SAND2005-7453P

Finite Element Method I:

FEM for One-Dimensional Heat Conduction

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Introductory Info

Evacuation Procedures:

- Exits are located…
- Restrooms out back

Classification:

- Absolutely no classified discussions
- If you have a concern, let us know
- Some material may be OUO, it will be marked as such

Summary for Finite Element Method

Begin with:

- Variational (weak) form of IBVP

and end with:

- Computational method for one-dimensional, steady heat conduction equation

Additional References:

E. B. Becker, G. F. Carey & J. T. Oden,
"Finite Elements, An Introduction, Volume I," Prentice-Hall, Englewood Cliffs, NJ (1981)

Questions for Finite Element Method I:

- What is a finite element method?
- What are shape functions, trial functions, test functions and weight functions?
- What is the form of the typical discrete equations for a heat conduction problem ?

Variational Form – Heat Conduction

Recall the weighted integral form for the steady heat conduction equation

$$\int_{\Omega} w(x_i)\left[\frac{\partial}{\partial x_i}\left(k_{ij}\frac{\partial T}{\partial x_j}\right)\right]d\Omega+\int_{\Omega} w(x_i)\,Q\,d\Omega=0$$

where w is a suitable weighting function. When integrated by parts (divergence theorem) this becomes a weak or variational form

$$\int_{\Omega}\frac{\partial w}{\partial x_i}k_{ij}\frac{\partial T}{\partial x_j}d\Omega=\int_{\Omega} wQ\,d\Omega+\int_{\Gamma} w\,k_{ij}\frac{\partial T}{\partial x_j}n_i\,d\Gamma$$

Solution Methods for Variational Forms (1)

We are going to use the MWR/variational form to solve the boundary value problem. Following the usual procedure, assume a functional form for the temperature

$$\overline{T}(x_i) = \sum_{i=0}^{N} c_i \, f_i(x_i)$$

Different methods are produced depending on the choice of weighting function and the integral form.

Collocation, subdomain, Galerkin and least squares are the common choices as you have seen from the previous examples.

Solution Methods for Variational Forms (2)

What are the drawbacks to these methods of solution for general heat conduction (boundary value) problems?

For general multi-dimensional applications, the correct choice of the approximating function proves to be very difficult. For a successful solution

- The approximating (trial) function should be computationally convenient, *i.e.,* easy to integrate and differentiate
- The trial function should be reasonably “close” to the true solution
- The approximation should converge to the true solution

Solution Methods for Variational Forms (3)

To make the variational form generally useful, we need to be able to easily select approximating (trial) functions that are simple to manipulate and compute with on complex domains.

Trial functions that can be defined piecewise on subregions of the domain provide the answer to the above problem. This type of function approximation is the essence of the finite element method.

Finite Element Method

There are several ways to develop the ideas and equations for a finite element method.

Many texts, especially in solid mechanics, begin by subdividing the domain into elements and developing the equations for an individual element. This is the so-called *direct stiffness method* and is historically motivated from the early aircraft structures applications. (Local Method)

We will begin with the Method of Weighted Residuals route and a more global approach that avoids some of the subtle technical questions of the direct stiffness method. (Global Method)

One-Dimensional Conduction (1)

Consider a one-dimensional domain, with the variational form for steady conduction

$$\int_0^L \frac{dw}{dx} k \frac{dT}{dx} dx = \int_0^L w Q \, dx + w_b \, q_n \Big|_0^L$$

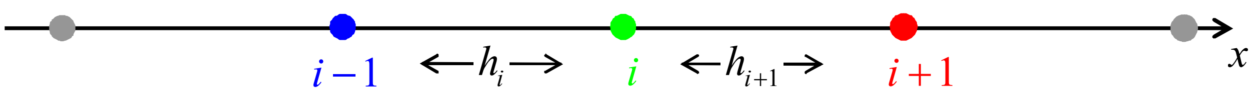
Subdivide the domain 0 to L into N (non-overlapping) intervals and define a temperature approximation

$$T(x) = \sum_{i=0}^{N} c_i \psi_i(x)$$

This still looks like our previous MWR method, except that the ψ_i functions will be defined to be nonzero only on the individual subdomains (local support).

One-Dimensional Conduction (2)

Finite Element Mesh:



The N intervals are labeled as *elements* and the $N+1$ points joining the elements are labeled *nodes*. The spatial coordinates for the nodes are x_i and h_i is the element length, $h_i = x_{i+1} - x_i$. The element lengths are not necessarily uniform.

One-Dimensional Conduction (3)

$$T(x) = \sum_{i=0}^{N} c_i \psi_i(x)$$

The trial function ψ_i must satisfy certain criteria

- The trial functions are defined piecewise, element-by-element with local support (Convenience)
- The trial functions must be complete and sufficiently smooth (Completeness)
- The trial functions are interpolative – the c_i in the approximation are the values of T at the nodes (Compatibility)

One-Dimensional Conduction (4)

A particularly simple (and admissible) set of shape (trial) functions for the domain can be written as

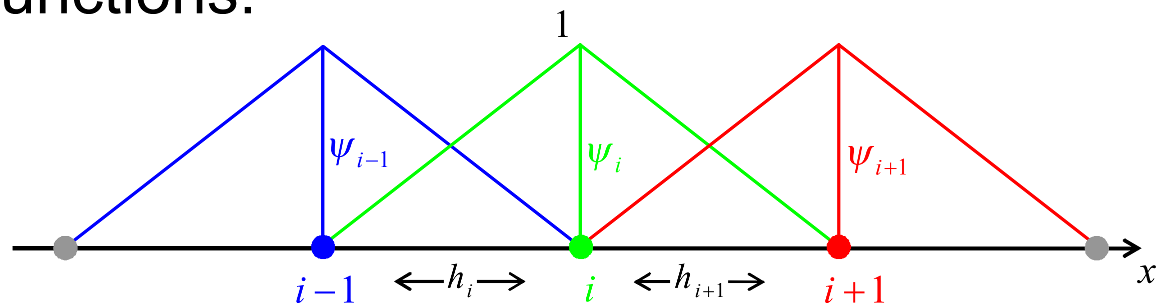
$$\psi_i = \begin{cases} (x - x_i)/h_i & for \;\; x_{i-1} \leq x \leq x_i \\ (x_{i+1} - x)/h_{i+1} & for \;\; x_i \leq x \leq x_{i+1} \\ 0 & for \;\; x \leq x_{i-1} \;\; and \;\; x \geq x_{i+1} \end{cases}$$

The derivatives of the shape (trial) functions are

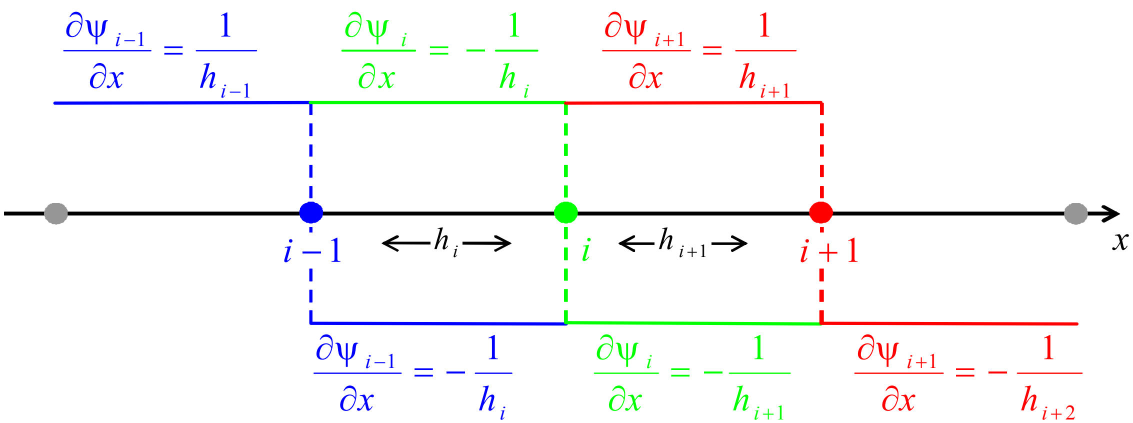
$$\frac{d\psi_i}{dx} = \begin{cases} 1/h_i & for \;\; x_{i-1} \leq x \leq x_i \\ -1/h_{i+1} & for \;\; x_i \leq x \leq x_{i+1} \\ 0 & for \;\; x \leq x_{i-1} \;\; and \;\; x \geq x_{i+1} \end{cases}$$

One-Dimensional Conduction (5)

Trial Functions:



Trial Function Derivatives:



One-Dimensional Conduction (6)

To complete the one-dimensional weak form, a weighting (test) function must be selected. For 2nd order, elliptic problems the optimal choice of weighting is a Galerkin method where

$$w(x) = \sum_{j=0}^{N} b_j \psi_j(x)$$

and b_j are arbitrary coefficients and ψ_j are the previously defined piecewise functions.

The specified test and trial (shape) functions are then inserted into the variational or weak form to produce a useful set of equations.

One-Dimensional Conduction (7)

With the assumed shape (trial and test) functions, the variational equation becomes

$$\int_0^L \left[\frac{d}{dx}\left(\sum_{i=0}^{N} b_i \psi_i\right)\right] k \left[\frac{d}{dx}\left(\sum_{j=0}^{N} c_j \psi_j\right)\right] dx = \int_0^L \left(\sum_{i=0}^{N} b_i \psi_i\right) Q(x)\, dx$$

The flux boundary conditions have been ignored at the ends of the domain; only essential boundary conditions will be used in this example

One-Dimensional Conduction (8)

Note that the b_i and c_j are independent of x and the summations may be moved outside the integrals

$$\sum_{i=0}^{N} b_i \left[\sum_{j=0}^{N} \int_0^L \left(\frac{d\psi_i}{dx} \right) k \left(\frac{d\psi_j}{dx} \right) dx \, c_j \right] = \sum_{i=0}^{N} b_i \left[\int_0^L \psi_i \, Q(x) dx \right]$$

The b_i were arbitrary constants and the above equation is really a set of equations for the c_j coefficients. We can rewrite the above in a more familiar form

One-Dimensional Conduction (9)

$$\sum_{j=0}^{N} K_{ij} c_j = F_i \qquad \textit{for each} \;\; i = 0,1,2,\ldots. N$$

with

$$K_{ij} = \int_0^L \frac{d\psi_i}{dx} k \frac{d\psi_j}{dx} dx \qquad F_i = \int_0^L \psi_i \, Q(x) \, dx$$

The entries in the diffusion matrix K_{ij} and the source vector F_i can be computed directly from the definitions of ψ_i and simple integrals over the domain.

One-Dimensional Conduction (10)

What would you expect the equation for the coefficients to look like for the case of the linear "hat" functions?

Working through the integrals for a node produces the equation for the *i* th node

$$-\frac{k}{h_i}c_{i-1} + \left(\frac{k}{h_i}+\frac{k}{h_{i+1}}\right)c_i - \frac{k}{h_{i+1}}c_{i+1} = \left(\frac{h_i}{2}+\frac{h_{i+1}}{2}\right)Q$$

If a uniform subdivision (equal elements) is assumed, the nodal equation becomes a standard centered difference relation

$$-\frac{k}{h}c_{i-1} + \frac{2k}{h}c_i - \frac{k}{h}c_{i+1} = hQ$$

One-Dimensional Conduction (9)

The matrix form for the assembled set of equations is

$$\begin{bmatrix} 2k/h & -k/h & 0 & 0 & 0 \\ -k/h & 2k/h & -k/h & 0 & 0 \\ 0 & -k/h & 2k/h & -k/h & 0 \\ 0 & 0 & -k/h & 2k/h & -k/h \\ 0 & 0 & 0 & -k/h & 2k/h \end{bmatrix} \begin{Bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} = \begin{Bmatrix} hQ \\ hQ \\ hQ \\ hQ \\ hQ \end{Bmatrix}$$

which can be solved for the coefficients when Q is given and appropriate boundary conditions are specified.

FEM Solution (1)

As an example, assume that Q has a constant value of 2 on each element and that $k = 1$ for the previous one-dimensional problem. Also, let the temperature at $x = 0$ be set to 1 and the temperature at $x = L = 1$ be set to 2 with a uniform 4 element mesh. The matrix problem is then

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -4 & 8 & -4 & 0 & 0 \\ 0 & -4 & 8 & -4 & 0 \\ 0 & 0 & -4 & 8 & -4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1/2 \\ 1/2 \\ 1/2 \\ 2 \end{Bmatrix}$$

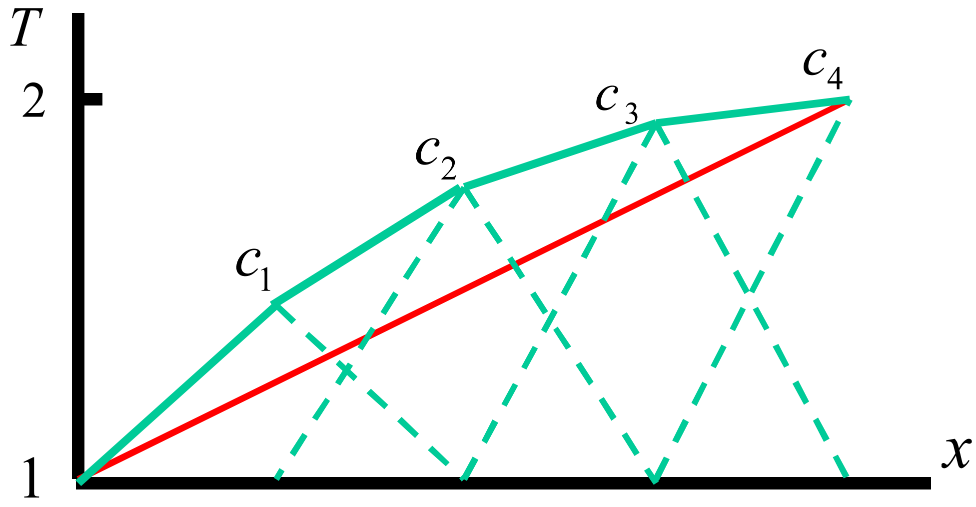
FEM Solution (2)

The matrix problem can be solved by elimination to produce the coefficients

$$c_0 = 8/8 \;;\; c_1 = 11.5/8 \;;\; c_2 = 14/8$$

$$c_3 = 15.5/8 \;;\; c_4 = 16/8$$

which are the temperatures at the nodes.



FEM Solution (3)

The FEM solution to the one-dimensional problem produces a piecewise continuous function for the temperature

$$T(x) = \psi_0 c_0 + \psi_1 c_1 + \psi_2 c_2 + \psi_3 c_3 + \psi_4 c_4$$

This can be evaluated at any point x in the domain. A comparison with the analytic solution at the nodes (x_i) produces

x/L :	0	¼	½	¾	1
FE Temp:	8/8	8.5/8	10/8	12.5/8	16/8
Analytic :	8/8	8.5/8	10/8	12.5/8	16/8

FEM Solution (4)

Likewise, the heat flux can be evaluated at any point in the domain using the derivative of the shape functions

$$q(x) = -k\frac{dT}{dx} = -\frac{d\psi_0}{dx}c_0 - \frac{d\psi_1}{dx}c_1 - \frac{d\psi_2}{dx}c_2 - \frac{d\psi_3}{dx}c_3 - \frac{d\psi_4}{dx}c_4$$

As the derivative is constant on the subdomains (elements), it is usual to determine the flux at the midpoint (centroid) of the element

x/L :	1/8	3/8	5/8	7/8
FE Flux:	-14/8	-10/8	-6/8	-2/8
Analytic :	-14/8	-10/8	-6/8	-2/8

Finite Element Matrices (1)

Return to the form of the general matrix problem. Each entry in the global stiffness matrix and load vector is computed from

$$K_{ij} = \int_0^L \frac{d\psi_i}{dx} k \frac{d\psi_j}{dx} dx \qquad F_i = \int_0^L \psi_i Q(x) dx$$

where the sums on i and j are implied and run over the number of nodes. The integral over the domain can be written as a series of integrals because the integration process is additive.

Finite Element Matrices (2)

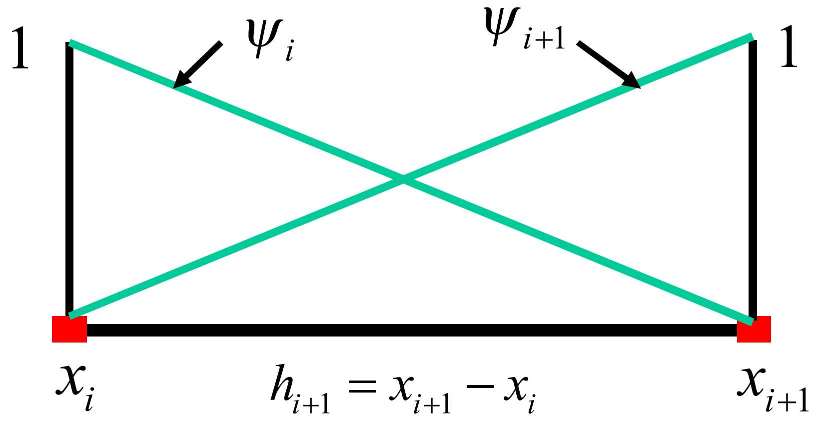
For our one-dimensional, N element problem

$$K_{ij} = \sum_{e=1}^{N} K_{ij}^{e} = \int_{x_0}^{x_1} \frac{d\psi_i}{dx} k \frac{d\psi_j}{dx} dx + \int_{x_1}^{x_2} \frac{d\psi_i}{dx} k \frac{d\psi_j}{dx} dx + ... + \int_{x_{N-1}}^{x_N} \frac{d\psi_i}{dx} k \frac{d\psi_j}{dx} dx$$

and similarly for the load vector

$$F_i = \sum_{e=1}^{N} F_i^e = \int_{x_0}^{x_1} \psi_i Q(x)\, dx + \int_{x_1}^{x_2} \psi_i Q(x)\, dx + ... + \int_{x_{N-1}}^{x_N} \psi_i Q(x)\, dx$$

element $i+1$



Finite Element Matrices (3)

Because the shape and weight functions are defined piecewise on each element, the individual element matrices and vectors can be constructed separately. For example, element e in our mesh, with coordinates x_i to x_{i+1} has the following element diffusion matrix and source vector

$$K^e c^e = \begin{bmatrix} k/h & -k/h \\ -k/h & k/h \end{bmatrix} \begin{Bmatrix} c_i \\ c_{i+1} \end{Bmatrix} = F^e = \begin{Bmatrix} hQ/2 \\ hQ/2 \end{Bmatrix}$$

where $h = x_{i+1} - x_i$

Finite Element Matrices

Important characteristics of global element matrices

- The global matrix (nodal equations) can be assembled from element level matrices and load vectors
- The global matrix is sparse (lots of zeros) due to the local support of the shape functions
- The global matrix is symmetric, which is to be expected since the diffusion operator is symmetric
- Note the possible arbitrary spatial variations of k and Q which are handled automatically by the method

Shape Functions (1)

The use of the actual spatial coordinates to define the element integrals is rather inconvenient. The element stiffness and load vector computations are repetitive and depend only on the shape function definition and limits of integration. We can take the first step in developing a “generic” element definition by normalizing the shape functions as

$$\psi_1(\xi) = 1 - \xi / h \quad ; \quad \psi_2(\xi) = \xi / h$$

where 1 and 2 refer to the local node numbers in the one-dimensional element, ξ is the coordinate along the element (limits 0 to 1) and $h = x_{i+1} - x_i$ is the length

Shape Functions (2)

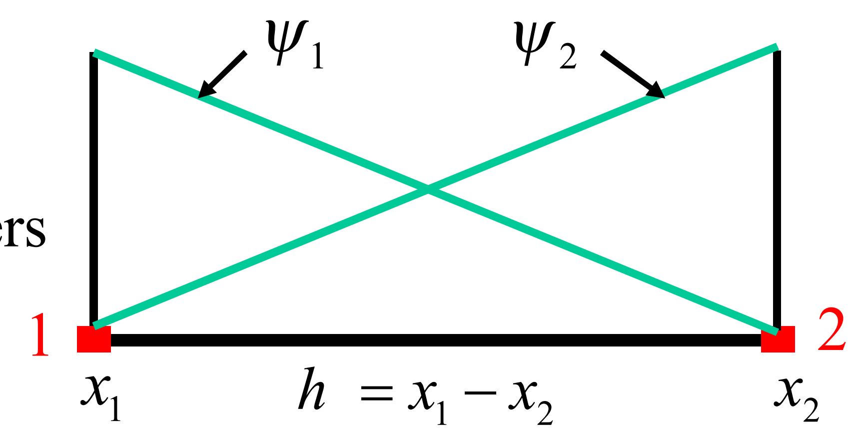
Further, using vector notation

$$\boldsymbol{\Psi} = \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix} = \begin{Bmatrix} 1-\xi/h \\ \xi/h \end{Bmatrix} \quad ; \quad T = \boldsymbol{\Psi}^{\mathrm{T}}\mathbf{c} = \{1-\xi/h\ ,\quad \xi/h\}\begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}$$

and the derivatives

$$\frac{d\boldsymbol{\Psi}}{d\xi} = \begin{Bmatrix} \dfrac{d\psi_1}{d\xi} \\ \dfrac{d\psi_2}{d\xi} \end{Bmatrix} = \begin{Bmatrix} -1/h \\ 1/h \end{Bmatrix} \quad ; \quad \frac{dT}{d\xi} = \frac{d\boldsymbol{\Psi}^{\mathrm{T}}}{d\xi}\mathbf{c} = \{-1/h\ ,\quad 1/h\}\begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}$$

element e – local node numbers



Element Matrix

Using the vector notation and the normalized shape functions the element matrix and load vector can be defined

$$\mathbf{K} = \int_0^1 \frac{d\,\boldsymbol{\Psi}}{d\xi} k \frac{d\,\boldsymbol{\Psi}^{\mathrm{T}}}{d\xi} d\xi \qquad \mathbf{F} = \int_0^1 \boldsymbol{\Psi}\, Q(\xi)\, d\xi$$

After evaluation, these forms will give the same element matrix as before.

Answers for FEM for 1D Conduction

- A finite element method is a weighted residual formulation of an IBVP where the weighting and trial functions are defined piecewise on subdomains.

- For FEM applications, shape (trial, approximating, etc) functions and weight (test) functions are simple, interpolating functions defined on subdomains (elements) with specific requirements on completeness and compatibility.

- The FEM equations for steady heat conduction form a matrix of algebraic relations that is sparse, banded and symmetric

Homework #2

- Re-do the four element problem with a specified heat flux q at $x = 0$ and a heat transfer coefficient h_c and reference temperature T_c at $x = L$.
- Derive the matrix problem
- Solve the matrix problem for the case of $q = 2$, $h_c = 10$ and $T_c = 2$
- Compare the solution with the analytic solution