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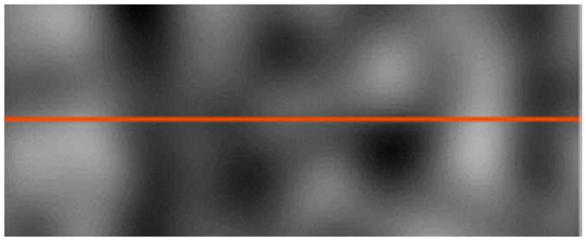

Flame Propagation and Burgers Turbulence: The Theoretical Connection

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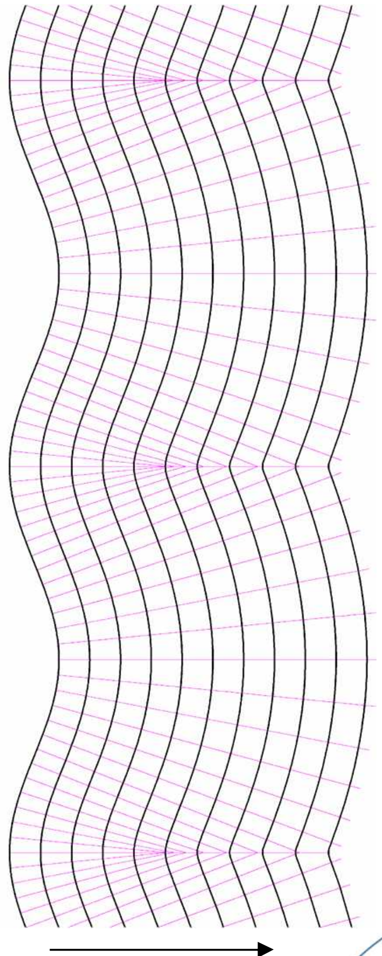
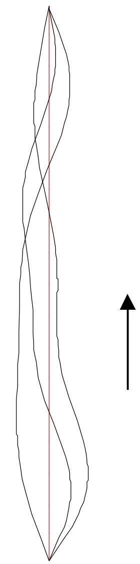
Huygens' principle idealizes the physics of front propagation

- Many phenomena (light, sound, combustion) spread at a characteristic speed
- At each instant t, a "front" marks the farthest progress
- The front comprises points to which the fastest path (first passage) takes time t
- The leading paths are "rays" perpendicular to the front
- Initial concave regions shrink to "cusps" that consume rays and flatten the front

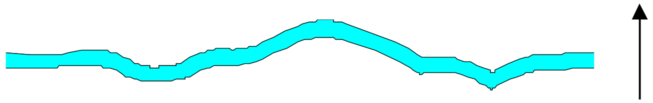


Medium fluctuations wrinkle the front, allowing faster passage

- First argument: Straight paths take (on average) the same time as in a uniform medium; allowing curved paths to take advantage of fluctuations can only shorten the average first-passage time



- Second argument: A wrinkled front has greater surface area and thus sweeps over more volume per unit time, resulting in faster propagation



Refraction and advection: Equivalent when weak, distinct when strong

- Refraction: A "quenched" medium with local speed $v(\mathbf{x})$

$$t = \int_C ds\, \frac{1}{v(\mathbf{x})}$$

- Advection: Propagation at fixed speed u_L in the local comoving frame of a fluid; for weak flow ($|\mathbf{u}| \ll u_L$) the effective local speed is $v = u_L + u_{||}$

Weakly random optics/acoustics/ solid combustion → Strongly random optics/acoustics/ solid combustion

Weak advection of premixed flamelets → Strong advection of premixed flamelets

Application to premixed combustion neglects thermal expansion and diffusive-thermal instability

In turbulent combustion, strong advection is primary, but weak also matters

- For laminar flame speed u_L and flow intensity u', dimensional analysis constrains the turbulent flame speed

$$u_T = u_L\, F\left(\frac{u'}{u_L}, \mathrm{Re}, \dots\right)$$

- Strong advection: Intuition and experiment show $F \propto u'/u_L$ and $u_T \propto u'$ for $u'/u_L \to \infty$
- The dependence on dimensionless flow parameters (Re, …) is not well understood
- Weak advection is a testing ground for flow-structure dependence
- A general flame-speed theory should match results obtained in the weak limit
- The weak limit's equivalence to other problems provides additional insights and tests

Weakly perturbed fronts relate to Burgers' compressible fluid model

- Take a near-uniform medium with $1/v = 1 + \sigma(\mathbf{x})$ and a near-straight path $\mathbf{x}_\perp(x_{||})$

$$t \ddagger \; [illegible] \; ds \frac{1}{v} \; [illegible] \; dx \; [illegible] \; \frac{1}{2} \mathbf{x}_\perp \; [illegible]$$

- Up to a constant, t is the action for a classical particle in the potential $-\sigma$
- First passage ↔ least action: The particles (rays) follow Newton's law $d^2\mathbf{x}_\perp / dx_{||}{}^2 = \nabla_\perp \sigma$ until they collide and disappear at shocks (cusps)
- Thus Huygens propagation is equivalent to a pressure-free fluid obeying the inviscid Burgers equation
- The Burgers fluid lives in one fewer spatial dimension than we started
- Because the front "tilt" is the Burgers velocity $\mathbf{w}$, the speedup (increase in surface area) is the Burgers energy density $w^2/2$

Adding a small viscosity is useful physically and mathematically

- The viscous Burgers equation smooths the shocks, returning to the inviscid limit at high Reynolds number

w + [illegible] ×**Ñ**¦ [illegible] nÑ¦^{2} **w** + **Ñ**¦ s

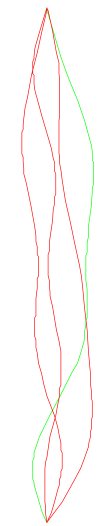
- Finite ν modifies Huygens propagation in a physical way, corresponding to finite wavelength (optics/acoustics) or Markstein length (flamelets)

- To describe Huygens propagation, we must take $\nu \rightarrow 0$ *before* the weak-perturbation limit; ν can be considered a mathematical regulator
- Formal advantage: The viscous Burgers equation relates to the Schrödinger equation for a quantum-mechanical wave function, which Feynman solved with an integral over *all* possible particle paths

“Path integrals” accumulate not only paths but medium realizations

- When the viscous Burgers equation is solved using a Feynman path integral, the least-action (fastest) path C^* is a “saddle point”

$$\int \mathcal{D}C \exp\left(-\frac{t(C)}{2\nu}\right) \approx \exp\left(-\frac{t(C^*)}{2\nu}\right)$$



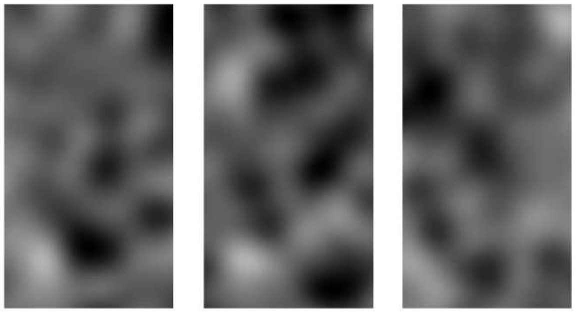
$t(C^*) \le t(C_i)$

- The relation becomes exact as $\nu \to 0$

$$t(C^*) = \lim_{\nu \to 0}\left[-2\nu \ln \int \mathcal{D}C \exp\left(-\frac{t(C)}{2\nu}\right)\right]$$

- We must next average $t(C^*)$ over the ensemble of random media

$$\langle t(C^*) \rangle = \int \mathcal{D}M \, t(C^*_M)$$



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Strategy: Reduce first passage to the white-noise Burgers equation, then analyze this equation

Analysis steps

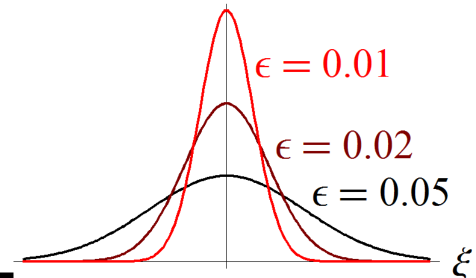
- Show that weak-perturbation first passage follows a white-noise process that fixes the dependence on the noise amplitude ϵ
- Analyze the noise-spectrum dependence by applying the replica method to the white-noise Burgers equation

Previous contributions

- White (1984) obtained white-noise ray deflections, but did not account for ray disappearance at cusps
- Kerstein and Ashurst (1992) argued heuristically that $O(\epsilon)$ fluctuations speed up front propagation by $O(\epsilon^{4/3})$
- Blum (1994) explicitly applied the replica method to an equivalent "directed polymer"
- Fedotov (1995) formally applied the replica method to weak-advection first passage, but assumed white noise *a priori*

A front rushes through weak perturbations and sees white noise

- Intuition: For advection, instead of $u' \to 0$, equivalently take $u_L \to \infty$; then each fluctuation affects the front briefly, and white noise is obtained
- Derivation: In Newton's law, rescale the fluctuations $\sigma(\mathbf{x}) \to \epsilon\gamma(\mathbf{x})$ and the longitudinal coordinate $x_{||} \to \epsilon^{-2/3}\xi$



$\frac{d^2\mathbf{x}}{dx^2}$ [illegible]

$\frac{d^2\mathbf{x}}{dx^2}$ [illegible] $^{-1/3}$ [illegible] $^{2/3}$ [illegible]

[illegible] $^{-1}$ [illegible]

[illegible] $^{2/3}$ [illegible] $^{2/3}$ [illegible]

- The noise is now white in the "slow time" ξ but correlated in the space $\mathbf{x}_\perp$
- Only the second moment matters since white noise is Gaussian
- The viscosity rescales as $\nu_{\text{old}} \to \epsilon^{2/3}\nu_{\text{new}}$; the white-noise and zero-viscosity limits are now interchangeable by a nontrivial rigorous result

"Directed polymers" provide a thermal interpretation of the model

- Apply rescaling to the travel time (renaming $\mathbf{x}_\perp \to \mathbf{x}$) and find the speedup δ

$$t = \int_0 dz \ldots \tfrac{1}{2}\mathbf{x} \ldots \mathbf{x} \ldots$$

$$-\delta = \frac{t}{l} - 1 = \frac{\epsilon^{2/3}}{l}\int_0 dz \ldots \tfrac{1}{2}\mathbf{x} \ldots \mathbf{x} \ldots = \epsilon^{2/3} \ldots$$

- The path integral gives the *first-passage* speedup Δ

$$-\Delta = -\epsilon^{4/3}\frac{2\nu}{\lambda}\ln \int D\mathbf{x} \ldots \frac{1}{2\nu}\int_0 dz \ldots \tfrac{1}{2}\mathbf{x} \ldots \mathbf{x} \ldots$$

$$= -\epsilon^{4/3}\frac{T\ln Z}{\lambda}$$

(we recognize a thermodynamic partition function)

- Thus $-\Delta$ is $\epsilon^{4/3}$ times the equilibrium free energy per unit length of a directed polymer (path) in the random potential η at temperature $T = 2\nu$

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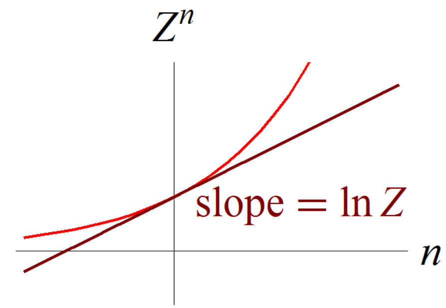
A "replicated" path integral allows exact averaging over the noise

- The ensemble average $\langle \Delta \rangle$ involves $\langle \ln Z \rangle$, an intractable quantity
- An identity comes to the rescue

$$\ln Z = \lim_{n \to 0} \frac{Z^n - 1}{n}$$



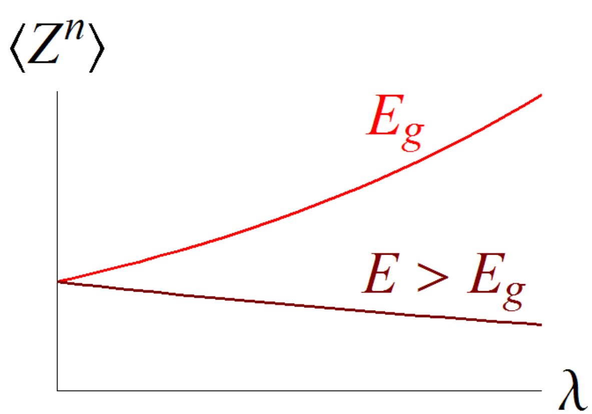
- Since Z^n depends on the noise η exponentially, we can average using the Gaussian identity $\langle \exp \eta \rangle = \exp(\langle \eta^2 \rangle / 2)$
- The nth power introduces n "replicas" of the polymer, which interact after averaging (indices a, b, c range over the n replicas)

Z^n … D … $\frac{1}{2n}$ … d … a … $\mathbf{x}_a^2 - \frac{1}{4n}$ … ab … V … $a - \mathbf{x}_b$ …

… $(0, 0$ …

The calculation reduces to the quantum mechanics of zero particles

- The quantity $\langle Z^n \rangle$ is the Feynman path integral for n nonrelativistic quantum particles with *static* pair potential $-V/4\nu$ (and $\hbar = 2\nu$)
- The wave function evolves by the "imaginary-time" Schrödinger equation and projects onto the ground state (energy $E_g < 0$) as $\lambda \to \infty$



$$\langle Z^n \rangle \propto \exp\left[-\frac{E_g(\lambda)}{2\nu}\right]$$

$$\langle \ln Z \rangle \approx -\frac{\lambda}{2\nu} \lim_{n \to 0} \frac{E_g(\lambda)}{n}$$

$$-D(\lambda) \propto \lambda^{3} \lim_{n \to 0} \frac{E_g(\lambda)}{n}$$

- We cannot numerically simulate $n \to 0$ particles; we must somehow analytically continue from positive integer n

A special variational method gives a bound on the answer

- General quantum variational method (used in atomic/molecular physics): Invent an arbitrary family of "trial" wave functions $|\psi\rangle$ and minimize $\langle\psi|H|\psi\rangle$; the result is an upper bound on the ground-state energy E_g
- If the family is expressed analytically in n, we can continue $\langle\psi|H|\psi\rangle$ to $n = 0$ particles

Counterintuitive, nonrigorous operations in the $n \to 0$ limit:

- There are now negative degrees of freedom; we *maximize* $\langle\psi|H|\psi\rangle$ to obtain a *lower* bound on E_g and thus an upper bound on the speedup $\langle\Delta\rangle$
- We use Gaussian trial wave functions that break the permutation symmetry among n replicas by dividing them into blocks and possibly sub-blocks, sub-sub-blocks, etc. (Parisi hierarchical symmetry breaking)

Explicit formulas generalize replica bounds to arbitrary spectra

- The general variational analysis is complex but becomes tractable in the inviscid limit ($\nu \to 0$) corresponding to Huygens propagation
- Take a noise spectrum $D(k)$ in N transverse dimensions [$D(k) \leftrightarrow V(x)$]
- "One-step" replica symmetry breaking yields the simplest bound

$$\langle\Delta\rangle \le \frac{3}{2^{5/3}} \epsilon^{4/3} N^{1/3} \left[\int \frac{d^N k}{(2\pi)^N} D(k)\right]^{1/3} \left[\int \frac{d^N k}{(2\pi)^N} k^2 D(k)\right]^{1/3}$$

- "Full" symmetry breaking often gives a tighter bound for $N = 1$ [the expression is valid below a critical N that can be calculated given $D(k)$]

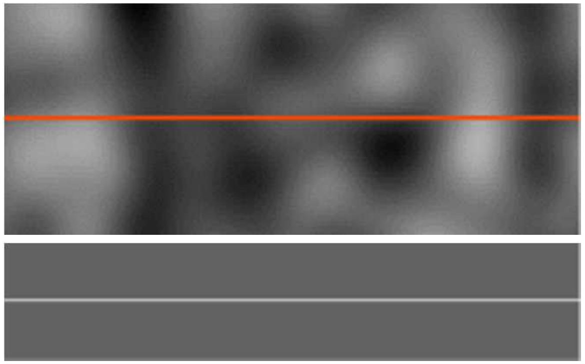
$$\langle\Delta\rangle \le \frac{3}{2^{10/3}} \epsilon^{4/3} N^{1/3} \left(\int_0^\infty dz \left[\int \frac{d^N k}{(2\pi)^N} k\, e^{-zk^2/2} D(k)\right]^{1/2} \right)^{2/3}$$

- The special form of these bounds for a "Gaussian" medium correlator $\langle \sigma(\mathbf{x})\sigma(\mathbf{x}+\mathbf{r})\rangle = \epsilon^2 \exp(-r^2/a^2)$ is implied by Blum (1994)—one-step: $\langle\Delta\rangle \le 1.744\epsilon^{4/3}N^{2/3}$; full ($N = 1$): $\langle\Delta\rangle \le 1.714\epsilon^{4/3}$

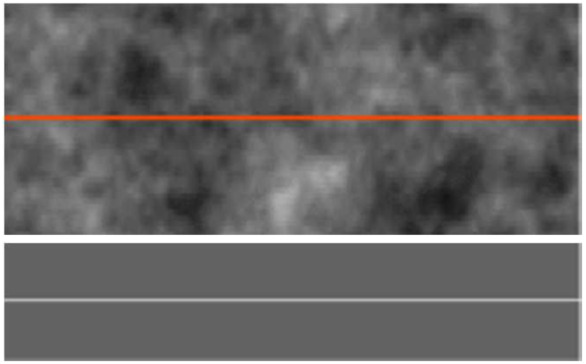
Multiscale media can produce divergent forcing but finite speedup

- For $N = 1$ and a 2D "exponential" correlator $\langle\sigma\sigma\rangle = \epsilon^2 \exp(-r/a)$, the Burgers force spectrum is $k^2 D(k) \sim 1/k$ and so its integral, the rate of energy input (and thus dissipation), is divergent at $k \to \infty$
- The "one-step" upper bound on the Burgers energy density $\langle\Delta\rangle$ is also divergent (uninformative), but "full" breaking gives $\langle\Delta\rangle \leq 2.038\,\epsilon^{4/3}$
- The infinite dissipation rate requires an infinite density of cusps
- The same considerations apply to weak advection by developed Navier–Stokes turbulence (corresponding to Burgers forcing $\sim 1/k^{2/3}$); the speedup remains finite as $\mathrm{Re} \to \infty$, despite cusp densification

Gaussian



Exponential



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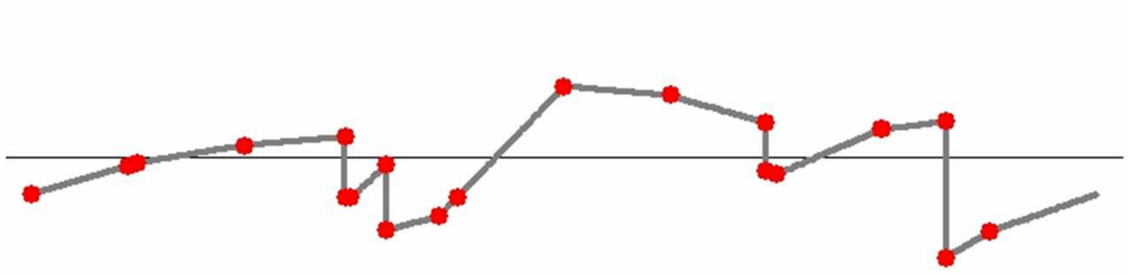
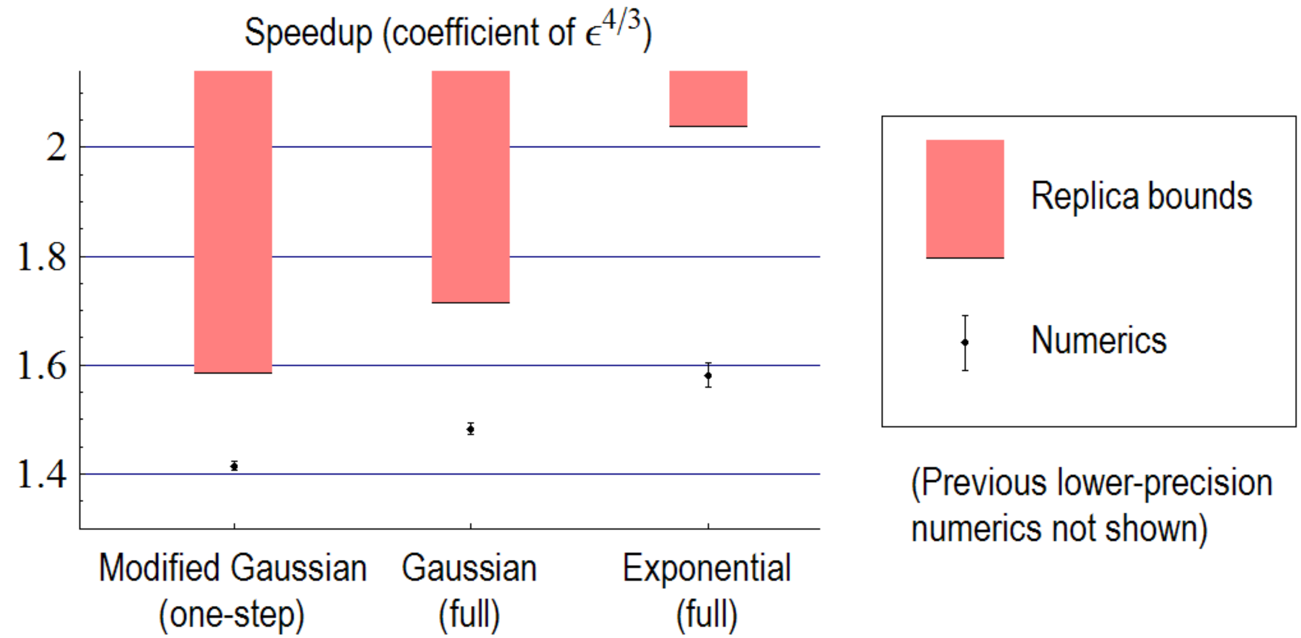
A Lagrangian numerical method allows systematic testing of replica results for 2D propagation

- For $N = 1$, represent an inviscid Burgers velocity field by piecewise linear sections, with discontinuities at shocks
- Evolve freely (retaining exact piecewise linearity) for a timestep, then “kick” the fluid with an impulsive force
- The force is synthesized from a given spectrum and taken as piecewise linear on a fixed grid
- The number of marker points increases as kicks occur, but stabilizes as shocks form and merge
- Convergence of the steady-state energy density is observed with the timestep, the forcing grid, and the length of the periodic domain



Numerical simulations confirm that 2D replica bounds are valid and reasonably sharp



(Previous lower-precision numerics not shown)

- The modified Gaussian is an alternate smooth medium [$D(k) \sim k^2 \exp(-k^2)$] for which one-step symmetry breaking applies
- We find significant and consistent dependence on the perturbation spectrum for fixed ϵ
- The agreement raises confidence in replica predictions for 3D

Synopsis: Links among diverse realms of physics contribute to the analysis

Physical systems	*Branches of physics*	*Meanings of ν*
Propagation of flames	Combustion	Markstein length
↓ Weak perturbations		
First passage of rays	Geometrical optics	Wavelength
↓ White noise		
The Burgers equation	Fluid dynamics	Viscosity
↓ Reinterpretation		
Directed polymers	Statistical mechanics	Temperature
↓ Replicas		
The Schrödinger equation	Quantum mechanics	Planck's constant

Conclusion: Weak-perturbation first passage is now well understood theoretically

- A weakly perturbed Huygens front, such as a premixed flamelet, can be reduced to an inviscid Burgers fluid driven by white noise (or to the low-temperature limit of an equivalent directed polymer)
- The white-noise reduction applies to random media with arbitrary (even non-Gaussian) statistics, provided the central limit theorem is obeyed
- In the process, the $\epsilon^{4/3}$ scaling of the front speedup is extracted
- The coefficient of $\epsilon^{4/3}$ for a given perturbation spectrum can be bounded above using the replica method—an illustration of the versatility of field theory
- Replica results for 2D propagation match within $\sim$ 20% the speedup values obtained numerically
- The success of the replica method implies direct applications to weakly random optics and acoustics (e.g., seismology)

Conclusion: The results contribute to understanding of turbulent combustion

- Because of the white-noise reduction, weakly random propagation senses only the second moment of fluctuations
- Even so, the magnitude of the speedup is nonuniversal and depends on the perturbation spectrum
- Strong advection of premixed flamelets exhibits no such reduction and should be even less universal, depending on arbitrary moments of the flow
- A general flame-speed theory should capture this nonuniversality and reproduce our results at weak perturbations
- The widely used flame-speed theory of Yakhot (1988) predicts universality for both strong and weak advection, and predicts ϵ^2 instead of $\epsilon^{4/3}$ dependence on weak perturbations
- Our analysis of the weak-perturbation limit can provide both inspiration and quantitative guidance for improved modeling of turbulent combustion