Title: Well-Balancing Gravity in Eulerian Hydrodynamics

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Intended for: presentation to external colleagues

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WELL-BALANCING GRAVITY IN EULERIAN HYDRODYNAMICS

Philipp Edelmann (XCP-1)

EAP Colloquium
December 14, 2020

EULER EQUATIONS WITH GRAVITY

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} + \frac{\partial \mathbf{G}(\mathbf{U})}{\partial y} + \frac{\partial \mathbf{H}(\mathbf{U})}{\partial z} = \mathbf{S}(\mathbf{U})$$

$$\mathbf{U} = (\rho, \rho u, \rho v, \rho w, E)^T \quad E = \rho\epsilon + \frac{1}{2}\rho\,|\,\mathbf{v}\,|^{\,2} + \rho\phi$$

HYDROSTATIC EQUILIBRIUM

Static ($\mathbf{v} = 0, \dfrac{\partial \mathbf{U}}{\partial t} = 0$) solution to the Euler equations with gravity

$$\nabla p = \rho \mathbf{g}$$

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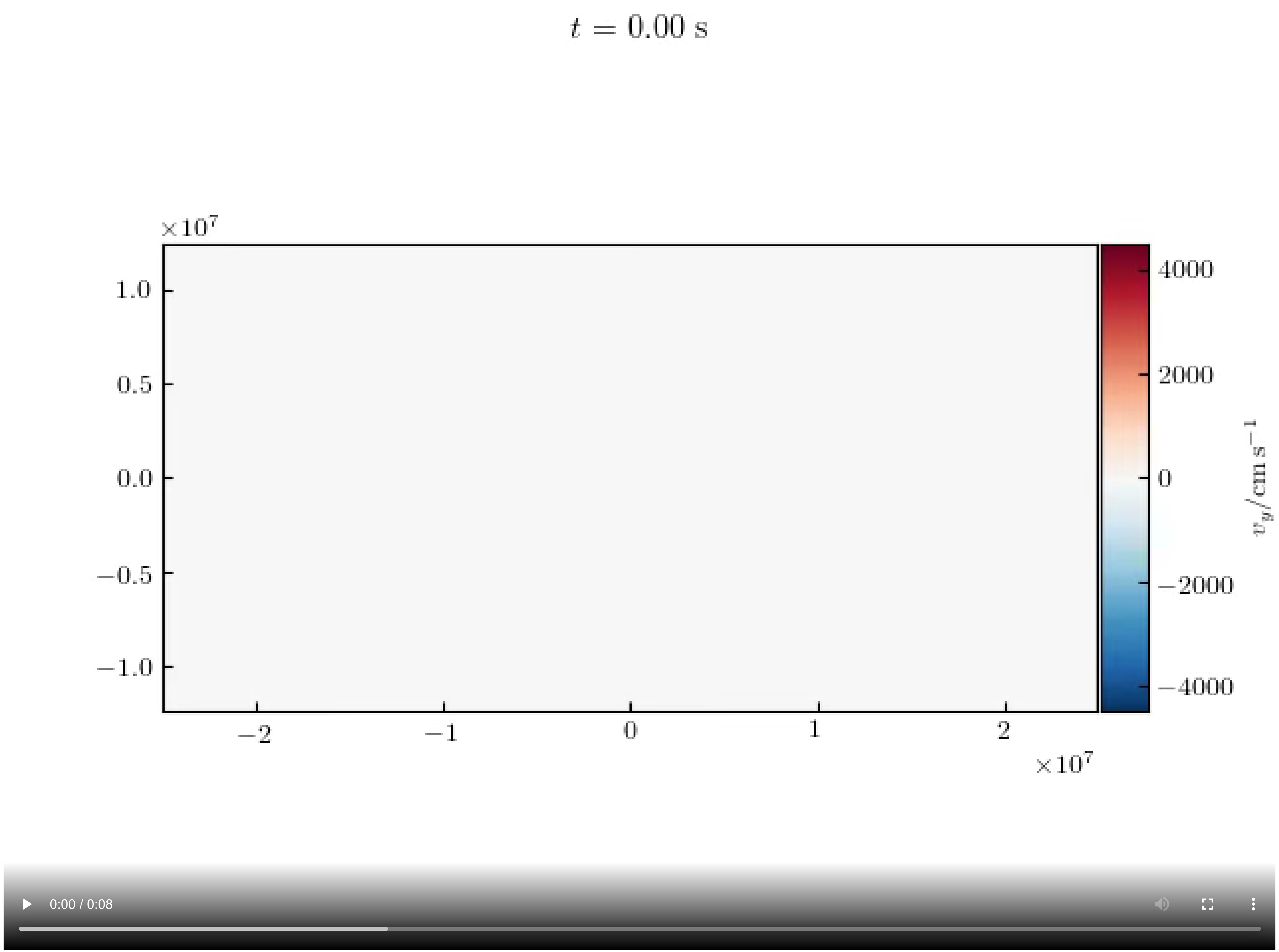
- $N^2 > 0$ (subadiabatic): small perturbations oscillate around equilibrium
- $N^2 < 0$ (superadiabatic): exponential growth of small perturbations forms turbulent convection

$N^2 > 0$

$N^2 < 0$

$N^2 > 0$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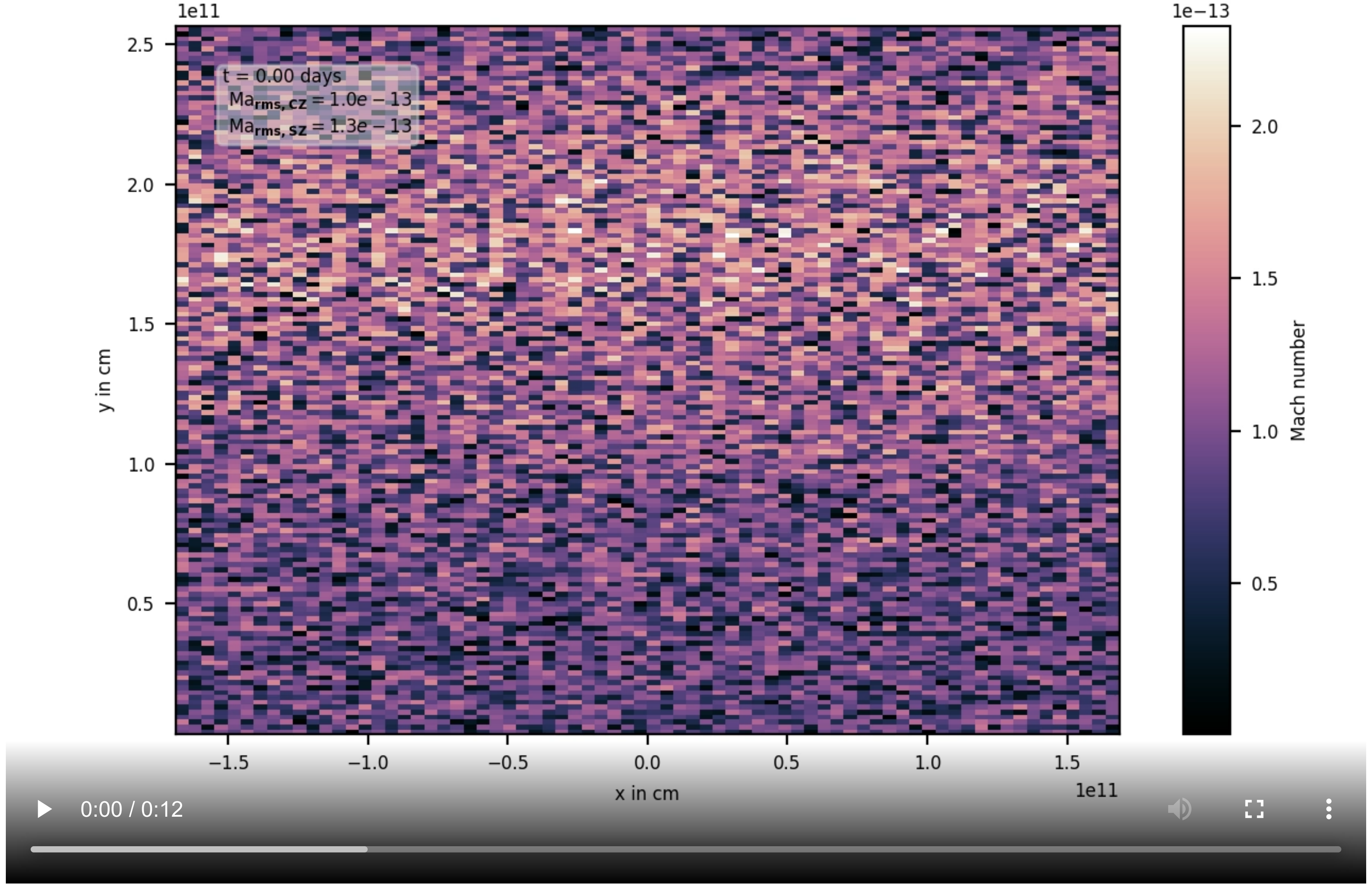
MOTIVATION: CONVECTION AT VERY LOW MACH NUMBERS

massive stars during core hydrogen burning typically have Mach numbers of 10^{-4}



no well-balancing

courtesy of Leo Horst (HITS)

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with well-balancing

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Well-Balanced Treatment of Gravity in Astrophysical Fluid Dynamics Simulations at Low Mach Numbers

P. V. F. Edelmann[1], L. Horst[2], J. P. Berberich[3], R. Andrassy[2], J. Higl[2], C. Klingenberg[3], and F. K. Röpke[2, 4]

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to be submitted by end of 2020

THE CONCEPT OF WELL-BALANCING

illustrate using single forward Euler step in 1D

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$$\rho_i^1 = \rho_i^0 - \frac{\Delta t}{\Delta x}\left(\left(\hat{\mathbf{F}}^0_{i+\frac{1}{2}}\right)_1 - \left(\hat{\mathbf{F}}^0_{i-\frac{1}{2}}\right)_1\right)$$

$$(\rho u)_i^1 = (\rho u)_i^0 - \frac{\Delta t}{\Delta x}\left(\left(\hat{\mathbf{F}}^0_{i+\frac{1}{2}}\right)_2 - \left(\hat{\mathbf{F}}^0_{i-\frac{1}{2}}\right)_2\right) + \Delta t\left(\hat{\mathbf{S}}_i\right)_2$$

$$E_i^1 = E_i^0 - \frac{\Delta t}{\Delta x}\left(\left(\hat{\mathbf{F}}^0_{i+\frac{1}{2}}\right)_3 - \left(\hat{\mathbf{F}}^0_{i-\frac{1}{2}}\right)_3\right)$$

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CARGO–LE ROUX METHOD

Cargo & Le Roux (1994); Le Roux (1999)

start out from 1D Euler eqs. with gravity

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho \\ \rho u \\ E^{'}\end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix}\rho u \\ \rho u^2 + p \\ u(E^{'} + p)\end{pmatrix} = \begin{pmatrix}0 \\ \rho g \\ \rho g u\end{pmatrix}; \qquad E^{'} = \rho\epsilon + \frac{1}{2}\rho\,|\,\mathbf{v}\,|^{\,2}$$

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introduce potential q with properties: $\frac{\partial q}{\partial x}=\rho g,\quad \frac{\partial q}{\partial t}=-\rho u g$

introduce new pressure and energy: $\Pi=p-q,\quad F'=E'+q$

CARGO–LE ROUX METHOD

same form as Euler equations without any source terms

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho \\ \rho u \\ F^{'}\end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix}\rho u \\ \rho u^2 + \Pi \\ u(F^{'} + \Pi)\end{pmatrix} = \begin{pmatrix}0 \\ 0 \\ 0\end{pmatrix}$$

modified equation of state: $\Pi = p - q, \quad F^{'} = E^{'} + q$

q corresponds to hydrostatic pressure profile

q is evolved using advection equation: $\frac{\partial(\rho q)}{\partial t} + \frac{\partial(\rho q u)}{\partial x} = 0$

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$$\mathbf{U} = (\rho, \rho u, \rho v, \rho w, E)^T \quad E = \rho \epsilon + \frac{1}{2} \rho |\mathbf{v}|^2 + \rho \phi$$

$$\nabla q = \rho_0 \mathbf{g}; \quad \frac{\partial q}{\partial t} = 0; \quad F = E + q$$

αβ METHOD

Berberich+ (2018)

modify the reconstruction step

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convert back after reconstruction

DEVIATION METHOD

Berberich+ (2020)

known stationary solution $\tilde{\mathbf{U}}$ ($\mathbf{v}$ can be nonzero): $\frac{\partial \tilde{\mathbf{U}}}{\partial t} = 0$

$$\frac{\partial \mathbf{F}(\tilde{\mathbf{U}})}{\partial x} + \frac{\partial \mathbf{G}(\tilde{\mathbf{U}})}{\partial y} + \frac{\partial \mathbf{H}(\tilde{\mathbf{U}})}{\partial z} = \mathbf{S}(\tilde{\mathbf{U}})$$

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subtract equilibrium eq. from Euler eq. for arbitrary $\mathbf{U}$, expressed using $\Delta\mathbf{U} = \mathbf{U} - \tilde{\mathbf{U}}$

DEVIATION METHOD (CONTINUED)

$\Delta\mathbf{U}$ at next step is calculated via:

$$\frac{\partial(\Delta\mathbf{U})_{i,j,k}}{\partial t} = \mathbf{F}^{\mathrm{dev}}_{i-\frac{1}{2},j,k} - \mathbf{F}^{\mathrm{dev}}_{i+\frac{1}{2},j,k} + \mathbf{G}^{\mathrm{dev}}_{i,j-\frac{1}{2},k} - \mathbf{G}^{\mathrm{dev}}_{i,j+\frac{1}{2},k} + \mathbf{H}^{\mathrm{dev}}_{i,j,k-\frac{1}{2}} - \mathbf{H}^{\mathrm{dev}}_{i,j,k+\frac{1}{2}} + \mathbf{S}^{\mathrm{dev}}_{i,j,k}$$

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perform reconstruction of $\Delta\mathbf{U}$ only

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SEVEN-LEAGUE HYDRO CODE

- solves the compressible Euler equations in 1-, 2-, 3-D
- explicit and implicit time integration
- flux preconditioning to ensure correct behavior at low Mach numbers
- other low Mach number schemes (e.g. AUSM$^+$-up)
- works for low and high Mach numbers on the same grid
- hybrid (MPI, OpenMP) parallelization (works up to 458 752 cores)
- several solvers for the linear system:
  BiCGSTAB, GMRES, Multigrid, (direct)
- arbitrary curvilinear meshes
  using a rectangular computational mesh
- gravity solver (monopole, Multigrid)
- radiation in the diffusion limit
- general equation of state
- general nuclear reaction network

LOW MACH NUMBER FLUXES

Typical flux functions in finite-volume schemes have Mach number dependent dissipation

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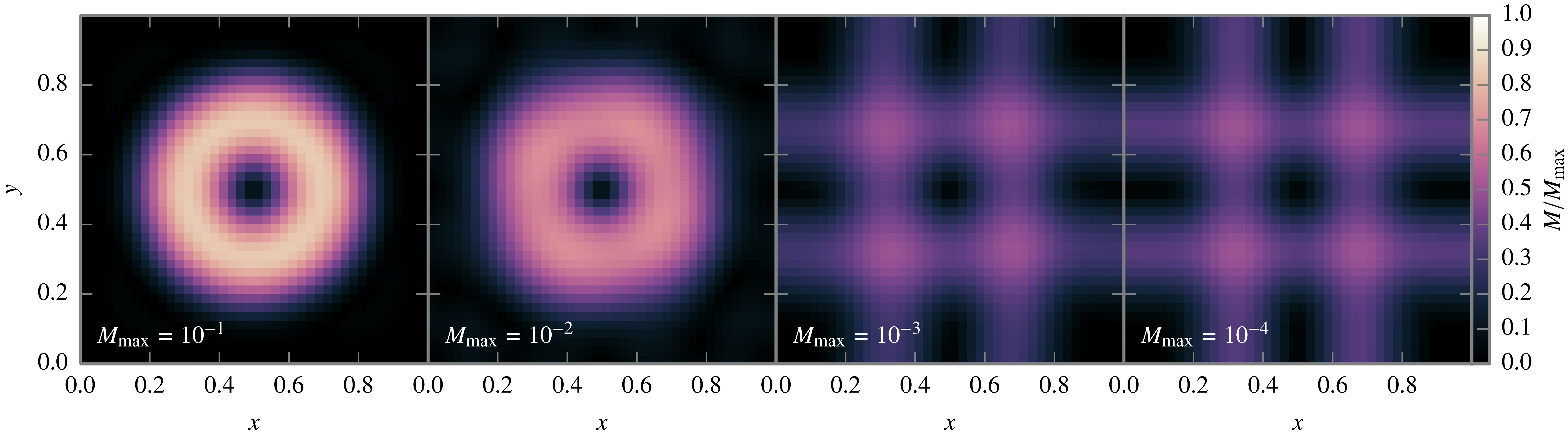
example: Gresho vortex with variable Mach number (Miczek+, 2015)



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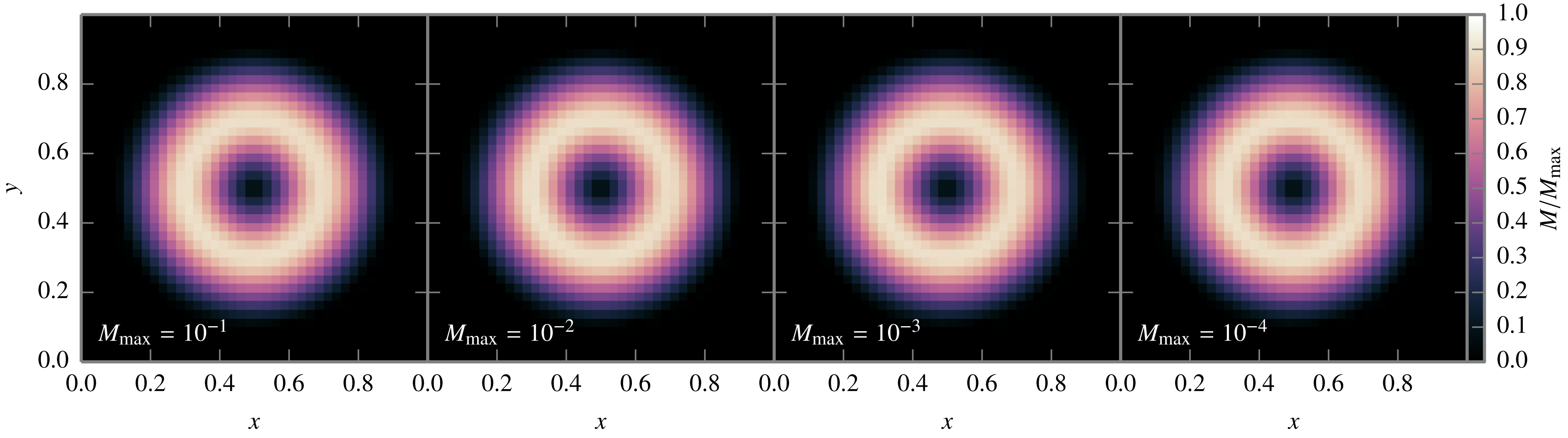
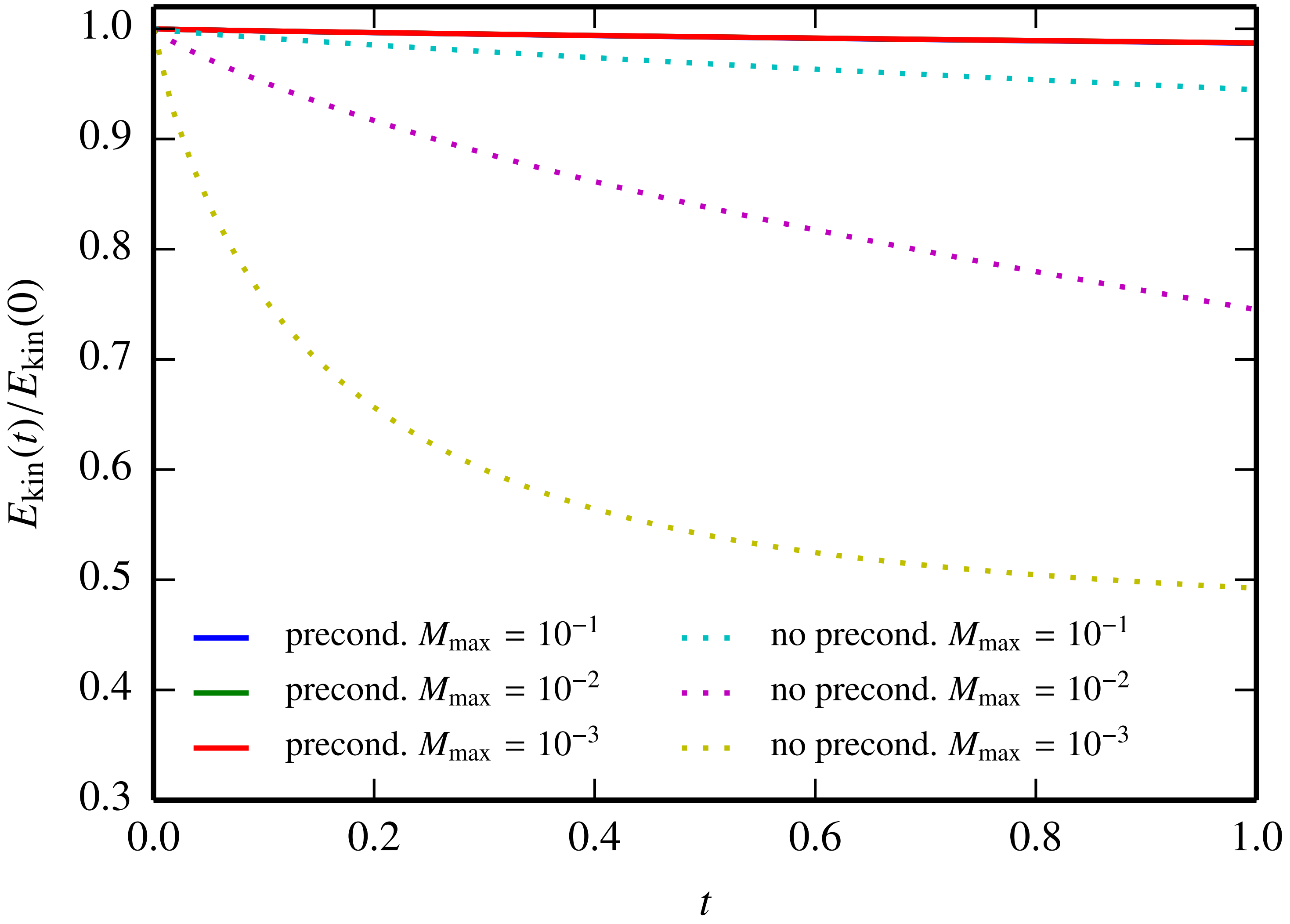


low Mach flux



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TIME STEP

$$M = \frac{|u|}{c}$$

explicit: $\Delta t \leq \dfrac{\Delta x}{|u| + c} \overset{M \ll 1}{=} \dfrac{\Delta x}{c}$

implicit: $\Delta t \leq \dfrac{\Delta x}{|u|}$

DETAILS OF IMPLICIT TIME-STEPPING

- typically use ESDIRK scheme (Explicit first stage, Singly Diagonally Implicit Runge–Kutta)
- a nonlinear system of eqs. needs to be solved at each (sub)step
- solved using Newton–Raphson method
- good initial guess: previous step
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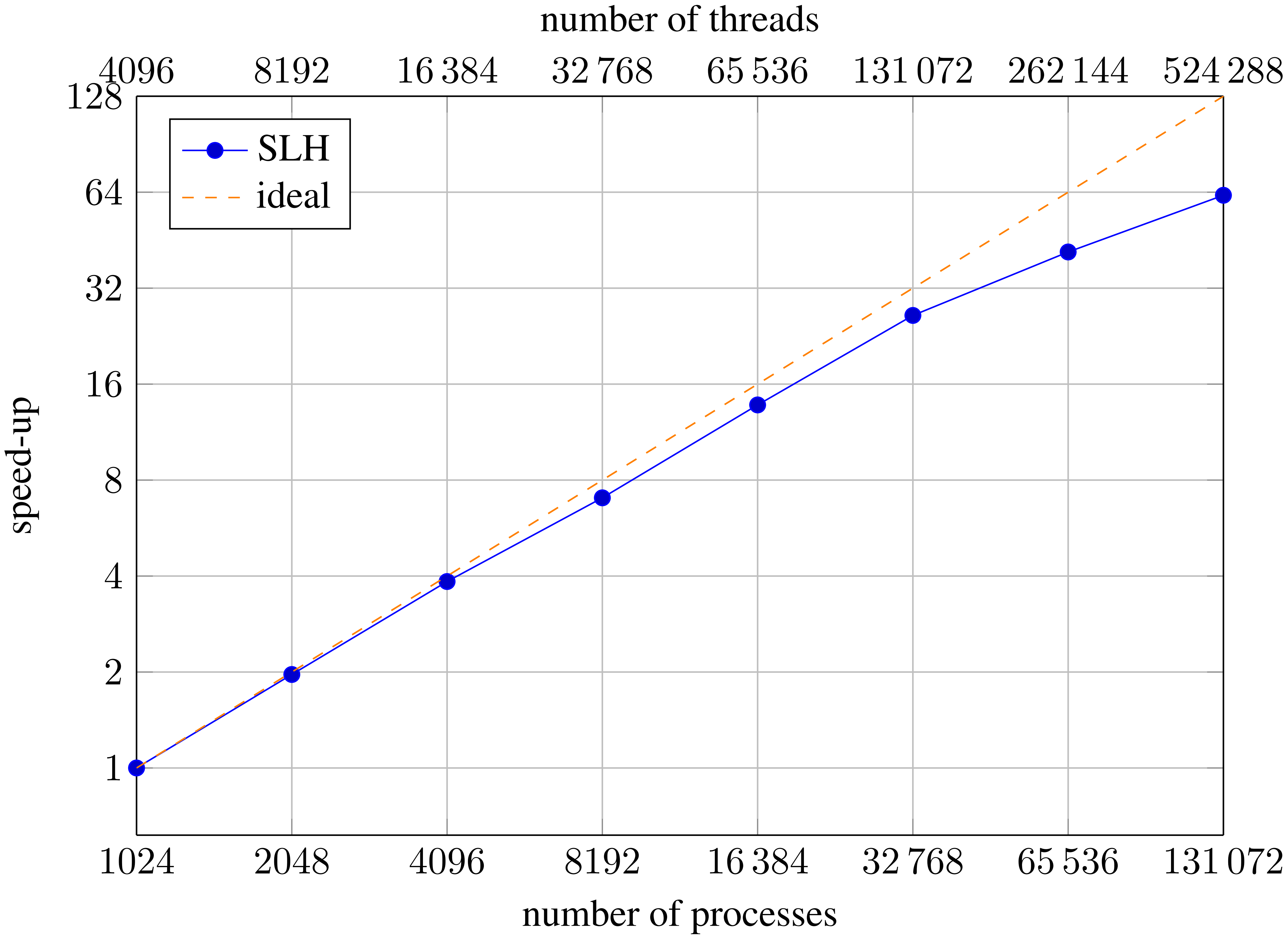
SIZE OF THE SYSTEM (512^3 EXAMPLE)

- size: $5N_xN_yN_z$ equations (6.7×10^8)
- Jacobian: $(5N_xN_yN_z)^2$ matrix ($4.5 \times 10^{17} \hat{=} 3.1\,\mathrm{EiB}$)
- nonzero entries: $325N_xN_yN_z$ ($4.3 \times 10^{10} \hat{=} 325\,\mathrm{GiB}$)
- sparseness: $9.7 \times 10^{-6}\%$

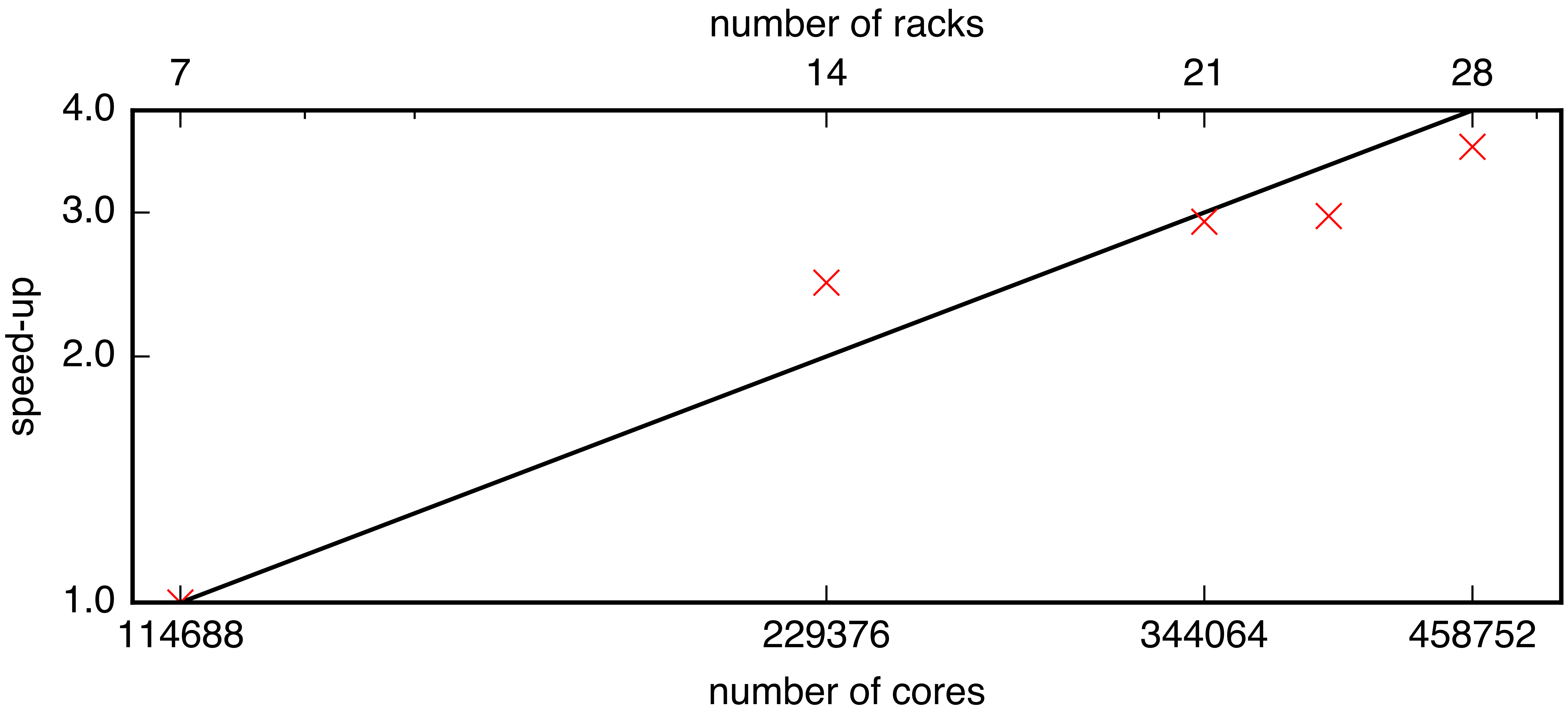
STRONG SCALING

512^3 grid on JUQUEEN
(Bluegene/Q at FZJ Jülich, Germany)



STRONG SCALING

1344^3 grid on JUQUEEN
(Bluegene/Q at FZJ Jülich, Germany)

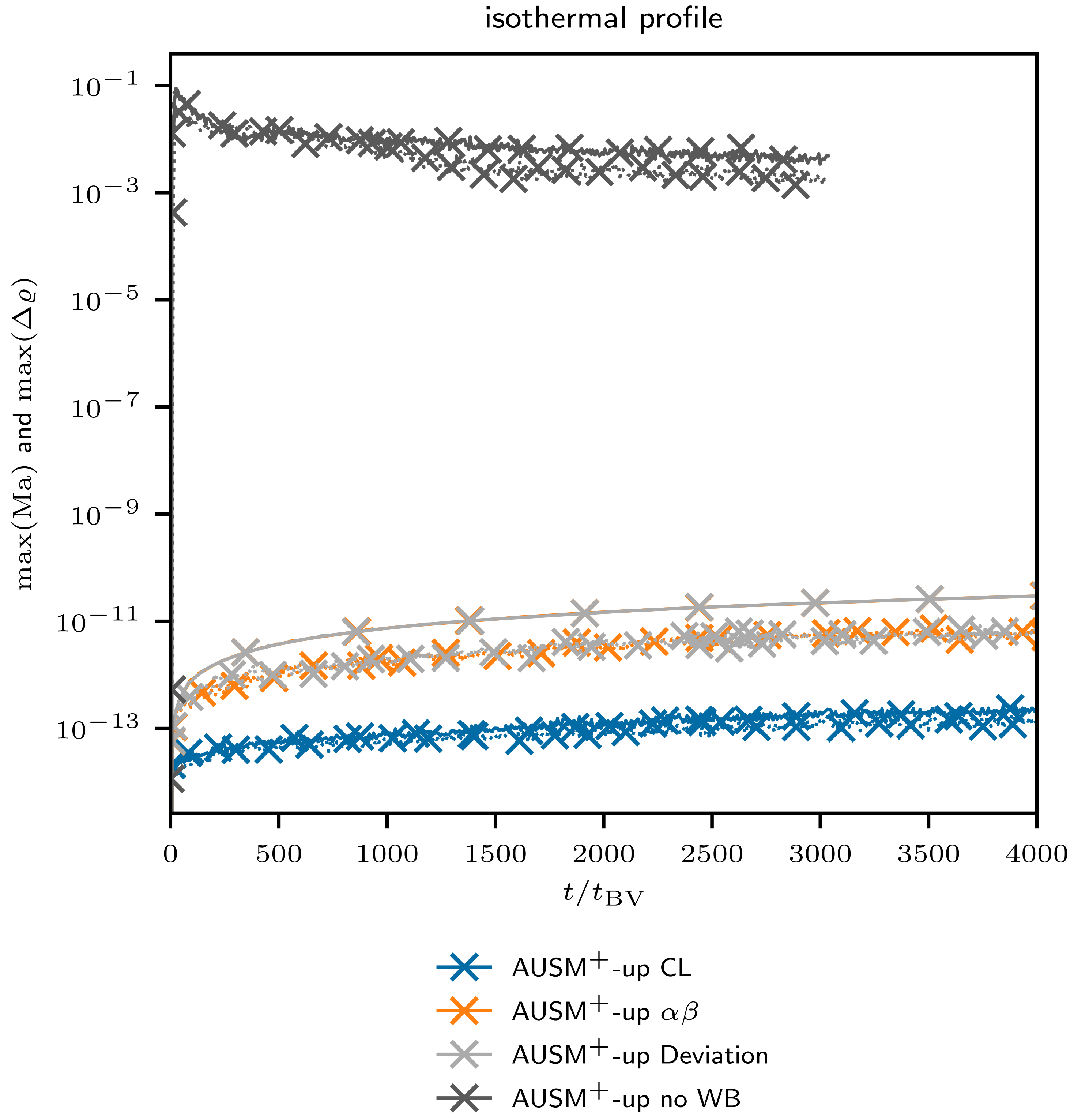


NUMERICAL TESTS

- all run with the SLH code
- 2D Cartesian box
- AUSM $^+$-up all Mach number flux (Liou 2006)
- second-order MUSCL scheme
- ESDIRK23 time-stepping
- reconstruction either in $\rho - P$ or $\rho - T$ variables
- well-balancing: Cargo–Le Roux (CL), αβ, Deviation

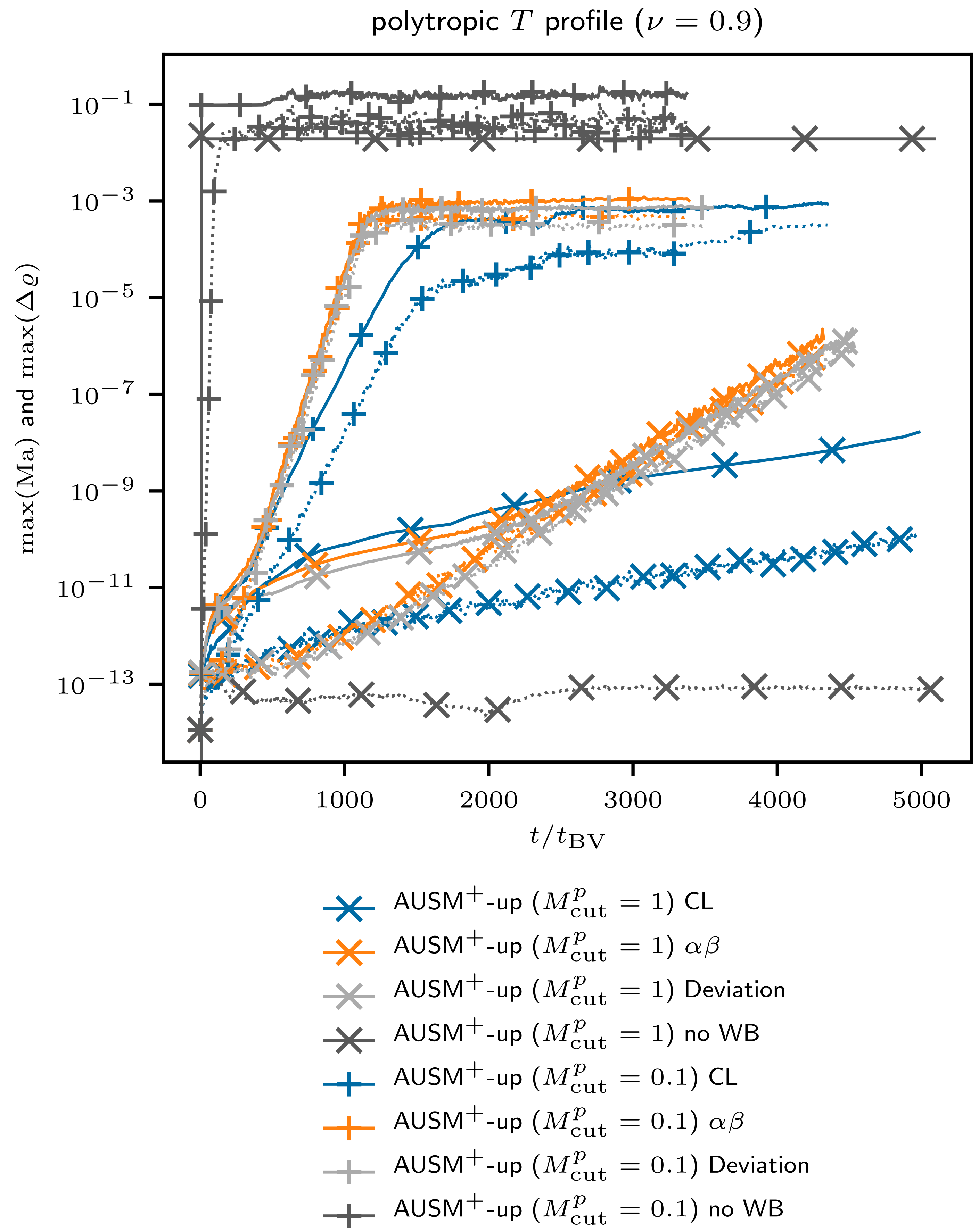
HYDROSTATIC TEST (ISOTHERMAL)

- sinusoidal gravity: $\phi = -s_0 \sin(2\pi y), \quad \mathbf{g} = -\nabla\phi$
- periodic boundaries on all sides
- isothermal stratification (convectively stable)
- $$\rho = \rho_0 \exp\left(-\frac{\rho_0}{p_0}\phi\right), \quad p = p_0 \exp\left(-\frac{\rho_0}{p_0}\phi\right).$$
- constant reconstruction



HYDROSTATIC TEST (POLYTROPIC)

- sinusoidal gravity: $\phi = -s_0 \sin(2\pi y), \quad \mathbf{g} = -\nabla\phi$
- periodic boundaries on all sides
- polytropic stratification
-
$$\theta = 1 - \frac{\nu - 1}{\nu}\frac{\rho_0}{p_0}\phi,$$
$$\rho = \rho_0 \theta^{\frac{1}{\nu - 1}}, \quad p = p_0 \theta^{\frac{\nu}{\nu - 1}}, \quad T = \frac{p_0 \mu}{\rho_0 R}\theta = T_0 \theta.$$
- $\nu = 0.9$ (convectively stable, $\nu < \gamma = 1.4$)
- constant reconstruction



HOT BUBBLE

- single rising, high-entropy bubble to study the most basic process of convection
- isentropic background stratification
- local entropy perturbation

$$A = A_0 \left[1 + \left(\frac{\Delta A}{A} \right)_{t=0} \cos \left(\frac{\pi}{2} \frac{r}{r_0} \right)^2 \right],$$

- periodic on all sides to avoid boundary effects
- gravity: $g_y = g_0 \sin(k_y y)$

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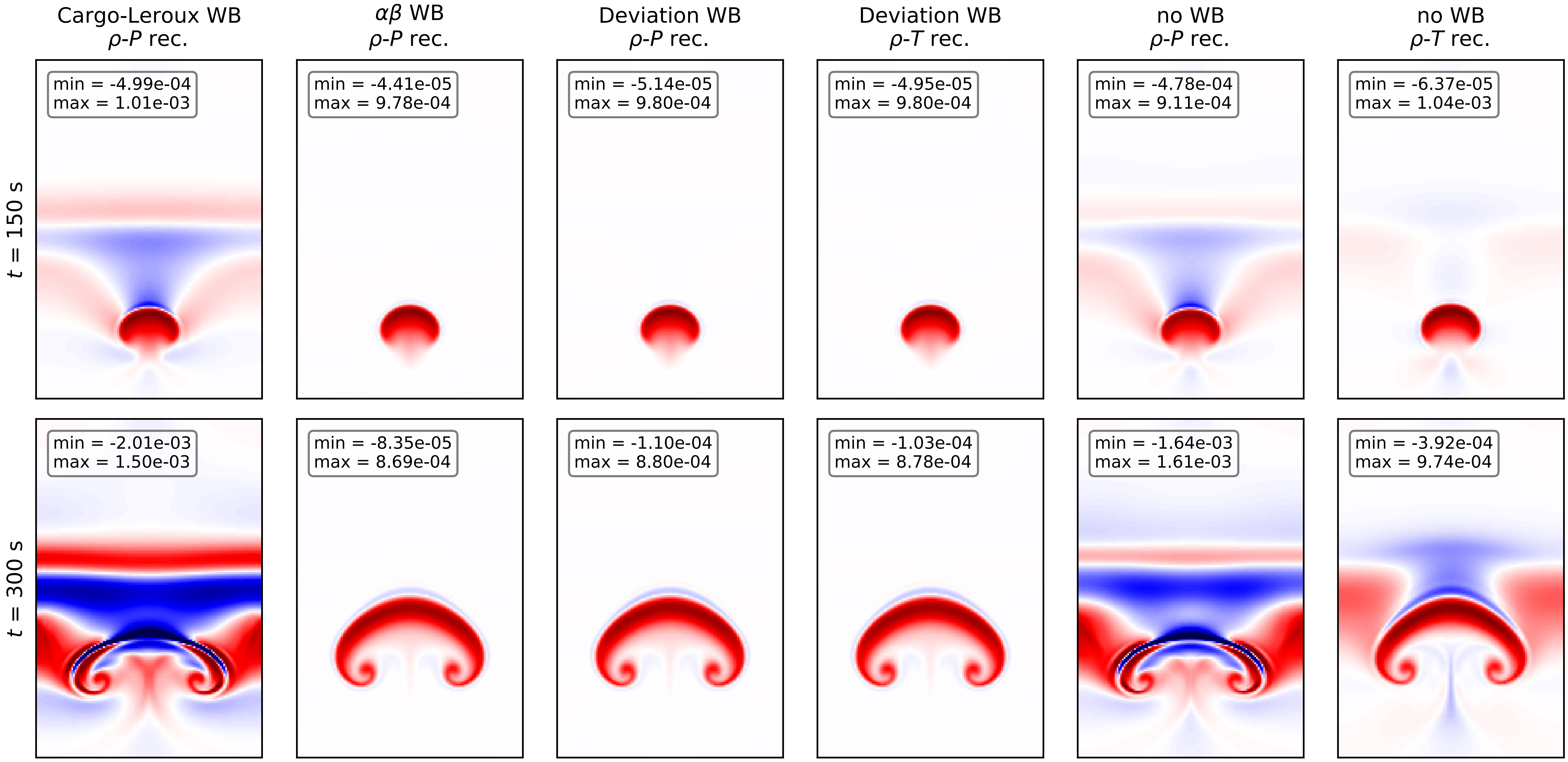
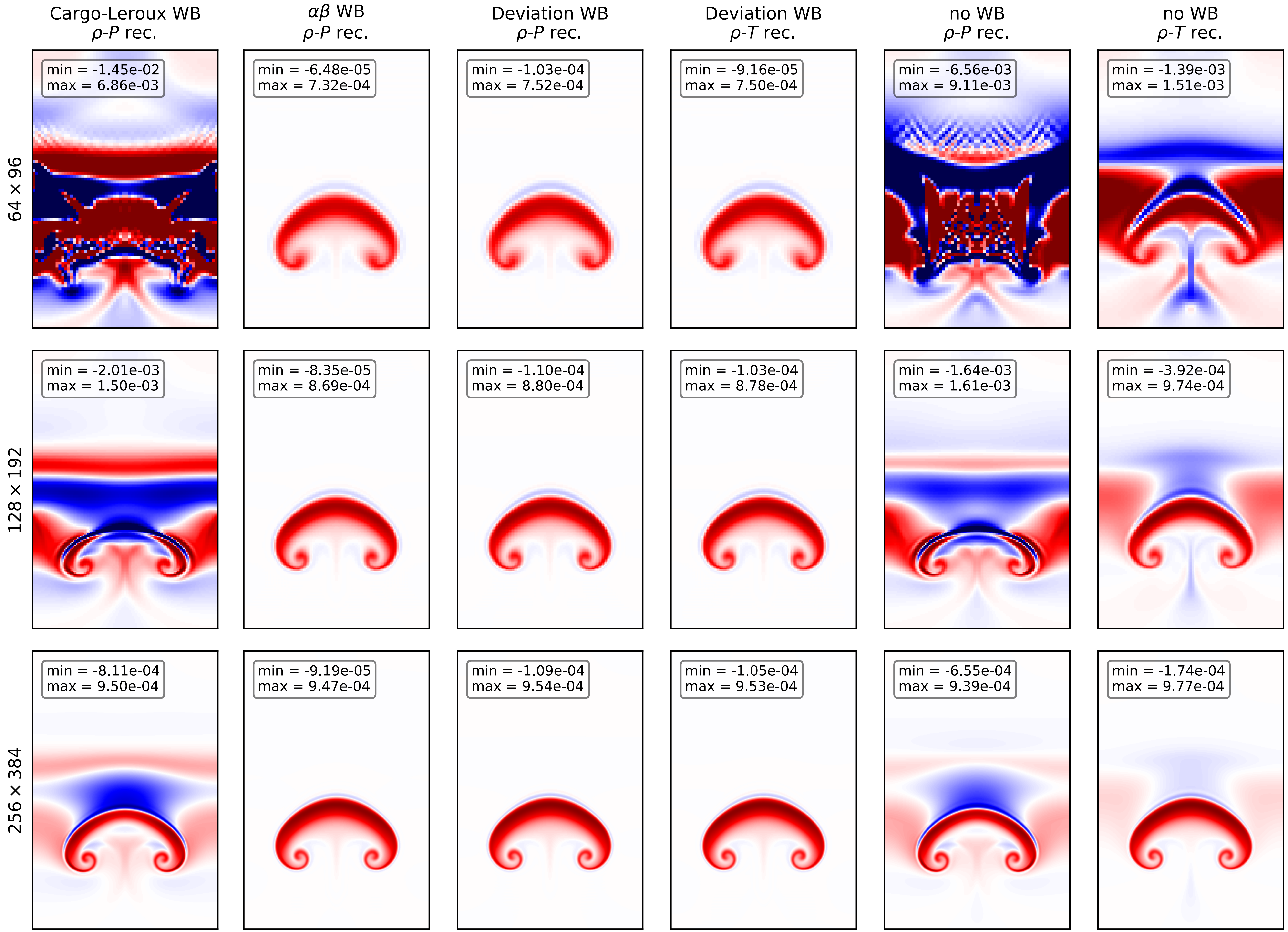
We expect the bubble (region with $\Delta A/A > 0$) to rise without any region going to negative $\Delta A/A$.

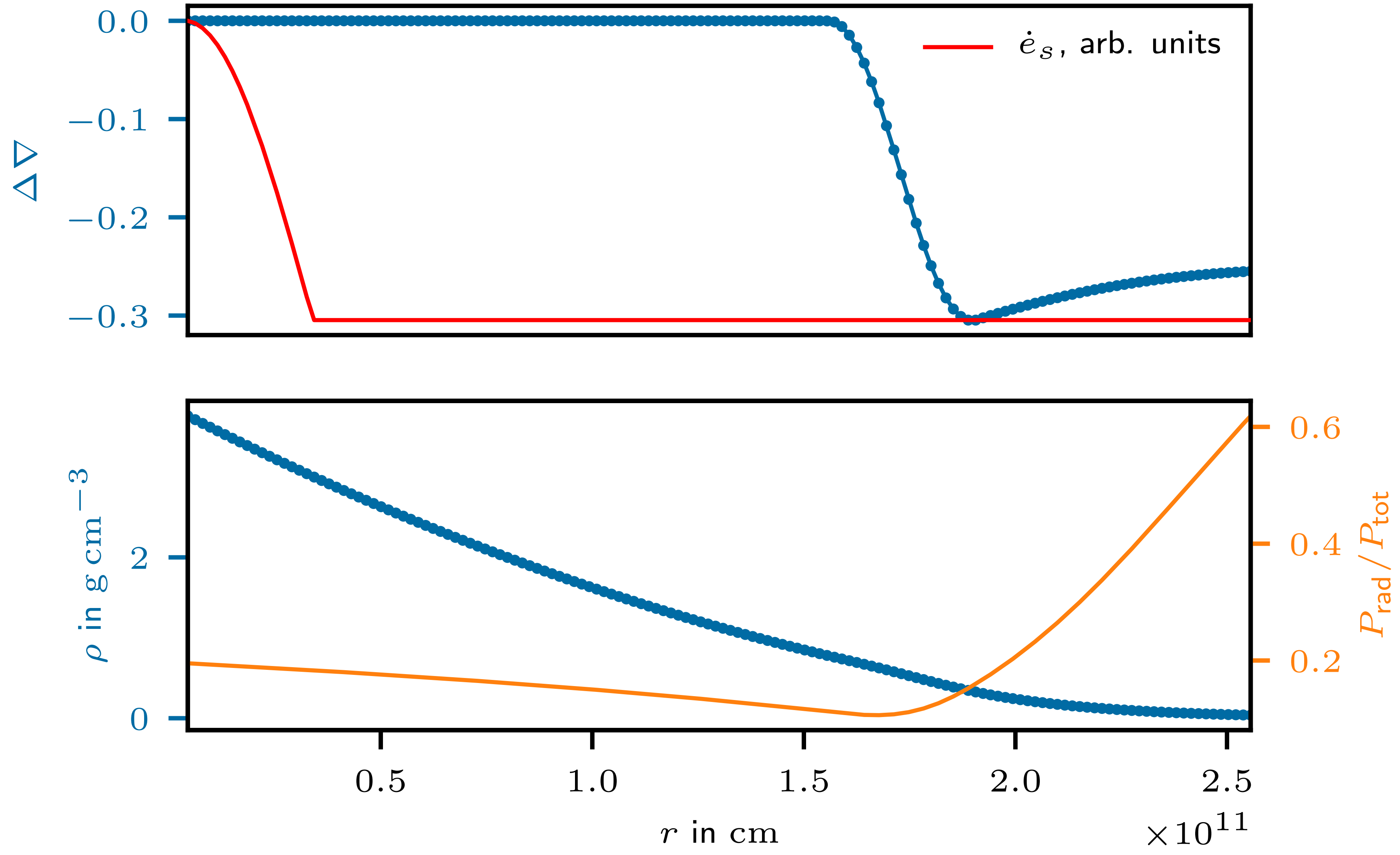


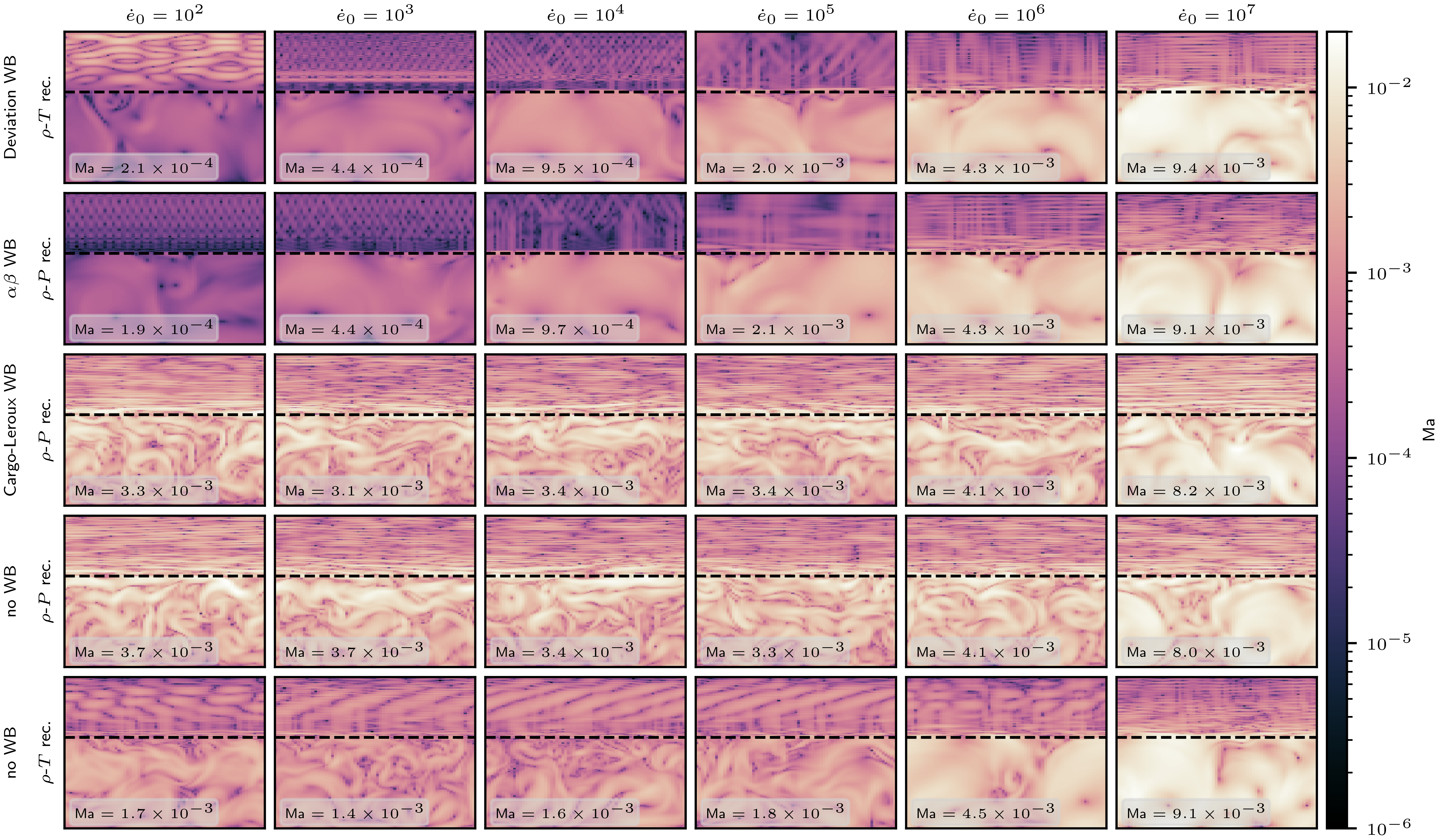
Cargo-Leroux WB ρ-P rec.
$\alpha\beta$ WB ρ-P rec.
Deviation WB ρ-P rec.
Deviation WB ρ-T rec.
no WB ρ-P rec.
no WB ρ-T rec.
t = 150 s
min = -4.99e-04 max = 1.01e-03
min = -4.41e-05 max = 9.78e-04
min = -5.14e-05 max = 9.80e-04
min = -4.95e-05 max = 9.80e-04
min = -4.78e-04 max = 9.11e-04
min = -6.37e-05 max = 1.04e-03
t = 300 s
min = -2.01e-03 max = 1.50e-03
min = -8.35e-05 max = 8.69e-04
min = -1.10e-04 max = 8.80e-04
min = -1.03e-04 max = 8.78e-04
min = -1.64e-03 max = 1.61e-03
min = -3.92e-04 max = 9.74e-04


Cargo-Leroux WB ρ-P rec.
$\alpha\beta$ WB ρ-P rec.
Deviation WB ρ-P rec.
Deviation WB ρ-T rec.
no WB ρ-P rec.
no WB ρ-T rec.
64×96
min = -1.45e-02 max = 6.86e-03
min = -6.48e-05 max = 7.32e-04
min = -1.03e-04 max = 7.52e-04
min = -9.16e-05 max = 7.50e-04
min = -6.56e-03 max = 9.11e-03
min = -1.39e-03 max = 1.51e-03
128×192
min = -2.01e-03 max = 1.50e-03
min = -8.35e-05 max = 8.69e-04
min = -1.10e-04 max = 8.80e-04
min = -1.03e-04 max = 8.78e-04
min = -1.64e-03 max = 1.61e-03
min = -3.92e-04 max = 9.74e-04
256×384
min = -8.11e-04 max = 9.50e-04
min = -9.19e-05 max = 9.47e-04
min = -1.09e-04 max = 9.54e-04
min = -1.05e-04 max = 9.53e-04
min = -6.55e-04 max = 9.39e-04
min = -1.74e-04 max = 9.77e-04

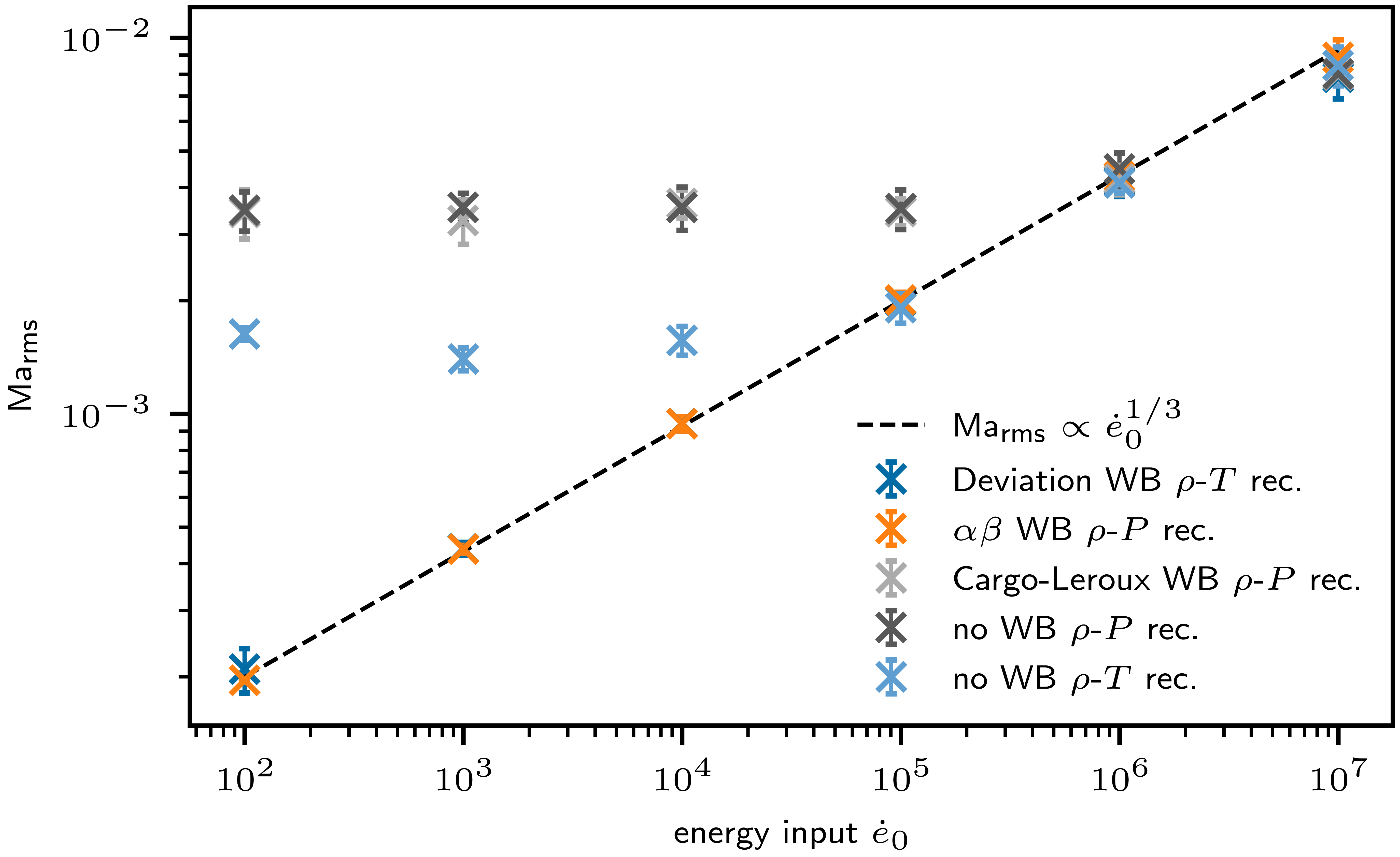
CONVECTIVE BOX

- Cartesian box with convective region (CR) at bottom, stable region at top
- resolved heating term at the bottom of the CR
- heating can be adjusted reach given Mach number: $\mathrm{Ma} \propto \dot{e}^{1/3}$




$\dot{e}_0 = 10^2$
$\dot{e}_0 = 10^3$
$\dot{e}_0 = 10^4$
$\dot{e}_0 = 10^5$
$\dot{e}_0 = 10^6$
$\dot{e}_0 = 10^7$
Deviation WB
ρ-T rec.
Ma = 2.1×10^{-4}
Ma = 4.4×10^{-4}
Ma = 9.5×10^{-4}
Ma = 2.0×10^{-3}
Ma = 4.3×10^{-3}
Ma = 9.4×10^{-3}
$\alpha\beta$ WB
ρ-P rec.
Ma = 1.9×10^{-4}
Ma = 4.4×10^{-4}
Ma = 9.7×10^{-4}
Ma = 2.1×10^{-3}
Ma = 4.3×10^{-3}
Ma = 9.1×10^{-3}
Cargo-Leroux WB
ρ-P rec.
Ma = 3.3×10^{-3}
Ma = 3.1×10^{-3}
Ma = 3.4×10^{-3}
Ma = 3.4×10^{-3}
Ma = 4.1×10^{-3}
Ma = 8.2×10^{-3}
no WB
ρ-P rec.
Ma = 3.7×10^{-3}
Ma = 3.7×10^{-3}
Ma = 3.4×10^{-3}
Ma = 3.3×10^{-3}
Ma = 4.1×10^{-3}
Ma = 8.0×10^{-3}
no WB
ρ-T rec.
Ma = 1.7×10^{-3}
Ma = 1.4×10^{-3}
Ma = 1.6×10^{-3}
Ma = 1.8×10^{-3}
Ma = 4.5×10^{-3}
Ma = 9.1×10^{-3}
Ma
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}

theoretical scaling of Mach number with heating term: $\mathrm{Ma} \propto \dot{e}^{1/3}$



KEPLERIAN DISK

setup from Gaburro+ (2018)

- rotating disk
- partially stabilized by centrifugal force
- example of stationary, but not static setup
- $$u(x,y) = -\sin[\alpha(x,y)]\sqrt{\frac{Gm_s}{r(x,y)}}$$
  $$v(x,y) = \cos[\alpha(x,y)]\sqrt{\frac{Gm_s}{r(x,y)}},$$
- radial density profile is supposed to stay constant

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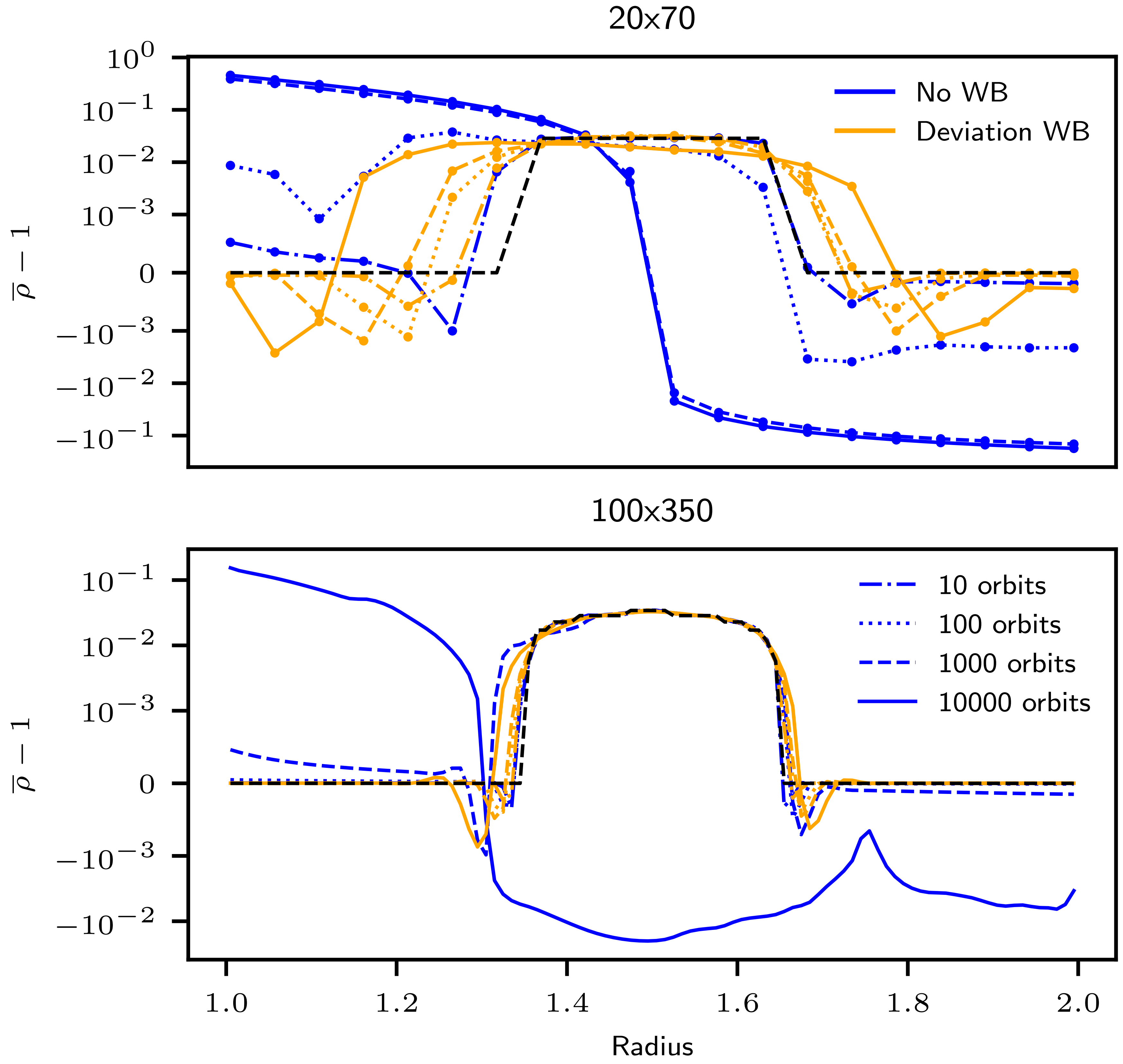
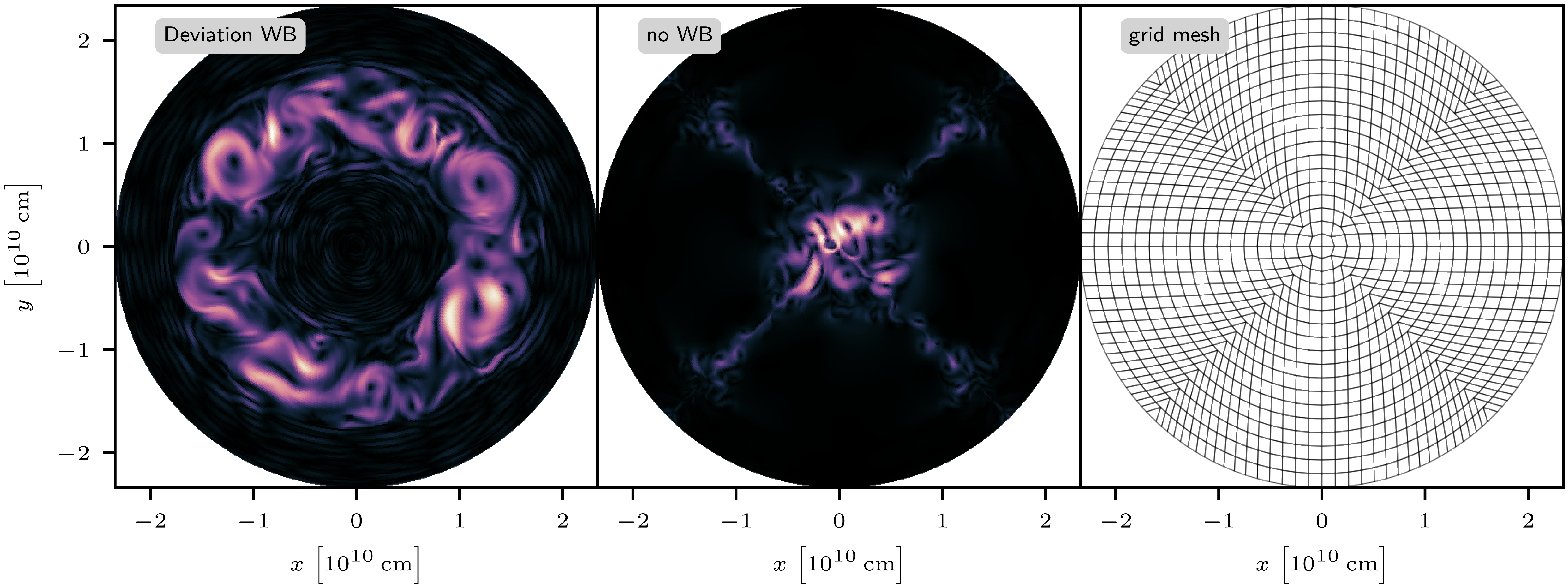


CURVILINEAR MESH

logically rectangular cubed sphere



CONCLUSIONS

- Setups close to hydrostatic equilibrium can suffer from large errors due to discretization discrepancies
- Errors can cause stable regions to become unstable and prevent reaching low Mach numbers.
- Well-balancing (WB) methods prevents discretization errors.
- Different approaches: change EoS, reconstruction
- Choice of reconstruction variables is crucial for low Mach convection ($\rho - T$ better than $\rho - P$)
- Inclusion of potential energy in the total energy improves results.
- WB can prevent unphysical results on complex meshes.

Questions? Ask now or email pedelmann@lanl.gov.