SAND2005-7705P

Finite Element Method III:

FEM for Multidimensional Heat Conduction

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Introductory Info

Evacuation Procedures:

- Exits are located…
- Restrooms out back

Classification:

- Absolutely no classified discussions
- If you have a concern, let us know
- Some material may be OUO, it will be marked as such

Summary for Finite Element Method III

Begin with:

- General form for element diffusion matrix and source/flux vectors

and end with:

- General procedure for computing element matrices and vectors in multi-dimensions

Questions for Finite Element Method III:

- What complications are encountered in developing general element matrices?
- What are simplex elements and simplex (area) coordinates?
- What are natural coordinates for non-simplex elements?
- What are typical shape functions for multi-dimensional elements?
- How and why is numerical quadrature performed?

Comments on Computing Element Matrices

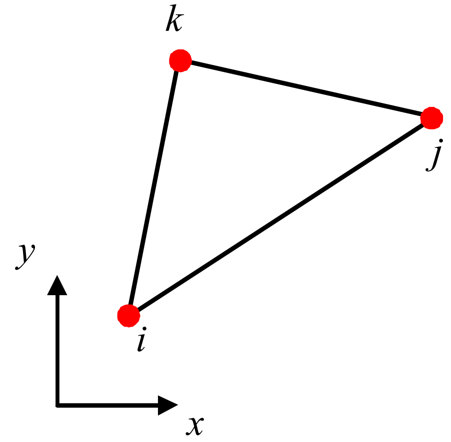
Recall again the definitions of the basic diffusion matrix and load vector

$$\mathbf{K}^e = \int_{\Omega^e} \frac{\partial \boldsymbol{\Psi}}{\partial x_i} k_{ij} \frac{\partial \boldsymbol{\Psi}^{\mathrm{T}}}{\partial x_j} d\Omega \qquad \mathbf{F}^e = \int_{\Omega^e} \boldsymbol{\Psi} Q \, d\Omega + \int_{\Gamma^e} \boldsymbol{\Psi} q_n d\Gamma$$

The shape functions $\boldsymbol{\Psi}$ could be defined directly in terms of the problem coordinates. This was done last time for the linear triangle. We will revisit that formulation to include the surface flux and to emphasize the need to change coordinate descriptions.

2-D Linear Triangle (1)

The shape functions for the linear triangle were developed from



$$T(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y$$

which produced

$$T(x,y) = \mathbf{\Psi}^{\mathrm{T}}\,\mathbf{T}$$

$$\mathbf{\Psi} = \begin{Bmatrix} \psi_i \\ \psi_j \\ \psi_k \end{Bmatrix} = \frac{1}{2A} \begin{Bmatrix} a_i + b_i x + c_i y \\ a_j + b_j x + c_j y \\ a_k + b_k x + c_k y \end{Bmatrix}$$

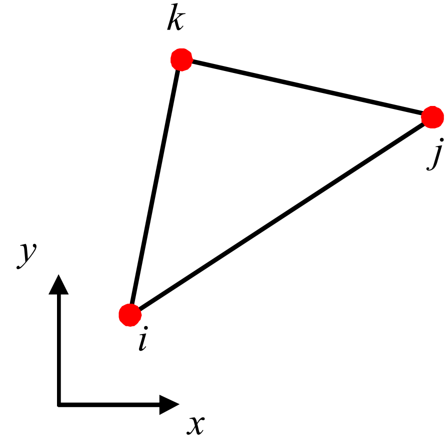
$$\frac{\partial T(x,y)}{\partial x} = \frac{\partial \mathbf{\Psi}^{\mathrm{T}}}{\partial x}\mathbf{T}$$

$$\frac{\partial T(x,y)}{\partial y} = \frac{\partial \mathbf{\Psi}^{\mathrm{T}}}{\partial y}\mathbf{T}$$

2-D Linear Triangle (2)

The element stiffness matrix can then be written as

$$\mathbf{K}^e\,\mathbf{T} = \int_{A^e} \left\{ \frac{\partial \boldsymbol{\Psi}}{\partial x} \quad \frac{\partial \boldsymbol{\Psi}}{\partial y} \right\} k \left\{ \begin{matrix} \dfrac{\partial \boldsymbol{\Psi}^{\mathrm{T}}}{\partial x}\mathbf{T} \\ \dfrac{\partial \boldsymbol{\Psi}^{\mathrm{T}}}{\partial y}\mathbf{T} \end{matrix} \right\} dA$$



and substituting

$$\mathbf{K}^e\,\mathbf{T} = \frac{1}{4A^2}\left(\int_{A^e} k \begin{bmatrix} b_i & c_i \\ b_j & c_j \\ b_k & c_k \end{bmatrix} \begin{bmatrix} b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix} dA \right) \left\{ \begin{matrix} T_i \\ T_j \\ T_k \end{matrix} \right\}$$

2-D Linear Triangle (3)

The element stiffness matrix can be written as

$$\mathbf{K}^e\,\mathbf{T} = \frac{1}{4A^2}\left(\int_{A^e} k \begin{bmatrix} b_i^2 + c_i^2 & b_i b_j + c_i c_j & b_i b_k + c_i c_k \\ b_j b_i + c_j c_i & b_j^2 + c_j^2 & b_j b_k + c_j c_k \\ b_k b_i + c_k c_i & b_k b_j + c_k c_j & b_k^2 + c_k^2 \end{bmatrix} dA\right)\begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix}$$

and integrating (if everything is constant)

$$\mathbf{K}^e\,\mathbf{T} = \frac{k}{4A}\begin{bmatrix} b_i^2 + c_i^2 & b_i b_j + c_i c_j & b_i b_k + c_i c_k \\ b_j b_i + c_j c_i & b_j^2 + c_j^2 & b_j b_k + c_j c_k \\ b_k b_i + c_k c_i & b_k b_j + c_k c_j & b_k^2 + c_k^2 \end{bmatrix}\begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix}$$

2-D Linear Triangle (4)

The load vector for the volume source is

$$\mathbf{F}_Q^e = \int_{A^e} \boldsymbol{\Psi} Q \, dA = \frac{1}{2A} \int_{A^e} \begin{Bmatrix} a_i + b_i x + c_i y \\ a_j + b_j x + c_j y \\ a_k + b_k x + c_k y \end{Bmatrix} Q(x,y) \, dA$$

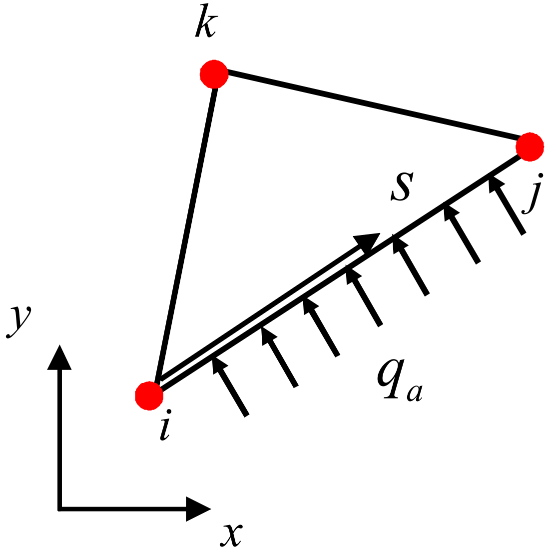
which for constant Q is

$$\mathbf{F}_Q^e = \frac{Q}{2A} \begin{Bmatrix} \left(a_i + b_i \bar{x} + c_i \bar{y}\right) A \\ \left(a_j + b_j \bar{x} + c_j \bar{y}\right) A \\ \left(a_k + b_k \bar{x} + c_k \bar{y}\right) A \end{Bmatrix} = \frac{QA}{3} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

2-D Linear Triangle (5)

For an element with an edge on the domain boundary the flux vector for a specified flux q_a is

$$\mathbf{F}^e_{q_a} = \int_{S^e} \mathbf{\Psi}_s \, q_a(s)\, ds$$



where s is the coordinate along the edge and $\mathbf{\Psi}_s$ is the element shape function restricted to (evaluated on) the element edge.

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2-D Linear Triangle (6)

For the linear triangle, the edge shape function is a linear function of distance along the edge or

$$\mathbf{\Psi}_s = \begin{Bmatrix} 1 - s/h \\ s/h \end{Bmatrix} \qquad \text{and} \qquad h = \sqrt{\left(x_j - x_i\right)^2 + \left(y_j - y_i\right)^2}$$

where s varies from 0 to h. For constant q_a

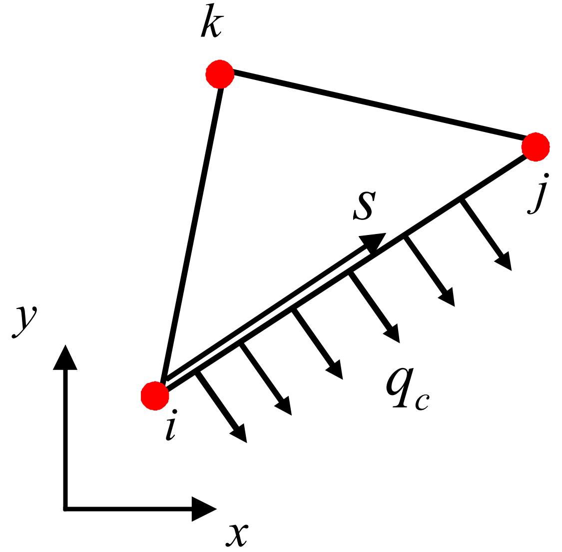
$$\mathbf{F}^e_{q_a} = \int_0^h \mathbf{\Psi}_s(s)\, q_a\, ds = q_a \begin{Bmatrix} h/2 \\ h/2 \end{Bmatrix}$$

2-D Linear Triangle (7)

For an element with an edge on the domain boundary the flux vector for a convective flux q_c is

$$\mathbf{F}^e_{q_c} = -\int_{S^e} \mathbf{\Psi}_s\, h_c (T - T_c)\, ds$$

$$\mathbf{F}^e_{q_c} = -\int_{S^e} \mathbf{\Psi}_s\, h_c (\mathbf{\Psi}_s^{\mathrm{T}} \mathbf{T} - T_c)\, ds$$



where s is the coordinate along the edge and $\mathbf{\Psi}_s$ is the element shape function restricted to (evaluated on) the element edge.

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2-D Linear Triangle (8)

For the linear triangle, the edge shape function is a linear function of distance along the edge or

$$\boldsymbol{\Psi}_{\mathrm{s}} = \begin{Bmatrix} 1 - s/h \\ s/h \end{Bmatrix} \quad \text{and} \quad h = \sqrt{\left(x_j - x_i\right)^2 + \left(y_j - y_i\right)^2}$$

where s varies from 0 to h. For constant h_c and T_c

$$\mathbf{F}^e_{q_c} = -\int_0^h \boldsymbol{\Psi}_{\mathrm{s}}\, h_c \boldsymbol{\Psi}^{\mathrm{T}}_{\mathrm{s}}\, ds\, \mathbf{T} + \int_0^h \boldsymbol{\Psi}_{\mathrm{s}}\, h_c T_c\, ds$$

$$\mathbf{F}^e_{q_c} = -h_c \begin{bmatrix} h/3 & h/6 \\ h/6 & h/3 \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \end{Bmatrix} + h_c T_c \begin{Bmatrix} h/2 \\ h/2 \end{Bmatrix}$$

Summary for 2-D Linear Triangle (9)

$$\mathbf{K}^e\,\mathbf{T} = \frac{k}{4A}\begin{bmatrix} b_i^2 + c_i^2 & b_ib_j + c_ic_j & b_ib_k + c_ic_k \\ b_jb_i + c_jc_i & b_j^2 + c_j^2 & b_jb_k + c_jc_k \\ b_kb_i + c_kc_i & b_kb_j + c_kc_j & b_k^2 + c_k^2 \end{bmatrix}\begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix}$$

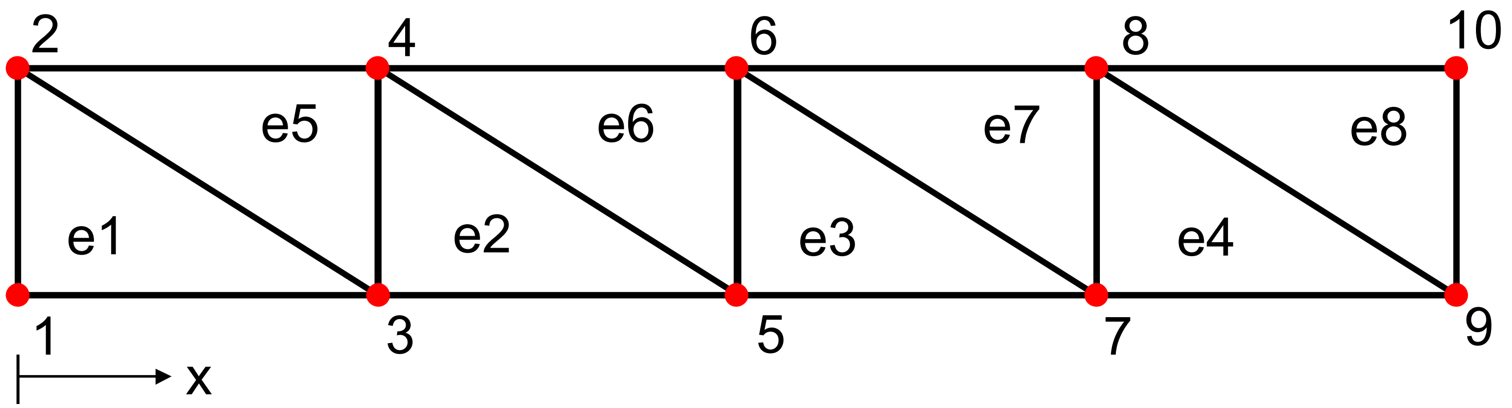
$$\mathbf{F}_q^e = q_a\begin{Bmatrix} h/2 \\ h/2 \end{Bmatrix}$$

$$\mathbf{F}_q^e = -h_c\begin{bmatrix} h/3 & h/6 \\ h/6 & h/3 \end{bmatrix}\begin{Bmatrix} T_i \\ T_j \end{Bmatrix} + h_cT_c\begin{Bmatrix} h/2 \\ h/2 \end{Bmatrix}$$

$$\mathbf{F}_Q^e = \frac{QA}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

Example Problem Using 2-D Linear Triangles

Consider our continuing example problem



For a total length of L,

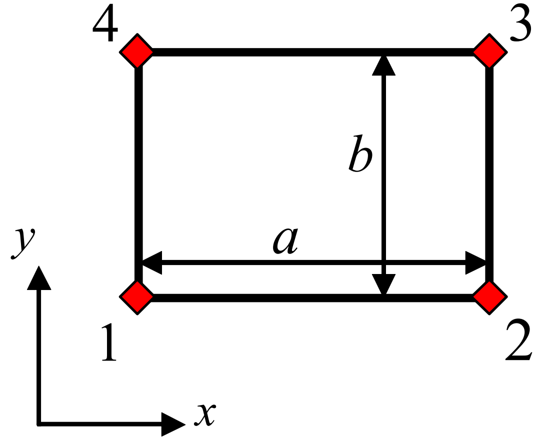
Each element is w=L/4 wide and h=L/4 high

Recall, Q=2, L=k=1 with q(0)=2 and q(L)=10 (T-2)

2-D Linear Rectangle (1)

The shape functions for a linear rectangle element can be derived from

$$T(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$



which produces

$$T(x,y) = \mathbf{\Psi}^{\mathrm{T}}\,\mathbf{T}$$

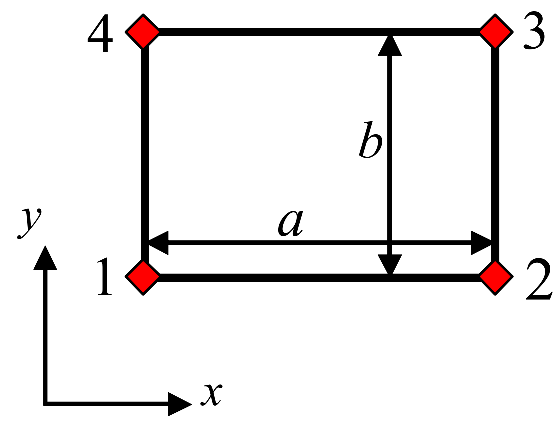
$$\mathbf{\Psi} = \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{Bmatrix} = \begin{Bmatrix} (1 - x/a)(1 - y/b) \\ (x/a)(1 - y/b) \\ (x/a)(y/b) \\ (1 - x/a)(y/b) \end{Bmatrix}$$

$$\frac{\partial T(x,y)}{\partial x} = \frac{\partial \mathbf{\Psi}^{\mathrm{T}}}{\partial x}\mathbf{T}$$

$$\frac{\partial T(x,y)}{\partial y} = \frac{\partial \mathbf{\Psi}^{\mathrm{T}}}{\partial y}\mathbf{T}$$

2-D Linear Rectangle (2)

The element stiffness matrix can then be written as



$$\mathbf{K}^e\,\mathbf{T} = \int_{A^e} \left\{ \frac{\partial \boldsymbol{\Psi}}{\partial x} \quad \frac{\partial \boldsymbol{\Psi}}{\partial y} \right\} k \left\{ \begin{matrix} \dfrac{\partial \boldsymbol{\Psi}^{\mathrm{T}}}{\partial x}\mathbf{T} \\ \dfrac{\partial \boldsymbol{\Psi}^{\mathrm{T}}}{\partial y}\mathbf{T} \end{matrix} \right\} dA$$

and substituting

$$\mathbf{K}^e\,\mathbf{T} = \int_0^b \int_0^a k \begin{bmatrix} -(1/a)(1-y/b) & -(1/b)(1-x/a) \\ (1/a)(1-y/b) & -(1/b)(x/a) \\ (1/a)(y/b) & (1/b)(x/a) \\ -(1/a)(y/b) & (1/b)(1-x/a) \end{bmatrix} \bullet$$

$$\begin{Bmatrix} -(1/a)(1-y/b) & (1/a)(1-y/b) & (1/a)(y/b) & -(1/a)(y/b) \\ -(1/b)(1-x/a) & -(1/b)(x/a) & (1/b)(x/a) & (1/b)(1-x/a) \end{Bmatrix} dxdy \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix}$$

2-D Linear Rectangle (3)

After integration the stiffness is

$$\mathbf{K}^e\,\mathbf{T} = \frac{k}{6ab}\begin{bmatrix} 2b^2+2a^2 & -2b^2+a^2 & -b^2-a^2 & b^2-2a^2 \\ -2b^2+a^2 & 2b^2+2a^2 & b^2-2a^2 & -b^2-a^2 \\ -b^2-a^2 & b^2-2a^2 & 2b^2+2a^2 & -2b^2+a^2 \\ b^2-2a^2 & -b^2-a^2 & -2b^2+a^2 & 2b^2+2a^2 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix}$$

for a constant k. The volume source vector is

$$\mathbf{F}_Q^e = \frac{Qab}{4}\begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$$

2-D Linear Rectangle (4)

The load vector for the applied flux and convective heat flux are the same as for the linear triangle.

$$\mathbf{F}^e_{q_a} = q_a \begin{Bmatrix} h/2 \\ h/2 \end{Bmatrix}$$

$$\mathbf{F}^e_{q_c} = -h_c \begin{bmatrix} h/3 & h/6 \\ h/6 & h/3 \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \end{Bmatrix} + h_c T_c \begin{Bmatrix} h/2 \\ h/2 \end{Bmatrix}$$

where $h = a$ or b depending on the edge and i, j are the nodes on the edge.

Summary for
2-D Linear Rectangle (5)

$$\mathbf{K}^e\mathbf{T} = \frac{k}{6ab}\begin{bmatrix} 2b^2+2a^2 & -2b^2+a^2 & -b^2-a^2 & b^2-2a^2 \\ -2b^2+a^2 & 2b^2+2a^2 & b^2-2a^2 & -b^2-a^2 \\ -b^2-a^2 & b^2-2a^2 & 2b^2+2a^2 & -2b^2+a^2 \\ b^2-2a^2 & -b^2-a^2 & -2b^2+a^2 & 2b^2+2a^2 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix}$$

$$\mathbf{F}^e_{q_a} = q_a\begin{Bmatrix} h/2 \\ h/2 \end{Bmatrix}$$

$$\mathbf{F}^e_{q_c} = -h_c\begin{bmatrix} h/3 & h/6 \\ h/6 & h/3 \end{bmatrix}\begin{Bmatrix} T_i \\ T_j \end{Bmatrix} + h_c T_c\begin{Bmatrix} h/2 \\ h/2 \end{Bmatrix}$$

$$\mathbf{F}^e_Q = \frac{Qab}{4}\begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$$

Observations on 2-D Elements

- Successfully developed required element matrices and vectors to represent steady heat conduction in 2D
- We should have observed some obvious limitations on these elements and their descriptions (coordinates)
  - For triangles, integration limits cause a problem when the integrands are not constant
  - For rectangles, the element geometry is not very useful as it leads to a structured mesh and any generalization of the geometry would lead to integration problems
- The conclusion from this exercise is that a more general method of element description and matrix construction is required

Simplex Elements (1)

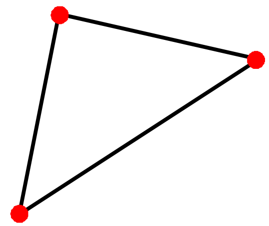
Simplex elements for N-space are defined “as the minimal possible nontrivial geometric figure in that space; it is always a figure defined by N+1 vertices.” In 1-D, this is a line, in 2-D this is a triangle and in 3-D this is a tetrahedron.

Shape functions for the 1-D element were previously developed using the natural coordinates for the line. For 2-D and 3-D elements we need to develop shape functions in the “natural” coordinates for the simplexes.

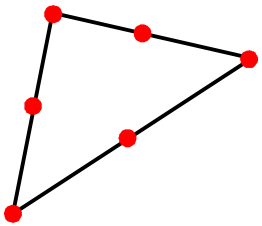
Simplex Elements (2)

General families of shape functions for 2D and 3D simplex elements can be derived using Pascal's triangle and its 3D counterpart. For example, the family of triangles

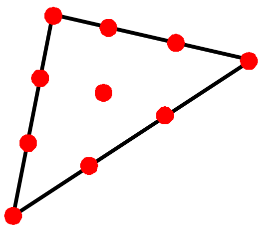
$$T(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y$$



$$T(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy + \alpha_5 x^2 + \alpha_6 y^2$$



$$T(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy + \alpha_5 x^2 + \alpha_6 y^2 + \alpha_7 x^3 + \alpha_8 x^2 y + \alpha_9 y^2 x + \alpha_{10} y^3$$

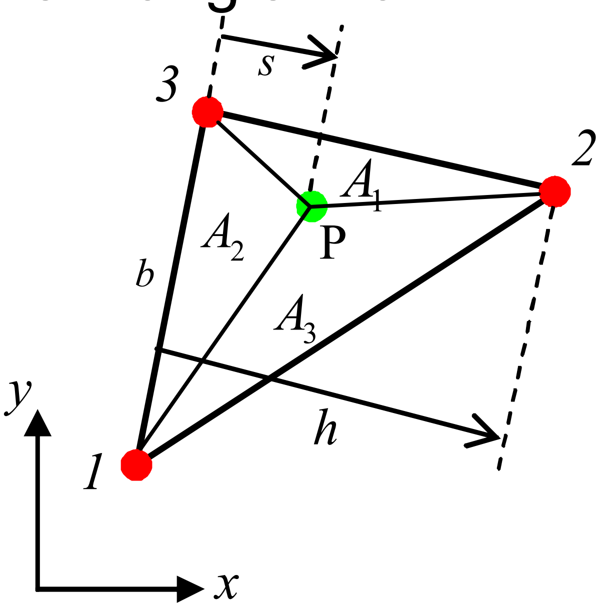


Simplex (Area) Coordinates (1)

A natural (or local) coordinate system for a triangle is defined by the simplex or area coordinates. For a point P located within a triangle, the area coordinates are given by L_i, L_j, L_k . These coordinates are the ratios of the areas of the sub-triangles formed by point P and any two vertices and the area of the triangle. For example, the second coordinate

$$A = \frac{bh}{2} \quad ; \quad A_2 = \frac{bs}{2} \quad ; \quad L_2 = \frac{A_2}{A} = \frac{s}{h}$$

$$\frac{A_i}{A} + \frac{A_j}{A} + \frac{A_k}{A} = 1 = L_i + L_j + L_k$$



Area Coordinates (2)

The problem coordinates for the triangle are related to the area coordinates by

$$x = \sum_{i=1}^{N} L_i x_i \quad ; \quad y = \sum_{i=1}^{N} L_i y_i \qquad \sum_{i=1}^{N} L_i = 1$$

Using these relations, the inverse transformation can be derived, which defines the linear shape function in area coordinates

$$\begin{Bmatrix} L_i \\ L_j \\ L_k \end{Bmatrix} = \frac{1}{2A} \begin{Bmatrix} a_i + b_i x + c_i y \\ a_j + b_j x + c_j y \\ a_k + b_k x + c_k y \end{Bmatrix} = \begin{Bmatrix} \psi_i \\ \psi_j \\ \psi_k \end{Bmatrix} = \mathbf{\Psi}$$

Area Coordinates (3)

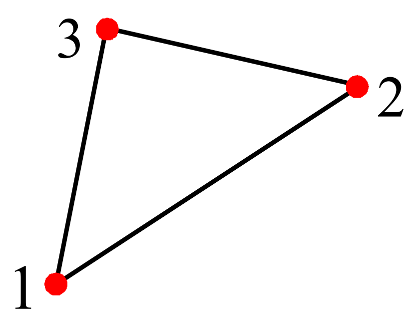
- Area coordinates vary from 0 to 1
- The area coordinates are not independent since there are three coordinates to describe two spatial dimensions
- The relation $L_1 + L_2 + L_3 = 1$ allows the third coordinate to be written in terms of the first two.

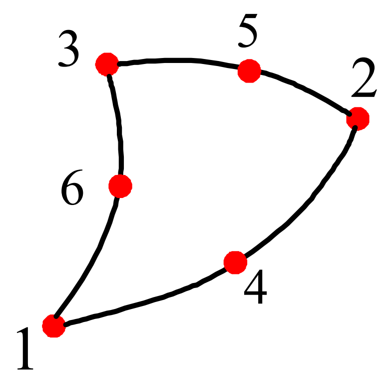
Simplex Elements (3)

Return to the writing of shape functions for the simplex elements using the area coordinates

$$T(L_i) = \mathbf{\Psi}^{\mathrm{T}}\,\mathbf{T}$$

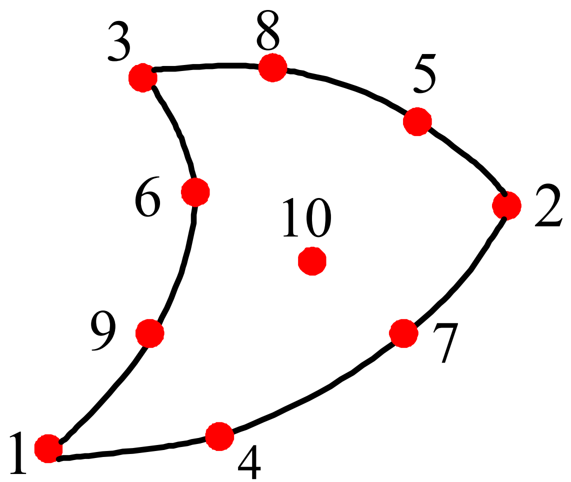
$$\mathbf{\Psi}_{linear} = \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} \qquad \mathbf{\Psi}_{quadratic} = \begin{Bmatrix} L_1(2L_1 - 1) \\ L_2(2L_2 - 1) \\ L_3(2L_3 - 1) \\ 4L_1L_2 \\ 4L_2L_3 \\ 4L_3L_1 \end{Bmatrix}$$





Simplex Elements (4)

$$\mathbf{\Psi}_{cubic} = \frac{1}{2}\begin{Bmatrix} L_1(3L_1-1)(3L_1-2) \\ L_1(3L_2-1)(3L_2-2) \\ L_1(3L_3-1)(3L_3-2) \\ 9L_1L_2(3L_1-1) \\ 9L_2L_3(3L_2-1) \\ 9L_3L_1(3L_3-1) \\ 9L_1L_2(3L_2-1) \\ 9L_2L_3(3L_3-1) \\ 9L_3L_1(3L_1-1) \\ 54L_1L_2L_3 \end{Bmatrix}$$



Isoparametric Triangle (1)

With the shape functions defined in area coordinates, we want to return to the construction of the element matrices and vectors. In order to allow a general element shape, we will use the isoparametric mapping defined previously to describe the element geometry.

Linear and quadratic shape functions are the usual choices and the only ones considered here. We will write the functions and their derivatives and then construct the general form of the matrices.

Isoparametric Triangle (2)

Linear :

$$\boldsymbol{\Psi}_l = \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} \qquad \frac{\partial \boldsymbol{\Psi}_l}{\partial L_1} = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} \qquad \frac{\partial \boldsymbol{\Psi}_l}{\partial L_2} = \begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix}$$

Quadratic:

$$\boldsymbol{\Psi}_q = \begin{Bmatrix} L_1(2L_1 - 1) \\ L_2(2L_2 - 1) \\ L_3(2L_3 - 1) \\ 4L_1L_2 \\ 4L_2L_3 \\ 4L_3L_1 \end{Bmatrix} \qquad \frac{\partial \boldsymbol{\Psi}_q}{\partial L_1} = \begin{Bmatrix} (4L_1 - 1) \\ 0 \\ 4L_1 + 4L_2 - 3 \\ 4L_2 \\ -4L_2 \\ 8L_1 - 4L_2 + 4 \end{Bmatrix} \qquad \frac{\partial \boldsymbol{\Psi}_q}{\partial L_2} = \begin{Bmatrix} 0 \\ (4L_2 - 1) \\ 4L_2 + 4L_1 - 3 \\ 4L_1 \\ -8L_2 - 4L_1 + 4 \\ -4L_1 \end{Bmatrix}$$

Isoparametric Triangle (3)
Element Equation

The weighted integral statement for steady conduction

$$\int_{\Omega^e} \frac{\partial \boldsymbol{\Psi}}{\partial x_i} k_{ij} \frac{\partial \boldsymbol{\Psi}^{\mathrm{T}}}{\partial x_j} d\Omega \; \mathbf{T} = \int_{\Omega^e} \boldsymbol{\Psi} Q \, d\Omega + \int_{\Gamma^e} \hat{\boldsymbol{\Psi}} q_n d\Gamma$$

which will produce

$$\left[\mathbf{K}^e_{\mathbf{xx}} + \mathbf{K}^e_{\mathbf{xy}} + \mathbf{K}^e_{\mathbf{yx}} + \mathbf{K}^e_{\mathbf{yy}} \right] \mathbf{T}^e = \mathbf{F}^e_Q + \mathbf{F}^e_q$$

We will consider how to construct each term in this equation in preparation for assembly into the global system.

Diffusion Matrix (1)
Isoparametric Triangle

To compute the matrix components we will use the parametric mapping where the element geometry is defined by the same shape functions as the dependent variable interpolation. Thus, for the triangle

$$x = x(L_i) \qquad ; \qquad y = y\left(L_i\right)$$

or

$$x = \boldsymbol{\Psi}^{\mathrm{T}}\,\mathbf{x} \qquad ; \qquad y = \boldsymbol{\Psi}^{\mathrm{T}}\,\mathbf{y}$$

where the $\mathbf{x}, \mathbf{y}$ vectors contain the coordinates of the nodal points.

Diffusion Matrix (2)
Isoparametric Triangle

The spatial derivatives of the shape functions are

$$\begin{Bmatrix} \dfrac{\partial \boldsymbol{\Psi}}{\partial L_1} \\ \dfrac{\partial \boldsymbol{\Psi}}{\partial L_2} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial L_1} & \dfrac{\partial y}{\partial L_1} \\ \dfrac{\partial x}{\partial L_2} & \dfrac{\partial y}{\partial L_2} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \boldsymbol{\Psi}}{\partial x} \\ \dfrac{\partial \boldsymbol{\Psi}}{\partial y} \end{Bmatrix} = [J] \begin{Bmatrix} \dfrac{\partial \boldsymbol{\Psi}}{\partial x} \\ \dfrac{\partial \boldsymbol{\Psi}}{\partial y} \end{Bmatrix}$$

or inverting the Jacobian matrix

$$\begin{Bmatrix} \dfrac{\partial \boldsymbol{\Psi}}{\partial x} \\ \dfrac{\partial \boldsymbol{\Psi}}{\partial y} \end{Bmatrix} = \frac{1}{\det[J]} \begin{bmatrix} \dfrac{\partial y}{\partial L_2} & -\dfrac{\partial y}{\partial L_1} \\ -\dfrac{\partial x}{\partial L_2} & \dfrac{\partial x}{\partial L_1} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \boldsymbol{\Psi}}{\partial L_1} \\ \dfrac{\partial \boldsymbol{\Psi}}{\partial L_2} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial \boldsymbol{\Psi}}{\partial L_1} \\ \dfrac{\partial \boldsymbol{\Psi}}{\partial L_2} \end{Bmatrix}$$

Diffusion Matrix (3)
Isoparametric Triangle

Substituting the mapping functions

$$\begin{Bmatrix} \dfrac{\partial \boldsymbol{\Psi}}{\partial x} \\ \dfrac{\partial \boldsymbol{\Psi}}{\partial y} \end{Bmatrix} = \frac{1}{\det[J]} \begin{bmatrix} \dfrac{\partial \boldsymbol{\Psi}^{\mathrm{T}}}{\partial L_2}\mathbf{y} & -\dfrac{\partial \boldsymbol{\Psi}^{\mathrm{T}}}{\partial L_1}\mathbf{y} \\ -\dfrac{\partial \boldsymbol{\Psi}^{\mathrm{T}}}{\partial L_2}\mathbf{x} & \dfrac{\partial \boldsymbol{\Psi}^{\mathrm{T}}}{\partial L_1}\mathbf{x} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \boldsymbol{\Psi}}{\partial L_1} \\ \dfrac{\partial \boldsymbol{\Psi}}{\partial L_2} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial \boldsymbol{\Psi}}{\partial L_1} \\ \dfrac{\partial \boldsymbol{\Psi}}{\partial L_2} \end{Bmatrix}$$

and

$$\det[J] = |J| = \frac{\partial x}{\partial L_1}\frac{\partial y}{\partial L_2} - \frac{\partial x}{\partial L_2}\frac{\partial y}{\partial L_1} = \frac{\partial \boldsymbol{\Psi}^{\mathrm{T}}}{\partial L_1}\mathbf{x}\frac{\partial \boldsymbol{\Psi}^{\mathrm{T}}}{\partial L_2}\mathbf{y} - \frac{\partial \boldsymbol{\Psi}^{\mathrm{T}}}{\partial L_2}\mathbf{x}\frac{\partial \boldsymbol{\Psi}^{\mathrm{T}}}{\partial L_1}\mathbf{y}$$

Diffusion Matrix (4)
Isoparametric Triangle

The area integration over x, y in the triangle must be transformed to the area coordinates L_1, L_2. The needed relation is

$$dxdy = d\Omega_{x,y} = \left|J\right| dL_1 dL_2 = \left|J\right| d\Omega_{L_1 L_2}$$

which states that the ratio of incremental areas between the physical element and the master element is given by the determinant of the Jacobian matrix for the coordinate transformation.

Diffusion Matrix (5)
Isoparametric Triangle

Diffusion matrix for anistropic case

$$\mathbf{K}^e = \int_{\Omega_e} \left(\frac{\partial \mathbf{\Psi}}{\partial x} k_{xx} \frac{\partial \mathbf{\Psi}^{\mathrm{T}}}{\partial x} + \frac{\partial \mathbf{\Psi}}{\partial x} k_{xy} \frac{\partial \mathbf{\Psi}^{\mathrm{T}}}{\partial y} + \frac{\partial \mathbf{\Psi}}{\partial y} k_{yx} \frac{\partial \mathbf{\Psi}^{\mathrm{T}}}{\partial x} + \frac{\partial \mathbf{\Psi}}{\partial y} k_{yy} \frac{\partial \mathbf{\Psi}^{\mathrm{T}}}{\partial y} \right) d\Omega_{x,y}$$

which is the sum of four matrices.

Diffusion Matrix (5)
Isoparametric Triangle

Consider in detail one of the matrices and transform to the master element

$$\mathbf{K}_{\mathbf{xy}}^{e} = \int_{\Omega_e} \left(\frac{\partial \mathbf{\Psi}}{\partial x} k_{xy} \frac{\partial \mathbf{\Psi}^{\mathrm{T}}}{\partial y} \right) d\Omega_{x,y}$$

$$\mathbf{K}_{\mathbf{xy}}^{e} = \int_0^1 \int_0^{1-L_1} \frac{1}{|J|} \left(\frac{\partial \mathbf{\Psi}^{\mathrm{T}}}{\partial L_2} \mathbf{y} \frac{\partial \mathbf{\Psi}}{\partial L_1} - \frac{\partial \mathbf{\Psi}^{\mathrm{T}}}{\partial L_1} \mathbf{y} \frac{\partial \mathbf{\Psi}}{\partial L_2} \right) k_{xy} \Box$$

$$\frac{1}{|J|} \left(\frac{\partial \mathbf{\Psi}^{\mathrm{T}}}{\partial L_2} \mathbf{y} \frac{\partial \mathbf{\Psi}}{\partial L_1} - \frac{\partial \mathbf{\Psi}^{\mathrm{T}}}{\partial L_1} \mathbf{y} \frac{\partial \mathbf{\Psi}}{\partial L_2} \right) |J| \, dL_2 dL_1$$

$$|J| = \frac{\partial \mathbf{\Psi}^{\mathrm{T}}}{\partial L_1} \mathbf{x} \frac{\partial \mathbf{\Psi}^{\mathrm{T}}}{\partial L_2} \mathbf{y} - \frac{\partial \mathbf{\Psi}^{\mathrm{T}}}{\partial L_2} \mathbf{x} \frac{\partial \mathbf{\Psi}^{\mathrm{T}}}{\partial L_1} \mathbf{y}$$

Diffusion Matrix (6)
Isoparametric Triangle

All of the diffusion matrices are of the same form

$$\mathbf{K}_{ij}^{e} = \int_{0}^{1} \int_{0}^{1-L_1} f(L_1, L_2)\, dL_2 dL_1$$

where f is generally a complex function of the area coordinates. For the case of area coordinates, there are simple rules (formulas) for the integration of polynomials. However, in general we prefer to perform this integration numerically to retain consistency and commonality with the non-simplex elements.

Volume Source Vector Isoparametric Triangle

Consider next the volume source term

$$\mathbf{F}_Q^e = \int\limits_{\Omega^e} \mathbf{\Psi}\, Q \, d\Omega$$

Transforming to the parent element and using area coordinates

$$\mathbf{F}_Q^e = \int_0^1 \int_0^1 \mathbf{\Psi}\, Q(L_i) \left| J \right| dL_1 dL_2$$

This has the same functional form as the diffusion matrix and is also treated via numerical integration.

Surface Flux Vector (1)
Isoparametric Triangle

Two components of the surface flux vector for elements with an edge on the boundary are

$$\mathbf{F}_q^e = \int_{\Gamma^e} \hat{\mathbf{\Psi}}\, q_n \, d\Gamma = \int_{\Gamma^e} \hat{\mathbf{\Psi}}\, q_a \, d\Gamma - \int_{\Gamma^e} \hat{\mathbf{\Psi}}\, q_c \, d\Gamma$$

or

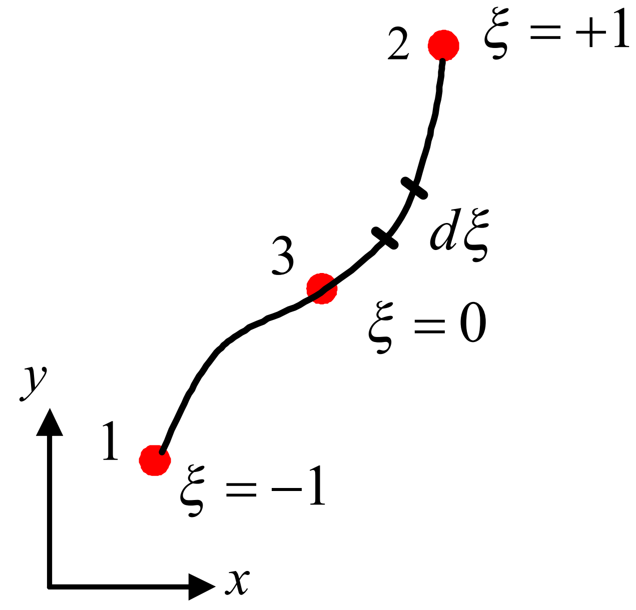
$$\mathbf{F}_q^e = \int_{\Gamma^e} \hat{\mathbf{\Psi}}\, q_a \, d\Gamma - \int_{\Gamma^e} \hat{\mathbf{\Psi}}\, h_c (\hat{\mathbf{\Psi}}^{\mathrm{T}} \hat{\mathbf{T}} - T_c)\, d\Gamma$$

Surface Flux Vectors (2)
Isoparametric Triangle

As before, we want to transform to a master element surface (edge) description . The linear and quadratic edge shape functions were given previously as

$$\hat{\mathbf{\Psi}}_{linear} = \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix} = \frac{1}{2}\begin{Bmatrix} (1-\xi) \\ (1+\xi) \end{Bmatrix}$$

$$\hat{\mathbf{\Psi}}_{quadratic} = \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{Bmatrix} = \frac{1}{2}\begin{Bmatrix} (1-\xi)\xi \\ (1+\xi)\xi \\ 2(1-\xi^2) \end{Bmatrix}$$



Surface Flux Vector (3) Isoparametric Triangle

The transformation of the line integral is similar to the area transformation

$$d\Gamma = \left|J_{\xi}\right| d\xi \qquad ; \qquad \left|J_{\xi}\right| = \Delta = \left[\left(\frac{\partial x}{\partial \xi} \right)^2 + \left(\frac{\partial y}{\partial \xi} \right)^2 \right]^{1/2}$$

Basically, the edge determinant (of the Jacobian) provides the ratio of the line lengths between the physical edge and the master edge.

Surface Flux Vector (4) Isoparametric Triangle

The surface flux vectors are then

$$\mathbf{F}_q^e = \int_{-1}^{+1} \hat{\boldsymbol{\Psi}} q_a \Delta \, d\xi - \int_{-1}^{+1} \hat{\boldsymbol{\Psi}} h_c (\hat{\boldsymbol{\Psi}}^{\mathrm{T}} \hat{\mathbf{T}} - T_c) \Delta \, d\xi$$

or in matrix form

$$\mathbf{F}_q^e = \mathbf{F}_{q_a}^e - \mathbf{C}\hat{\mathbf{T}} + \mathbf{F}_{hc}$$

The edge integrals will be computed using numerical integration. Note that the $\mathbf{C}$ matrix will be moved to the left-hand-side of the equation and combined with the diffusion matrix since the term contains unknown nodal point temperatures.

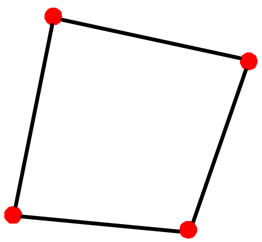
Non-Simplex Elements (1) Quadrilaterals

Shape functions for non-simplex, quadrilateral elements can be generated using tensor products of one-dimensional polynomials. These products are most conveniently done in the natural coordinate system for the master element. The natural coordinate system, as designed for the line element, is used in each direction for the quadrilateral. The coordinates, ξ, η run from -1 to +1 and are centered in the master element.

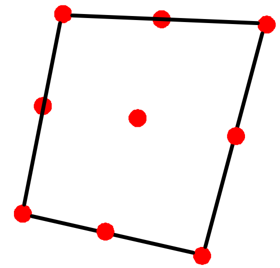
Quadrilateral Elements (2)

General families of shape functions for 2D quadrilaterals can be derived directly using polynomial expansions of the following forms

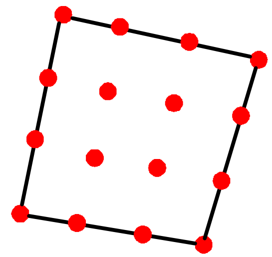
$$T(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$



$$T(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy + \alpha_5 x^2 + \alpha_6 y^2 + \alpha_7 x^2 y + \alpha_8 xy^2 + \alpha_9 x^2 y^2$$



$$T(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy + \alpha_5 x^2 + \alpha_6 y^2 + \alpha_7 x^3 + \alpha_8 x^2 y + \alpha_9 y^2 x + \alpha_{10} y^3 + \ldots \alpha_{16} x^3 y^3$$

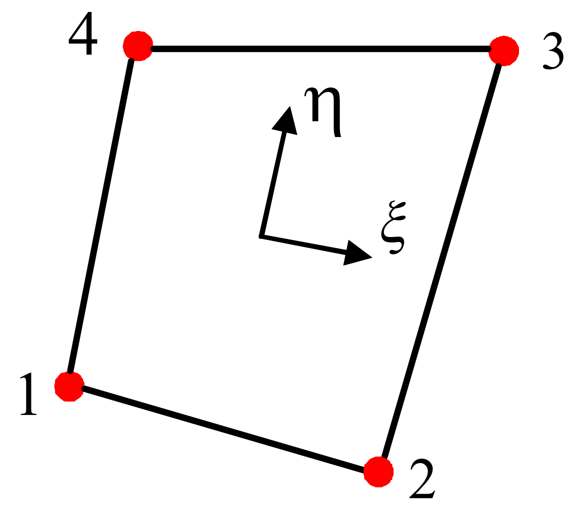


Quadrilateral Elements (3)

Return to the writing of shape functions for quadrilateral elements using the natural coordinates

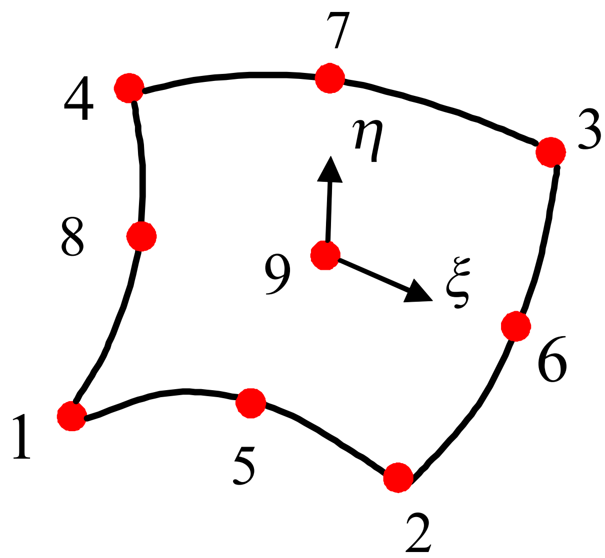
$$T(\xi,\eta) = \mathbf{\Psi}^{\mathrm{T}}\,\mathbf{T}$$

$$\mathbf{\Psi}_{linear} = \frac{1}{4}\begin{Bmatrix} (1-\xi)(1-\eta) \\ (1+\xi)(1-\eta) \\ (1+\xi)(1+\eta) \\ (1-\xi)(1+\eta) \end{Bmatrix}$$



Quadrilateral Elements (4)

$$\mathbf{\Psi}_{quadratic} = \frac{1}{4}\begin{Bmatrix} (\xi^2-\xi)(\eta^2-\eta) \\ (\xi^2+\xi)(\eta^2-\eta) \\ (\xi^2+\xi)(\eta^2+\eta) \\ (\xi^2-\xi)(\eta^2+\eta) \\ 2(1-\xi^2)(1-\eta^2) \\ 2(\xi^2+\xi)(\eta^2-\eta) \\ 2(1-\xi^2)(\eta^2+\eta) \\ 2(\xi^2-\xi)(1-\eta^2) \\ 4(1-\xi^2)(1-\eta^2) \end{Bmatrix}$$



Diffusion Matrix (1)
Isoparametric Quadrilateral

We will only consider one component of the diffusion matrix in detail, as everything done for the isoparametric triangle carries over to the quadrilateral. The only change that is required is a change in the element coordinates from L_1, L_2 to ξ, η and a change in the limits of integration. A cross component of the diffusion matrix then is

$$\mathbf{K}_{\mathbf{xy}}^{e} = \int_{\Omega_e} \left(\frac{\partial \boldsymbol{\Psi}}{\partial x} k_{xy} \frac{\partial \boldsymbol{\Psi}^{\mathrm{T}}}{\partial y} \right) d\Omega_{x,y}$$

Diffusion Matrix (2)
Isoparametric Quadrilateral

Transform to the master element

$$\mathbf{K}^{e}_{\mathbf{xy}} = \int_{-1}^{+1}\int_{-1}^{+1} \frac{1}{|J|}\left(\frac{\partial \mathbf{\Psi}^{\mathrm{T}}}{\partial \eta}\mathbf{y}\frac{\partial \mathbf{\Psi}}{\partial \xi} - \frac{\partial \mathbf{\Psi}^{\mathrm{T}}}{\partial \xi}\mathbf{y}\frac{\partial \mathbf{\Psi}}{\partial \eta} \right) k_{xy} \square$$

$$\frac{1}{|J|}\left(\frac{\partial \mathbf{\Psi}^{\mathrm{T}}}{\partial \eta}\mathbf{y}\frac{\partial \mathbf{\Psi}}{\partial \xi} - \frac{\partial \mathbf{\Psi}^{\mathrm{T}}}{\partial \xi}\mathbf{y}\frac{\partial \mathbf{\Psi}}{\partial \eta} \right) |J|\, d\xi\, d\eta$$

$$|J| = \frac{\partial \mathbf{\Psi}^{\mathrm{T}}}{\partial \xi}\mathbf{x}\frac{\partial \mathbf{\Psi}^{\mathrm{T}}}{\partial \eta}\mathbf{y} - \frac{\partial \mathbf{\Psi}^{\mathrm{T}}}{\partial \eta}\mathbf{x}\frac{\partial \mathbf{\Psi}^{\mathrm{T}}}{\partial \xi}\mathbf{y}$$

Volume Source Vector & Surface Flux Vector

The load vectors are of the same form as for the isoparametric triangle. The coordinates for the area integrals must be changed for the quadrilateral but the coordinates and shape functions for the surface (edge) integrals are the same.

Numerical Integration (1)

The integrals of interest for computing element matrices and vectors are generally complex functions of the element coordinates, property variations, etc. The limits of integration are simplified by use of the natural coordinates for element shape functions and the (isoparametric) mapping of the element into a master element.

Numerical integration (quadrature) is generally used to compute the area and boundary integrals. For some element types this type of integration is mandatory. Element libraries are most easily constructed when quadrature is is used on all element types.

Numerical Integration (2)

Consider the general area integral for a quadrilateral

$$I = \int_{-1}^{+1}\int_{-1}^{+1} f(\xi,\eta)\, d\xi\, d\eta$$

This can be evaluated using a product Gauss-Legendre quadrature formula

$$I = \sum_{I=1}^{M}\sum_{J=1}^{N} f(\xi_I,\eta_J)\, W_I\, W_J$$

where M, N are the number of quadrature points, ξ_I, η_J are the quadrature points in the interval at which f is evaluated and W_I, W_J are weights associated with each quadrature point.

Numerical Integration (3)

For quadrilateral elements, $M = N$ because the interpolating functions are the same degree in each direction. The number of points M is selected such that

$$M = \text{int}\left[\frac{(p+1)}{2}\right] + 1$$

where p is the order of the highest polynomial in f. The minimum value of M is the number of points required to integrate the area of the element exactly; this ensures convergence of the method in the limit of the mesh size going to zero.

Numerical Integration (4)

For simplex elements, the basic area integral is of the form

$$I = \int_0^1 \int_0^{1-L_1} f(L_1, L_2)\, dL_2\, dL_1$$

This is typically integrated using a non-product rule where

$$I = \sum_{I=1}^{M} f(L_1^I, L_2^I) W_I$$

Numerical Integration (5)

Integration for boundary integrals proceeds in the same manner as the area integrals. The quadrature rules are single sums since the integration is one-dimensional.

Answers for Finite Element Method III:

- Complications encountered in developing general element matrices include limits of integration, integration of complex functions and describing element geometries
- Simplex elements are basically triangles and tetrahedrons and simplex (area) coordinates are the natural coordinates for simplex elements
- Natural coordinates for non-simplex elements are normalized coordinates running from -1 to +1 in each coordinate direction
- Shape functions for multi-dimensional elements?
- How and why is numerical quadrature performed?

Answers for Finite Element Method III:

- Shape functions for multi-dimensional elements included Lagrange polynomials and tensor products of polynomials written in natural coordinates for each element type
- Numerical quadrature is performed to evaluate general area and boundary integrals that are usually too complex for analytic evaluation. Both product rules and non-product formulas are used.