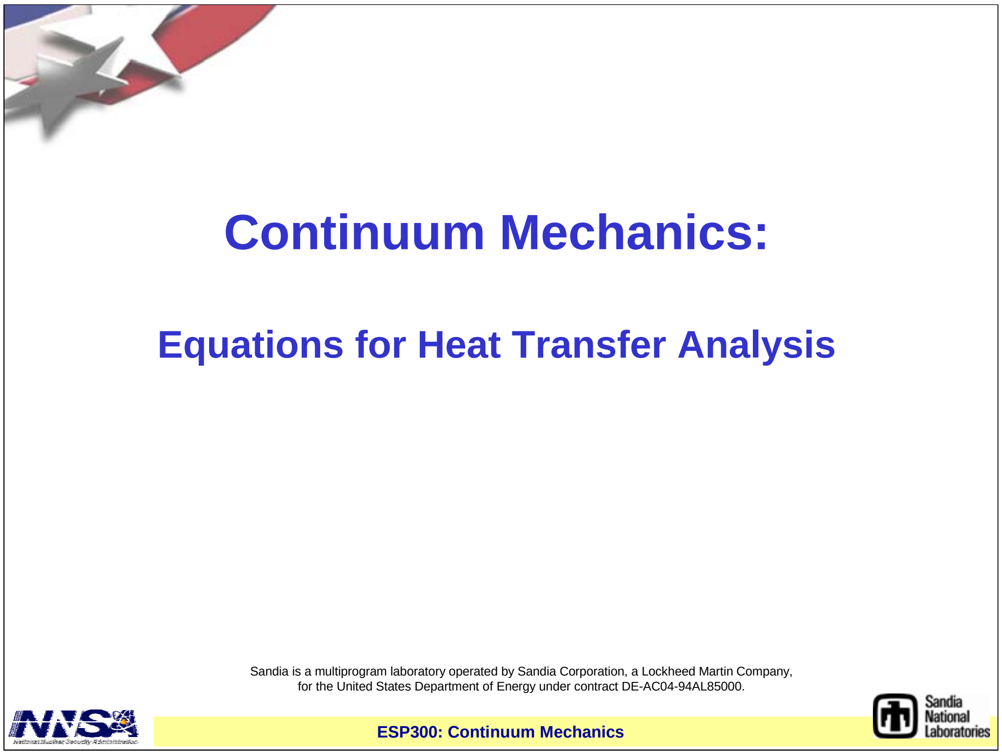

ESP = Engineering Sciences Program

ESP100 is a course on computational solid mechanics

ESP200 is a course on digital signal processing with MATLAB

ESP300 is a course on heat transfer analysis using the finite element method

There are plans to offer additional courses in the future.

All of these courses are intended to provide a continuing education opportunity – in the spirit of the INTEC courses some years ago

Introductory Info

Evacuation Procedures:

- Exits are located…
- Restrooms out back

Classification:

- Absolutely no classified discussions
- If you have a concern, let us know
- Some material may be OUO, it will be marked as such

Summary for Continuum Mechanics

Begin with:

- Continuum mechanics and conservation laws

and end with:

- General boundary value problem for heat conduction

Additional References:

W. Prager,
"Introduction to the Mechanics of Continua," Ginn & Co., Boston, MA (1961)

L. Malvern,
"Introduction to the Mechanics of a Continuous Medium," Prentice-Hall, Englewood Cliffs, NJ (1969)

It is important to recognize where heat transfer fits into the overall area of continuum mechanics; that is why we are going through the general mechanics setup. This is especially important when considering the multiphysics applications that are becoming more prevalent. Most current finite element codes are designed to be used in both single and multiphysics problems.

Most heat transfer books will skip right to a derivation of some form of the energy equation or a discussion of the first law of thermodynamics. When the introduction to this class said that there would be a review of heat transfer, it was not our intent to do a review of standard mechanical engineering heat transfer but rather review heat transfer from a general mechanics point of view.

Questions for Continuum Mechanics:

- What are the general conservation equations ?
- What are the conservation equations relevant to heat transfer ?
- What are the important partial differential equations for heat transfer analysis ?

2 - 4

ESP300: Continuum Mechanics

The first part of this section is really about definitions, the various areas of mechanics and how they are related to each other.

The fundamental question for this section is … What are the basic partial differential equations we are going to solve (via computation) for heat transfer applications? This is the **first** part of stating an initial, boundary value problem (IBVP); the **second** part involves boundary and initial conditions and this will be covered in another lecture.

The answer to the fundamental question will be … It depends on the type of problem. But we will end up with descriptions of energy transfer that we can use in a very wide variety of problems.

We are not going to derive too much here – mostly, just write things down and define some terms. Still, this first class may seem like too much detail on esoteric and peripheral topics. However, the basis for almost all of computational mechanics is in the conservation laws including computational heat transfer.

Continuum Mechanics

- Continuum mechanics deals with the equilibrium and/or motion of condensed matter (gases, liquids, solids) which is defined to have a continuously distributed mass.
- Not all physical situations can be described in terms of a continuum BUT almost everything we will discuss here will make the continuum approximation
- Radiation will be an exception

Continuum mechanics underlies much of what is done in modern engineering analysis for macroscopic systems. Increasing interest in nanoscale systems and devices will require alteration and extension of traditional continuum descriptions and in some cases a switch to noncontinuum and atomistic methods.

The principles of continuum mechanics provide for a **mathematical** description of physical behavior.

The assumed continuity of physical properties allows **differential** equations to be used to describe and solve problems in mechanics.

Continuum Mechanics

- Continuum mechanics theory is usually developed in two/three main parts:
  - General principles which are applied to all continuous media
  - Constitutive equations that apply to specific media
  - Special theories
- Engineering study of continuum mechanics usually splits into discipline oriented subjects, *i.e.,* solids, fluids, rheology
- Heat transfer spans all the major disciplines but is usually more important and prevalent in fluid dynamics

2 - 6

ESP300: Continuum Mechanics

Continuum mechanics has a long history and is associated with most of the famous names in mechanics, *e.g*. Euler, Navier, Cauchy, Lagrange, Stokes,…

The fundamental modern treatises are by Truesdell and colleagues:

C. Truesdell and R. Toupin, "The Classical Field Theories," Handbuch der Physik (Encyclopedia of Physics) Vol. III/1 (S. Flugge, Ed.) Springer-Verlag, Berlin (1960)

C. Truesdell and W. Noll, "The Non-Linear Field Theories of Mechanics," Handbuch der Physik (Encyclopedia of Physics) Vol. III/3 (S. Flugge, Ed.) Springer-Verlag, Berlin (1965)

Truesdell also has a very complete book on the history of mechanics which is quite good.

The main reasons for the division and different approaches between solid and fluid are:

a. Differences in the fundamental descriptions of material motion between solids and fluids; solid mechanics is concerned with deformation (displacement) and fluid mechanics is concerned with rate-of-displacement (velocity). Each type of motion is most easily described in different coordinates.

b. Differences in emphasis on constitutive relations; solid mechanics has a strong focus on constitutive relations to describe the multitude of material behavior while fluid mechanics has a single constitutive relation for most common fluids

c. Boundary condition issues are emphasized in fluid mechanics and receive less emphasis in solids (contact excepted)

Heat Transfer & Continuum Mechanics

- Heat transfer is a core discipline within continuum mechanics that is concerned with the transfer of energy due to temperature differences
- Heat transfer is one of the two types of energy interactions that appear in the First Law of Thermodynamics - work transfer being the other type
- For a closed system, the First Law provides

$$dE = \delta Q - \delta W$$

where E – energy, Q – heat transfer, W – work transfer

2 - 7

ESP300: Continuum Mechanics

Though heat transfer and thermodynamics are essential to continuum mechanics, most continuum mechanics texts do not focus on energy issues – to a large extent traditional continuum mechanics would appear to be isothermal.

We are not going to review thermodynamics – I'm sure you will recall everything you need to know about definitions, closed systems, properties, path independent quantities, reversible processes, etc.

We will use the first law as a conservation statement and relate it to other areas of mechanics.

The delta notation indicates that the variations are path dependent.

Continuum Mechanics - Kinematic Descriptions

Two descriptions of material motion are common :

- Lagrangian Description (also known as a Material, Convected or Referential Description)
  - Preferred in solid mechanics
- Eulerian Description (also known as a Spatial Description)
  - Preferred in fluid mechanics

2 - 8

ESP300: Continuum Mechanics

Generally there are a lot of prerequisites to a serious study of continuum mechanics – we are not going to be very serious and will skip most all of the preliminaries dealing with vectors, tensors, strain, stress, etc.

Material motion however, cannot be skipped as it influences the types of equations used to describe mechanics.

Heat transfer may use either of these descriptions depending on the material motion.

Though the modes of heat transfer have not yet been defined:

Heat conduction problems are usually in a Lagrangian description.

Convection problems are usually in an Eulerian description

Radiation problems are independent of the kinematic description

Continuum Mechanics – Kinematic Descriptions

- Lagrangian
  - Follows the motion of material particles
  - Independent variables are (initial) material position and time
  - Time derivative is

$$\left.\frac{\partial}{\partial t}\right|_X$$

- Eulerian
  - Observes the motion at a fixed location
  - Independent variables are spatial location and time
  - Time derivative is a material derivative in spatial coordinates

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i} = \frac{\partial}{\partial t} + \mathrm{u} \cdot \nabla$$

where u_i is the material velocity

2 - 9

ESP300: Continuum Mechanics

In the Lagrangian form, the material position is given by a mapping (function) from the reference state or position to the current position x=x(X,t)

In the Eulerian form, the material derivative is also called a convected or substantial derivative.

Note that when there is no advective velocity (no material motion) the two descriptions are the same.

NOTE: The standard summation (indicial) notation will be the standard used to describe vectors, tensors and vector/tensor operations. The indices run from 1 to 3 and a Cartesian coordinate system is adequate for everything we need to describe. Some vector notation may be used where convenient or needed.

Transport Theorem

The transport theorem is useful for developing the conservation laws. Consider the volume integral of some (tensorial) property of the continuum taken over a fixed mass

$$\mathrm{F}(t) = \int_{\Omega(t)} f(x_i, t)\, d\Omega$$

The rate of change of this quantity, following the material motion is the material derivative

$$\frac{D\mathrm{F}}{Dt} = \int_{\Omega(t)} \frac{\partial f}{\partial t} d\Omega + \int_{\Omega(t)} \frac{\partial f u_i}{\partial x_i} d\Omega = \int_{\Omega(t)} \frac{\partial f}{\partial t} d\Omega + \int_{\Gamma(t)} f u_i n_i \, d\Gamma$$

2 - 10

ESP300: Continuum Mechanics

Dealing with a continuum involves material volumes and surfaces as a function of time in three dimensions.

The volume integral of some property over a fixed mass is important for stating the conservation laws. In following the motion of the fixed mass, the volume and the integrand will change with time. A derivative that accounts for this changing volume is needed. Note that for a fixed volume the time derivative will commute with the integral so that the time derivative of the integral is the integral of the time derivative.

The second form of the derivative involving the surface integral is obtained from the first form by use of Gauss's theorem, which relates the flux through the surface of a region to the divergence of the flux within the volume. Gauss's theorem is also know as the Green-Gauss or divergence theorem. Vector and tensor transformations, such as the divergence theorem, are used throughout continuum mechanics and its discrete counterpart, the finite element method.

The transport theorem is usually attributed to O.Reynolds.

Conservation Laws

Two Thermodynamic Conservation Laws

- Conservation of Mass
- Conservation of Energy

Two Mechanical Conservation Laws

- Conservation of Linear Momentum
- Conservation of Angular Momentum

2 - 11

ESP300: Continuum Mechanics

Special relativity would collapse the two thermodynamic laws to one but we are not going to go there – we will only consider Newtonian mechanics.

Conservation of electric charge and Maxwell's equations would have to be added to obtain a complete description of electromagnetic (EM) materials and effects.

We may point out additional forces and sources due to EM effects but will not really go into detail. The EM coupling with mechanics is becoming more important, especially for manufacturing, MEMS, and nanotechnology applications.

Conservation of Mass (1)

Define the mass for a region of the continuum as the integral of the (continuous) density over the material volume

$$\mathrm{M} = \int_{\Omega(t)} \rho(x_i, t)\, d\Omega$$

Conservation of mass requires that the time rate of change (material derivative) of M is zero or

$$\frac{D\mathrm{M}}{Dt} = \int_{\Omega(t)} \left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} \right) d\Omega = 0$$

2 - 12

ESP300: Continuum Mechanics

We are using an Eulerian description (reference frame) for stating the conservation laws, which implies the application is fluid mechanics. The material is moving with velocity *u* (components given by u_i).

The use of the transport theorem is immediate; the density is a scalar function in the general transport theorem.

The equation for mass conservation in solid mechanics using a Lagrangian frame will be defined, without derivation, in a subsequent slide.

Conservation of Mass (2)

Since the selected volume is arbitrary the integrand must hold pointwise within the continuum and

$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho u_i}{\partial x_i}=0 \;=\frac{\partial \rho}{\partial t}+\rho\frac{\partial u_i}{\partial x_i}+u_i\frac{\partial \rho}{\partial x_i}=\frac{D\rho}{Dt}+\rho\frac{\partial u_i}{\partial x_i}$$

where the material derivative D/Dt has been used.

Also, the continuity equation in vector notation is

$$\frac{\partial \rho}{\partial t}+\nabla\cdot\rho\mathrm{u} = 0$$

2 - 13

ESP300: Continuum Mechanics

The conservation of mass or continuity equation is an equation for the thermodynamic variable, density.

In general, the density will be a function of two thermodynamic variables (given by an equation of state), such as the pressure and temperature, as well as the independent variables, spatial location and time.

Conservation of Mass (3)

Special cases of importance include a constant density flow and the flow of an incompressible material where

$$\frac{D\rho}{Dt} = 0 \qquad \text{and} \qquad \frac{\partial u_i}{\partial x_i} = 0$$

Conservation of mass in a Lagrangian frame is

$$\rho = \rho_0 \frac{1}{J}$$

where J is the determinant of the deformation gradient

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ESP300: Continuum Mechanics

A constant density or **incompressible material** is a thermodynamic term; an **incompressible flow** is a fluid dynamic term and defines when density variations are negligible.

For an incompressible material note that the density (thermodynamic variable) is no longer a part of the continuity equation. The mass conservation equation is now a constraint on the material motion; the motion of an incompressible material or flow is isochoric and the velocity field is solenoidal.

The deformation gradient is the spatial derivative of the motion; the determinant of the deformation gradient measures the change in volume between the reference configuration and the current configuration.

Conservation of Momentum (1)

Define the momentum for a region of the continuum as

$$\mathrm{P}_i = \int_{\Omega(t)} \rho(x_j,t)\, u_i(x_j,t)\, d\Omega$$

Newton's Second Law of Motion requires that the time rate of change (material derivative) of momentum is balanced by the sum of the body and surface forces

$$\frac{D\mathrm{P}_i}{Dt} = F_i \quad \text{or} \quad \int_{\Omega(t)} \frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j}\, d\Omega = \int_{\Omega(t)} \rho b_i\, d\Omega + \int_{\Gamma(t)} \mathrm{T}_i\, d\Gamma$$

where b_i is a body force, $\mathrm{T}_i = \tau_{ij} n_j$ is the traction, τ_{ij} is the stress tensor and n_j is the normal

2 - 15

ESP300: Continuum Mechanics

The transport theorem is again used to define the material derivative of the volume integral.

The surface force is defined in terms of the traction (or stress vector) which is a force per unit. The traction depends on the orientation of the surface which is defined in terms of its outward normal vector, n_j.

The relation between the traction and the surface normal vector produces the definition of the stress tensor (see any standard text on mechanics for a derivation).

Conservation of Momentum (2)

Transforming the surface integral by Gauss' theorem

$$\int_{\Omega(t)} \frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} \, d\Omega = \int_{\Omega(t)} \rho b_i \, d\Omega + \int_{\Omega(t)} \frac{\partial \tau_{ij}}{\partial x_j} d\Omega$$

Again, since the volume is arbitrary, the integrand must hold pointwise,

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = \rho b_i + \frac{\partial \tau_{ij}}{\partial x_j}$$

2 - 16

ESP300: Continuum Mechanics

Note that this form of the PDE is considered a conservative form of the momentum equation – no simplification or assumption regarding continuity (conservation of mass) has been made (see next slide). Some computational methods rely on this form.

Conservation of Momentum (3)

Simplify by the continuity equation to produce

$$\rho \frac{Du_i}{Dt} = \rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = \rho b_i + \frac{\partial \tau_{ij}}{\partial x_j}$$

This is Cauchy's first law of motion. For completion, the stress tensor will have to be related to the rate of deformation for a particular material (constitutive equation).

2 - 17

ESP300: Continuum Mechanics

To simplify the equation, expand the derivatives, collect terms and recognize the occurrence of the conservation of mass equation.

Cauchy's first law of motion is also referred to as the momentum equation or the equation of motion.

Conservation of Momentum (4)

For solid mechanics, change to a Lagrangian description and use displacement (u_i^*) instead of velocity.

$$\rho \frac{\partial^2 u_i^*}{\partial t^2} = \rho b_i + \frac{\partial \tau_{ij}}{\partial x_j}$$

This is still Cauchy's first law of motion. For completion, the stress tensor will have to be related to the deformation for a particular material (constitutive equation).

2 - 18

ESP300: Continuum Mechanics

Note that when the acceleration is negligible, the equation reduces to the sum of the forces being zero. This is the equilibrium equation for quasi-statics in solid mechanics. The full equation is used to describe solid dynamics.

Conservation of Angular Momentum

The time rate of change of angular momentum (moment of momentum) is balanced by the moment of the forces.

Without derivation, this implies

$$\tau_{ij} = \tau_{ji}$$

under standard assumptions of no couple stresses and no body couples.

This Cauchy's second law of motion.

The assumption of no couple stresses is valid for most common materials, both fluids and solids. There is a large literature on the mechanics of micropolar materials for application to biological materials, heterogeneous mixtures and composites and materials at the nanoscale.

Conservation of Energy (1)

Define the internal and kinetic energies as components of the total energy for a fixed mass of the continuum

$$\mathrm{E} = \int_{\Omega(t)} \rho e \, d\Omega + \int_{\Omega(t)} \tfrac{1}{2} \rho u_i u_i \, d\Omega$$

The first law of thermodynamics then states that the time rate of change of the total energy is balanced by the heat and work transfer to the system

$$\frac{D\mathrm{E}}{Dt} = \frac{\delta Q}{\delta t} - \frac{\delta W}{\delta t}$$

2 - 20

ESP300: Continuum Mechanics

This conservation equation will be somewhat more complex than the previous balance laws because it couples the thermal and mechanical processes.

The internal energy is composed of the microscopic energy modes in the material, i.e., molecular translations, vibrations and rotations.

The nomenclature is often confusing for these equations – standard thermodynamic texts will use u for the internal energy and e for the total energy. Because u is being used for velocity (and displacement) we will not follow this convention and instead will define e as the internal energy and E as the total energy.

Conservation of Energy (2)

Define the heat transfer and work transfer rates as

$$\frac{\delta Q}{\delta t} = -\int_{\Gamma(t)} q_i n_i \, d\Gamma + \int_{\Omega(t)} \mathrm{Q} \, d\Omega = \int_{\Omega(t)} \frac{\partial q_i}{\partial x_i} d\Omega + \int_{\Omega(t)} \mathrm{Q} \, d\Omega$$

$$\frac{\delta W}{\delta t} = -\int_{\Omega(t)} \rho b_i u_i \, d\Omega - \int_{\Gamma(t)} u_i \tau_{ij} n_j \, d\Gamma = -\int_{\Omega(t)} \rho b_i u_i \, d\Omega - \int_{\Omega(t)} \frac{\partial u_i \tau_{ij}}{\partial x_j} d\Omega$$

where q_i is the heat flux vector and Q is a volumetric energy source.

2 - 21

The divergence theorem is again used to change the surface integrals to volume integrals.

Work rate is defined as a force acting through a distance per unit time (velocity).

Work done by the system is positive; work done on the system is negative.

The volume heat source is due to any other non-mechanical processes that may be present, e.g. chemical, electromagnetic

Conservation of Energy (3)

Putting these relations together then

$$\int_{\Omega(t)} \frac{\partial(\rho e + \frac{1}{2}\rho u_i u_i)}{\partial t} d\Omega + \int_{\Omega(t)} \frac{\partial(\rho e + \frac{1}{2}\rho u_i u_i) u_j}{\partial x_j} d\Omega =$$

$$-\int_{\Omega(t)} \frac{\partial q_i}{\partial x_i} d\Omega + \int_{\Omega(t)} \mathrm{Q}\, d\Omega + \int_{\Omega(t)} \rho b_i u_i \, d\Omega + \int_{\Omega(t)} \frac{\partial u_i \tau_{ij}}{\partial x_j} d\Omega$$

which must hold pointwise within the continuum

$$\frac{\partial(\rho e + \frac{1}{2}\rho u_i u_i)}{\partial t} + \frac{\partial(\rho e + \frac{1}{2}\rho u_i u_i) u_j}{\partial x_j} = -\frac{\partial q_i}{\partial x_i} + \mathrm{Q} + \rho b_i u_i + \frac{\partial u_i \tau_{ij}}{\partial x_j}$$

2 - 22

The transport theorem is used to rewrite the material derivative of the total energy. The stress work term contains two parts which can be seen when the derivative is expanded. The term with the velocity times the divergence of the stress can be combined with the body force and rewritten using the conservation of momentum equation. This term contains expresses the stress work done in changing the kinetic energy of the material. The second term with the stress times the velocity gradient represents the stress work done in deforming the material and is associated with a change in internal energy. Once a constitutive equation for the stress is defined, this second term is usually rewritten in terms of a dissipation function (always positive) and a pressure work term, which may be reversible.

Again, this PDE is considered a conservative form since no simplification has occurred. Some computational methods rely on this form.

Conservation of Energy (4)

After some manipulation, the total energy equation can be simplified to a thermal energy equation

$$\rho \frac{De}{Dt} = \rho \frac{\partial e}{\partial t} + \rho u_i \frac{\partial e}{\partial x_i} = -\frac{\partial q_i}{\partial x_i} + \mathrm{Q} + \tau_{ij} \frac{\partial u_i}{\partial x_j}$$

For completion, the heat flux and internal energy must be related to the temperature for a particular material (constitutive and thermodynamic relations).

The simplification of the total energy equation may be accomplished in two different ways. In the first approach, the total energy terms and stress work terms are expanded, rewritten and combined into a form that has both the continuity and momentum equations included as coefficients. Dropping out these terms leads to the thermal energy equation. A second approach recognizes that the kinetic energy equation is not an independent equation but comes from taking the (dot) product of the velocity with the momentum equation. If the kinetic energy equation is subtracted from the total energy equation the result is the thermal energy equation.

Coupling between mechanical and thermal processes is still present in the last term, the stress work. The stress work term is a contraction between the stress tensor and the velocity gradient tensor. The velocity gradient tensor is usually decomposed into two parts, the rate-of-deformation (stretching) tensor and the spin (vorticity) tensor. Only the rate-of-deformation tensor contributes to the stress work.

Conservation of Energy (5)

In a Lagrangian description the energy equation becomes

$$\rho \frac{\partial e}{\partial t} = -\frac{\partial q_i}{\partial x_i} + \mathrm{Q} + \tau_{ij}\,\varepsilon^{*}_{ij}$$

Constitutive and thermodynamic relations are still required for completion.

2 - 24

ESP300: Continuum Mechanics

Note that the stress work term has been written in terms of the strain rate tensor. This is not the same as the rate-of-deformation tensor used in the Eulerian description. The particular form of the dissipation term will depend on the strain measure used in the constitutive relation. For linear elastic materials this term is usually assumed to provide only reversible work and is neglected. For large strains and nonlinear materials the term is not negligible.

Summary of Conservation Equations

The conservation equations for the nonisothermal motion of a continuum have now been derived (stated) using the two main frames of reference. We do not yet have a complete mathematical description of the mechanics problem because material specific constitutive relations and boundary/initial conditions have not yet been defined.

A summary of the conservation equations for fluids and solids will help to define missing relations.

Summary of Conservation Equations

For a fluid using an Eulerian description:

Mass: $$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0$$ 1 Unknown

Momentum: $$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = \rho b_i + \frac{\partial \tau_{ij}}{\partial x_j}$$ 9 Unknowns

Energy: $$\rho \frac{\partial e}{\partial t} + \rho u_i \frac{\partial e}{\partial x_i} = -\frac{\partial q_i}{\partial x_i} + \mathrm{Q} + \tau_{ij} \frac{\partial u_i}{\partial x_j}$$ 4 Unknowns

Total equations – 5 14 Unknowns

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ESP300: Continuum Mechanics

The unknowns are density, 3 velocity components, 6 stress components (reduced from 9 by symmetry and angular momentum), internal energy and 3 components of the heat flux.

Summary of Conservation Equations

For a solid using a Lagrangian description:

Mass:	$\rho = \rho_0 \dfrac{1}{J}$	1 Unknown
Momentum:	$\rho \dfrac{\partial^2 u_i^*}{\partial t^2} = \rho b_i + \dfrac{\partial \tau_{ij}}{\partial x_j}$	9 Unknowns
Energy:	$\rho \dfrac{\partial e}{\partial t} = -\dfrac{\partial q_i}{\partial x_i} + Q + \tau_{ij} \dfrac{\partial u_i^*}{\partial x_j}$	4 Unknowns
Total equations – 5		14 Unknowns

2 - 27

The unknowns are density, 3 components of displacement, 6 stress components (reduced from 9 by symmetry and angular momentum), internal energy and 3 components of the heat flux.

Constitutive Equations

The topic of constitutive equations, especially for solid mechanics, is far too extensive and complex to be considered here. Constitutive relations must meet a series of criteria related to invariance (frame, material, dimensional, etc) and generally relate fluxes/forces to the dependent variables. For our purposes, simple mechanical and thermal constitutive relations for some common materials will be used as examples. This will allow the equation system to be closed and a mathematical description to be completed. Some thermodynamic relations are also needed (such as an equation of state) and will be stated without derivation.

For more information on constitutive relations for solid mechanics any standard text can be consulted or take the ESP100 course.

Constitutive Equations – Fluids (1)

For many fluids of interest, the Newtonian and Fourier constitutive relations are sufficient. In this case,

$$\tau_{ij} = -P\,\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\delta_{ij}$$

and

$$q_i = -k\frac{\partial T}{\partial x_i}$$

where μ is the viscosity and k is the thermal conductivity

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ESP300: Continuum Mechanics

A Stokesian fluid postulates that the stress is a continuous function of only the rate-of-deformation and thermodynamic variables. A Newtonian fluid is a linear Stokesian fluid, which says that the stress is linearly dependent on the rate-of-deformation. Note that the third term in the stress relation vanishes for an incompressible fluid. In the general derivation of the constitutive relation, two viscosity coefficients appear; Stokes assumption relates the two coefficients.

Fourier's law simply states that the heat flux is proportional to the temperature gradient and may be a function of the thermodynamic state through the conductivity. Conductivity is a scalar for most fluids.

Constitutive Equations – Fluids (2)

From thermodynamics, an equation of state, is required. Typical examples are the perfect gas law, a constant density assumption or a Boussinesq fluid assumption

$$P = \rho R T$$

or

$$\rho = \rho_0$$

or

$$\rho = \rho_0 \left[1 - \beta (T - T_0)\right]$$

where R is the gas constant and β is the coefficient of thermal expansion.

2 - 30

Other equations of state are possible.

The Boussinesq approximation is normally used for natural convection problems with small temperature differences.

Constitutive Equations – Fluids (3)

Another thermodynamic relation is needed to relate the internal energy to the temperature. There are many possibilities for this relation but one of the most general has the form

$$\rho \frac{De}{Dt} = \rho C_{\mathrm{V}} \frac{DT}{Dt} + \left[-P + T \left. \frac{\partial P}{\partial T} \right|_{\rho} \right] \frac{\partial u_i}{\partial x_i}$$

2 - 31

ESP300: Continuum Mechanics

Note that for an incompressible flow or material, the second term in the internal energy definition is zero by continuity.

Also, for a perfect gas, the second term is zero but for a different reason.

For most liquids, C_V is very close to C_P

Constitutive Equations – Solids (1)

The simplest constitutive relations for a solid are Hooke's law and Fourier's law. In this case

$$\tau_{ij} = 2\,\mu_E\,\varepsilon_{ij} + \lambda_E\,\varepsilon_{kk}\,\delta_{ij} \qquad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i^*}{\partial X_j} + \frac{\partial u_j^*}{\partial X_i}\right)$$

and

$$q_i = -k_{ij}\,\frac{\partial T}{\partial x_j}$$

where μ_E, λ_E are the Lame constants and k_{ij} is the conductivity tensor

Hooke's law describes linearly elastic materials. The stress tensor is written in terms of the small strain tensor; the small strain tensor is defined in terms of the displacement gradients. Though the displacement gradients are shown as derivatives with respect to the material coordinates, in the small strain limit the material and spatial coordinates are the same.

The Lame constants are usually replaced by (related to) the shear modulus, Young's modulus and Poisson's ratio.

Constitutive Equations – Solids (2)

From thermodynamics a relation between the internal energy and the temperature must be obtained. In most cases, the required relation is

$$\rho \frac{\partial e}{\partial t} = \rho C_{\mathrm{V}} \frac{\partial T}{\partial t}$$

General PDE's for Fluids

Mass:

$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho u_i}{\partial x_i}=0$$

Momentum:

$$\rho \frac{\partial u_i}{\partial t}+\rho u_j \frac{\partial u_i}{\partial x_j}=\rho b_i-\frac{\partial P}{\partial x_j}\delta_{ij}+\frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)-\frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\delta_{ij}\right]$$

Energy:

$$\rho C_V \frac{\partial T}{\partial t}+\rho C_V u_i \frac{\partial T}{\partial x_i}=\frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right)+Q-P\frac{\partial u_k}{\partial x_k}+\Phi$$

State:

$$\rho=\rho(P,T)$$

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There are many simplifications that can be made for various applications. The field equations are fairly complex for fluid applications; the constitutive behavior for common liquids and gases is straightforward. Note that these equations are in nonconservative form.

This is a complete mathematical system with 6 equations in the 6 unknowns: density, 3 components of velocity, pressure and temperature.

General PDE's for Solids

Mass:

$$\rho = \rho_0 \frac{1}{J}$$

Momentum:

$$\rho \frac{\partial^2 u_i^*}{\partial t^2} = \rho b_i + \frac{\partial \tau_{ij}}{\partial x_j}$$

Energy:

$$\rho C_V \frac{\partial T}{\partial t} = \frac{\partial}{\partial x_i}\left(k_{ij} \frac{\partial T}{\partial x_j} \right) + \Phi_S$$

Constitutive:

$$\tau_{ij} = \tau_{ij}(\varepsilon_{ij}(u_i^*), T)$$

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This is a general statement but hides most of the complexity found in solid dynamics, nonlinear materials and quasi-statics. The field equations are relatively simple with all of the complexity in the constitutive relation and strain measures.

This is a complete mathematical system with 5 equations in the 5 unknowns: density, 3 components of displacement and temperature. If the constitutive relation is a function of another thermodynamic variable, say pressure, then an equation of state would have to be included in the description.

Modes of Heat Transfer

The three basic modes of heat transfer are:

- Conduction - transfer of heat through a material due to molecular motion; a diffusion process that follows Fourier’s law
- Convection - transfer of heat due to relative motion of the material
- Radiation – transfer of heat due to electromagnetic radiation

Continuum mechanics defines the equations for conduction and convection

General PDE's for Heat Transfer

Heat transfer in the continuum is described by the conservation of energy equation which may be applied to

Conduction (Lagrangian description):

$$\rho C_V \frac{\partial T}{\partial t} = \frac{\partial}{\partial x_i}\left(k_{ij} \frac{\partial T}{\partial x_j} \right) + \Phi$$

Convection (Eulerian description):

$$\rho C_V \frac{\partial T}{\partial t} + \rho C_V u_i \frac{\partial T}{\partial x_i} = \frac{\partial}{\partial x_i}\left(k \frac{\partial T}{\partial x_i} \right) + Q - P \frac{\partial u_k}{\partial x_k} + \Phi$$

Heat Transfer Equations

The conduction and convection equations are both scalar equations for the temperature. Both equations will be considered in the remainder of this course, though the primary emphasis will be on the conduction equation. The obvious coupling of the convection equation with material motion makes it more difficult to consider as a standalone equation. The general coupling of the energy equation to other mechanics will be discussed after the full boundary value problem is described.

Answers for Continuum Mechanics

- The general conservation equations are mass, linear momentum, angular momentum and energy.
- The primary conservation equation for heat transfer is the energy balance, though mass conservation must also be respected.
- The partial differential equations that describe heat transfer are the conduction equation (Lagrangian description) and the convection equation (Eulerian description)

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These are the brief answers to the questions that were posed at the beginning of the class. Hopefully, these are now familiar ideas.