SAND2019-10028R

Approximating Joint Chance Constraints in Two-Stage Stochastic Programs

David Mildebrath

Sandia National Laboratories

August 15, 2019

Abstract: We present various approximations to joint chance constraints arising in two-stage stochastic programming models. Our approximations are derived from three classical inequalities: Markov's inequality, Chebysev's inequality, and Chernoff's bound. We provide preliminary computational results illustrating the quality of our approximation using a two-stage joint-chance-constrained stochastic program from the literature. We also briefly introduce other alternatives for constructing approximations for joint-chance-constrained two-stage programs.

---

1 Some Classical Concentration Inequalities

We primarily study the following three classical concentration inequalities:

Theorem 1 (Markov's Inequality)**.** *For any non-negative random variable X and any scalar $a > 0$,*

$$\Pr[X \geq a] \leq \frac{\mathbb{E}[X]}{a}.$$

Theorem 2 (Chebyshev's Inequality)**.** *Let X be a random variable with finite expected value μ and non-zero variance σ^2. Then for any scalar $t > 0$,*

$$\Pr\left[|X - \mu| \geq t\sigma\right] \leq \frac{1}{t^2}.$$

Note that in the case $0 < t \leq 1$, Chebyshev's inequality is trivial. Moreover, Chebyshev's inequality remains valid (albeit weaker) if the absolute value bars are excluded.

Theorem 3 (Chernoff's Bound)**.** *Let X be a random variable and let a be a scalar. Then for any $t > 0$,*

$$\Pr[X \geq a] \leq \frac{\mathbb{E}[\exp(tX)]}{\exp(ta)}.$$

In particular,

$$\Pr[X \geq a] \leq \inf_{t>0} \frac{\mathbb{E}[\exp(tX)]}{\exp(ta)}.$$

Note that Chernoff's bound does not require the assumption that X is non-negative and/or $a > 0$ (as in Markov's inequality). The function $M_X(t) = \mathbb{E}[\exp(tX)]$ is the *moment-generating function* of the random variable X.

Chebyshev's inequality can directly bound $\Pr[X \geq a]$ for scalar a as follows:

Sandia National Laboratories is a multimission laboratory managed and operated by National Technology & Engineering Solutions of Sandia, LLC, a wholly owned subsidiary of Honeywell International Inc., for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-NA0003525.

Theorem 4 (Modified Chebyshev's Inequality)**.** *Let X be a random variable with finite expected value μ and non-zero variance σ^2. Then for any $a > \mu$,*

$$\Pr[X \geq a] \leq \frac{\sigma^2}{(a-\mu)^2}.$$

Similarly, for any $a < \mu$,

$$\Pr[X \leq a] \leq \frac{\sigma^2}{(a-\mu)^2}.$$

Proof. Suppose first that $a > \mu$. By Chebyshev's inequality,

$$\Pr\left[|X-\mu| \geq t\sigma\right] \leq \frac{1}{t^2} \quad \implies \quad \Pr\left[X \geq \mu + t\sigma\right] \leq \frac{1}{t^2}.$$

Substituting $t = (a-\mu)/\sigma > 0$ gives the desired result. The case $a < \mu$ follows similarly, using the value $t = (\mu - a)/\sigma > 0$. □

2 Model Problem

For the remainder of this note, we will use the bounds introduced in the previous section to provide bounds on the optimal values of two-stage stochastic programs with joint chance constraints. We use as a prototype the two-stage joint-chance-constrained program from [3]:

$$\begin{aligned}
z^* := \max_{x,y} \; & \sum_{t\in T} \left(R_t x_t - B_t \mathbb{E}[y_t^\omega]\right) && \text{(1a)}\\
\text{s.t. } & \Pr[x_t \leq y_t^\omega + w_t^\omega \; \forall\, t \in T] \geq 1 - \varepsilon && \text{(1b)}\\
& 0 \leq y_t^\omega \leq \Delta \; \forall\, t \in T, \omega \in \Omega && \text{(1c)}\\
& x_t \geq 0 \; \forall\, t \in T && \text{(1d)}
\end{aligned}$$

where T is a finite index set, Ω is a finite set of scenarios, x and y are decision variables, w is a random variable, and R_t and B_t are non-negative deterministic parameters. Assuming that all $N = |\Omega|$ scenarios are equally likely, an equivalent big-M formulation is given by

$$\begin{aligned}
z^* := \max_{x,y,z} \; & \sum_{t\in T} \left(R_t x_t - B_t \cdot \tfrac{1}{N} \sum_{\omega\in\Omega} y_t^\omega\right) && \text{(2a)}\\
\text{s.t. } & x_t \leq y_t^\omega + w_t^\omega + M_t^\omega z^\omega \; \forall\, t \in T, \omega \in \Omega && \text{(2b)}\\
& \frac{1}{N} \sum_{\omega\in\Omega} z^\omega \leq \varepsilon && \text{(2c)}\\
& 0 \leq y_t^\omega \leq \Delta \; \forall\, t \in T, \omega \in \Omega && \text{(2d)}\\
& x_t \geq 0 \; \forall\, t \in T && \text{(2e)}\\
& z^\omega \in \{0,1\} \; \forall\, \omega \in \Omega && \text{(2f)}
\end{aligned}$$

where $M_t^\omega > 0$ are sufficiently large positive constants. We can obtain lower bounds on z^* by approximating the joint chance constraint using classical tail inequalities.

For the remainder of this note, we replace the joint chance constraint (1b) with the equivalent individual chance constraint

$$\Pr\left[\max_{t\in T}\{x_t - y_t^\omega - w_t^\omega\} \le 0\right] \ge 1-\varepsilon. \tag{3}$$

We will explore both relaxations and conservative approximations to the joint chance constraint (1b) (and its individual equivalent (3)).

3 Relaxations Based on Classical Inequalities

We begin by exploring relaxations of the constraint (3) using classical concentration inequalities. We will see in the sequel that these relaxations are typically very poor approximations. Intuitively, this can be explained by the fact that the classical inequalities in Section 1 provide *upper* bounds on the tail of the probability distribution, while formulating a relaxation of (3) requires a *lower* bound.

Proposition 1 (Markov Relaxation). *Let $F(z,\xi)$ be a function of a vector of decision variables z and a random vector ξ. Suppose that for all $\xi \in \Xi$ there exists $L(\xi) > 0$ such that $F(z,\xi) \le L(\xi)$ for all z (i.e., F is bounded above for all $\xi \in \Xi$). If $\Pr[F(z,\xi) \le 0] \ge 1-\varepsilon$, then*

$$\mathbb{E}\left[\left(\frac{F(z,\xi)}{L(\xi)}\right)_+\right] \le \varepsilon.$$

Proof. Follows from Proposition 1 of [1] with $g(z,\xi) = -F(z,\xi)$ and $\phi(t) = \min\{1,t\}$. □

Proposition 2 (Chebyshev Relaxation). *Let $F(z,\xi)$ be a function of a vector of decision variables z and a random vector ξ. If $\Pr[F(z,\xi) \le 0] \ge 1-\varepsilon$ then*

$$\mu - \frac{1}{\sqrt{1-\varepsilon}}\sigma \le 0,$$

where $\mu = \mathbb{E}_\xi[F(z,\xi)]$ and $\sigma^2 = \mathrm{Var}(F(z,\xi))$.

Proof. Apply the Chebyshev inequality with $X = -F(z,\xi)$ and $a = 0$. □

It is clear by inspection that this is a very, very poor approximation, as it is dominated by the inequality $\mu \le 0$. Moreover, as $\varepsilon \to 1$, the inequality reduces to $\mu \le \infty$.

Proposition 3 (Chernoff Relaxation). *Let $F(z,\xi)$ be a function of a vector of decision variables z and a random vector ξ. If $\Pr[F(z,\xi) \le 0] \ge 1-\varepsilon$ then, for any $t > 0$, $\mathbb{E}[\exp(-t \cdot F(z,\xi))] \ge 1-\varepsilon$.*

Proof. Apply the Chernoff bound with $X = -F(z,\xi)$ and $a = 0$. □

4 Conservative Approximations Based on Classical Inequalities

We next introduce three conservative approximations to (3) based classical inequalities. We find these conservative approximations to be much better than the relaxations introduced in Section 3.

We note that the approximations obtained in this section may be derived using the framework of [2], by particular choices of the so-called "generating function" ψ. In this section, however, we provide independent proofs that do not depend on the generating function framework.

4.1 Markov Approximation

We state the conservative approximation obtained via the Markov inequality in more generality, then specify it to the model (1).

Proposition 4 (Markov Conservative Approximation)**.** *Let $F(z, \xi)$ be a function of a vector of decision variables z and a random vector ξ. Then for any $\alpha > 0$,*

$$\mathbb{E}\left[\left(1 + \frac{F(z,\xi)}{\alpha}\right)_+\right] \leq \varepsilon \tag{4}$$

implies $\Pr[F(z, \xi) \leq 0] \geq 1 - \varepsilon$*.*

Proof. Fix $\alpha > 0$. Then

$$\varepsilon \geq \mathbb{E}\left[\left(1 + \frac{F(z,\xi)}{\alpha}\right)_+\right] \geq \Pr\left[\left(1 + \frac{F(z,\xi)}{\alpha}\right)_+ \geq 1\right] \geq \Pr\left[1 + \frac{F(z,\xi)}{\alpha} \geq 1\right]$$
$$= \Pr\left[\frac{F(z,\xi)}{\alpha} \geq 0\right] = \Pr[F(z,\xi) \geq 0] \geq \Pr[F(z,\xi) > 0],$$

where the second inequality follows from Markov's inequality with $X = (1+F(z, \xi)/\alpha)_+$ and $a = 1$. Hence $\Pr[F(z, \xi) > 0] \leq \varepsilon$ which implies $\Pr[F(z, \xi) \leq 0] \geq 1 - \varepsilon$. □

By applying Proposition 4 to (3), we obtain

$$\sum_{\omega \in \Omega} p_\omega \left(1 + \frac{1}{\alpha} \max_{t \in T} \left\{x_t - y_t^\omega - w_t^\omega\right\}\right)_+ \leq \varepsilon.$$

This may be linearized by introducing an auxiliary variable z^ω for all $\omega \in \Omega$:

$$\sum_{\omega \in \Omega} p_\omega z^\omega \leq \varepsilon$$
$$z^\omega \geq 1 + \tfrac{1}{\alpha}(x_t - y_t^\omega - w_t^\omega) \text{ for all } \omega \in \Omega,\ t \in T$$
$$z^\omega \geq 0 \text{ for all } \omega \in \Omega.$$

The parameter $\alpha > 0$ can be transformed into an optimization variable by using the substitution $z^\omega \mapsto \alpha z^\omega$:

$$\begin{aligned}
&\sum_{\omega\in\Omega} p_\omega z^\omega \leq \varepsilon\alpha \\
&z^\omega \geq \alpha + x_t - y_t^\omega - w_t^\omega \text{ for all } \omega \in \Omega,\ t \in T \\
&z^\omega \geq 0 \text{ for all } \omega \in \Omega \\
&\alpha \geq 0.
\end{aligned}$$

The complete approximation to (1) is

$$z^{\text{Markov}} := \max_{x,y,z,\alpha} \sum_{t\in T} \left(R_t x_t - B_t \cdot \tfrac{1}{N} \sum_{\omega\in\Omega} y_t^\omega\right) \tag{5a}$$

$$\text{s.t. } z^\omega \geq \alpha + x_t - y_t^\omega - w_t^\omega \ \forall\, t \in T, \omega \in \Omega \tag{5b}$$

$$\frac{1}{N}\sum_{\omega\in\Omega} z^\omega \leq \varepsilon\alpha \tag{5c}$$

$$0 \leq y_t^\omega \leq \Delta \ \forall\, t \in T, \omega \in \Omega \tag{5d}$$

$$x_t \geq 0 \ \forall\, t \in T \tag{5e}$$

$$z^\omega \geq 0 \ \forall\, \omega \in \Omega \tag{5f}$$

$$\alpha \geq 0. \tag{5g}$$

Note that this model is a linear program, and thus highly tractable.

4.2 Chebyshev Approximation

We turn next to Chebyshev's inequality.

Proposition 5 (Chebyshev Conservative Approximation)**.** *Let $F(z, \xi)$ be a function of a vector of decision variables z and a random vector ξ. Let $\mu = \mathbb{E}_\xi[F(z, \xi)]$ and $\sigma^2 = \mathrm{Var}(F(z, \xi))$. If $\mu < 0$, then*

$$\mu + \frac{1}{\sqrt{\varepsilon}}\sigma \leq 0$$

implies $\Pr[F(z, \xi) \leq 0] \geq 1 - \varepsilon$.

Proof. Suppose $\mu < 0$ and $\mu + \varepsilon^{-1/2}\sigma \leq 0$. The second inequality can be re-arranged to give $\sigma^2/\mu^2 \leq \varepsilon$. We have that

$$\Pr[F(z, \xi) > 0] \leq \Pr[F(z, \xi) \geq 0] \leq \frac{\sigma^2}{(0 - \mu)^2} \leq \varepsilon,$$

where the second inequality follows from the modified Chebyshev inequality with $X = F(z, \xi)$ and $a = 0$. Hence $\Pr[F(z, \xi) > 0] \leq \varepsilon$, and thus $\Pr[F(z, \xi) \leq 0] \geq 1 - \varepsilon$. □

This approximation has a close relationship with the normal approximation to a chance constraint, in which the term $1/\sqrt{\varepsilon}$ is replaced by $\Phi^{-1}(1 - \varepsilon)$, where $\Phi(\cdot)$ is the CDF of the standard normal distribution.

The Chebyshev bound can be applied to the joint chance constraint (1b). This requires computing the mean and variance of the random variable $Z = \max_t\{x_t - y_t^\omega - w_t^\omega\}$. The mean can be computed using the law of the unconscious statistician (LOTUS):

$$\mu = \frac{1}{N} \sum_{\omega \in \Omega} z^\omega$$

where $z^\omega := \max_{t \in T} \left\{x_t - y_t^\omega - w_t^\omega\right\}$ for all $\omega \in \Omega$. The variance σ^2 is given by

$$\sigma^2 = z^{\mathrm{T}} \Sigma z,$$

where the symmetric positive semi-definite matrix Σ is given by $\Sigma := (1/N)I - (1/N^2)J$, where I is the identity matrix, and J is the square matrix of all ones. Hence, the Chebyshev approximation for (1b) can be summarized:

$$z^\omega = \max_{t \in T}\{x_t - y_t^\omega - w_t^\omega\} \text{ for all } \omega \in \Omega, \tag{6a}$$

$$\mu + \sqrt{(1/\varepsilon) z^{\mathrm{T}} \Sigma z} \leq 0. \tag{6b}$$

Note that the constraint (6a) cannot be relaxed into a $\geq$ constraint—that is, the formulation (6) is not convex. Its solution would require the solution of a QMIP (where the quadratic constraints are convex).

The complete Chebyshev approximation is

$$z^{\text{Chebyshev}} := \max_{x,y,z} \sum_{t \in T} \left(R_t x_t - B_t \cdot \tfrac{1}{N} \sum_{\omega \in \Omega} y_t^\omega\right) \tag{7a}$$

$$\text{s.t. } z^\omega = \max_{t \in T}\{x_t - y_t^\omega - w_t^\omega\} \; \forall\, \omega \in \Omega \tag{7b}$$

$$\frac{1}{N} \sum_{\omega \in \Omega} z^\omega + \text{const.} \sqrt{\sum_{\omega \in \Omega} (z^\omega)^2} \leq 0 \tag{7c}$$

$$0 \leq y_t^\omega \leq \Delta \; \forall\, t \in T, \omega \in \Omega \tag{7d}$$

$$x_t \geq 0 \; \forall\, t \in T \tag{7e}$$

where const. $= [(1/\varepsilon)(1/N - 1/N^2)]^{1/2}$. The constraints (7b) can be enforced by introducing binary variables. Consequently, the model (7) is a quadratic mixed-integer program (QMIP).

4.3 Chernoff Approximation

Finally, we present the approximation based on Chernoff's bound.

Proposition 6 (Chernoff Conservative Approximation)**.** *Let $F(z, \xi)$ be a function of a vector of decision variables z and a random vector ξ. For any $\alpha > 0$,*

$$\mathbb{E}\left[\exp(\tfrac{1}{\alpha} F(z, \xi))\right] \leq \varepsilon$$

implies $\Pr[F(z, \xi) \leq 0] \geq 1 - \varepsilon$.

Proof. The result follows directly from Chernoff's bound with $t = 1/\alpha$, $X = F(z, \xi)$ and $a = 0$. □

Specializing this proposition to the joint chance constraint (1b) gives

$$\frac{1}{N}\sum_{\omega\in\Omega}\exp(z^\omega/\alpha)\le\varepsilon$$
$$z^\omega\ge x_t-y_t^\omega-w_t^\omega \text{ for all } t\in T,\omega\in\Omega.$$

As discussed above, the scalar $\alpha>0$ may be transformed into a decision variable. In this setting, such a transformation may be more reasonable, as it the structure of the above inequalities is already non-linear (unlike in the case of the Markov approximation).

The complete Chernoff approximation is .

$$z^{\text{Chernoff}} := \max_{x,y,z,\alpha}\sum_{t\in T}\left(R_t x_t - B_t\cdot\tfrac{1}{N}\sum_{\omega\in\Omega}y_t^\omega\right) \tag{8a}$$
$$\text{s.t. } z^\omega\ge x_t-y_t^\omega-w_t^\omega\ \forall\, t\in T,\omega\in\Omega \tag{8b}$$
$$\frac{1}{N}\sum_{\omega\in\Omega}\alpha\exp(z^\omega/\alpha)\le\varepsilon\alpha \tag{8c}$$
$$0\le y_t^\omega\le\Delta\ \forall\, t\in T,\omega\in\Omega \tag{8d}$$
$$x_t\ge 0\ \forall\, t\in T \tag{8e}$$
$$\alpha\ge 0. \tag{8f}$$

The constraint (8c) is jointly convex in z and α due to the theory of perspective functions, and thus (8) is a convex program, which can be solved efficiently.

Finally, we compare the Chernoff relaxation (Proposition 3) with the Chernoff conservative approximation (Proposition 6). These two approximations coincide if

$$1-\mathbb{E}[\exp(tF(z,\xi))]=\mathbb{E}[\exp(-tF(z,\xi))]$$

for some z and t. We will show that this is impossible. For brevity, let $Z=F(z,\xi)$. From the definition of moment-generating functions (which we are implicitly assuming exist), we have that

$$\mathbb{E}[\exp(tZ)]=1+t\mathbb{E}[Z]+\frac{t^2}{2!}\mathbb{E}[Z^2]+\frac{t^3}{3!}\mathbb{E}[Z^3]+\cdots$$

Similarly,

$$\mathbb{E}[\exp(-tZ)]=1-t\mathbb{E}[Z]+\frac{t^2}{2!}\mathbb{E}[Z^2]-\frac{t^3}{3!}\mathbb{E}[Z^3]+\cdots$$

Summing these two terms gives

$$\mathbb{E}[\exp(tZ)]+\mathbb{E}[\exp(-tZ)]=2\left(1+\frac{t^2}{2!}\mathbb{E}[Z^2]+\frac{t^4}{4!}\mathbb{E}[Z^4]+\cdots\right).$$

Hence, if $\mathbb{E}[\exp(tZ)]+\mathbb{E}[\exp(-tZ)]=1$, this implies

$$\frac{t^2}{2!}\mathbb{E}[Z^2]+\frac{t^4}{4!}\mathbb{E}[Z^4]+\cdots=-\frac{1}{2}.$$

The convergent sum on the left contains only positive terms, while the right-hand side is strictly negative, a contradiction. Hence, the relaxation obtained by applying the Chernoff bound to $X=-F(z,\xi)$ is *not* equivalent to the conservative approximation obtained by applying the Chernoff bound to $X=+F(z,\xi)$.

	Lower Bound		Upper Bound	
$z^* = 8634.1$	ARMA, 250 scenarios, $\varepsilon = 1\%$			
Markov	8339.7	(3.4%)	9300.2	(7.7%)
Chebyshev	7941.4	(8.0%)	$\geq$ 12029.3	(39.3%)
Chernoff	8339.7	(3.4%)	13056.2	(51.2%)
$z^* = 9154.9$	ARMA, 250 scenarios, $\varepsilon = 3\%$			
Markov	8339.7	(8.9%)	10873.7	(18.8%)
Chebyshev	7954.1	(13.1%)	$\geq$ 12029.3	(31.4%)
Chernoff	8339.7	(8.9%)	14141.5	(54.5%)
$z^* = 9353.2$	Gaussian, 250 scenarios, $\varepsilon = 1\%$			
Markov	9092.3	(2.8%)	9989.7	(6.8%)
Chebyshev	8941.2	(4.4%)	$\geq$ 12672.1	(35.5%)
Chernoff	9092.3	(2.8%)	13594.0	(45.3%)
$z^* = 9884.0$	Gaussian, 250 scenarios, $\varepsilon = 3\%$			
Markov	9092.3	(8.0%)	11542.7	(16.8%)
Chebyshev	9009.4	(8.8%)	$\geq$ 12672.1	(28.2%)
Chernoff	9092.3	(8.0%)	14329.0	(45.0%)

Table 1: Preliminary computational results for the approximations introduced in Section 3 and Section 4.

5 Preliminary Computational Results

We now present some preliminary computational results on the model (1), using the data from [3]. The objective value and optimality gaps for various values of ε and w are given in Table 1.

Some observations: in all cases, the Markov and Chernoff lower bounds produce the same result, equal to the objective value of the model if we take $\varepsilon = 0$ (*i.e.*, we enforce the chance constraint as a hard constraint). It is well-known that the Markov lower bound is equivalent to the CVaR approximation to the chance constraint. This raises the general question: under what conditions is the CVaR approximation equivalent to the enforcing the chance constraint as a hard constraint?

The Chebyshev lower bounds require the solution of a QMIP—the reported bounds are the best obtained after 2100s. In all cases, the optimality gap for the QMIP solution was less than 1.2%.

It is proved in [1] that the Markov relaxation (lower bound) for a finite number of scenarios is equivalent to the LP relaxation of the big-M formulation of the model. Consequently, the obtained bound depends on the values of M chosen. The values of M used here are those given in Proposition 1 of [3].

Finally, the Chebyshev relaxation (upper bound) is dominated by the relaxation $\mu \leq 0$. The reported value is that obtained using the $\mu \leq 0$ relaxation, which dominates the actual Chebyshev relaxation bound. Note that this bound is independent of ε—it is roughly equivalent to replacing the chance constraint with its expectation.

6 Other Approximation Techniques

We briefly explore two other approximation techniques for the JCC (1b).

6.1 Bounds Based on the Union of Violations

An alternative viewpoint to approximating the JCC (1b) is to bound the quantity

$$1 - \Pr[x_t \le \tilde{y}_t + \tilde{w}_t \; \forall \, t \in T] = \Pr\big[\bigcup_{t \in T} A_t\big], \tag{9}$$

where the event $A_t = \{x_t - \tilde{y}_t - \tilde{w}_t > 0\}$. Many classical bounds for (9) exist involving the quantities

$$S_1 = \sum_{t \in T} \Pr[A_t], \quad S_2 = \sum_{t \le t'} \Pr[A_t \cap A_{t'}].$$

We first relate S_1 and S_2 to the first and second moments of the random variable $Z := \sum_{t \in T} X_t$, where $X_t = \mathbf{1}\{A_t\}$ is the indicator variable on the event A_t. Simple calculation gives $\mathbb{E}[Z] = S_1$. Similarly,

$$Z^2 = \Big(\sum_{t \in T} X_t\Big)^2 = \sum_{t \in T} X_t^2 + 2 \sum_{t < t'} X_t X_{t'}.$$

Because $\mathbb{E}[X_t^2] = \mathbb{E}[X_t] = \Pr[A_t]$ and $\mathbb{E}[X_t X_{t'}] = \Pr[A_t \cap A_{t'}]$, it follows that $\mathbb{E}[Z^2] = S_1 + 2S_2$.

We can now use the equivalence

$$\Pr\big[\bigcup_{t \in T} A_t\big] = \Pr\big[\sum_{t \in T} X_t \ge 1\big]$$

to derive bounds on (9). For example, we can apply Markov's inequality to obtain

$$\Pr\big[\bigcup_{t \in T} A_t\big] = \Pr\big[\sum_{t \in T} X_t \ge 1\big] \le \mathbb{E}\big[\sum_{t \in T} X_t\big] = S_1. \tag{10}$$

The inequality (10) is equivalent to Boole's inequality. We can also use Chebyshev's inequality

$$\Pr[Z \ge 1] \le \frac{\mathrm{Var}(Z)}{(1 - \mathbb{E}[Z])^2},$$

which is valid provided $S_1 = \mathbb{E}[Z] < 1$ (it's not obvious that this condition will necessarily hold, particularly for larger values of ε). Anyway, this bound is equivalent to

$$\Pr\big[\bigcup_{t \in T} A_t\big] \le \frac{S_1(1 - S_1) + 2S_2}{(1 - S_1)^2}. \tag{11}$$

Unfortunately, for all $0 \leq S_1 \leq 1$ and $S_2 \geq 0$,

$$S_1 \leq \frac{S_1(1-S_1)+2S_2}{(1-S_1)^2},$$

and thus the inequality (10) dominates the inequality (11) for all values of S_1 and S_2 for which (11) is valid (put plainly, the inequality (11) is useless). Finally, one could also attempt to use the Chernoff bound, yielding

$$\Pr\big[\bigcup_{t\in T} A_t\big] \leq \inf_{\alpha>0} \exp(-\alpha)\mathbb{E}[\exp(\alpha Z)].$$

Unfortunately, computing $\mathbb{E}[\exp(\alpha Z)]$ (the MGF of Z) is virtually impossible, because the random variables X_t are not independent.

6.2 LP-Based Bounds of Yang et al. (2016)

Consider the standard joint chance constraint (JCC) formulation, where the event

$$A_t = \{\omega : \text{ constraint } t \text{ is violated}\}.$$

Suppose our model contains binary variables $u_t^\omega \in \{0,1\}$ for $t \in T$, $\omega \in \Omega$, and $v_{tt'}^\omega \in \{0,1\}$ for $t, t' \in T$, $t > t'$ and $\omega \in \Omega$ such that

$$u_t^\omega = \begin{cases} 1, & \text{constraint } t \text{ is violated in scenario } \omega, \\ 0, & \text{else,} \end{cases}$$

$$v_{tt'}^\omega = \begin{cases} 1, & \text{both constraints } t \text{ and } t' \text{ are violated in scenario } \omega, \\ 0, & \text{else.} \end{cases}$$

The paper [4] provides the following lower bound:

$$\Pr\left[\bigcup_{t\in T} A_t\right] \geq z(\alpha,\gamma),$$

where $\alpha, \gamma \in \mathbb{R}^{|T|}$ are given by

$$\alpha_t = \Pr[A_t] = \sum_{\omega\in\Omega} \Pr(\omega) u_t^\omega,$$
$$\gamma_t = \sum_{t'\in T} \Pr[A_t \cap A_{t'}] = \sum_{t'\in T}\sum_{\omega\in\Omega} \Pr(\omega) v_{tt'}^\omega,$$

and z is the value function of the following linear program:

$$z(\alpha,\gamma) := \min_{a\geq 0} \sum_{i=1}^{|T|}\sum_{k=1}^{|T|} \frac{1}{k} a_{ik} \tag{12a}$$

$$\text{s.t.} \sum_{k=1}^{|T|} a_{tk} = \alpha_t \text{ and } \sum_{k=1}^{|T|} k a_{tk} = \gamma_t \;\forall\, t \in T \tag{12b}$$

$$-k a_{tk} + \sum_{i=1}^{|T|} a_{ik} \geq 0 \;\forall\, (t,k) \in T \times T. \tag{12c}$$

Hence, the inequality $z(\alpha, \gamma) \leq \varepsilon$ is a relaxation of the original joint chance constraint. The inequality $z(\alpha, \gamma) \leq \varepsilon$ can be enforced by taking the dual of the above linear program, and adding inequalities of the form

$$\pi^{\mathrm{T}}\alpha + \sigma^{\mathrm{T}}\gamma \leq \varepsilon$$

to the master problem, where π and σ are dual multipliers corresponding to the constraints (12b). This implies the following simple cutting plane algorithm:

1. Solve the master problem to obtain (α, γ).
2. Solve (12) with α and γ fixed to obtain dual multipliers $(\hat{\pi}, \hat{\sigma})$.
3. If $\hat{\pi}^{\mathrm{T}}\alpha + \hat{\sigma}^{\mathrm{T}}\gamma \leq \varepsilon$, terminate.
4. Add the inequality $\hat{\pi}^{\mathrm{T}}\alpha + \hat{\sigma}^{\mathrm{T}}\gamma \leq \varepsilon$ to the master problem. Go to step 1.

Preliminary computational experience indicates that this method produces very good bounds, but is *very* slow to converge. Specifically, a typical master solve takes longer than solving the true big-M formulation itself.

7 Conclusions and Future Work

Our results indicate that classical inequalities can be effective for approximating two-stage stochastic programs with joint chance constraints. Future work includes constructing better relaxations of such inequalities, and an analytical assessment of the comparative strength of such approximations.

Acknowledgments: The author would like to thank Dr. Bismark Singh and Dr. Jean-Paul Watson for helpful insights and tutelage.

Sandia National Laboratories is a multimission laboratory managed and operated by National Technology and Engineering Solutions of Sandia, LLC., a wholly owned subsidiary of Honeywell International, Inc., for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-NA-0003525. This paper describes objective technical results and analysis. Any subjective views or opinions that might be expressed in the paper do not necessarily represent the views of the U.S. Department of Energy or the United States Government.

References

[1] S. Ahmed. Convex relaxations of chance constrained optimization problems. *Optimization Letters*, 8(1):1–12, 2014.

[2] A. Nemirovski and A. Shapiro. Convex approximations of chance constrained programs. *SIAM Journal on Optimization*, 17(4):969–996, 2006.

[3] B. Singh and J.-P. Watson. Approximating two-stage chance-constrained programs with classical probability bounds. *Optimization Letters*, 13(6):1403–1416, 2019.

[4] J. Yang, F. Alajaji, and G. Takahara. Lower bounds on the probability of a finite union of events. *SIAM Journal on Discrete Mathematics*, 30(3):1437–1452, 2016.