How linear is a linear system?

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1. Abstract

Often, when testing structures, engineers assume the experimental system only exhibits linear behavior. This linear assumption means that the modal frequency and damping of the structure do not change with response level. In many assembled structures, components are connected through bolted joints. These systems behave in a weakly nonlinear fashion due to frictional contact at these interfaces, but often these structures are still treated linearly at low excitation levels. This work contains a case study where an assumed linear system exhibits nonlinear behavior. Because of this nonlinearity, if the force applied to the structure during *linear* testing is not sufficiently low then the test may capture a nonlinear frequency or damping instead of the true linear parameters. The errors associated with this linearization causes inaccuracy when simulating a system response. In particular, a linear substructuring problem is presented in which true linear frequencies and damping ratios are compared to slightly nonlinear counterparts to observe the error caused in the assembled response. This paper documents lessons learned and heuristics to be considered when capturing true linear parameters from a weakly nonlinear structure.

Keywords – linear modal analysis, nonlinear systems, structural dynamics, heuristics, best practices

2. Introduction

Traditional experimental modal analysis transforms a set of measured responses into single degree-of-freedom (DOF) modal responses. Linear modal analysis is a useful tool for updating finite element models, performing low excitation level system predictions, and obtaining modal information about a mechanical system (i.e. natural frequencies and mode shapes). Many industries manufacture mechanical systems assembled using bolted joints. The frictional interfaces that occur due to these joints often introduce nonlinearity into an otherwise linear system. This type of weakly nonlinear response is often observed experimentally as a small change in frequency and a large change in damping [1, 2, 3]. Often this nonlinearity is overlooked when predicting system response which can lead to erroneous results.

This work shows the errors that can arise when one assumes linearity of a weakly nonlinear system and documents this practice from the previous work [4]. This is shown through an experimental-analytical substructuring problem and highlights the errors which arise when the wrong linear parameters are identified due to a weak nonlinearity. The assembly of interest contains multiple bolted joints which add nonlinearity to the system. This system was previously thoroughly tested for nonlinearity in [3, 5, 4] where in [4] the authors completed nonlinear substructuring on the same structure of interest. In [4] the linear substructuring errors for the system were significantly lower than previous works using similar techniques. This study documents the methods used to minimize these substructuring errors through the use of nonlinear theory and testing, providing heuristics and tools to ensure an experimentalist is obtaining the linear dynamics of a system.

This paper is organized as follows. Section 3 details the substructuring example covered in this work. Section 4 provides the theory behind the substructuring method used and the tools suggested to obtain true linear parameters of the system. Section 5 steps through a case study on the experimental system. Finally, Section 6 contains conclusions from this study.

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3. Subcomponent Definitions

The structure of interest consists of 5 main parts: the nose-beam, plate, cylinder, aft-plate, and tail-beam. The beam is bolted to the plate which is attached to the cylinder using eight bolts. The aft plug threads into the cylinder and has a flange that seats on the aft of the cylinder. Finally, the tail is attached to the plug by two bolts. A finite element mesh of the structure is shown in Fig. 1 and the physical hardware is shown in Fig. 2.

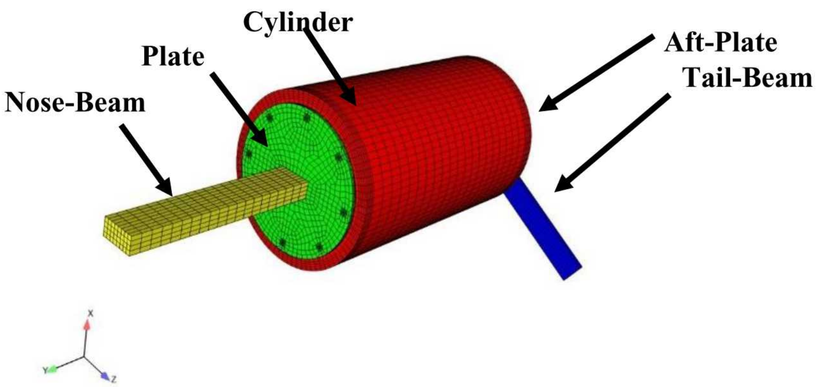

Fig. 1. Truth assembly solid model and structure identification

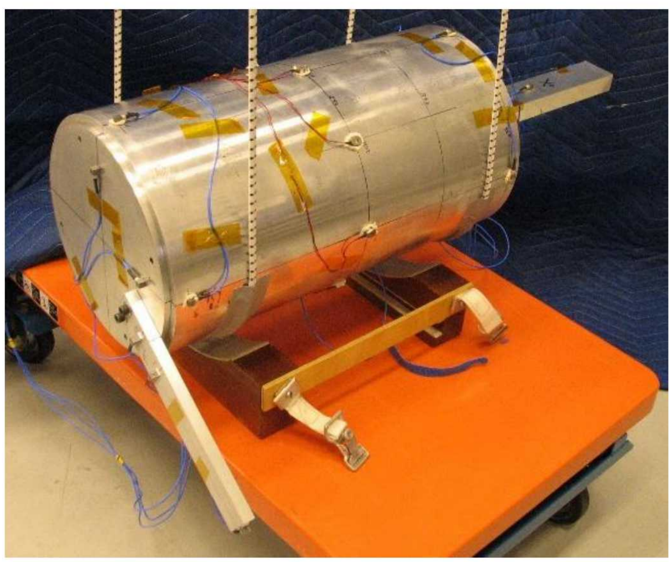

Fig. 2. Truth assembly test hardware

To complete substructuring predictions for this structure, two subcomponent assemblies were tested. The first, subcomponent A, consists of the nose-beam, plate, cylinder, and aft-plate, with no tail-beam. The second subcomponent, B, consists of the plate, cylinder, aft-plate and tail-beam. Finally, a finite element model of the cylinder with end plates is used to connect the two structures. This work stems from [4] where more details on the experimental set-up and substructuring configurations can be found.

4. Theory

This section provides an overview for the theory used in this work. First, the dynamic substructuring theory of the TS method is presented. Second, a nonlinear analysis tool the Hilbert Transform is presented as a means to verify the linear modal parameters extracted from measurements.

4.1 Transmission Simulator Method

The Transmission Simulator method was first introduced by Allen and Mayes in [6]. This method provides a quality modal and best simulates the boundary conditions between subcomponents by mass-loading the interface between subcomponents. In this study, experimental Subcomponent A and B are connected a central cylinder which acts as a mass-loading of the jointed interface. The theory for the TS method is briefly discussed here for convenience.

First, each subcomponent is written as a set of uncoupled modal equations of motion. Modal parameters ω, ζ, and ϕ are the **linear** natural frequency, damping ratio, and mode shapes of a subcomponent model. The modal acceleration, velocity and displacement and external force are represented by $\ddot{q}$, $\dot{q}$, q, and F.

$$\begin{bmatrix} I_A & 0 & 0 \\ 0 & I_B & 0 \\ 0 & 0 & -I_{TS} \end{bmatrix} \begin{Bmatrix} \ddot{q}_A \\ \ddot{q}_B \\ \ddot{q}_{TS} \end{Bmatrix} + \begin{bmatrix} [\ddots 2\zeta_A\omega_A \ddots] & 0 & 0 \\ 0 & [\ddots 2\zeta_B\omega_B \ddots] & 0 \\ 0 & 0 & -[\ddots 2\zeta_{TS}\omega_{TS} \ddots] \end{bmatrix} \begin{Bmatrix} \dot{q}_A \\ \dot{q}_B \\ \dot{q}_{TS} \end{Bmatrix} + \begin{bmatrix} [\ddots \omega_A^2 \ddots] & 0 & 0 \\ 0 & [\ddots \omega_B^2 \ddots] & 0 \\ 0 & 0 & -[\ddots \omega_{TS}^2 \ddots] \end{bmatrix} \begin{Bmatrix} q_A \\ q_B \\ q_{TS} \end{Bmatrix} = \begin{Bmatrix} \phi_A^T F_A \\ \phi_B^T F_B \\ \phi_{TS}^T F_{TS} \end{Bmatrix} \tag{1}$$

To couple the subcomponents, constraints must be enforced on these equations of motion. The constrains are first written in the form of Eqn. (2). Here, x represents physical displacements of each subcomponent, and $\tilde{B}$ is a Boolean matrix such that the motion of shared DOFs between multiple subcomponents is equal.

$$\tilde{B} \begin{Bmatrix} x_A \\ x_B \\ x_{TS} \end{Bmatrix} = 0 \tag{2}$$

Using a modal approximation, this constraint equation can be transformed into modal coordinates as shown in Eqn. (3). The constraints have been softened using the pseudo-inverse of the TS shapes as it would be difficult to enforce them strictly when using measured data.

$$\begin{bmatrix} \phi_{TS}^+ & 0 \\ 0 & \phi_{TS}^+ \end{bmatrix} \begin{bmatrix} \phi_A & 0 & -\phi_{TS} \\ 0 & \phi_B & -\phi_{TS} \end{bmatrix} \begin{Bmatrix} q_A \\ q_B \\ q_{TS} \end{Bmatrix} = \tilde{B} \begin{Bmatrix} q_A \\ q_B \\ q_{TS} \end{Bmatrix} = 0 \tag{3}$$

A coordinate transformation $q = L\eta$ is used to enforce these constraints on the uncoupled equations of motion. Rewriting the constraints in terms of the new generalized coordinate is shown in Eqn. (4).

$$\tilde{B} L \eta = 0 \tag{4}$$

Because $\bar{B}$ is known and because η is arbitrary, L must reside in the nullspace of $\bar{B}$.

$$L = null(\tilde{B}) \tag{5}$$

Using the transformation matrix L, the uncoupled equations of motion can be synthesized into a new set of equations of motion which describe the dynamics of the fully coupled structure. Solving the eigenvalue problem of these equations of motion provides predictions of linear natural frequencies and modes shapes of the assembled structure. Note, the predictions of the assembly involve the linear natural frequencies and damping ratios of the subcomponent system. Therefore, if those are inaccurately measured due to nonlinearity the system level predictions will be inaccurate. This study will explore the magnitude of such errors when the nonlinearity of the system is activated during linear testing.

$$L^T \begin{bmatrix} I_A & 0 & 0 \\ 0 & I_B & 0 \\ 0 & 0 & -I_{TS} \end{bmatrix} L \begin{Bmatrix} \ddot{\eta}_A \\ \ddot{\eta}_B \\ \ddot{\eta}_{TS} \end{Bmatrix} + L^T \begin{bmatrix} [\ddots 2\zeta_A\omega_A \ddots] & 0 & 0 \\ 0 & [\ddots 2\zeta_B\omega_B \ddots] & 0 \\ 0 & 0 & -[\ddots 2\zeta_{TS}\omega_{TS} \ddots] \end{bmatrix} L \begin{Bmatrix} \dot{\eta}_a \\ \dot{\eta}_B \\ \dot{\eta}_{TS} \end{Bmatrix} + L^T \begin{bmatrix} [\ddots \omega_A^2 \ddots] & 0 & 0 \\ 0 & [\ddots \omega_B^2 \ddots] & 0 \\ 0 & 0 & -[\ddots \omega_{TS}^2 \ddots] \end{bmatrix} L \begin{Bmatrix} \eta_A \\ \eta_B \\ \eta_{TS} \end{Bmatrix} = L^T \begin{Bmatrix} \phi_A^T F_A \\ \phi_B^T F_B \\ \phi_{TS}^T F_{TS} \end{Bmatrix} \tag{6}$$

4.2 The Hilbert Transform

Low-level testing is used to capture the linear modal parameters of each subcomponent. To ensure the linear test is not exhibiting any nonlinear behavior the authors implemented the Hilbert Transform [7]. To use this tool a high-level excitation is performed to obtain ringdown data from the structure. Then, the physical measurements from this excitation are transformed to a modal response using the mode shape matrix as a spatial filter.

The Hilbert Transform assumes the modal response can be rewritten in an exponential form as shown in Eqn. (7), where $\psi_r(t)$ is the Hilbert envelope and $\psi_i(t)$ is the unwrapped phase.

$$\ddot{q} = e^{\psi_r(t)+i\psi_i(t)} \tag{7}$$

This amplitude and phase can be fit and used to calculate a time varying frequency and damping ratio for the measured response.

$$\omega_d(t) = \frac{d\psi_i}{dt} \tag{8}$$

$$-\zeta\omega_n(t) = \frac{d\psi_r}{dt} \tag{9}$$

These values can then be plotted against response amplitude to obtain amplitude dependent relationships for frequency and dissipation of each mode. Knowing that at low levels a linear response should have a stationary frequency and damping ratio. Therefore, obtaining ringdown data, performing the Hilbert Transform, and then calculating these amplitude dependent parameters can be used to reveal a true linear frequency and damping ratio during subcomponent testing.

5. Linear Substructuring Tests and Predictions

In this section the linear substructuring predictions from [4] are investigated. The linear predictions are performed with two sets of data. First, slightly nonlinear modal parameters are used from a low-level modal test. Next, the Hilbert Transform is used to determine a shaker setting which is low enough to extract true linear parameters. The predictions are updated with these new parameters in order to highlight the importance of ensuring a linear test is not exceeding linear response levels.

5.1 Original Measurements

Low excitation level impact and shaker tests were performed on subcomponents A and B. Linear natural frequencies, damping values, and mode shapes were obtained using the Synthesize Modes and Correlate (SMAC) [8] program developed by Mayes and Hensley. The results for substructures A and B are shown Table 1 and Table 2 respectively. Here, the subscript o represents the parameters found from initial testing (which was performed at slightly nonlinear levels).

Table 1 – Subcomponent *A* Original Linear Parameters

Mode	$f_{n,o}$ [Hz]	ζ_o [%cr]	Shape Description
7	128.03	0.38	1st bend of Beam in soft direction
8	170.00	0.28	1st bend of Beam in stiff direction
9	548.36	0.37	Axial mode
10	863.75	1.15	(2,0) ovaling of Cylinder
11	878.25	1.11	(2,0) ovaling of Cylinder
12	980.80	0.45	(3,0) ovaling of Cylinder
13	987.70	0.46	(3,0) ovaling of Cylinder
14	1084.20	0.12	2nd bend of Beam in soft direction

Table 2 – Subcomponent *B* Original Linear Parameters

Mode	$f_{n,o}$ [Hz]	ζ_o [%cr]	Shape Description
7	232.23	0.174	1st bend of Tail in soft direction
8	607.00	0.869	1st bend of Tail in stiff direction
9	879.75	0.996	(2,0) ovaling of Cylinder
10	886.06	1.058	(2,0) ovaling of Cylinder
11	981.82	0.437	(3,0) ovaling of Cylinder
12	990.10	0.390	(3,0) ovaling of Cylinder
13	1148.75	1.406	Axial Mode
14	1279.50	0.590	2nd bend of Tail in soft direction

Linear substructuring predictions were calculated following Eqn. (6). There are many realizations of substructuring predictions to consider. In this case, the target range for predictions was up to 600 Hz. Thus, following general component mode synthesis heuristics, modes up to 1200 Hz were included in the subcomponent models. This results in 14 modes for both substructures A and B and only 10 modes for the TS. The predictions from this substructuring exercise are shown in Table 3 compared to the results obtained from an assembled truth test. Note, the frequency error in these predictions is low (under 2%), but the damping error is high (over 30% on multiple modes). These results are typical of component mode synthesis practices, but in [4] the authors were trying to observe the nonlinearity in the system damping. Therefore, a minimization of this damping error was desired.

Table 3 – Original linear Substructuring Predictions

Mode	Predicted $f_{n,o}$ [Hz]	Measured f_n [Hz]	% Diff.	Predicted ζ_o [%cr]	Measured ζ [%cr]	% Diff.	MAC
7	128.50	128.22	0.22%	0.381	0.276	38.22%	1.00
8	170.30	169.46	0.46%	0.282	0.176	60.53%	0.99
9	232.29	233.37	-0.46%	0.168	0.176	-0.70%	1.00
10	548.37	551.39	-0.55%	0.375	0.245	52.97%	0.99
11	606.18	616.25	-1.63%	0.860	0.426	101.57%	1.00

5.2 Example Hilbert Transform Analysis

High-level excitations were performed using a windowed sinusoidal excitation technique discussed in [5]. Using the Hilbert Transform as discussed in Section 4.2 amplitude depenedent relations for frequency and damping ratio were obtained for the first mode of the subcomponent A and are shown in Fig. 3. Here, the blue curve shows the measured amplitude dependent relationship, the circular marker shows the linear results from Section 5.1. The red line shows the true linear parameter as identified from the testing described in Section 5.3. This analysis was repeated for all modes showing a nonlinear response until linear parameters were identified for all modes used in the substructuring predictions.

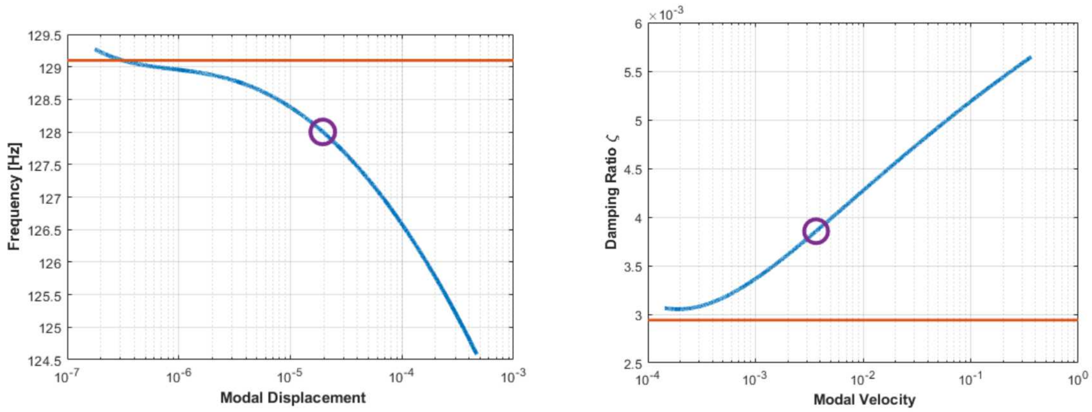

Fig. 3. Hilbert Transform derived amplitude dependent relationships for frequency and damping ratio

5.3 Revised Measurements

Updated linear modal parameters were obtained by continually lowering the excitation level until the extracted parameters were not sensitive to such changes. The final updated parameters for Subcomponents A and B are contained in Table 4 and Table 5. Here, the subscript o represents the original measured linear parameters and the subscript u denotes the updated parameters. For both subcomponents the frequency error from the original measurements was relatively small (less than 1%) but the damping ratio errors were large (many modes over 50%). This is expected as the amplitude dependence of damping is usually more sensitive than that of frequency in weakly nonlinear structures.

Table 4 – Subcomponent A Updated Linear Parameters

Mode	$f_{n,o}$ [Hz]	ζ_o [%cr]	$f_{n,u}$ [Hz]	ζ_u [%cr]	f_n % Error	ζ % Error
7	128.03	0.384	129.2	0.294	-0.91%	30.61%
8	170.00	0.284	171.1	0.170	-0.64%	67.06%
9	548.36	0.373	552.0	0.241	-0.66%	54.77%
10	863.75	1.149	867.8	1.122	-0.47%	2.41%
11	878.25	1.113	883.3	0.930	-0.57%	19.68%
12	980.80	0.451	980.8	0.447	0.00%	0.89%
13	987.70	0.456	987.7	0.462	0.00%	-1.30%
14	1084.20	0.123	1031.8	0.090	-0.10%	37.17%

Table 5 – Subcomponent *B* Updated Linear Parameters

Mode	$f_{n,o}$ [Hz]	ζ_o [%cr]	$f_{n,u}$ [Hz]	ζ_u [%cr]	f_n % Error	ζ % Error
7	232.23	0.174	232.5	0.168	-0.12%	3.87%
8	607.00	0.869	608.9	0.415	-0.31%	109.25%
9	879.75	0.996	882.5	1.138	-0.31%	-12.47%
10	886.06	1.058	892.4	0.812	-0.71%	30.35%
11	981.82	0.437	981.8	0.448	0.00%	-2.49%
12	990.10	0.390	990	0.393	0.01%	-0.64%
13	1148.75	1.406	1171	1.075	-1.90%	30.79%
14	1279.50	0.590	1284.5	0.480	-0.39%	22.87%

Table 6 contains the substructuring predictions using the updated linear subcomponent parameters. The results showed great improvement in the damping ratio predictions with only one mode having an error higher than 5%. The frequency errors remained mostly constant. The first two elastic modes increased in error, from below 0.5% error to 1% error, while the following three modes all decreased in frequency error. This case study shows that it is prudent to ensure that linear testing is truly linear!

Table 6 – Updated linear substructuring predictions

Mode	Predicted $f_{n,u}$ [Hz]	Measured f_n [Hz]	% Diff.	Predicted ζ_u [%cr]	Measured ζ [%cr]	% Diff.	MAC
7	129.62	128.22	1.09%	0.295	0.276	6.94%	1.00
8	171.47	169.46	1.19%	0.171	0.176	-2.90%	0.99
9	232.56	233.37	-0.35%	0.168	0.176	-4.47%	1.00
10	552.01	551.39	0.11%	0.241	0.245	-1.43%	0.99
11	608.08	616.25	-1.32%	0.415	0.426	-2.56%	1.00

6. Conclusions and Remarks

This paper documents lessons learned from a substructuring exercise previously presented in [4]. In this previous study the authors completed experimental-analytical dynamic substructuring predictions at linear levels with very low damping ratio error. To achieve this accuracy level the authors performed multiple tests on each subcomponent to assure that the systems were being tested at low, linear excitation levels. With the original measurements damping ratio errors were observed as high as 100% error. After using a few heuristics to ensure the quality of the extracted linear model, this error was reduced to under 7%, showing a drastic improvement.

It is recommended that for future test plans, where linear information is desired, that test engineers include test points at high-level and use the Hilbert Transform to determine if the level set for linear testing is sufficient for capture linear structural dynamic behavior. An alternative recommendation to high-level testing would be to make multiple measurements decreasing the amplitude of excitation in each case until the extracted modal parameters are shown to not change.

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