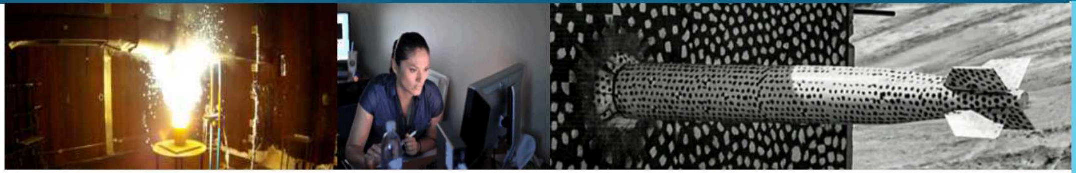
SAND2020-0145C

Solving IPDEs on Spiking Neuromorphic Hardware



Presented by

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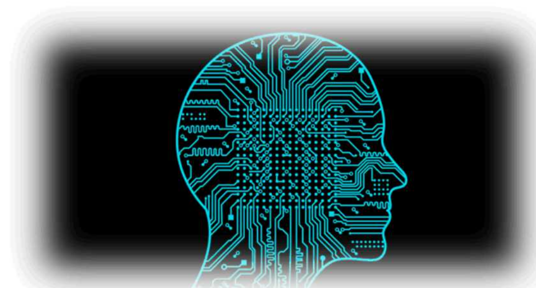
Overview

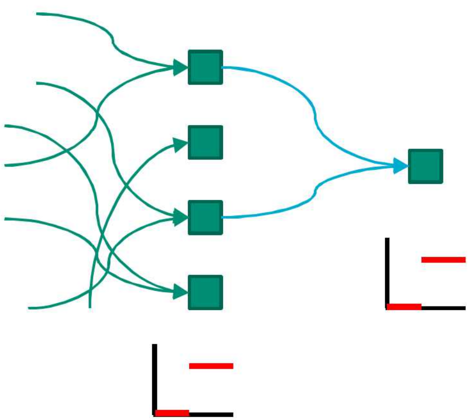
- Introduction to Spiking Neuromorphic Hardware
- Connections between SDEs and PDEs
- Generalized Feynman-Kac
- Application to Particle Transport
- Implementation Accuracy
- Conclusions

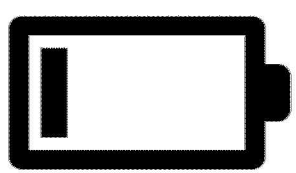
What is “Spiking Neuromorphic Hardware”?

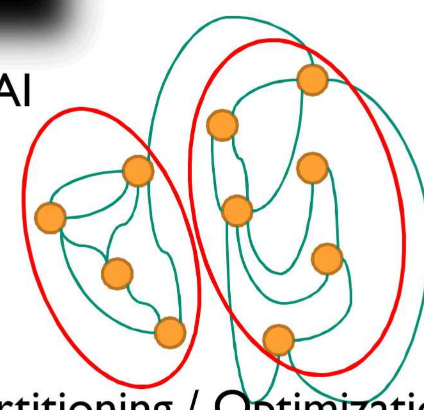
Biologically Inspired Computing



Machine Learning / AI







Graph Partitioning / Optimization

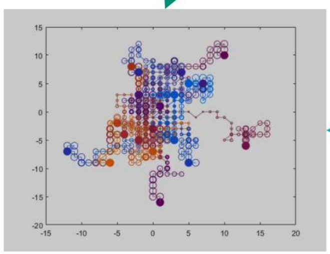
Connecting PDEs and Random Walks

$$u_t = d u_{xx}$$

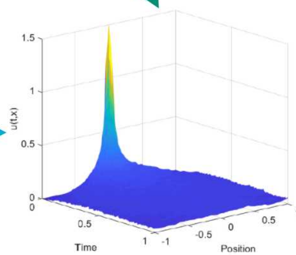
$$u(0,x) = \begin{cases} -150x^2 + \frac{3}{2}, & x \in \left[-\frac{1}{10}, \frac{1}{10}\right] \\ 0, & \text{otherwise} \end{cases}$$

$\mathbb{P}$

Traditional



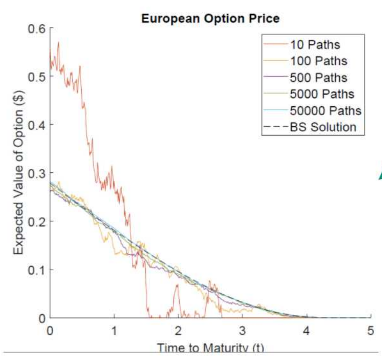
$\mathbb{E}$



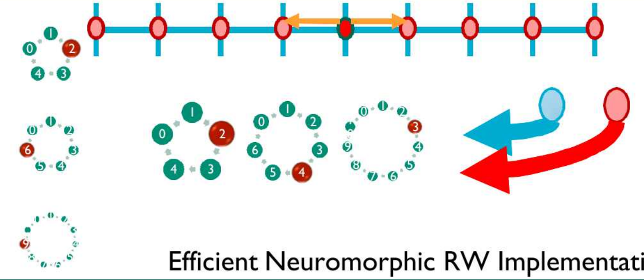
Known

$$-V_t + rV = \frac{1}{2}\sigma^2 x^2 V_{xx} + rxV_x$$

Deduction from underlying stock process



Challenging Applications, Low Power Implementation



Efficient Neuromorphic RW Implementation

+

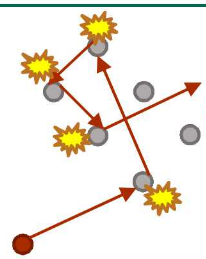
$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$$

$$u_t = f(u_x, u_{xx}, t)$$

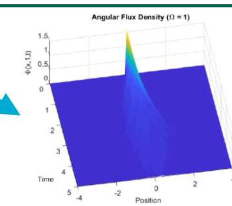
$$u(t,x) = \mathbb{E}[g(X_t)|X_0 = x]$$

Rigorous Probabilistic Connections

=



$$\mathbb{E}[e^{-\sigma_a t} B(X_t, Y_t)|X_0, Y_0]$$



Low Power Particle Transport Simulation

A Class of Integro-PDEs with a Probabilistic Interpretation

The IPDE-IVP

$$u_t(t,\boldsymbol{x}) = \frac{1}{2}\sum_{i,j} a_{i,j}(t,\boldsymbol{x})u_{x_i x_j}(t,\boldsymbol{x}) + \sum_i b_i(t,\boldsymbol{x})u_{x_i}(t,\boldsymbol{x})$$
$$+\lambda(t,\boldsymbol{x})\int \Big(u\big(t,\boldsymbol{x}+h(t,\boldsymbol{x},q)\big) - u(t,\boldsymbol{x})\Big)\,\phi_Q(q;t,\boldsymbol{x})\mathrm{d}q$$
$$+c(t,\boldsymbol{x})u(t,\boldsymbol{x}) + f(t,\boldsymbol{x})$$
$$u(t,\boldsymbol{x}) = g(\boldsymbol{x})$$

has solution

$$u(t,\boldsymbol{x}) = \mathbb{E}\left[g(\boldsymbol{X}_t)\exp\left(\int_0^t c(s,\boldsymbol{X}_s)ds\right) + \int_0^t f(s,\boldsymbol{X}_s)\exp\left(\int_0^s c(u,\boldsymbol{X}_u)du\right)ds \,\middle|\, \boldsymbol{X}_0 = \boldsymbol{x}\right]$$

where

$$\mathrm{d}\boldsymbol{X}_t = b(t,\boldsymbol{X}_t)\mathrm{d}t + \sigma(t,\boldsymbol{X}_t)\mathrm{d}W_t + h(t,\boldsymbol{X}_t,Q)\mathrm{d}P_{t;,Q,t,\boldsymbol{X}_t}$$

and $a, b, c, g, h,$ and f are all real valued, $\lambda < 0$; further for each t and $\boldsymbol{x}$ that $\phi_Q \geq 0$ and $\int \phi_Q(q)\mathrm{d}q$ so that $P(t;Q,t,\boldsymbol{x})$ is a Poisson process with rate $-\int_0^t \lambda(s,\boldsymbol{x})\mathrm{d}s$. We further require that $a = \sigma\sigma^\top, b,$ and h are all defined so that the stochastic process $\boldsymbol{X}_t$ has a unique solution that belongs almost surely to the domain of g.

A Class of Integro-PDEs with a Probabilistic Interpretation

The IPDE-IVP

$$u_t(t,\boldsymbol{x}) = \frac{1}{2}\sum_{i,j} a_{i,j}(t,\boldsymbol{x}) u_{x_i x_j}(t,\boldsymbol{x}) + \sum_i b_i(t,\boldsymbol{x}) u_{x_i}(t,\boldsymbol{x})$$

$$+\lambda(t,\boldsymbol{x}) \int \Big(u\big(t,\boldsymbol{x}+h(t,\boldsymbol{x},q)\big) - u(t,\boldsymbol{x})\Big)\phi_Q(q;t,\boldsymbol{x})\mathrm{d}q$$

$$+c(t,\boldsymbol{x})u(t,\boldsymbol{x}) + f(t,\boldsymbol{x})$$

$$u(t,\boldsymbol{x}) = g(\boldsymbol{x})$$

$$u(t,\boldsymbol{x}) = \mathbb{E}\left[g(\boldsymbol{X}_t)\exp\left(\int_0^t c(s,\boldsymbol{X}_s)ds\right) + \int_0^t f(s,\boldsymbol{X}_s)\exp\left(\int_0^s c(u,\boldsymbol{X}_u)du\right)ds \,\middle|\, \boldsymbol{X}_0 = \boldsymbol{x}\right]$$

Non-Zero Terms	Application	SDE Example
a	Heat Equation	$\mathrm{d}X_t = \sigma \mathrm{d}W_t$
a, b, f	European Option Pricing	$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t \mathrm{d}W_t$
b, λ, h, c, f	Particle Transport	$dX_t = -vY_t dt;\ dY_t = \omega_{Y_t} dP_{t;Y_t}$
a, f	Electrostatic Scalar Potential*	$\mathrm{d}X_t^{(i)} = \sqrt{\varepsilon}\mathrm{d}W_t$
b, c	Pollutant Source Deterioration	$\mathrm{d}X_t = v\mathrm{d}t$^

*steady state solution, ^actually an ODE

Boltzmann Particle Transport Equation

- In one-dimension with constant energy, the **angular flux density** $\Phi(x,\Omega,t)$ is the product of particle speed v and the angular density of particles at position x traveling in direction Ω at time t, $N(x,\Omega,t)$.
- The angular flux density is assumed to satisfy the Boltzmann transport equation:

$$\frac{1}{v}\frac{\partial\Phi}{\partial t}+\Omega\frac{\partial\Phi}{\partial x}+(\sigma_s+\sigma_a)\Phi=\int\Phi(x,\Omega',t)\sigma_s(x,t)p(\Omega'\to\omega)\mathrm{d}\Omega'+S.$$

- Traditionally, the method of characteristics is employed to reduce this equation to an integral equation of the second kind:

$$\Phi=\boldsymbol{K}\Phi+S',$$

where $\boldsymbol{K}$ is an integral operator.

- This is then rewritten as the Neumann series

$$\Phi=\sum\Phi_i,\qquad \Phi_0=S',\qquad \Phi_i=\boldsymbol{K}\Phi_{i-1},$$

with the physical interpretation that Φ_i is the angular flux of particles that have undergone exactly *i* collisions.

Probabilistic Boltzmann Transport Equation

- The Boltzmann transport equation can be written in the general IPDE form through the change of variable $\omega = \Omega' - \Omega$:

$$\frac{\partial \Phi}{\partial t} = -v\Omega \frac{\partial \Omega}{\partial x} + 0 \cdot \frac{\partial \Phi}{\partial \Omega} + v\sigma_s(x,t) \int \big(\Phi(x, \Omega + \omega, t) - \Phi(x, \Omega, t)\big) p(\omega \to 0|\Omega) d\omega$$
$$-v\sigma_a(x,t)\Phi + vS.$$

- The generalized Feynman-Kac allows us to write the solution to the Boltzmann equation with initial condition $\Phi(x, \Omega, 0) = B(x, \Omega)$ as

$$\Phi(x, \Omega, t) = \mathbb{E}\left(B(X_t, Y_t) \exp\left(-v \int_0^t \sigma_a(X_s, s) \mathrm{d}s\right) \middle| X_0 = x, Y_0 = \Omega\right)$$
$$+\mathbb{E}\left(v \int_0^t S(X_s, Y_z, s) \exp\left(-v \int_0^t \sigma_a(X_u, u) \mathrm{d}u\right) \mathrm{d}s \middle| X_0 = x, Y_0 = \Omega\right),$$

where the underlying stochastic process is given by

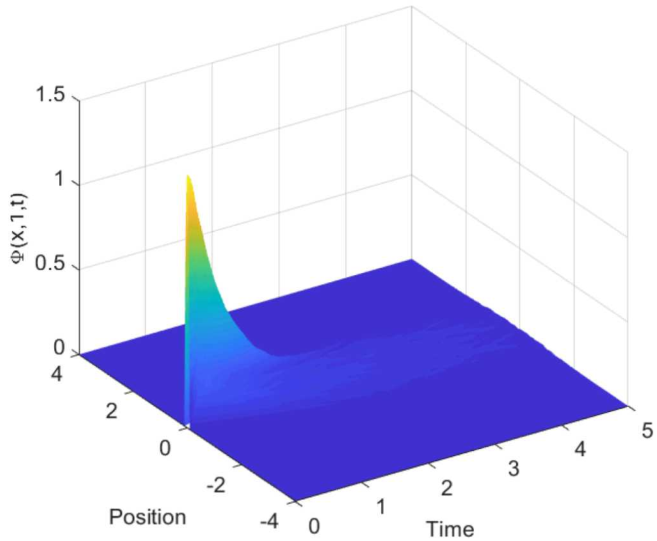
$$dX_t = -vY_t dt$$
$$dY_t = \omega_{Y_t} dP_{t;Y_t}.$$

Proof of Concept Example

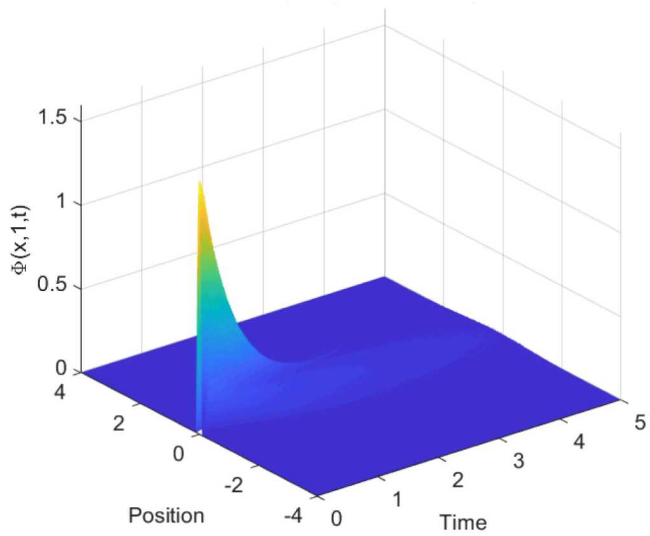
$$\frac{\partial\Phi}{\partial t} = -\frac{1}{2}\Omega\frac{\partial\Omega}{\partial x} + \frac{5}{2}\int\big(\Phi(x,\Omega+\omega,t) - \Phi(x,\Omega,t)\big)p(\omega\to 0|\Omega)\mathrm{d}\omega - \frac{1}{4}\Omega$$

$$\Phi(x,\Omega,0) = B(x,\pm 1) = \begin{cases} -150x^2 + \frac{3}{2} & x\in\left[-\frac{1}{10},\frac{1}{10}\right] \\ 0 & \text{otherwise} \end{cases}$$

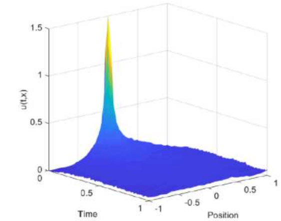
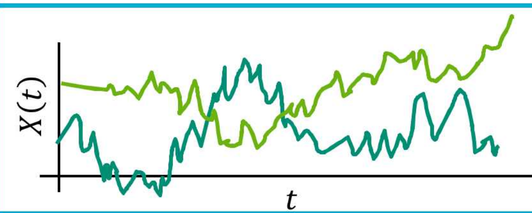
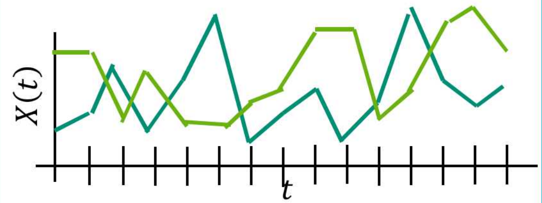
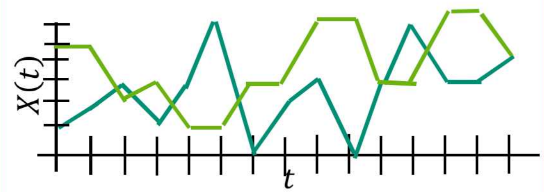
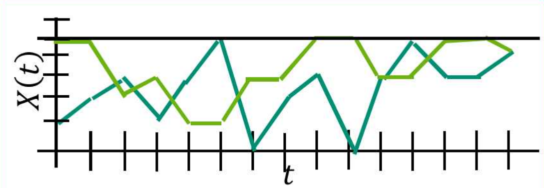
$\Phi(x,1,t)$ using SDE



$\Phi(x,1,t)$ using Neumann Series



Accuracy Stack for Neuromorphic Implementation

$u_t = f(t, u, u_x, u_{xx})$ $u(t,x) = \mathbb{E}[g(t, X_t) \mid X_0 = x]$	PDE and Solution Ground Truth		
$u(t,x) \approx \frac{1}{M}\sum_{i=1}^{M} g(t, X_t^i);\ \ X_0^{(i)} = x$	We must sample paths from the stochastic process, incurring sampling error on the order of $\frac{1}{\sqrt{M}}$.		$\frac{1}{\sqrt{M}}$
$u(j\Delta t, x) \approx \frac{1}{M}\sum_{i=1}^{M} g(j\Delta t, X_{j\Delta t}^i);\ \ X_0^{(i)} = x$	We cannot sample the continuous paths of X_t, so we approximate by a discretization scheme. We incur error based on this scheme.		$\sqrt{\Delta t}$
$u(j\Delta t, x_k) \approx \frac{1}{M}\sum_{i=1}^{M} g(j\Delta t, \hat{X}_{j\Delta t}^i);\ \ \hat{X}_0^{(i)} = x_k$	Further we limit the values the walk can take. Denote the process by $\hat{X}$ which assumes its values on a grid size of Δs. In the best case scenario, the error accrued in each time step is proportional to $\Delta s/2$		$\frac{1}{2} j\Delta t\Delta s$
$u(j\Delta t, x_k) \approx \frac{1}{M}\sum_{i=1}^{M} g(j\Delta t, \check{X}_{j\Delta t}^i);\ \ \check{X}_0^{(i)} = x_k$	Finally, we may need to set a max value for the process, causing dependent on the function g, the process, and the max value set.		varies

Conclusions and Directions

PDEs

Linear, Elliptic IPDEs

Use Feynman-Kac style equations to approximate solutions via random walks.

Utilize application specific, bespoke random walk methods to solve specialized problems.

Example: Chemotaxis

$$u_t = \nabla(D_u \nabla u - \chi_u u \nabla v)$$
$$v_t = D_v \Delta v + \mu u - \delta v$$

Existing random walk and Monte Carlo Methods will work in our framework.

Explore a variety of machine learning, deep learning and other neuromorphic suited methods to solve Riemann problems.

Example: Sod Shock Tube

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho \\ \rho U \\ E\end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix}\rho U \\ \rho U^2 + \rho \\ (E+\rho)U\end{pmatrix} = 0$$

Will require new techniques to solve for specific subclasses.

Use Ito Calculus to relate certain linear hyperbolic PDEs to SDEs through clever averaging.

Example: Wave Equations

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Investigate new Feynman-Kac style formulas for vector valued linear PDEs/systems of connected linear PDEs.

Example: Linear Riemann Problem

$$\frac{\partial W}{\partial t} = -\begin{bmatrix} 0 & p_0 \\ \frac{a^2}{p_0} & 0 \end{bmatrix} \frac{\partial W}{\partial x}$$

Non-Linear Example: Steady Flow for Inviscid Fluids
- Not solvable with linear techniques, but insight will be gained.

Example: Schrodinger operators

$$Hf = -\Delta f + Qf$$

where Q is not diagonal

Under current investigation.

Selected References

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