SAND2005-7669P

Finite Element Method II:

FEM for Multi-Dimensional Heat Conduction

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Introductory Info

Evacuation Procedures:

- Exits are located…
- Restrooms out back

Classification:

- Absolutely no classified discussions
- If you have a concern, let us know
- Some material may be OUO, it will be marked as such

Summary for
Finite Element Method

Begin with:

- General integrals for element “stiffness” and “load” vectors

and end with:

- Global system of equations for one-dimensional and two-dimensional steady heat conduction equation with selected element types

Questions for Finite Element Method II:

- How do we compute “element” matrices in a general way for more complex, multi-dimensional elements?
- What are problem coordinates, natural coordinates?
- What are isoparametric mappings?
- What are the “Jacobian matrix” and the “Jacobian?”
- How do we “assemble” these element matrices into a global matrix?

Preview

Today we will cover the following topics:

- Coordinate transformations between problem coordinates and natural coordinates (master elements)
- 1-D linear element using natural coordinates
- 1-D quadratic element using problem and natural
- 2-D triangle using problem coordinates

Comments on Computing Element Matrices

Recall from last time that we need to compute element matrices of the following form.

$$K_{ij}^{e} = \int_{x_1}^{x_2} \frac{d\psi_i}{dx} k \frac{d\psi_j}{dx} dx \qquad F_i = \int_{x_1}^{x_2} \psi_i \, Q(x) \, dx$$

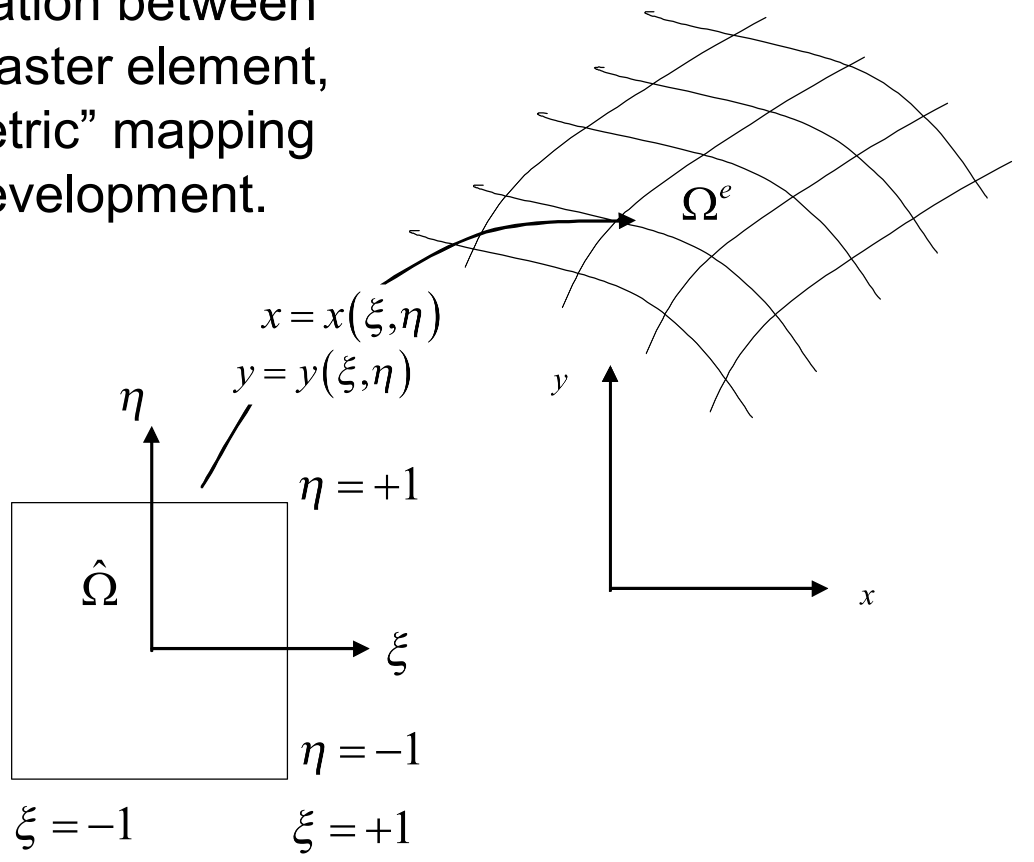
We need to define shape functions and derivatives of shape functions in a way that is convenient in a more general sense.

To do so, we will consider "mapping" elements in physical space into a "master" element or a (-1 to +1) space. We will also have to use numerical integration to compute the element matrices in the general case.

Numerical integration of element matrices for general element shapes is the primary driver!

Mapping Coordinate Transformations

To develop the transformation between actual element and the master element, we will use an "isoparametric" mapping and will consider a 2-D development.



Mapping Coordinate Transformations (2)

To begin, we change coordinates from (x, y) to (ξ, η)

$$x = x(\xi, \eta) \qquad y = y(\xi, \eta)$$

Now

$$\psi(x, y) = \psi\{x(\xi, \eta), y(\xi, \eta)\}$$

To compute the derivative of the shape function, use the chain rule

$$\frac{\partial \Psi}{\partial \xi} = \frac{\partial \Psi}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial \Psi}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial \Psi}{\partial \eta} = \frac{\partial \Psi}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial \Psi}{\partial y}\frac{\partial y}{\partial \eta}$$

Mapping Coordinate Transformations (3)

Expressed in matrix form

$$\begin{Bmatrix} \dfrac{\partial \boldsymbol{\Psi}}{\partial \xi} \\ \dfrac{\partial \boldsymbol{\Psi}}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \boldsymbol{\Psi}}{\partial x} \\ \dfrac{\partial \boldsymbol{\Psi}}{\partial y} \end{Bmatrix} = [J] \begin{Bmatrix} \dfrac{\partial \boldsymbol{\Psi}}{\partial x} \\ \dfrac{\partial \boldsymbol{\Psi}}{\partial y} \end{Bmatrix}$$

Where $[J]$ is the Jacobian matrix of the transformation.

For an isoparametric mapping, we use the same coordinate functions as for the shape functions

$$x(\xi,\eta) = \sum_{i=1}^{N} x_i \, \psi_i(\xi,\eta) \qquad y(\xi,\eta) = \sum_{i=1}^{N} y_i \, \psi_i(\xi,\eta)$$

Mapping Coordinate Transformations (4)

For an isoparametric coordinate mapping, the entries in the Jacobian matrix can be computed using the assumed coordinate functions.

$$\begin{Bmatrix} \dfrac{\partial \boldsymbol{\Psi}}{\partial \xi} \\ \dfrac{\partial \boldsymbol{\Psi}}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \boldsymbol{\Psi}}{\partial x} \\ \dfrac{\partial \boldsymbol{\Psi}}{\partial y} \end{Bmatrix} = [J] \begin{Bmatrix} \dfrac{\partial \boldsymbol{\Psi}}{\partial x} \\ \dfrac{\partial \boldsymbol{\Psi}}{\partial y} \end{Bmatrix}$$

Which yields

$$\frac{\partial x}{\partial \xi} = \sum_{i=1}^{N} x_i \frac{\partial \psi_i(\xi,\eta)}{\partial \xi} \qquad \frac{\partial y}{\partial \xi} = \sum_{i=1}^{N} y_i \frac{\partial \psi_i(\xi,\eta)}{\partial \xi}$$

$$\frac{\partial x}{\partial \eta} = \sum_{i=1}^{N} x_i \frac{\partial \psi_i(\xi,\eta)}{\partial \eta} \qquad \frac{\partial y}{\partial \eta} = \sum_{i=1}^{N} y_i \frac{\partial \psi_i(\xi,\eta)}{\partial \eta}$$

Mapping Coordinate Transformations (5)

With the entries in the Jacobian matrix known, then the inverse can be computed and the derivatives with respect to the problem coordinates (which we need for our element integrals) are written in matrix form

$$\begin{Bmatrix} \dfrac{\partial \mathbf{\Psi}}{\partial x} \\ \dfrac{\partial \mathbf{\Psi}}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial \mathbf{\Psi}}{\partial \xi} \\ \dfrac{\partial \mathbf{\Psi}}{\partial \eta} \end{Bmatrix}$$

To compute the inverse of the Jacobian matrix requires that the Jacobian matrix is non-singular; the "Jacobian" or the determinant of the Jacobian matrix is non-zero.

$$"Jacobian" = J = \det[J] = \det \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}$$

Mapping Coordinate Transformations (6)

Putting it all together to compute our “element” stiffness matrix and load vectors, we need to consider the other terms in the integral.

In addition to the terms involving the derivatives with respect to the problem coordinates, we need to also transform the differential area associated the integrals. In 2-D the result is

$$dA = dx\,dy \; = \det\left[J\right] d\xi\, d\eta$$

The Jacobian provides the mapping of areas (or volumes in 3-D) from the actual element to the master element.   In the long run, this approach greatly simplifies the computation of these element matrices.

Element Matrices for 1-D Linear Element using Natural Coordinates

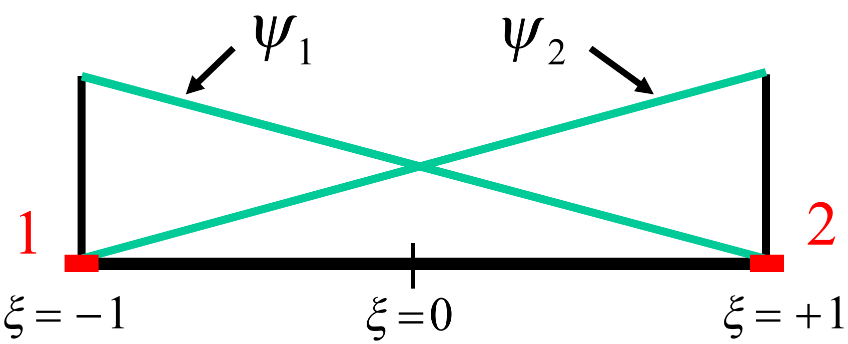
Further, using vector notation

$$\boldsymbol{\Psi} = \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix} = \begin{Bmatrix} (1-\xi)/2 \\ (1+\xi)/2 \end{Bmatrix} \quad ; \quad T = \boldsymbol{\Psi}^{\mathrm{T}}\,\mathbf{T} = \left\{ (1-\xi)/2 \;,\quad (1+\xi)/2 \right\} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

and the derivatives

$$\frac{d\,\boldsymbol{\Psi}}{d\xi} = \begin{Bmatrix} \dfrac{d\psi_1}{d\xi} \\[2ex] \dfrac{d\psi_2}{d\xi} \end{Bmatrix} = \begin{Bmatrix} -1/2 \\ 1/2 \end{Bmatrix} \quad ; \quad \frac{d\,T}{d\xi} = \frac{d\,\boldsymbol{\Psi}^{\mathrm{T}}}{d\xi}\mathbf{T} = \left\{ -1/2 \;,\quad 1/2 \right\} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

element e –
local node numbers



Element Matrices for 1-D Linear Element using Natural Coordinates (2)

Using the same function for the coordinate transformation

$$x = \boldsymbol{\Psi}^{\mathrm{T}}\,\mathbf{x} = \left\{(1-\xi)/2\ ,\quad (1+\xi)/2\right\}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

Then the Jacobian is

$$J = \frac{dx}{d\xi} = \frac{\partial \boldsymbol{\Psi}^{\mathrm{T}}}{\partial \xi}\mathbf{x} = \left\{\ -1/2\ ,\quad 1/2\ \right\}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \frac{1}{2}(x_2 - x_1)$$

And the temperature gradient is

$$\frac{dT}{dx} = \frac{d\xi}{dx}\frac{dT}{d\xi} = \frac{1}{J}\frac{dT}{d\xi} = \frac{1}{J}\frac{d\,\boldsymbol{\Psi}^{\mathrm{T}}}{d\xi}\mathbf{T} = \frac{1}{J}\left\{-1/2\ ,\quad 1/2\right\}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$\frac{dT}{dx} = \frac{1}{(x_j - x_i)/2}\left\{-1/2\ ,\quad 1/2\right\}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

Element Matrices for 1-D Linear Element using Natural Coordinates (3)

Now, evaluate the two integrals for the element stiffness and load vector

$$\mathbf{K}^e\,\mathbf{T} = \left(\int_{x_1}^{x_2} \frac{d\,\mathbf{\Psi}}{dx} k \frac{d\,\mathbf{\Psi}^{\mathrm{T}}}{dx}\, dx \right) \mathbf{T} = \left(\int_{-1}^{+1} k \frac{1}{J} \frac{d\,\mathbf{\Psi}}{d\xi} \frac{d\,\mathbf{\Psi}^{\mathrm{T}}}{d\xi} \frac{1}{J}\, J\, d\xi \right) \mathbf{T}$$

$$\mathbf{K}^e\,\mathbf{T} = \left(\int_{-1}^{+1} k \frac{1}{J} \begin{Bmatrix} -1/2 \\ 1/2 \end{Bmatrix} \left\{ -1/2\,, \quad 1/2 \right\} \frac{1}{J}\, J\, d\xi \right) \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$\mathbf{K}^e\,\mathbf{T} = \left(\frac{k}{4J} \int_{-1}^{+1} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \left\{ -1 \quad 1 \right\} d\xi \right) \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \left(\frac{k}{4J} \int_{-1}^{+1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} d\xi \right) \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$\mathbf{K}^e\,\mathbf{T} = \frac{k}{(x_2 - x_1)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

Element Matrices for 1-D Linear Element using Natural Coordinates (4)

Now, evaluate the two integrals for the element stiffness and load vector

$$\mathbf{F}^e = \int_{x_1}^{x_2} \boldsymbol{\Psi}\, Q(x)\, dx = \int_{-1}^{+1} \boldsymbol{\Psi}\, Q\; J\, d\xi$$

$$\mathbf{F}^e = \int_{-1}^{+1} \boldsymbol{\Psi}\, Q\; J\, d\xi = \int_{-1}^{+1} \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix} Q\; J\, d\xi$$

$$\mathbf{F}^e = \int_{-1}^{+1} \begin{Bmatrix} (1-\xi)/2 \\ (1+\xi)/2 \end{Bmatrix} Q\; J\, d\xi$$

$$\boxed{\mathbf{F}^e = \frac{Q\left(x_2 - x_1\right)}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}}$$

1-D Quadratic Elements Using Problem Coordinates

Element Matrices for 1-D Quadratic Element using Problem Coordinates

We will develop the element matrices using problem coordinates by assuming a quadratic temperature function with unknown constants, α's

$$T(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2$$

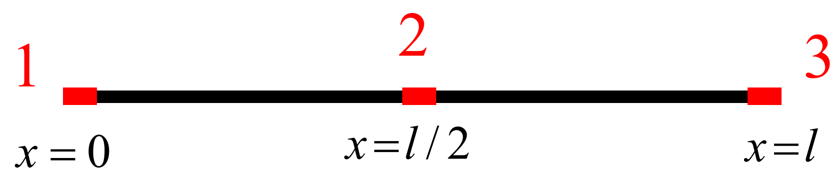
To determine the constants, evaluate the temperature function at each node (3 eqs, 3 unknowns)

$$T(0) = T_1 = \alpha_1$$

$$T(l/2) = T_2 = \alpha_1 + \alpha_2 \left(l/2\right) + \alpha_3 \left(l/2\right)^2$$

$$T(l) = T_3 = \alpha_1 + \alpha_2 l + \alpha_3 l^2$$

element e –
local node numbers



Element Matrices for 1-D Quadratic Element using Problem Coordinates (2)

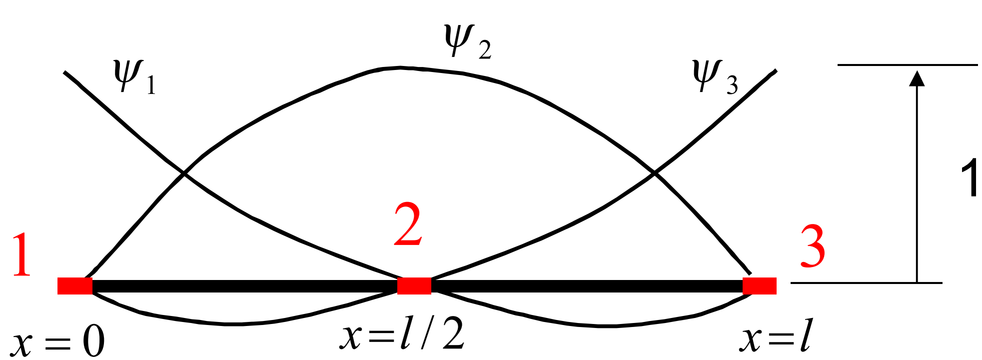
Solving for the three unknowns, results in

$$\alpha_1 = T_1 \qquad \alpha_2 = \left(-3T_1 + 4T_2 - T_3\right)/L \qquad \alpha_3 = 2\left(T_1 - 2T_2 + T_3\right)/l^2$$

So

$$T = \mathbf{\Psi}^{\mathrm{T}}\,\mathbf{T} = \{\ \psi_1 \quad \psi_2 \quad \psi_3\ \}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} \qquad \mathbf{\Psi} = \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{Bmatrix} = \begin{Bmatrix} 1 - 3(x/l) + 2(x/l)^2 \\ 4(x/l) - 4\left(x/l\right)^2 \\ -(x/l) + 2(x/l)^2 \end{Bmatrix}$$

element e – local node numbers



Element Matrices for 1-D Quadratic Element using Problem Coordinates (3)

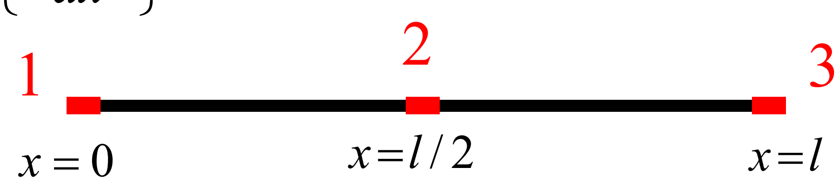
We need the temperature and shape function derivatives,

$$\frac{dT}{dx} = \frac{d\mathbf{\Psi}^{\mathrm{T}}}{dx}\mathbf{T} = \begin{Bmatrix} \dfrac{d\psi_1}{dx} & \dfrac{d\psi_2}{dx} & \dfrac{d\psi_3}{dx} \end{Bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}$$

and

$$\frac{d\mathbf{\Psi}}{dx} = \begin{Bmatrix} \dfrac{d\psi_1}{dx} \\ \dfrac{d\psi_2}{dx} \\ \dfrac{d\psi_1}{dx} \end{Bmatrix} = \begin{Bmatrix} -3/l + 4x/l^2 \\ 4/l - 8x/l^2 \\ -1/l + 4x/l^2 \end{Bmatrix}$$

element e –
local node numbers



Element Matrices for 1-D Quadratic Element using Problem Coordinates (4)

Now, evaluate the two integrals for the element stiffness and load vector

$$\mathbf{K}^e\ \mathbf{T} = \left(\int_0^l \frac{d\,\boldsymbol{\Psi}}{dx} k \frac{d\,\boldsymbol{\Psi}^{\mathrm{T}}}{dx} dx \right) \mathbf{T}$$

$$\mathbf{K}^e\ \mathbf{T} = \left(\int_0^l k \begin{Bmatrix} \dfrac{d\psi_1}{dx} \\ \dfrac{d\psi_2}{dx} \\ \dfrac{d\psi_3}{dx} \end{Bmatrix} \begin{Bmatrix} \dfrac{d\psi_1}{dx} & \dfrac{d\psi_2}{dx} & \dfrac{d\psi_3}{dx} \end{Bmatrix} dx \right) \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}$$

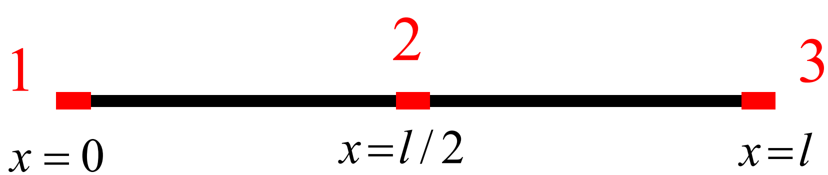
Element Matrices for 1-D Quadratic Element using Problem Coordinates (5)

Or

$$\mathbf{K}^e\,\mathbf{T} = \left(\int_0^l k \begin{bmatrix} \left(\frac{d\psi_1}{dx}\frac{d\psi_1}{dx}\right) & \left(\frac{d\psi_1}{dx}\frac{d\psi_2}{dx}\right) & \left(\frac{d\psi_1}{dx}\frac{d\psi_3}{dx}\right) \\ \left(\frac{d\psi_2}{dx}\frac{d\psi_1}{dx}\right) & \left(\frac{d\psi_2}{dx}\frac{d\psi_2}{dx}\right) & \left(\frac{d\psi_2}{dx}\frac{d\psi_3}{dx}\right) \\ \left(\frac{d\psi_3}{dx}\frac{d\psi_1}{dx}\right) & \left(\frac{d\psi_3}{dx}\frac{d\psi_2}{dx}\right) & \left(\frac{d\psi_3}{dx}\frac{d\psi_3}{dx}\right) \end{bmatrix} dx \right) \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}$$

We have everything in terms of x, “just integrate it out!”

$$\mathbf{K}^e\,\mathbf{T} = \frac{k}{6l}\begin{bmatrix} 14 & -16 & 2 \\ -16 & 32 & -16 \\ 2 & -16 & 14 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}$$



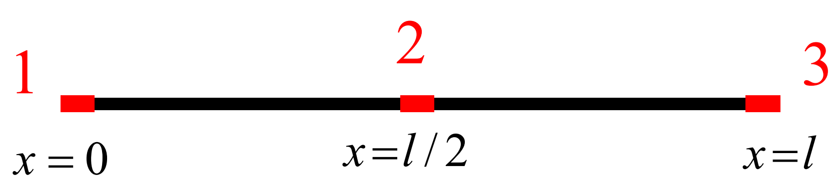
Element Matrices for 1-D Quadratic Element using Problem Coordinates (6)

Now, evaluate the two integrals for the element stiffness and load vector

$$\mathbf{F}^e = \int_0^l \mathbf{\Psi}\, Q(x)\, dx = \int_0^l \begin{Bmatrix} \dfrac{d\psi_1}{dx} \\ \dfrac{d\psi_2}{dx} \\ \dfrac{d\psi_3}{dx} \end{Bmatrix} Q(x)\, dx = \int_0^l \begin{Bmatrix} 1-3(x/l)+2(x/l)^2 \\ 4(x/l)-4\left(x/l\right)^2 \\ -(x/l)+2(x/l)^2 \end{Bmatrix} Q(x)dx$$

or

$$\boxed{\mathbf{F}^e = \frac{Q\,l}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix}}$$



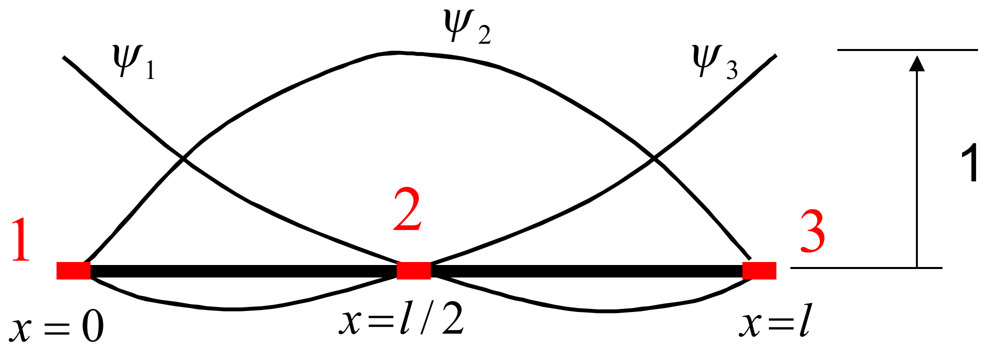
Element Matrices for 1-D Quadratic Element using Problem Coordinates (7)

To summarize the 1-D quadratic element, the element stiffness and load vector are

$$\mathbf{K}^e\,\mathbf{T} = \frac{k}{6l}\begin{bmatrix} 14 & -16 & 2 \\ -16 & 32 & -16 \\ 2 & -16 & 14 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}$$

And

$$\mathbf{F}^e = \frac{Q\,l}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix}$$



We will apply them to our example problem later.

1-D Quadratic Elements Using Natural Coordinates

Element Matrices for 1-D Quadratic Element using Natural Coordinates

We will develop the element matrices using natural coordinates using a quadratic temperature function. With natural coordinates, we will use a transformation that results in limits of our integrals being -1 to +1.

Recall that we will need integrals of the form

$$\mathbf{K}^e\ \mathbf{T} = \left(\int_{x_1}^{x_2} \frac{d\,\mathbf{\Psi}}{dx} k \frac{d\,\mathbf{\Psi}^{\mathrm{T}}}{dx} dx \right) \mathbf{T} = \left(\int_{-1}^{+1} k \frac{1}{J} \frac{d\,\mathbf{\Psi}}{d\xi} \frac{d\,\mathbf{\Psi}^{\mathrm{T}}}{d\xi} \frac{1}{J}\ J\, d\xi \right) \mathbf{T}$$

and

$$\mathbf{F}^e = \int_{x_1}^{x_2} \mathbf{\Psi}\, Q(x)\, dx = \int_{-1}^{+1} \mathbf{\Psi}\, Q\ J\, d\xi$$

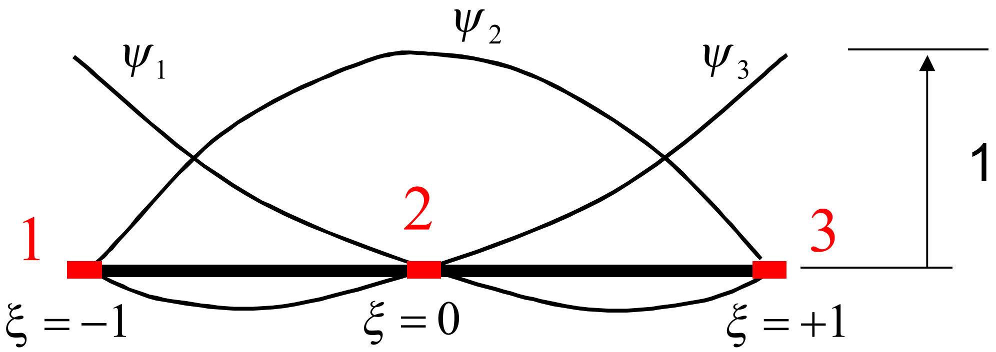
Just what terms will we need?

Element Matrices for 1-D Quadratic Element using Natural Coordinates (2)

We begin with shape functions for the temperature function written in natural coordinates

$$T = \mathbf{\Psi}^{\mathrm{T}}\,\mathbf{T} = \{\, \psi_1 \quad \psi_2 \quad \psi_3 \,\} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} \qquad \mathbf{\Psi} = \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{Bmatrix} = \begin{Bmatrix} -\xi\,(1-\xi)/2 \\ 1-\xi^2 \\ \xi\,(1+\xi)/2 \end{Bmatrix}$$

element e – local node numbers



Element Matrices for 1-D Quadratic Element using Natural Coordinates (3)

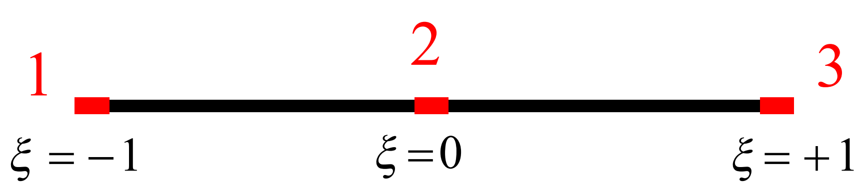
We will need the derivatives of the shape functions and the temperature function.

$$\frac{\partial T}{\partial \xi} = \frac{\partial \mathbf{\Psi}^{\mathrm{T}}}{\partial \xi}\mathbf{T} = \left\{ \frac{\partial \psi_1}{\partial \xi} \quad \frac{\partial \psi_2}{\partial \xi} \quad \frac{\partial \psi_3}{\partial \xi} \right\} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}$$

$$\mathbf{\Psi} = \begin{Bmatrix} \dfrac{\partial \psi_1}{\partial \xi} \\ \dfrac{\partial \psi_2}{\partial \xi} \\ \dfrac{\partial \psi_3}{\partial \xi} \end{Bmatrix} = \begin{Bmatrix} \xi - 1/2 \\ -2\xi \\ \xi + 1/2 \end{Bmatrix}$$

We will also need the Jacobian

element e – local node numbers



Element Matrices for 1-D Quadratic Element using Natural Coordinates (4)

We will also need to construct the Jacobian. For an isoparametric coordinate mapping, the entries in the Jacobian matrix can be computed using the assumed coordinate functions.

$$\begin{Bmatrix} \dfrac{\partial \boldsymbol{\Psi}}{\partial \xi} \\ \dfrac{\partial \boldsymbol{\Psi}}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \boldsymbol{\Psi}}{\partial x} \\ \dfrac{\partial \boldsymbol{\Psi}}{\partial y} \end{Bmatrix} = [J] \begin{Bmatrix} \dfrac{\partial \boldsymbol{\Psi}}{\partial x} \\ \dfrac{\partial \boldsymbol{\Psi}}{\partial y} \end{Bmatrix}$$

which, for 1-D simplifies to

$$\frac{\partial \boldsymbol{\Psi}}{\partial \xi} = \frac{\partial x}{\partial \xi} \frac{\partial \boldsymbol{\Psi}}{\partial x} = [J] \frac{\partial \boldsymbol{\Psi}}{\partial x} \qquad \text{or} \qquad \frac{\partial \boldsymbol{\Psi}}{\partial x} = \frac{1}{J} \frac{\partial \boldsymbol{\Psi}}{\partial \xi}$$

with

$$\frac{\partial x}{\partial \xi} = \sum_{i=1}^{N} x_i \frac{\partial \psi_i(\xi,\eta)}{\partial \xi}$$

Element Matrices for 1-D Quadratic Element using Natural Coordinates (5)

Using the same function for the coordinate transformation

$$x = \mathbf{\Psi}^{\mathrm{T}}\ \mathbf{x} = \left\{ -\xi\left(1-\xi\right)/2 \quad 1-\xi^{2} \quad \xi\left(1+\xi\right)/2 \right\} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

Then the Jacobian is

$$J = \frac{dx}{d\xi} = \frac{\partial \mathbf{\Psi}^{\mathrm{T}}}{\partial \xi}\mathbf{x} = \left\{ \left(\xi - 1/2\right) \ \left(-2\xi\right) \ \left(\xi + 1/2\ \right) \right\} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

$$J = \frac{dx}{d\xi} = \frac{\partial \mathbf{\Psi}^{\mathrm{T}}}{\partial \xi}\mathbf{x} = \left(\xi - 1/2\right)x_1 - 2\xi x_2 + \left(\xi + 1/2\ \right)x_3$$

Note that this time, the Jacobian is a function of position ξ

Element Matrices for 1-D Quadratic Element using Natural Coordinates (6)

So, we now have determined all the terms necessary to evaluate the element matrices

$$\mathbf{K}^e\,\mathbf{T} = \left(\int_{-1}^{+1} k \frac{1}{J(\xi)} \frac{d\,\boldsymbol{\Psi}(\xi)}{d\xi} \frac{d\,\boldsymbol{\Psi}^{\mathrm{T}}(\xi)}{d\xi} \frac{1}{J(\xi)} \, J(\xi)\, d\xi \right) \mathbf{T}$$

$$\mathbf{F}^e = \int_{-1}^{+1} \boldsymbol{\Psi}(\xi)\, Q\; J(\xi)\, d\xi$$

Note that for the element stiffness, the integrand is a ratio of polynominals. We will resort to numerical integration to evaluate these. Extrapolate that thought to multi-D problems, because that's where we are heading. We will defer the discussion of numerical integration for later.

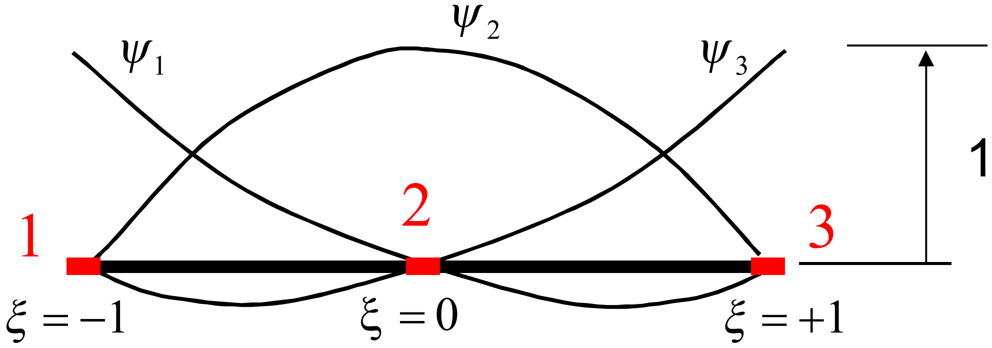
Element Matrices for 1-D Quadratic Element using Natural Coordinates (7)

To summarize the 1-D quadratic element, the element stiffness and load vector are (again)

$$\mathbf{K}^e\,\mathbf{T} = \frac{k}{6l}\begin{bmatrix} 14 & -16 & 2 \\ -16 & 32 & -16 \\ 2 & -16 & 14 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}$$

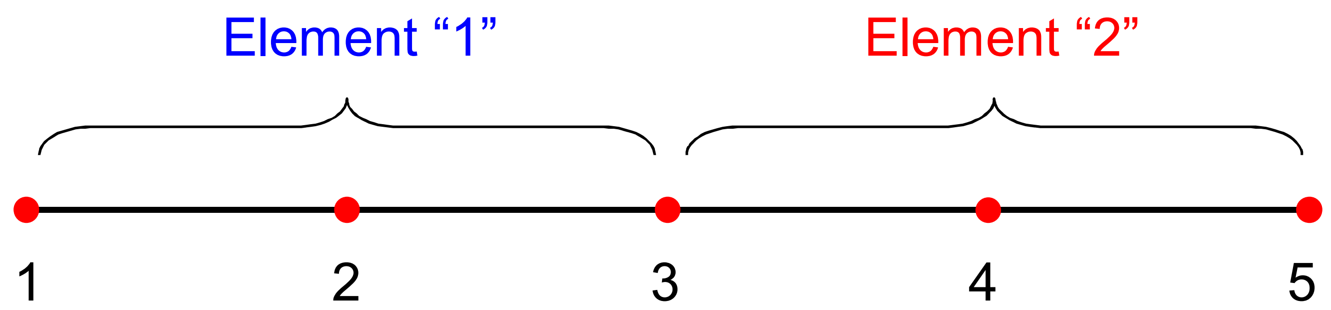
And

$$\mathbf{F}^e = \frac{Q\,l}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix}$$



We will apply them to our example problem later.

Revisit the 5-node Example Problem



For a total length of L, each element is L/2

Element 1 has global nodes 1, 2, and 3

Element 2 has global nodes 3, 4, and 5

Recall, Q=2, L=k=1 with T(0)=1 and T(L)=2

Contributions of Element 1 to the Global System of Equations

For element 1

$$\mathbf{K}^e\,\mathbf{T} = \frac{k}{6(L/2)}\begin{bmatrix} 14 & -16 & 2 \\ -16 & 32 & -16 \\ 2 & -16 & 14 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}$$

$$\mathbf{F}^e = \frac{Q(L/2)}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix}$$

“Assembly” of element 1 contributions in the global system of equations

$$\frac{k}{6(L/2)}\begin{bmatrix} 14 & -16 & 2 & 0 & 0 \\ -16 & 32 & -16 & 0 & 0 \\ 2 & -16 & 14 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \frac{Q(L/2)}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \\ 0 \\ 0 \end{Bmatrix}$$

Contributions of Element 2 to the Global System of Equations

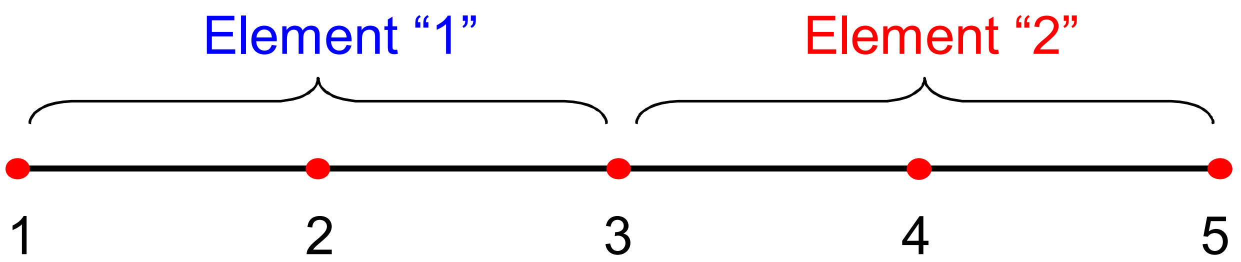
For element 2

$$\mathbf{K}^e\,\mathbf{T} = \frac{k}{6(L/2)}\begin{bmatrix} 14 & -16 & 2 \\ -16 & 32 & -16 \\ 2 & -16 & 14 \end{bmatrix}\begin{Bmatrix} T_3 \\ T_4 \\ T_5 \end{Bmatrix} \qquad \mathbf{F}^e = \frac{Q(L/2)}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix}$$

"Assembly" of element 2 contributions in the global system of equations

$$\frac{k}{6(L/2)}\begin{bmatrix} 14 & -16 & 2 & 0 & 0 \\ -16 & 32 & -16 & 0 & 0 \\ 2 & -16 & 14+14 & -16 & 2 \\ 0 & 0 & -16 & 32 & -16 \\ 0 & 0 & 2 & -16 & 14 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \frac{Q(L/2)}{6}\begin{Bmatrix} 1 \\ 4 \\ 1+1 \\ 4 \\ 1 \end{Bmatrix}$$

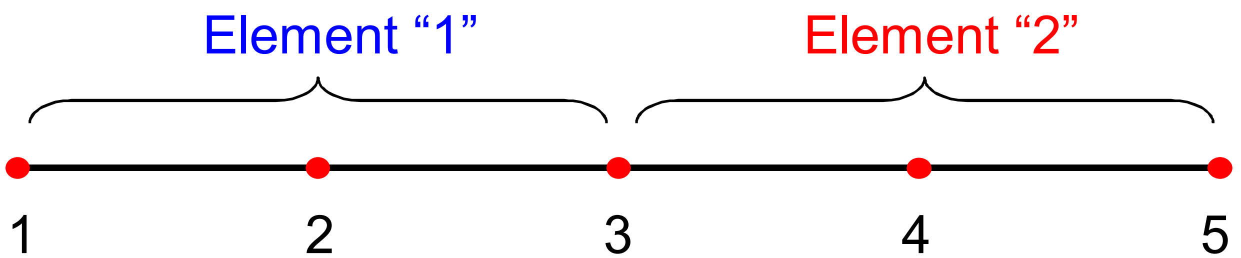
Modifications to the Global System of Equations for Specified Temperature



$$\frac{k}{6(L/2)}\begin{bmatrix} 14 & -16 & 2 & 0 & 0 \\ -16 & 32 & -16 & 0 & 0 \\ 2 & -16 & 14+14 & -16 & 2 \\ 0 & 0 & -16 & 32 & -16 \\ 0 & 0 & 2 & -16 & 14 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \frac{Q(L/2)}{6} \begin{Bmatrix} 1 \\ 4 \\ 1+1 \\ 4 \\ 1 \end{Bmatrix}$$

We can see how each of the element contributes to the global system of equations

Modifications to the Global System of Equations for Specified Temperature



Substituting in for Q, k, and L

$$\begin{bmatrix} 7 & -8 & 1 & 0 & 0 \\ -8 & 16 & -8 & 0 & 0 \\ 1 & -8 & 14 & -8 & 1 \\ 0 & 0 & -8 & 16 & -8 \\ 0 & 0 & 1 & -8 & 7 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \frac{1}{4} \begin{Bmatrix} 1 \\ 4 \\ 2 \\ 4 \\ 1 \end{Bmatrix}$$

We still need to impose boundary conditions T(0)=1 and T(L)=2

Modifications to the Global System of Equations for Specified Temperature

$$\begin{bmatrix} 7 & -8 & 1 & 0 & 0 \\ -8 & 16 & -8 & 0 & 0 \\ 1 & -8 & 14 & -8 & 1 \\ 0 & 0 & -8 & 16 & -8 \\ 0 & 0 & 1 & -8 & 7 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \frac{1}{4} \begin{Bmatrix} 1 \\ 4 \\ 2 \\ 4 \\ 1 \end{Bmatrix}$$

For boundary conditions T(0)=1 and T(L)=2

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -8 & 16 & -8 & 0 & 0 \\ 1 & -8 & 14 & -8 & 1 \\ 0 & 0 & -8 & 16 & -8 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \frac{1}{4} \begin{Bmatrix} 4(1) \\ 4 \\ 2 \\ 4 \\ 4(2) \end{Bmatrix}$$

Results for 1-D Quadratic Elements

The FEM solution and the analytical agree.

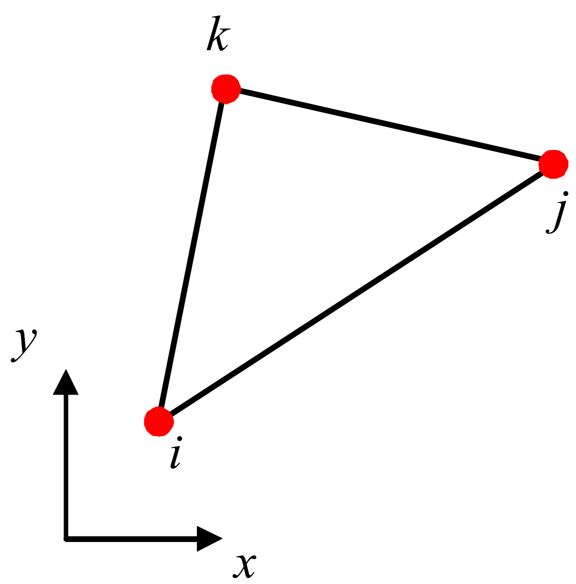
Because these are quadratic elements, the flux varies linearly from $q(0) = -2$ to $q(L) = 0$.

x/L :	0	¼	½	¾	1
FE Temp:	8/8	11.5/8	14/8	15.5/8	16/8
Analytic :	8/8	11.5/8	14/8	15.5/8	16/8

2-D Linear Elements Using Problem Coordinates

2-D Linear Triangles Using Problem Coordinates

We will develop the element matrices using problem coordinates by assuming a quadratic temperature function of the form



$$T(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y$$

The constants are determined by evaluating the function at the nodes results in 3 equations with 3 unknown constants

$$T_i = \alpha_1 + \alpha_2 x_i + \alpha_3 y_i$$

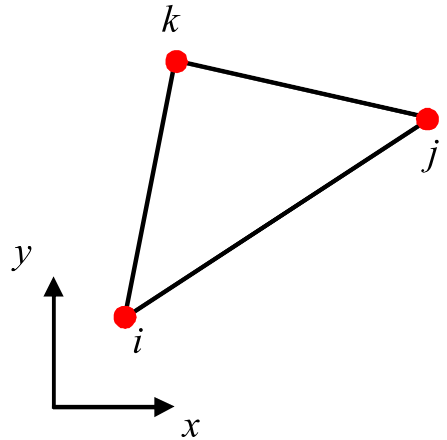
$$T_j = \alpha_1 + \alpha_2 x_j + \alpha_3 y_j$$

$$T_k = \alpha_1 + \alpha_2 x_k + \alpha_3 y_k$$

2-D Linear Triangles Using Problem Coordinates (2)

The set of equations can be rewritten as

$$\begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$$



Solving, results in the following constants

$$\alpha_1 = \frac{1}{2A}\left[\left(x_j y_k - x_k y_j\right)T_i + \left(x_k y_i - x_i y_k\right)T_j + \left(x_i y_j - x_j y_i\right)T_k\right]$$

$$\alpha_2 = \frac{1}{2A}\left[\left(y_j - y_k\right)T_i + \left(y_k - y_i\right)T_j + \left(y_i - y_j\right)T_k\right]$$

$$\alpha_1 = \frac{1}{2A}\left[\left(x_k - x_j\right)T_i + \left(x_i - x_k\right)T_j + \left(x_j - x_i\right)T_k\right]$$

where $2A = \det\begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix}$

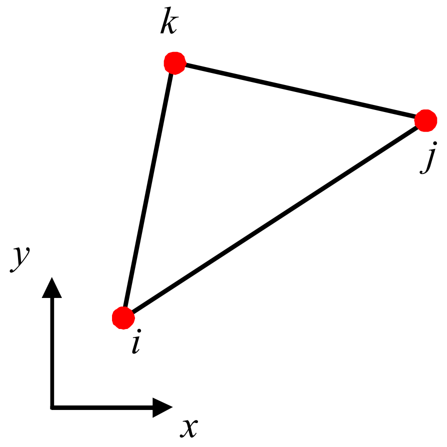
2-D Linear Triangles Using Problem Coordinates (3)

The shape functions are

$$\psi_i = \frac{1}{2A}\left(a_i + b_i x + c_i y\right)$$

$$\psi_j = \frac{1}{2A}\left(a_j + b_j x + c_j y\right)$$

$$\psi_k = \frac{1}{2A}\left(a_k + b_k x + c_k y\right)$$



where the constants are given by

$$a_i = x_j y_k - x_k y_j \qquad b_i = y_j - y_k \qquad c_i = x_k - x_j$$

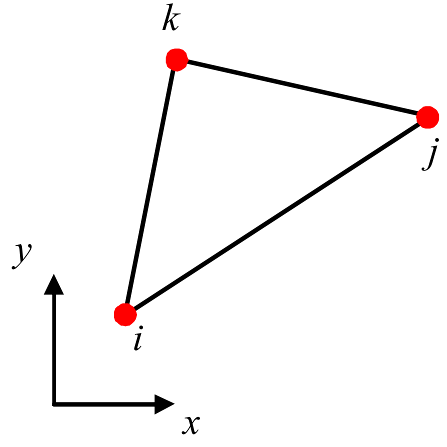
$$a_j = x_k y_i - x_i y_k \qquad b_j = y_k - y_i \qquad c_j = x_i - x_k$$

$$a_k = x_i y_j - x_j y_i \qquad b_k = y_i - y_j \qquad c_k = x_j - x_i$$

2-D Linear Triangles Using Problem Coordinates (4)

Using these shape functions, the temperature profile and derivatives can be written as

$$T(x,y) = \sum_{i=1}^{N} \psi_i(x,y) T_i$$



And the gradients as

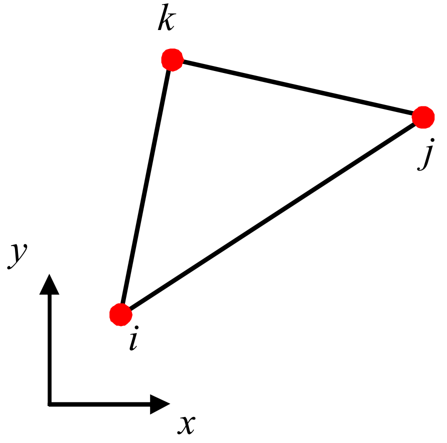
$$\frac{\partial T(x,y)}{\partial x} = \sum_{i=1}^{N} \frac{\partial \psi_i}{\partial x} T_i = \sum_{i=1}^{N} \frac{b_i}{2A} T_i$$

$$\frac{\partial T(x,y)}{\partial y} = \sum_{i=1}^{N} \frac{\partial \psi_i}{\partial y} T_i = \sum_{i=1}^{N} \frac{c_i}{2A} T_i$$

2-D Linear Triangles Using Problem Coordinates (5)

The derivatives needed in the element matrices can be written in matrix form

$$\mathbf{\Psi} = \begin{Bmatrix} \psi_i \\ \psi_j \\ \psi_k \end{Bmatrix} = \frac{1}{2A} \begin{Bmatrix} a_i + b_i x + c_i y \\ a_j + b_j x + c_j y \\ a_k + b_k x + c_k y \end{Bmatrix}$$



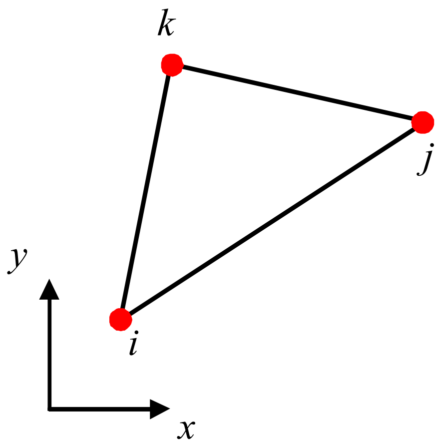
$$\begin{bmatrix} \dfrac{\partial \mathbf{\Psi}}{\partial x} & \dfrac{\partial \mathbf{\Psi}}{\partial y} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \psi_i}{\partial x} & \dfrac{\partial \psi_i}{\partial y} \\ \dfrac{\partial \psi_j}{\partial x} & \dfrac{\partial \psi_j}{\partial y} \\ \dfrac{\partial \psi_k}{\partial x} & \dfrac{\partial \psi_k}{\partial y} \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} b_i & c_i \\ b_j & c_j \\ b_k & c_k \end{bmatrix}$$

2-D Linear Triangles Using Problem Coordinates (6)

The derivatives of the temperature can be written in matrix form

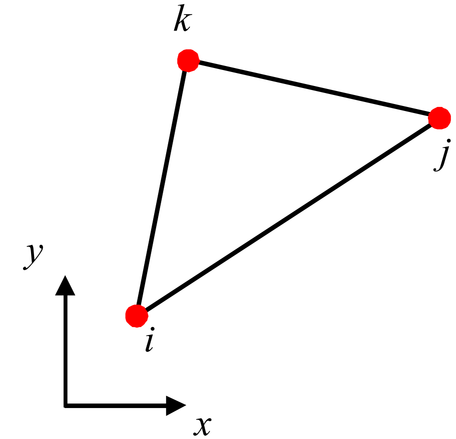
$$T = \mathbf{\Psi}^{\mathrm{T}}\,\mathbf{T} = \frac{1}{2A}\left\{a_i + b_i x + c_i y\ ,\quad a_j + b_j x + c_j y\ ,\quad a_k + b_k x + c_k y\right\}\begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial T}{\partial x} \\ \dfrac{\partial T}{\partial y} \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix}\begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix}$$



2-D Linear Triangles Using Problem Coordinates (7)

The element stiffness matrix can be written as



$$\mathbf{K}^e\,\mathbf{T} = \int_{A^e} \begin{bmatrix} \dfrac{\partial \boldsymbol{\Psi}}{\partial x} & \dfrac{\partial \boldsymbol{\Psi}}{\partial y} \end{bmatrix} k \begin{Bmatrix} \dfrac{\partial T}{\partial x} \\ \dfrac{\partial T}{\partial y} \end{Bmatrix} dA$$

$$\mathbf{K}^e\,\mathbf{T} = \frac{k}{4A^2} \left(\int_{A^e} \begin{bmatrix} b_i & c_i \\ b_j & c_j \\ b_k & c_k \end{bmatrix} \begin{bmatrix} b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix} dA \right) \begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix}$$

2-D Linear Triangles Using Problem Coordinates (8)

The element stiffness matrix can be written as

$$\mathbf{K}^e\,\mathbf{T} = \frac{k}{4A^2}\left(\int_{A^e}\begin{bmatrix} b_i^{\,2}+c_i^{\,2} & b_ib_j+c_ic_j & b_ib_k+c_ic_k \\ b_jb_i+c_jc_i & b_j^{\,2}+c_j^{\,2} & b_jb_k+c_jc_k \\ b_kb_i+c_kc_i & b_kb_j+c_kc_j & b_k^{\,2}+c_k^{\,2} \end{bmatrix} dA\right)\begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix}$$

Integrating (everything is constant)

$$\mathbf{K}^e\,\mathbf{T} = \frac{k}{4A}\begin{bmatrix} b_i^{\,2}+c_i^{\,2} & b_ib_j+c_ic_j & b_ib_k+c_ic_k \\ b_jb_i+c_jc_i & b_j^{\,2}+c_j^{\,2} & b_jb_k+c_jc_k \\ b_kb_i+c_kc_i & b_kb_j+c_kc_j & b_k^{\,2}+c_k^{\,2} \end{bmatrix}\begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix}$$

2-D Linear Triangles Using Problem Coordinates (9)

The volumetric energy generation (load vector) can be written as

$$\mathbf{F}^e = \frac{1}{2A}\int_{A^e} \begin{Bmatrix} a_i + b_i x + c_i y \\ a_j + b_j x + c_j y \\ a_k + b_k x + c_k y \end{Bmatrix} Q(x,y)\, dA$$

$$\int_{A^e} x\, dA = \overline{x}\, A$$

Integrating

$$\mathbf{F}^e = \frac{Q}{2A}\begin{Bmatrix} \int_{A^e}(a_i + b_i x + c_i y)\, dA \\ \int_{A^e}(a_j + b_j x + c_j y)\, dA \\ \int_{A^e}(a_k + b_k x + c_k y)\, dA \end{Bmatrix} = \frac{Q}{2A}\begin{Bmatrix} (a_i + b_i \overline{x} + c_i \overline{y})\, A \\ (a_j + b_j \overline{x} + c_j \overline{y})\, A \\ (a_k + b_k \overline{x} + c_k \overline{y})\, A \end{Bmatrix}$$

2-D Linear Triangles Using Problem Coordinates (10)

The load vector is

$$\mathbf{F}^e = \frac{Q}{2A}\begin{Bmatrix} \int_{A^e} \left(a_i + b_i x + c_i y\right) dA \\ \int_{A^e} \left(a_j + b_j x + c_j y\right) dA \\ \int_{A^e} \left(a_k + b_k x + c_k y\right) dA \end{Bmatrix} = QA\begin{Bmatrix} \frac{1}{2A}\left(a_i + b_i \overline{x} + c_i \overline{y}\right) \\ \frac{1}{2A}\left(a_j + b_j \overline{x} + c_j \overline{y}\right) \\ \frac{1}{2A}\left(a_k + b_k \overline{x} + c_k \overline{y}\right) \end{Bmatrix}$$

where

$$\int_{A^e} x\, dA = \overline{x}\, A \qquad \int_{A^e} y\, dA = \overline{y}\, A$$

and

$$\overline{x} = \frac{1}{3}\sum_{i=1}^{3} x_i \qquad \overline{y} = \frac{1}{3}\sum_{i=1}^{3} y_i$$

2-D Linear Triangles Using Problem Coordinates (11)

The load vector is

$$\mathbf{F}^e = QA\begin{Bmatrix} \frac{1}{2A}\left(a_i + b_i\overline{x} + c_i\overline{y}\right) \\ \frac{1}{2A}\left(a_j + b_j\overline{x} + c_j\overline{y}\right) \\ \frac{1}{2A}\left(a_k + b_k\overline{x} + c_k\overline{y}\right) \end{Bmatrix} = \frac{QA}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

2-D Linear Triangles Using Problem Coordinates (12)

Summarizing,

the element stiffness is given by

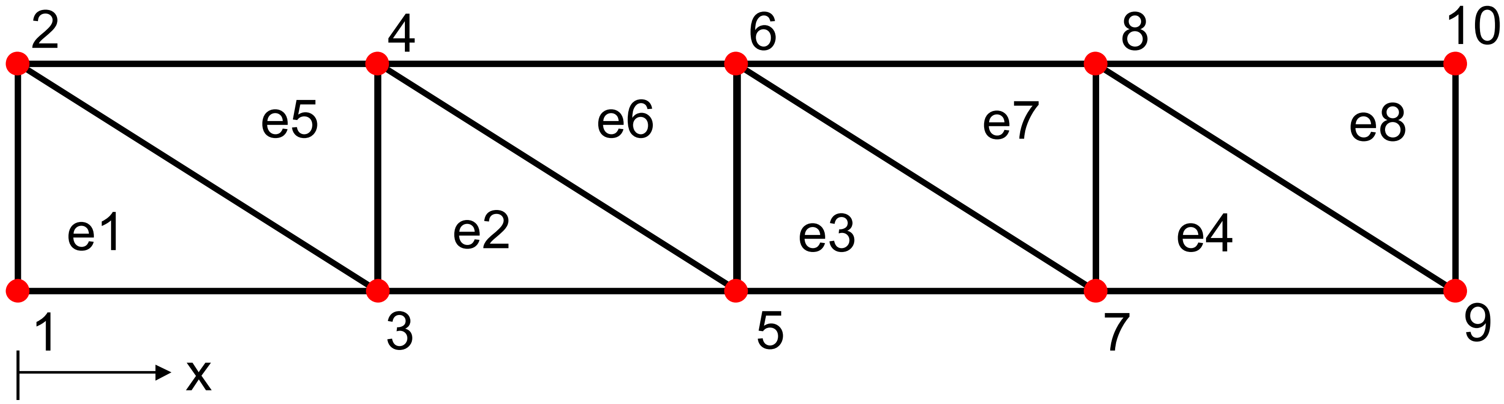
$$\mathbf{K}^e\,\mathbf{T} = \frac{k}{4A}\begin{bmatrix} b_i^2 + c_i^2 & b_i b_j + c_i c_j & b_i b_k + c_i c_k \\ b_j b_i + c_j c_i & b_j^2 + c_j^2 & b_j b_k + c_j c_k \\ b_k b_i + c_k c_i & b_k b_j + c_k c_j & b_k^2 + c_k^2 \end{bmatrix}\begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix}$$

and the load vector is

$$\mathbf{F}^e = \frac{QA}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

Example Problem Using 2-D Linear Triangles

We will consider our continuing example problem



For a total length of L,

Each element is w=L/4 wide and h=L/4 high

Recall, Q=2, L=k=1 with T(0)=1 and T(L)=2

Assume constant conductivity and volumetric energy generation

Example Problem Using 2-D Linear Triangles (2)

For element 1, global nodes i=1, j=3, k=2

$$a_i = x_j y_k - x_k y_j = hw \qquad b_i = y_j - y_k = -h \qquad c_i = x_k - x_j = -w$$

$$a_j = x_k y_i - x_i y_k = 0 \qquad b_j = y_k - y_i = h \qquad c_j = x_i - x_k = 0$$

$$a_k = x_i y_j - x_j y_i = 0 \qquad b_k = y_i - y_j = 0 \qquad c_k = x_j - x_i = w$$

$$\mathbf{K}^e\,\mathbf{T} = \frac{k}{4A}\begin{bmatrix} h^2 + w^2 & -h^2 & -w^2 \\ -h^2{}_i & h^2 & 0 \\ -w^2 & 0 & w^2 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_3 \\ T_2 \end{Bmatrix} \qquad \mathbf{F}^e = \frac{QA}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

Elements 2, 3, and 4 are identical

Example Problem Using 2-D Linear Triangles (3)

For element 1, global nodes i=1, j=3, k=2

$$\mathbf{K}^e\,\mathbf{T} = \frac{k}{4A}\begin{bmatrix} h^2+w^2 & -h^2 & -w^2 \\ -h^2{}_i & h^2 & 0 \\ -w^2 & 0 & w^2 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_3 \\ T_2 \end{Bmatrix} \qquad \mathbf{F}^e = \frac{QA}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

Substituting w = L/4, h = L/4, and A = hw/2 = $L^2/32$

$$\mathbf{K}^e\,\mathbf{T} = \frac{k}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_3 \\ T_2 \end{Bmatrix} \qquad \mathbf{F}^e = \frac{QL^2}{96}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

Elements 2, 3, and 4 are similar

Example Problem Using 2-D Linear Triangles (4)

For element 5, global nodes i=4, j=2, k=3

$$\mathbf{K}^e\,\mathbf{T} = \frac{k}{4A}\begin{bmatrix} h^2 + w^2 & -h^2 & -w^2 \\ -h^2{}_i & h^2 & 0 \\ -w^2 & 0 & w^2 \end{bmatrix}\begin{Bmatrix} T_4 \\ T_3 \\ T_3 \end{Bmatrix} \qquad \mathbf{F}^e = \frac{QA}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

Substituting w = L/4, h = L/4, and A = hw/2 = $L^2/32$

$$\mathbf{K}^e\,\mathbf{T} = \frac{k}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{Bmatrix} T_4 \\ T_2 \\ T_3 \end{Bmatrix} \qquad \mathbf{F}^e = \frac{QL^2}{96}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

Elements 6, 7, and 8 are similar

Example Problem Using 2-D Linear Triangles (5)

Assembly of the element stiffness for elements 1, 2, 3, and 4

$$\mathbf{K}^e\,\mathbf{T} = \frac{k}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_3 \\ T_2 \end{Bmatrix}^{e1} \qquad \begin{Bmatrix} T_3 \\ T_5 \\ T_4 \end{Bmatrix}^{e2} \quad \begin{Bmatrix} T_5 \\ T_7 \\ T_6 \end{Bmatrix}^{e3} \quad \begin{Bmatrix} T_7 \\ T_9 \\ T_8 \end{Bmatrix}^{e4}$$

And for elements 5, 6, 7, and 8

$$\mathbf{K}^e\,\mathbf{T} = \frac{k}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{Bmatrix} T_4 \\ T_2 \\ T_3 \end{Bmatrix}^{e5} \qquad \begin{Bmatrix} T_6 \\ T_4 \\ T_5 \end{Bmatrix}^{e6} \quad \begin{Bmatrix} T_8 \\ T_6 \\ T_7 \end{Bmatrix}^{e7} \quad \begin{Bmatrix} T_{10} \\ T_8 \\ T_9 \end{Bmatrix}^{e8}$$

Where the “exponent” refers to the corresponding element number

Example Problem Using 2-D Linear Triangles (6)

The global stiffness matrix is populated with element contributions

$$
\frac{k}{2}\begin{bmatrix}
2^{e1} & -1^{e1} & -1^{e1} & & & & & & & \\
-1^{e1} & 1^{e1}+1^{e5} & 0^{e1}+0^{e5} & -1^{e5} & & & & & & \\
-1^{e1} & 0^{e1}+0^{e5} & 1^{e1}+2^{e2}+1^{e5} & -1^{e2}-1^{e5} & -1^{e2} & & & & & \\
 & -1^{e5} & -1^{e2}-1^{e5} & 1^{e2}+2^{e5}+1^{e6} & 0^{e2}+0^{e6} & -1^{e6} & & & & \\
 & & -1^{e2} & 0^{e2}+0^{e6} & 1^{e2}+2^{e3}+1^{e6} & -1^{e3}-1^{e6} & -1^{e3} & & & \\
 & & & -1^{e6} & -1^{e3}-1^{e6} & 1^{e3}+2^{e6}+1^{e7} & 0^{e3}+0^{e7} & -1^{e7} & & \\
 & & & & -1^{e3} & 0^{e3}+0^{e7} & 1^{e3}+2^{e4}+1^{e7} & -1^{e4}-1^{e7} & -1^{e4} & \\
 & & & & & -1^{e7} & -1^{e4}-1^{e7} & 1^{e4}+2^{e7}+1^{e8} & 0^{e4}+0^{e8} & -1^{e8} \\
 & & & & & & -1^{e4} & 0^{e4}+0^{e8} & 1^{e4}+1^{e8} & -1^{e8} \\
 & & & & & & & -1^{e8} & -1^{e8} & 2^{e8}
\end{bmatrix}
\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \\ T_{10} \end{Bmatrix}
$$

where the “superscripts” refer to the corresponding element number

Example Problem Using 2-D Linear Triangles (7)

The global load vector is populated with element contributions

$$\frac{QL^2}{96}\begin{Bmatrix} 1^{e1} \\ 1^{e1}+1^{e5} \\ 1^{e1}+1^{e2}+1^{e5} \\ 1^{e2}+1^{e5}+1^{e6} \\ 1^{e2}+1^{e3}+1^{e6} \\ 1^{e3}+1^{e6}+1^{e7} \\ 1^{e3}+1^{e4}+1^{e7} \\ 1^{e4}+1^{e7}+1^{e8} \\ 1^{e4}+1^{e8} \\ 1^{e8} \end{Bmatrix} = \frac{QL^2}{96}\begin{Bmatrix} 1 \\ 2 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 2 \\ 1 \end{Bmatrix}$$

where the “superscripts” refers to the corresponding element number

Example Problem Using 2-D Linear Triangles (8)

So, the global system of equations is

$$\frac{k}{2}\begin{bmatrix} 2 & -1 & -1 & & & & & & & \\ -1 & 2 & 0 & -1 & & & & & & \\ -1 & 0 & 4 & -2 & -1 & & & & & \\ & -1 & -2 & 4 & 0 & -1 & & & & \\ & & -1 & 0 & 4 & -2 & -1 & & & \\ & & & -1 & -2 & 4 & 0 & -1 & & \\ & & & & -1 & 0 & 4 & -2 & -1 & \\ & & & & & -1 & -2 & 4 & 0 & -1 \\ & & & & & & -1 & 0 & 2 & -1 \\ & & & & & & & -1 & -1 & 2 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \\ T_{10} \end{Bmatrix} = \frac{QL^2}{96}\begin{Bmatrix} 1 \\ 2 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 2 \\ 1 \end{Bmatrix}$$

For this problem, we will apply temperature boundary conditions at the two ends of the domain.

Example Problem Using 2-D Linear Triangles (9)

Applying the boundary conditions at T(0)=1 and T(L)=2

$$\frac{k}{2}\begin{bmatrix} 2/k & 0 & 0 & & & & & & & \\ 0 & 2/k & 0 & 0 & & & & & & \\ -1 & 0 & 4 & -2 & -1 & & & & & \\ & -1 & -2 & 4 & 0 & -1 & & & & \\ & & -1 & 0 & 4 & -2 & -1 & & & \\ & & & -1 & -2 & 4 & 0 & -1 & & \\ & & & & -1 & 0 & 4 & -2 & -1 & \\ & & & & & -1 & -2 & 4 & 0 & -1 \\ & & & & & & 0 & 0 & 2/k & 0 \\ & & & & & & & 0 & 0 & 2/k \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \\ T_{10} \end{Bmatrix} = \frac{QL^2}{96} \begin{Bmatrix} \frac{96}{QL^2} \\ \frac{96}{QL^2} \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ \frac{2(96)}{QL^2} \\ \frac{2(96)}{QL^2} \end{Bmatrix}$$

Computed Results for 2-D Triangular Elements

The FEM solution and the analytical agree. However, because these are constant flux elements, the flux is discontinuous as it was with the linear 1-D element.

x/L :	0	¼	½	¾	1
FE Temp:	8/8	11.5/8	14/8	15.5/8	16/8
Analytic :	8/8	11.5/8	14/8	15.5/8	16/8