SAND2020-0566C

An Approximate Direct Inverse as a Preconditioner for Ill-conditioned Problems

Chung Hyun Lee and Jin-Fa Lee
Electrical and Computer Engineering Department,
The Ohio State University
Columbus, Ohio, USA
lee.4542@osu.edu, lee.1863@osu.edu

William L. Langston, Brian Zinser,
Vinh Q. Dang, Andy Huang, and Salvatore Campione
Sandia National Laboratories
Albuquerque, NM, USA
wllangs@sandia.gov, bzinser@sandia.gov,
vqdang@sandia.gov, ahuang@sandia.gov, sncampi@sandia.gov

***Abstract*—This paper implemented an approximate direct inverse for the surface integral equation including multilevel fast-multipole method. We apply it as a preconditioner to two examples suffering convergence problem with an iterative solver.**

I. Introduction

The surface integral equation (SIE) has been successfully applied to various problems for the last few decades. However, some problems may not be solved correctly, e.g. structures supporting high-quality factor resonances or extremely large problems compared to the wavelength, because the system matrix is extremely ill-conditioned. Herein, a robust preconditioner is proposed with Schur-complement [1] and randomized principal component analysis (PCA) [2]. In addition, because the PCA process needs to compute the transpose of the multilevel fast multipole method (MLFMM) [3], we also present the method to compute it.

II. Two level Preconditioner

We suggest two level preconditioners for a given matrix equation $Ax = b$ where A, x, and b are a SIE system matrix, a solution vector, and a right-hand side excitation vector, respectively. The matrix equation can be rewritten as a left preconditioned system equation, $Q_kP_lAx = Q_kP_lb$ where P_l and Q_k are a local and a global preconditioner, respectively. We will further discuss about the local and global preconditioner and their subscript k and l in the next subsections.

A. Local Preconditioner

To begin with, the given geometry is partitioned with the oct-tree of MLFMM. For clarification, the finest box is at the lowest level ($l = 0$). Also, between two oct-tree levels, two additional auxiliary levels are introduced to make a binary tree. From this point, the level refers to binary tree level if not otherwise specified. For the given box i at the given level l with two child boxes (j and k), the system matrix can be written as:

$$A_i^l = \begin{bmatrix} A_j^{l-1} & C_{jk}^{l-1} \\ C_{kj}^{l-1} & A_k^{l-1} \end{bmatrix} \tag{1}$$

where A_i^l and C_{jk}^l denote a block matrix of box i at level l and a coupling matrix between box j and k at the level l, respectively. We can remove $l-1$ from (1) without loss of generality. With Schur Complement, the inverse of A_i^l can be derived as:

$$(A_i^l)^{-1} = \begin{bmatrix} I_m & 0 \\ A_k^{-1}C_{kj} & I_n \end{bmatrix} \begin{bmatrix} (I_m - A_j^{-1}C_{jk}A_k^{-1}C_{kj})^{-1} & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} I_m & A_j^{-1}C_{jk} \\ 0 & I_n \end{bmatrix} \begin{bmatrix} A_j^{-1} & 0 \\ 0 & A_k^{-1} \end{bmatrix} \tag{2}$$

where I_m and I_n are the identity matrix with dimension of $m \times m$ and $n \times n$, respectively. The most important part is that A_j^{-1} and A_k^{-1} need to be recursively computed until the leaf level of the binary tree. Consequently, the block inverse implemented through a recursive subroutine.

In addition, the block inverse of the second block matrix of (2) can be compressed with PCA.

$$(I_m - A_j^{-1}C_{jk}A_k^{-1}C_{kj})^{-1} \cong (I - U\Sigma V)^{-1} = I - U(\Sigma^{-1} - VU)^{-1}V = I - USV \tag{3}$$

where U, Σ, and V are a $m \times q$ matrix with left singular vectors, a $q \times q$ diagonal matrix with singular values, and a $q \times m$ matrix with right singular vectors. Note that S is a $q \times q$ matrix and $q \ll m$ and $q \ll n$, so the S matrix can be calculated explicitely. The rank k is determined by the relative error being less than a predetermined tolerance, i.e. $\sigma_k/\sigma_1 \leq \epsilon$ where σ_i is a i^{th}singular value and ϵ is the tolerance.

Eventually, the local preconditioner P_l is defined as the following:

$$P_l := \begin{bmatrix} A_1^{-1} & 0 & \cdots & 0 \\ 0 & A_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_\xi^{-1} \end{bmatrix} \tag{4}$$

where ξ is the number of boxes at the level l. In our simulations, we fixed l as 6 and the finest oct-tree block size is 0.5λ. Effectively, when $l = 6$, P_l is a local preconditioner with approximate block inverse of 2λ block size.

B. Global Preconditioner

In order to take advantage of $N\log N$ computation resources of MLFMM, we decide to stop P_l at the given level l and add

Sandia National Laboratories is a multimission laboratory managed and operated by National Technology & Engineering Solutions of Sandia, LLC, a wholly owned subsidiary of Honeywell International Inc., for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-NA0003525.

the second level preconditioner as the global preconditioner. One can show that $(P_l A)^{-1} = (I+\delta)^{-1} = (I+LDR)^{-1} \cong I - L(RL)^{-1}R$ where δ, L, D, and R are a perturbation matrix after applying P_l to the system matrix A, $N \times k$ left singular vectors, a $k \times k$ diagonal matrix with singular values, and $k \times N$ right singular vectors, respectively. Herein, N is a degree of freedom and k is a rank of δ and $k \ll N$. Again, the PCA algorithm is applied to factorize δ with fixed k. Then, the global preconditioner Q_k is defined as:

$$Q_k := I - L(RL)^{-1}R \tag{5}$$

Note that one can adaptively increase k, such that the spectral radius is less than one to guarantee to the convergence of the iterative solver.

III. Transpose MLFMM

When computing both local and global preconditioners, the matrix-vector multiplication of transpose of the system matrix is needed. For self and near terms, the transpose operation is a typical task, but for the far interactions, it is troublesome because the terms are not explicitly computed. We propose a method to compute the transpose of MLFMM by changing the direction of displacement vectors and propagation vectors.

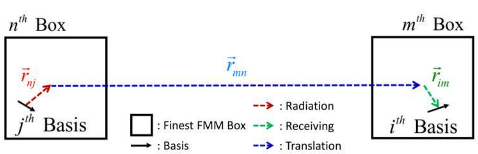

Fig. 1. Single Level FMM Example.

Let's consider the example in Fig. 1. Considering the electric field integral equation, the coupling matrix can be written as:

$$C_{mn} = \int V_s(\hat{k}, \vec{r}_{nj}) T_{mn}(\hat{k} \cdot \vec{r}_{mn}) V_f(\hat{k}, \vec{r}_{im}) d\hat{k}^2 \tag{6}$$

where $V_s(\hat{k}, \vec{r}_{nj})$, $T_{mn}(\hat{k} \cdot \vec{r}_{mn})$, and $V_f(\hat{k}, \vec{r}_{im})$ are radiation pattern, translation operator, and receiving pattern of converntional FMM, respectively. For the Transpose of C_{mn}, we can write:

$$C_{mn}^T = \int V_f(-\hat{k}, \vec{r}_{mi}) T_{mn}(-\hat{k} \cdot \vec{r}_{nm}) V_s(-\hat{k}, \vec{r}_{jn}) d\hat{k}^2 \tag{7}$$

IV. Numerical Results

We applied the approximate direct inverse as a preconditioner to discontinuous Galerkin integral equation (IEDG) [4] method. We assume $k = 50$, $l = 6$, $\epsilon = 10^{-4}$, and use the generalized conjugate residual (GCR) for all the following examples. The incident electric field is z-polarized and propagating along $-x$ direction. The first example in Fig. 2 is a high-quality factor slotted cylindrical cavity made of real metal simulated through the impedance boundary condition (IBC) to simulate finite conductivity. The second example is

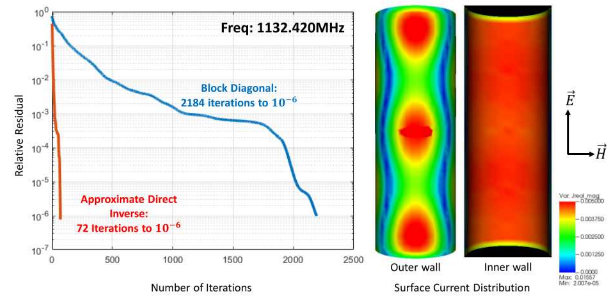

Fig. 2. IBC High-Q Slotted Cylinder.

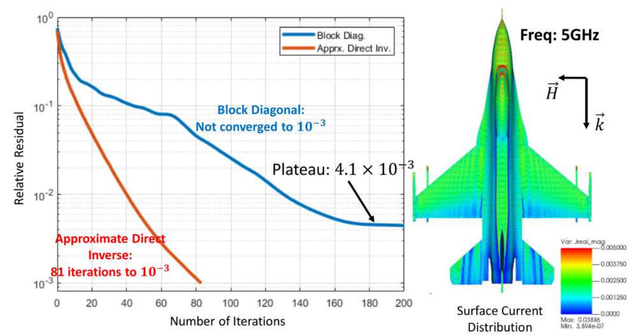

Fig. 3. PEC F-16.

PEC in Fig. 3 considers a perfect electric conductor (PEC) F-16. The advantage of using the proposed preconditioner is better convergece behaviors as shown in the above exaples.

V. Conclusion

In this paper, the approximate direct inverse is applied to the SIE for electromagnetic scattering. According to Fig 2 and Fig 3 which are two examples having convergence issues, we can conclude that the proposed preconditioner is effective even for ill-conditioned problems.

Acknowledgment

Sandia National Laboratories is a multimission laboratory managed and operated by National Technology and Engineering Solutions of Sandia, LLC, a wholly owned subsidiary of Honeywell International, Inc., for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-NA-0003525.

References

[1] Tzon-Tzer Lu and Sheng-Hua Shiou, "Inverses of 2x2 block matrices", *Computers & Mathematics with Applications*, 43(1-2):119–129, 2002.

[2] Vladimir Rokhlin, Arthur Szlam, and Mark Tygert, "A randomized algorithm for principal component analysis", *SIAM Journal on Matrix Analysis and Applications*, 31(3):1100–1124, 2009

[3] Jiming Song, Cai-Cheng Lu, and Weng Cho Chew, Multilevel Fast Multipole Algorithm for Electromagnetic Scattering", *IEEE Transactions on Antennas and Propagation*, VOL. 45, NO. 10, OCTOBER 1997

[4] Z. Peng, K.-H. Lim, and J.-F. Lee, "A Discontinuous Galerkin Surface Integral Equation Method for Electromagnetic Wave Scattering From Nonpenetrable Targets," *IEEE Transactions on Antennas and Propagation*, vol. 61, no. 7, pp. 3617–3628, jul 2013.