux of $\boldsymbol{u}$ across $\boldsymbol{f}_{ij}$, i.e.,

$$\int_{\boldsymbol{f}_{ij}} \boldsymbol{u} \cdot \boldsymbol{n}_{\boldsymbol{f}} dS \approx \boldsymbol{c}_{ij}(\boldsymbol{u}^h) \cdot \boldsymbol{\mu}_{ij}.$$

The second MMD operator retains the GMLS field moments but defines *virtual face* moments using computationally efficient weighted graph-Laplacian equations. This MMD instance does not require a background grid and is appropriate for applications where mesh generation creates a computational bottleneck. It allows one to trade an expensive mesh generation problem for a scalable algebraic one, without sacrificing compatibility with the divergence operator. We refer to [1] for further details.

Acknowledgments

Sandia National Laboratories is a multimission laboratory managed and operated by National Technology and Engineering Solutions of Sandia, LLC., a wholly owned subsidiary of Honeywell International, Inc., for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-NA-0003525. This paper describes objective technical results and analysis. Any subjective views or opinions that might be expressed in the paper do not necessarily represent the views of the U.S. Department of Energy or the United States Government.

This material is based upon work supported by the U.S. Department of Energy, Office of Science, Office of Advanced Scientific Computing Research under Award Number DE-SC-0000230927, and the Laboratory Directed Research and Development program at Sandia National Laboratories.

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