SAND2006-7330P

Variational Formulations & Methods for Heat Transfer

Sandia is a multiprogram laboratory operated by Sandia Corporation, a Lockheed Martin Company, for the United States Department of Energy's National Nuclear Security Administration under contract DE-AC04-94AL85000.

ESP = Engineering Sciences Program

ESP100 is a course on computational solid mechanics

ESP200 is a course on digital signal processing with MATLAB

ESP300 is a course on heat transfer analysis using the finite element method

There are plans to offer additional courses in the future.

All of these courses are intended to provide a continuing education opportunity – in the spirit of the INTEC courses some years ago

Introductory Info

Evacuation Procedures:

- Exits are located…
- Restrooms out back

Classification:

- Absolutely no classified discussions
- If you have a concern, let us know
- Some material may be OUO, it will be marked as such

Summary for Variational Principles

Begin with:

- General initial/boundary value problem for heat conduction

and end with:

- Variational formulation applied to heat conduction
- Weighted residual formulation applied to heat conduction

Additional References:

J.N. Reddy, "Energy and Variational Methods in Applied Mechanics," John Wiley & Sons, New York, NY (1984)

B. A. Finlayson, "The Method of Weighted Residuals and Variational Principles," Academic Press, New York, NY (1972)

Questions for Variational Formulations For Heat Transfer Analysis

- What is the “strong” form, the “weak” form? How are they different?
- Why is the weak form so important to us? How do we use it to obtain approximate solutions?
- What is a variational principle and what is the method of weighted residuals? How are they alike, how are they different?
- How are finite difference, finite volume, and finite element methods related? How are they different?

Remember the Initial/Boundary Value Problem (IBVP) Discussed Last Time

- An initial/boundary value problem is
  - a mathematical statement comprised of the field equation(s) describing the process of interest within a domain,
  - conditions describing the behavior of the dependent variables on the boundary of the domain (BC's) and
  - initial values (IC's) for the dependent variables.
- A boundary value problem (BVP) is an IBVP without time as an independent variable; IC's are not required.

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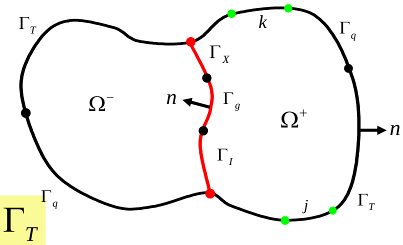
These are standard definitions. For the most part, we will consider time-dependent problems as they are the most general. Time independent problems will be singled out when it comes to Finite Element solution methods.

These definitions assume that all necessary material properties and source terms are known, i.e. in mathematical terms, the parameters and sources for the problem are given data with appropriate functional behavior (smoothness).

Summary of Heat Transfer - IBVP

For a region Ω with boundary Γ



$$\rho C_V \frac{\partial T}{\partial t} = \frac{\partial}{\partial x_i}\left(k_{ij}\frac{\partial T}{\partial x_j}\right) + Q + \Phi_S$$

$$T(s_i,t) = f_T(s_i,t) \qquad on \qquad \Gamma_T$$

$$-k_{ij}\frac{\partial T}{\partial x_j} n_i = f_q(s_i,t) \qquad on \qquad \Gamma_q$$

$$T(x_i,0) = T_0(x_i)$$

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This is the general mathematical setting for the equations that we will address with the FEM. It will be assumed that all relevant "data" is given and all we have to do is find T(x,t)

This is referred to as the "strong" form. We will figure out just what this means in today's class.

Heat Transfer Analysis (1)

A well-posed mathematical description of heat transfer has been developed. The objective is now to solve practical engineering problems described by the given equations. These problems usually involve-

- Complicated geometries
- Complex boundary conditions
- Complex material behavior
- Variety of time and length scales
- Coupled physical phenomena

Heat Transfer Analysis (2)

What does it mean to “solve” a heat transfer or mechanics problem?

Generally, we want

- to obtain an accurate description of the dependent variable as a function of time and space
- an accurate description of various derivatives (fluxes) of the dependent variable as a function of time and space
- an accurate description of various integrals of the dependent variable and/or its derivatives as a function of time and space

Solution Approaches (1)

- Analytic Solution Methods
  - Separation of variables methods
  - Green’s function methods
  - Integral transform methods
- Approximate Analytic Solution Methods
  - Integral methods
  - Rayleigh-Ritz methods (variational)
  - Weighted residual methods (Galerkin)

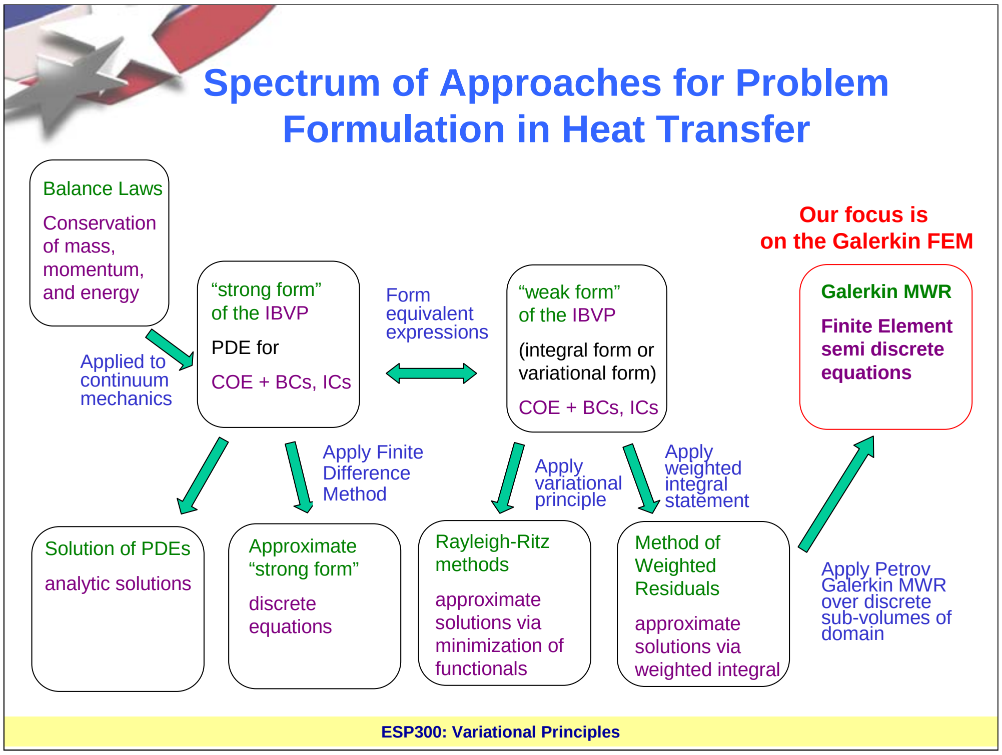
The analytic solution methods listed are classic methods for partial differential equations; their use in heat transfer is well covered in the books by Ozisik and Carslaw & Jaeger. The separation of variables techniques generally lead to Sturm-Liouville equations that are ultimately solved by special functions such as Bessel and Legrendre functions depending on the geometry.

Solution Approaches (2)

- Numerical Solution Methods
  - Differential equation methods
  - Integral equation methods

Because of the many limitations of analytic and approximate analytic solution methods, engineering practice has come to rely on numerical methods for the solution of realistic problems. Also, numerical algorithms based on differential equation methods have limitations that exclude them from general application. Integral equation methods, being very general, will be our focus.

Note that numerical solution methods are **all** approximate solution methods.


Spectrum of Approaches for Problem Formulation in Heat Transfer
Balance Laws
Conservation of mass, momentum, and energy
Applied to continuum mechanics
"strong form" of the IBVP
PDE for
COE + BCs, ICs
Form equivalent expressions
"weak form" of the IBVP
(integral form or variational form)
COE + BCs, ICs
Our focus is on the Galerkin FEM
Galerkin MWR
Finite Element semi discrete equations
Apply Finite Difference Method
Apply variational principle
Apply weighted integral statement
Solution of PDEs
analytic solutions
Approximate "strong form"
discrete equations
Rayleigh-Ritz methods
approximate solutions via minimization of functionals
Method of Weighted Residuals
approximate solutions via weighted integral
Apply Petrov Galerkin MWR over discrete sub-volumes of domain
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The "Strong Form" Representation of the Heat Conduction BVP

$$-\frac{\partial}{\partial x}\left(k_{xx}\frac{\partial T}{\partial x}\right)-\frac{\partial}{\partial y}\left(k_{yy}\frac{\partial T}{\partial y}\right)=Q$$

- Solutions must be satisfy this equation over the entire domain (all points, point by point)
- Analytical solution must be smooth and differentiable to at least 2^{nd} derivative
- Analytical solution must satisfy boundary conditions

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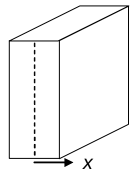
For most of the discussion today, we will focus on the time-independent equation for simplicity and avoidance of technical problems (details). We could write this as a time-dependent problem in which case the PDE must be satisfied for all times as well.

An “Example” Problem to Assess Different Solution Methods

Consider a case of 1-D conduction in a plane wall (2L thick) with uniform volumetric energy generation



$$-k\frac{d^2T}{dx^2} = Q$$

$$with \quad T(L) = 0 \quad and \quad \left.\frac{dT}{dx}\right|_{x=0} = 0$$

Integrating twice, results in $T(x) = \frac{Q}{2k}x^2 + c_1 x + c_2$

Evaluating the constants of integration using the boundary conditions

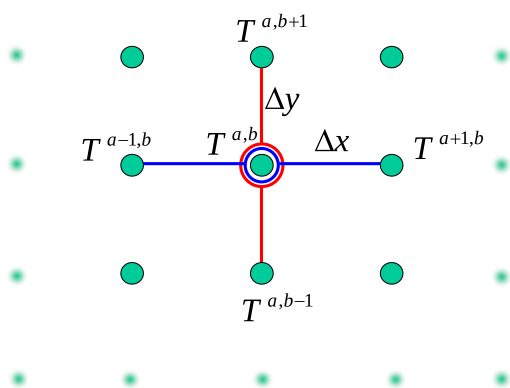
$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0 \quad gives \quad c_1 = 0 \qquad\qquad T(L) = 0 = \frac{Q}{2k}L^2 + \cancel{c_1}L + c_2 \quad gives \ \ c_2 = \frac{Q}{2k}L^2$$

For which the analytical solution is $T(x) = \frac{QL^2}{2k}\left(1 - x^2/L^2\right)$

This will be an “example” problem that we will use to apply different approximate solution methods. The analytical solution will allow us to compare the accuracy of these methods.

Typical Finite Difference Method

Finite difference methods directly approximate the strong form on **topologically regular grids**



spatial derivatives are approximated using a **difference stencil**, e.g. at node (a,b) :

$$\left.\frac{\partial T}{\partial y}\right|^{a,b} = \left(T^{a,b+1} - T^{a,b-1}\right) / 2\Delta y$$

$$\left.\frac{\partial T}{\partial x}\right|^{a,b} = \left(T^{a+1,b} - T^{a-1,b}\right) / 2\Delta x$$

$$\left.\frac{\partial^2 T}{\partial x^2}\right|^{a,b} = \left(T^{a+1,b} - 2T^{a,b} + T^{a-1,b}\right) / \Delta x^2$$

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The approximations in a finite difference method arise from two sources.

•The first is that spatial and temporal quantities are only know at discrete points (as opposed to continuously) in space and time.

•The second is that spatial and temporal derivatives of a function are approximated using a stencil – that is difference formulas are used on the function values defined only at discrete points.

These difference “stencils” for the derivatives may be developed using either of two approaches:

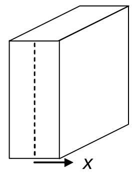
•using a Taylor series expansion of temperature

•Using the definition of a derivative in an approximate manner (without applying the limiting process)

Check out those boundary flux conditions and the need for “ghost nodes”!

Application the Finite Difference Method to the Plane Wall Example

Consider a case of 1-D conduction in a plane wall (2L thick) with uniform volumetric energy generation, Q

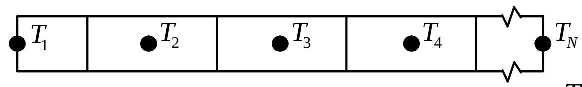


$$-k\frac{d^2T}{dx^2} = Q$$

$$with \quad T(L) = 0 \quad and \quad \left.\frac{dT}{dx}\right|_{x=0} = 0$$

Set up a 1-D "grid" of points at which we will solve the PDE

$$\left.\frac{dT}{dx}\right|_{x=0} = 0$$



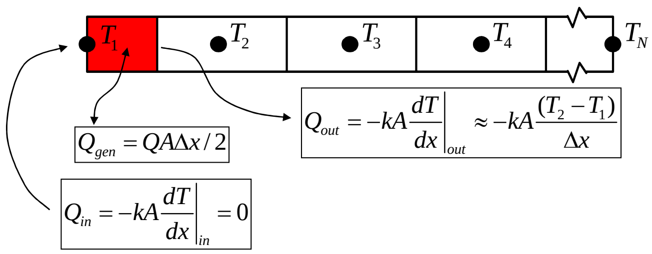
$$T(L) = T_L = 0$$

We will develop finite difference equations for each grid point by considering an energy balance over a finite volume

T

Application the Finite Difference Method to the Plane Wall Example (2)

Consider a control volume for the first grid point, grid point “1”



Conservation of energy is $Q_{out} - Q_{in} = Q_{gen}$

or $\left\{-kA\frac{(T_2 - T_1)}{\Delta x}\right\} - \{0\} = \frac{QA\Delta x}{2}$

$$T_1 - T_2 = \frac{Q\Delta x^2}{2k}$$

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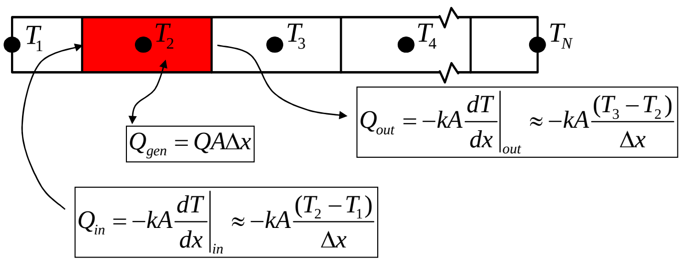
Application the Finite Difference Method to the Plane Wall Example (3)

Consider a control volume for an interior grid point, grid point “2”



Conservation of energy is $Q_{out} - Q_{in} = Q_{gen}$

or $\left\{-kA\frac{(T_3 - T_2)}{\Delta x}\right\} - \left\{-kA\frac{(T_2 - T_1)}{\Delta x}\right\} = QA\Delta x$

$$-T_1 + 2T_2 - T_3 = \frac{Q\Delta x^2}{k}$$

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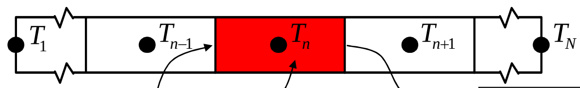
Application the Finite Difference Method to the Plane Wall Example (4)

Consider a “typical” interior grid point, call it grid point “n”



$$Q_{out} = -kA\left.\frac{dT}{dx}\right|_{out} \approx -kA\frac{(T_{n+1}-T_n)}{\Delta x}$$

$$Q_{gen} = QA\Delta x$$

$$Q_{in} = -kA\left.\frac{dT}{dx}\right|_{in} \approx -kA\frac{(T_n-T_{n-1})}{\Delta x}$$

Conservation of energy is $Q_{out} - Q_{in} = Q_{gen}$

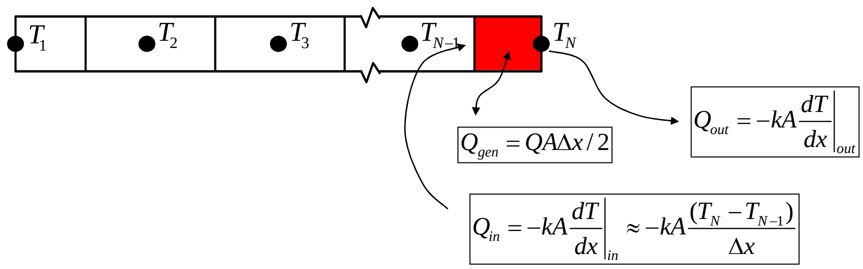
or $\left\{-kA\frac{(T_{n+1}-T_n)}{\Delta x}\right\} - \left\{-kA\frac{(T_n-T_{n-1})}{\Delta x}\right\} = QA\Delta x$

$$-T_{n-1} + 2T_n - T_{n+1} = \frac{Q\Delta x^2}{k}$$

T

Application the Finite Difference Method to the Plane Wall Example (5)

Consider a “typical” interior grid point, call it grid point “n”



If we were going to solve for T_N , then we would construct

$Q_{out} - Q_{in} = Q_{gen}$ as we have previously done

For the specified boundary condition, we will set $T_N = T(L) = 0$

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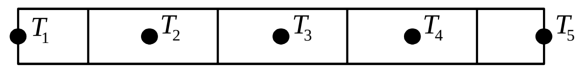
Application the Finite Difference Method to the Plane Wall Example (6)

For 5 grid points, the system of equations can be written as a matrix system of equations with the five unknown grid point temperatures,



$$\left.\frac{dT}{dx}\right|_{x=0} = 0 \qquad T_5 = T(L) = 0$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 0.5Q^{**} \\ Q^{**} \\ Q^{**} \\ Q^{**} \\ T(L)=0 \end{Bmatrix} \quad where \;\; Q^{**} = \frac{Q\Delta x^2}{k}$$

Remember that we developed this system of algebraic equations by applying the PDE at each grid point. Although, integral formulations will be different, they will also result in a set of algebraic equations.

For 2-dimensional structured grids, the coefficient matrix is also banded with five rows of non-zero coefficients.

For 3-dimensional structured grids, there are seven rows of non-zero coefficients.

Observations on the Finite Difference Method

- Method usually requires a topologically regular grid
- Method focuses on the “grid point” with no consideration of the temperature between the points
- Analogous to the “resistor-capacitor” network method
- Application of boundary conditions, in particular, flux boundary conditions can be difficult
- Can be simple to implement, can be computationally efficient, utilize customized solver algorithms
- Poor representation of complex geometry

Background on Variational Principles

General remarks on variational principles:

- Foundations in solid mechanics
- Typically associated with energy quantities (potential energy, etc)
- Involve minimization of functionals
  - Integral equivalents of some PDE's for COE (along with the essential and natural boundary conditions)
  - Produce an Euler-Lagrange equation that corresponds to a PDE of interest
  - Functionals are "functions of functions" and often an integral form

We will begin with a "model" problem corresponding to the BVP for COE

Much of the early application of variational principles was directed at solid mechanics and considered such concepts as virtual work and minimization of energy principles.

More on “Functionals”

$$I = \int_{x_1}^{x_2} F\left(x, T(x), T_x(x), \cdots\right) dx$$

Functional – function that takes a function as an argument, typically the integral of a function.

Functionals depend on the function, T, used in the integrand.

The objective is to make I stationary (usually a minimum),

So $\delta I = 0$ necessary condition

Use calculus of variations to obtain the Euler-Lagrange equation

Lots of functionals can be written down and made stationary, but the resulting Euler-Lagrange equations will probably not correspond to anything recognizable in mechanics.

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Calculus of variations was designed to work with functionals just the way calculus works with functions.

One of the shortcomings of applying the variational principle to mechanics problems is the limited number of problems that have appropriate functionals. “Appropriate functionals” being those that result in the Euler-Lagrange equation that represents meaningful mechanics.

On the other hand, there are a number of problems for which the functional represents some physical quantity (like potential or strain energy) that will seek a minimum potential.

Definitions for the Discussion of Variational Methods

Variational principle implies an extremum of a functional, which includes the governing equations, the boundary and/or initial conditions, and constraints.

Variational statement (or formulation, method) is more general and encompasses both variational principles and more general integral forms.

We need a variational statement or weak form
- Can start with a variational principle in some special cases
- Can develop a weak form for virtually any PDE

Why use a weak form?
- We want to solve an easier problem, without as many restrictions on the solution function (smoothness). Averaging (or weighted averaging) is one way to smooth or weaken the problem. Averaging functions implies integration over domains.

The Variational Principle Considers the BVP over a Finite Volume

$$\int_{\Omega} \delta T \left[-\frac{\partial}{\partial x}\left(k_{xx} \frac{\partial T}{\partial x} \right) - \frac{\partial}{\partial y}\left(k_{yy} \frac{\partial T}{\partial y} \right) + hT - Q \right] d\Omega = 0$$

- The first step is to multiply the energy equation by a variation of temperature and integrate over the domain
- In this formulation, the solution is satisfied over the arbitrary volume and is no longer, necessarily, satisfied on a point-by-point basis
- We can reduce the order of the equation by integrating by parts (or Green-Gauss theorem) to transfer one differentiation of temperature to the variation of temperature.

By performing a “weighted” integral over an arbitrary volume, we have expanded the set of acceptable temperature functions that will satisfy this integral expression. It will be satisfied on an integral sense, as opposed to at each point in the domain, which suggests a less restrictive set of possible solutions. Consequently, the integral representation is often called the “weak” form.

You may have noticed the additional term “hT” and be wondering where it came from. In a more general case, it could represent a convective heat transfer or it could be thought of as a temperature-dependent generation term. You can see how this term might occur by considering the development of the energy equation for the fin problem described on pg 75 of LNS (Lewis, Nithiarasu, and Seetharamu).

The Variational Principle (1)

- Having used the integration-by-parts on the second derivative

$$\int_{\Omega}\left[\frac{\partial \delta T}{\partial x}\left(k_{xx}\frac{\partial T}{\partial x}\right)+\frac{\partial \delta T}{\partial y}\left(k_{yy}\frac{\partial T}{\partial y}\right)+\delta T\, hT-\delta T\,\mathrm{Q}\right]d\Omega$$
$$-\oint_{\Gamma}\delta T\left(k_{xx}\frac{\partial T}{\partial x}n_x+k_{yy}\frac{\partial T}{\partial y}n_y\right)\, d\Gamma = 0$$

- Now, the solution only needs to be differentiable once (the acceptable solution space is broader than for the strong form)
- The essential and natural boundary conditions are satisfied through the boundary integral term

The Variational Principle (2)

Using the following boundary conditions

$$T = T_{specified} \quad on \ \Gamma_T$$

$$k_{xx} \frac{\partial T}{\partial x} n_x + k_{yy} \frac{\partial T}{\partial y} n_y = q_b \quad on \ \Gamma_q$$

results in the “weak” or “variational” form of the PDE

$$\int_\Omega \left[\frac{\partial \delta T}{\partial x} \left(k_{xx} \frac{\partial T}{\partial x} \right) + \frac{\partial \delta T}{\partial y} \left(k_{yy} \frac{\partial T}{\partial y} \right) + h \, \delta T \, T - \delta T \, \mathrm{Q} \right] d\Omega$$

$$- \oint_{\Gamma_q} \delta T \, q_b \, d\Gamma = 0$$

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On the specified temperature boundary, the variation of temperature “delta T” is zero.

On the remainder of the boundary, the flux is specified to be q_b.

The Variational Principle (3)

Because the order of the variation and the derivative can be switched*, then

$$\delta\left\{\int_{\Omega}\left\{\frac{1}{2}\left[k_{xx}\left(\frac{\partial T}{\partial x}\right)^2 + k_{yy}\left(\frac{\partial T}{\partial y}\right)^2 + h\,T^2\right] - T\,Q\right\} d\Omega - \oint_{\Gamma} T\,q_b\,d\Gamma\right\} = 0$$

or

$$\delta I = 0$$

where $I\left(T\right)$ is the “functional”

$$I(T) = \frac{1}{2}\int_{\Omega}\left[k_{xx}\left(\frac{\partial T}{\partial x}\right)^2 + k_{yy}\left(\frac{\partial T}{\partial y}\right)^2 + h\,T^2\right] d\Omega - \int_{\Omega} T\,Q\,d\Omega - \oint_{\Gamma_q} T\,q_b\,d\Gamma$$

It can be demonstrated that the temperature solution that makes this integral stationary, also satisfies the original PDE!

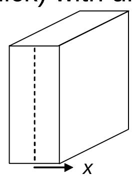
Actually, we can write this as a functional because the operator on *T* and *delta_T* is symmetric with respect to *T* and *delta_T*. Consequently, the result that we are able to express this as the first variation of the functional, *I*. This can be related to the minimum entropy principle – an extremum principle.

It can be shown using calculus of variations, that when the functional I, can be minimized (or made stationary), the Euler-Lagrange equation (which is our PDE in this case) is also satisfied.

Applicability of this approach is limited to problems in which there is a functional that corresponds to the correct PDE. In general, this is not always available. Consequently, we will seek other, more general methods in the long run. For now, we will consider an example problem that demonstrates this method.

Application of Variational Principle or Rayleigh-Ritz (1)

Recall our example problem which is1-D conduction in a plane wall (2L thick) with uniform volumetric energy generation



$$-k\frac{d^2T}{dx^2} = Q$$

$$with \quad T(L) = 0 \quad and \quad \left.\frac{dT}{dx}\right|_{x=0} = 0$$

Integrating twice, results in $T(x) = \frac{Q}{2k}x^2 + c_1 x + c_2$

Evaluating the constants of integration using the boundary conditions

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0 \quad gives \quad c_1 = 0 \qquad\qquad T(L) = 0 = \frac{Q}{2k}L^2 + \cancel{c_1} L + c_2 \quad gives \ \ c_2 = \frac{Q}{2k}L^2$$

For which the analytical solution is $T(x) = \frac{QL^2}{2k}\left(1 - {x^2}/{L^2}\right)$

Application of Variational Principle or Rayleigh-Ritz (2)

"Recipe" for applying variational formulation

1. Assume a temperature function (having unknown constants) that is adequately smooth and satisfies the boundary conditions. Denote the assumed temperature as "T-bar," $\bar{T}$
2. Substitute assumed temperature function into the functional for heat conduction and evaluate integral
3. Determine unknown constants in the assumed temperature function, $\bar{T}$, by minimization of functional

$$\delta I\left(\bar{T}\right) = 0 = \frac{\partial I\left(\bar{T}\right)}{\partial c_i}\delta c_i \quad \text{for arbitrary } c_i \text{ then}$$

$$\frac{\partial I\left(\bar{T}\right)}{\partial c_i} = 0 \quad \text{with respect to each unknown constant}$$

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The result here is a system of N equations with N (unknown constants) as unknowns.

Application of Variational Principle or Rayleigh-Ritz (3)

We will consider the following assumed temperature functions:

1. $\bar{T}(x) = c_0 + c_1 x$
2. $\bar{T}(x) = c_0 + c_1 x + c_2 x^2$
3. $\bar{T}(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$
4. $\bar{T}(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4$

and will consider our model problem using each of these assumed functions with the variational principle.

$$I(\bar{T}) = \frac{1}{2}\int_{\Omega}\left[k_{xx}\left(\frac{\partial \bar{T}}{\partial x}\right)^2 + k_{yy}\left(\frac{\partial \bar{T}}{\partial y}\right)^2 + h\bar{T}^2\right]d\Omega - \int_{\Omega}\bar{T}\,Q\,d\Omega - \oint_{\Gamma_q}\bar{T}q_b\,d\Gamma$$

or for our problem $I(\bar{T}) = \frac{1}{2}\int_{\Omega}\left[k\left(\frac{\partial \bar{T}}{\partial x}\right)^2\right]d\Omega - \int_{\Omega}\bar{T}\,Q\,d\Omega$

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This is the functional that, when made stationary, is equivalent to solving the our time-independent heat conduction equation.

There are some requirements on the assumed temperature functions:

•They need to be continuous and differentiable up to the derivatives in the functional

•They need to be "complete." That is, they have to support all orders of the polynomial.

Consider the Assumed Temperature Functions for our Plane-Wall Problem

Begin with the most general form.. The other functions are simplifications of this case.

$$\bar{T}(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4$$

For our boundary conditions

$$\left.\frac{d\bar{T}(x)}{dx}\right|_{x=0} = 0 = \left.\left(0 + c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3\right)\right|_{x=0} \qquad c_1 = 0$$

$$\bar{T}(L) = T_L = c_0 + c_1 L + c_2 L^2 + c_3 L^3 + c_4 L^4 \qquad c_0 = \cancel{T_L} - \left(c_1 L + c_2 L^2 + c_3 L^3 + c_4 L^4\right)$$

Using these simplifications, then

$$\bar{T}(x) = -c_2\left(L^2 - x^2\right) - c_3\left(L^3 - x^3\right) - c_4\left(L^4 - x^4\right)$$

$$\frac{d\bar{T}(x)}{dx} = 2c_2 x + 3c_3 x^2 + 4c_4 x^3 \qquad \frac{d^2\bar{T}(x)}{dx^2} = 2c_2 + 6c_3 x + 12c_4 x^2$$

For our plane-wall problem, we have specified that T_L is zero.

If we want to consider up to 2nd order, then we just set c_3 and c_4 to zero
If we want to consider up to 3rd order, then we set c_4 to zero.

Example 1 – Assumed Linear Temperature Function

Does this assumed function satisfy the boundary conditions?

$$\left.\frac{d\bar{T}}{dx}\right|_{x=0} = 0 + c_1 = 0 \qquad \bar{T}(L) = 0 = c_0 + c_1 0 = 0$$

For this assumed function, both of the coefficients must be zero to satisfy the boundary conditions.

Clearly, an assumed temperature function that is linear in temperature does not satisfy the boundary conditions and is not a viable assumed function for this problem.

Example 2 – Assumed Quadratic Temperature Function

For this assumed temperature function:

$$\bar{T}(x) = -c_2\left(L^2 - x^2\right) \qquad \frac{d\bar{T}(x)}{dx} = 2c_2 x$$

$$I(\bar{T}) = \frac{1}{2}\int_\Omega \left[k\left(\frac{\partial \bar{T}}{\partial x}\right)^2 \right] d\Omega - \int_\Omega \bar{T}\,Q\, d\Omega$$

$$I(\bar{T}) = \frac{1}{2}\int_0^L \left[k\left(2c_2 x\right)^2 \right] d\Omega - \int_0^L -c_2\left(L^2 - x^2\right)Q\, d\Omega \;=\; \frac{2}{3}kL^3c_2^2 + \frac{2}{3}QL^3c_2$$

$$\frac{dI(\bar{T})}{dc_2} = 0 = \frac{d}{dc_2}\left(\frac{2}{3}kL^3c_2^2 + \frac{2}{3}QL^3c_2\right) = 2kc_2 + Q \qquad c_2 = -\frac{Q}{2k}$$

Results in the exact solution $\bar{T}(x) = \frac{QL^2}{2k}\left(1 - \left(x/L\right)^2\right)$

Because the functional is a function of only one unknown constant, we differentiate it once with respect to c_2.

Example 3 – Assumed Cubic Temperature Function

For this assumed temperature function:

$$\bar{T}(x) = -c_2\left(L^2 - x^2\right) - c_3\left(L^3 - x^3\right) \qquad \frac{d\bar{T}(x)}{dx} = 2c_2 x + 3c_3 x^2$$

$$I(\bar{T}) = \frac{1}{2}\int_\Omega \left[k\left(\frac{\partial \bar{T}}{\partial x}\right)^2\right] d\Omega - \int_\Omega \bar{T}\,Q\,d\Omega$$

$$I(\bar{T}) = \frac{1}{2}\int_0^L \left[k\left(2c_2 x + 3c_3 x^2\right)^2\right] d\Omega - \int_0^L \left(-c_2\left(L^2 - x^2\right) - c_3\left(L^3 - x^3\right)\right) Q\,d\Omega$$

Solving $\frac{\partial I(\bar{T})}{\partial c_2} = 0 \quad \frac{\partial I(\bar{T})}{\partial c_3} = 0$ yields $c_2 = -\frac{Q}{2k} \quad c_3 = 0$

Results in the exact solution $\bar{T}(x) = \frac{QL^2}{2k}\left(1 - \left(x/L\right)^2\right)$

In this case, the functional, I, is a function of the two constants, c_2 and c_3. Taking the partial derivatives with respect to the two unknown constants results in two equations with the two unknown constants. The unknown constants are determined by solving this set of equations.

Example 3 – Assumed Quartic Temperature Function

For this assumed temperature function:

$$\bar{T}(x) = -c_2\left(L^2 - x^2\right) - c_3\left(L^3 - x^3\right) - c_4\left(L^4 - x^4\right) \qquad \frac{d\bar{T}(x)}{dx} = 2c_2x + 3c_3x^2 + 4c_4x^3$$

$$I(\bar{T}) = \frac{1}{2}\int_0^L \left[k\left(2c_2x + 3c_3x^2 + 4c_4x^3\right)^2 \right] d\Omega - \int_0^L \left(-c_2\left(L^2 - x^2\right) - c_3\left(L^3 - x^3\right) - c_4\left(L^4 - x^4\right)\right) Q \, d\Omega$$

Solving $\frac{\partial I(\bar{T})}{\partial c_2} = 0 \quad \frac{\partial I(\bar{T})}{\partial c_3} = 0 \quad \frac{\partial I(\bar{T})}{\partial c_4} = 0$ yields $c_2 = -\frac{Q}{2k}$, $c_3 = 0$, $c_4 = 0$

Results in the exact solution $\bar{T}(x) = \frac{QL^2}{2k}\left(1 - \left(\frac{x}{L}\right)^2\right)$

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In this case, the functional has three unknown constants. Taking the partial derivatives with respect to each unknown constant give us a set of three equations with three unknowns. Solving the set simultaneously, we get the three unknown constants. Again, the solution is exact and the constants for the higher order (higher than 2) are zero.

Observations on our Plane-Wall Example Problem

- For our boundary conditions, the lowest-order function that we assumed (second order) was also the analytical solution. Therefore, all the higher-order functions assumed produced the analytical solution.
- Higher order (than quadratic) functions resulted in the coefficients of the higher-order terms being zero and dropping those terms out of the solution.
- Had our assumed function not supported the analytical solution, we would not have seen this behavior. See LNS textbook, pages 75-84.
- We will embellish this plane-wall problem when we discuss the method of weighted residuals.

In the example in the Lewis, et.al. text, pgs. 75-84, the assumed solution did not support the analytical, so the solutions produced there are approximate solutions.

Observations on the Variational Principle

- Variational principle often referred to as “weak form”
- Needed an assumed profile that satisfied boundary conditions
- Problem was simple (1-D, steady-state)
- In general, it may be difficult to assume an appropriate profile for multi-D, with more complex boundary conditions
- An appropriate functional must exist for the PDE of interest. In many cases, it does not. We need a more general method.

Rather than continue to seek a solution with the variational principle (Rayleigh-Ritz) using minimization, we will take a different approach that has broader application.

The Method of Weighted Residuals (1)

If we assume a temperature function that does not exactly solve the PDE, then the "error" is termed the residual

$$R(\bar{T}, x_i) = \rho C \frac{\partial \bar{T}}{\partial t} - \frac{\partial}{\partial x}\left(k_{xx} \frac{\partial \bar{T}}{\partial x}\right) - \frac{\partial}{\partial y}\left(k_{yy} \frac{\partial \bar{T}}{\partial y}\right) - \mathrm{Q} \neq 0$$

The MWR allows you to determine approximate solutions for which the residual error is distributed over the domain through the use of a global "weighting function."

$$\int_{\Omega} w(x_i)\, R(\bar{T}, x_i)\, d\Omega = 0$$

For simplicity, we will continue our discussion of this concept using a one-dimensional formulation

Here the weighting function is written as a global function over the domain. We will see later how be might express this in terms of a series of functions.

The Method of Weighted Residuals (2)

Assuming a temperature function and a weighting function

$$\bar{T}(x) = \sum_{i=1}^{N} c_i \, f_i(x) \qquad\qquad w(x) = \sum_{j=1}^{N} b_j \, w_j(x)$$

Then the weighted residual statement can be rewritten as

$$\int_\Omega \sum_{j=1}^{N} b_j \, w_j(x) \, R(c_i, x) \, dx = 0$$

Or, because the b_j's are independent of x

$$\sum_{j=1}^{N} b_j \left[\int_\Omega w_j(x) \, R(c_i, x) \, dx \right] = 0$$

Remember here that we are assuming global functions for both the temperature function and the weighting function. We are writing them both in terms of a series with unknown coefficients.

The Method of Weighted Residuals (3)

$$\sum_{j=1}^{N} b_j \left[\int_\Omega w_j(x)\, R(c_i, x)\, dx \right] = 0$$

Now, expanding the sum, we get

$$b_1 \int_\Omega w_1(x)\, R(c_i, x)\, dx \;+ b_2 \int_\Omega w_2(x)\, R(c_i, x)\, dx \;+ \cdots\cdots$$
$$+\, b_{N-1} \int_\Omega w_{N-1}(x)\, R(c_i, x)\, dx \;+\; b_N \int_\Omega w_N(x)\, R(c_i, x)\, dx = 0$$

If the b_j's are arbitrary constants (and non-zero), then each of these *"N"* integrals must equal zero. We now have *"N"* integral equations with the *"N"* unknown constants, the c_i's

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To demonstrate this argument, assume that arbitrary constants, b_2 through b_N are zero. Then, if b_1 is non-zero, the integral it multiplies must be zero.

The Method of Weighted Residuals (4)

$$b_1 \int_\Omega w_1(x)\, R(c_i, x)\, dx \;+ b_2 \int_\Omega w_2(x)\, R(c_i, x)\, dx \;+ \cdots\cdots$$
$$+ b_{N-1} \int_\Omega w_{N-1}(x)\, R(c_i, x)\, dx \;+ b_N \int_\Omega w_N(x)\, R(c_i, x)\, dx = 0$$

The discussion in the textbook does not go into this much detail, but makes the leap to the *"N"* integral equations in the form of

$$\int_\Omega w_j(x)\, R(c_i, x)\, dx = 0 \qquad for \quad j = 1, N$$

Now, we understand where these integrals come from and why they are equal to zero.

The next issue is to consider functions we might use for the weighting function, w.

It is clear now (hopefully), that the weighted integral statement actually generates N equations from which we can determine the N unknown constants in the assumed temperature function.

We still have several choices for the weighting function which we will discuss in more detail.

We used "j" as the index here to make the distinction between the weighting function and the assumed temperature function. In the future, we will also use "*i*" as an index.

The Method of Weighted Residuals (5)

$$\int_{\Omega} w_i(x_i)\left[\rho C\frac{\partial \overline{T}}{\partial t} - \frac{\partial}{\partial x}\left(k_{xx}\frac{\partial \overline{T}}{\partial x}\right) - \frac{\partial}{\partial y}\left(k_{yy}\frac{\partial \overline{T}}{\partial y}\right) - Q\right] d\Omega = 0$$

Again, this integral representation is referred to as a “weighted integral” formulation. To get a “weak” form, we integrate by parts to reduce the order of the temperature derivative.

One significant difference is that we do NOT need a functional like we did with the variational formulation. We just need the PDE.

By integrating over an arbitrary volume, we “distribute” this residual error over the volume.

There are several different forms for this weighting function that distribute the error differently and may have different physical interpretations.

Here we have written the weighted integral in a form for a particular weighting function, the i^{th} weighting function.

The Method of Weighted Residuals (6)

For an assumed temperature profile $\bar{T}(x) = \sum_{i=1}^{N} c_i \, f_i(x)$

And applying the MWR

$$\int_{\Omega} w_i(x) R \, d\Omega = 0 \quad for \quad i = 1, N$$

Commonly used weighting functions include:

- Collocation method – for $w_i = \delta(x - x_i)$
- Sub-domain method – for $w_i = 1$
- Petrov-Galerkin method – for $w_i(x) = g_i(x)$ (in general)
  - Galerkin method – if the weighting function is the same as the assumed temperature function $w_i(x) = f_i(x)$
- Least Squares method – for $w_i(x) = \partial R / \partial c_i$

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We will see later that the sub-domain method can be interpreted as a finite volume formulation as well.

For the Petrov-Galerkin method the weighting function is more general than in the Galerkin method, in which it is the functions that multiply the unknown constants in the assume temperature profile.

For the least-squares method, we seek to minimize the square of the residual error.

Application of the Method of Weighted Residuals

To implement this method, you first assume a temperature profile in terms of "N" unknown constants. $\bar{T}(x) = \sum_{i=1}^{N} c_i \, f_i(x)$

The equation below can then be evaluated to determine the unknown constants.

Exactly how this is done depends on the weighting function and will be demonstrated for each method.

$$\int_{\Omega} w_i(x_i) \left[\rho C \frac{\partial \bar{T}}{\partial t} - \frac{\partial}{\partial x}\left(k_{xx} \frac{\partial \bar{T}}{\partial x} \right) - \frac{\partial}{\partial y}\left(k_{yy} \frac{\partial \bar{T}}{\partial y} \right) - Q \right] d\Omega = 0$$

We will continue our "plane wall" example here to demonstrate the use of the MWR with different weighting functions.

The terms shaded above require the assumed temperature function to be smooth and differentiable up to the second derivative. This may not be an issue, but we would like to relax that requirement if possible. One way to do that is to manipulate the residual using integration-by-parts and Gauss-Divergence theorem to reduce the order of the temperature derivatives and transfer one differentiation to the weighting function.

Transferring Derivatives Broadens the Admissible Solution Space

Often, we may want to simplify this integral equation and expand the acceptable temperature profiles by reducing the order of the temperature derivatives. This is accomplished using integration and the Gauss Divergence theorem. When done, the integral equation is

$$\int_{\Omega} \left[w_i \rho C \frac{\partial \overline{T}}{\partial t} + \left(\frac{\partial w_i}{\partial x} k_{xx} \frac{\partial \overline{T}}{\partial x} \right) + \left(\frac{\partial w_i}{\partial y} k_{yy} \frac{\partial \overline{T}}{\partial y} \right) - w_i \mathrm{Q} \right] d\Omega$$

$$- \int_{\Gamma_q} w_i \left(k_{xx} \frac{\partial \overline{T}}{\partial x} n_x + k_{yy} \frac{\partial \overline{T}}{\partial y} n_y \right) d\Gamma = 0$$

Now, admissible temperature solutions only need to be smooth and differentiable up to the first derivative (“broader solution space”).

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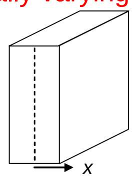
By transferring one differentiation from the temperature function to the weighting function (shaded box above), we have increased the potential functions that might be assumed for temperature by requiring only first derivatives to exist. In FEM speak, we expanded the admissible solution space. We also see that the boundary integral terms occur naturally from the weighted integral formulation.

For the sub-domain method, where w=1 over specific volumes (and dw=0), the equation above simplifies because the terms in the shaded box above are now zero. The remaining terms look like the heat capacitance and energy generation over the sub-domain volume and the boundary integral looks like the net heat flux out of the sub-domain across the boundaries. Does this look/sound like a finite volume method? Yes it does! It is at this point that we can see that finite volume methods are really a specific implementation of the sub-domain formulation of the method of weighted residuals.

Plane-Wall with Linearly Varying Volumetric Energy Generation

Consider a case of 1-D conduction in a plane wall (2L thick) with spatially varying volumetric energy generation, $Q = Q_0 + Q_1 x$



$$-k\frac{d^2T}{dx^2} = Q_0 + Q_1 x$$

$$with \quad T(L) = 0 \quad and \quad \left.\frac{dT}{dx}\right|_{x=0} = 0$$

Integrating twice, results in $T(x) = -\frac{Q_0}{2k}x^2 - \frac{Q_1}{6k}x^3 + c_1 x + c_2$

Evaluating the constants of integration using the boundary conditions

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0 \qquad c_1 = 0$$

$$T(L) = 0 = -\frac{Q_0}{2k}L^2 - \frac{Q_1}{6k}L^3 + c_1 L + c_2 \qquad c_2 = \frac{Q_0}{2k}L^2 + \frac{Q_1}{6k}L^3$$

For which the analytical solution is $T(x) = \frac{Q_0 L^2}{2k}\left(1 - \left(x/L\right)^2\right) + \frac{Q_1 L^3}{6k}\left(1 - \left(x/L\right)^3\right)$

Collocation Method – Assumed Quadratic Temperature Function

Recall

$$\int_{\Omega} w_i(x)\, R\, dx = 0 \qquad for \quad i = 1, N$$

$$\int_0^L \delta(x - x_i)\left(-k\frac{d^2\overline{T}}{dx^2} - Q_0 - Q_1 x\right) dx = 0$$

$$w_1 = \delta(x - x_1)$$

$$R = -k\frac{d^2\overline{T}}{dx^2} - Q_0 - Q_1 x$$

$$\frac{d^2\overline{T}(x)}{dx^2} = 2c_2$$

Because we only have one constant to evaluate, we will use one weighting function with a collocation point at $x_1 = L/2$

$$\int_0^L \delta(x - x_1)\left(-2k c_2 - Q_0 - Q_1 x\right) dx = \left(-2k c_2 - Q_0 - Q_1 x\right)\Big|_{x=L/2} = 0$$

evaluating and solving for c_2

$$c_2 = -\left(\frac{Q_0}{2k} + \frac{Q_1 L}{4k}\right)$$

results in an approximate solution

$$\overline{T}(x) = \frac{L^2}{2k}\left(Q_0 + \frac{Q_1 L}{2}\right)\left(1 - \left(\frac{x}{L}\right)^2\right)$$

Note that the Q terms in the approximate solution are the volumetric energy generation evaluated at the mid-point.

If there were more than one constant (say N) in our assumed temperature function, then we would choose N weighting functions with collocation points (probably equally spaced) over the 0→L domain.

So, for two unknown constants, we could use collocation points at L/3 and 2L/3. This would result in two equations from which the two constants can be determined.

Sub-domain Method – Assumed Quadratic Temperature Function

Recall

$$\int_{\Omega} w_i(x)\, R\, dx = 0 \qquad for \quad i = 1, N$$

$$\int_0^L 1.0\left(-k\frac{d^2\overline{T}}{dx^2} - Q_0 - Q_1\, x\right) dx = 0$$

$$w_1 = 1$$

$$R = -k\frac{d^2\overline{T}}{dx^2} - Q_0 - Q_1\, x$$

$$\frac{d^2\overline{T}(x)}{dx^2} = 2c_2$$

Because we only have one constant to evaluate, we will use one weighting function that is unity over the entire domain 0 → L

$$\int_0^L \left(-2k\,c_2 - Q_0 - Q_1\, x\right) dx = \left(-2k\,c_2 x - Q_0 x - Q_1\frac{x^2}{2}\right)\Bigg|_0^L = 0$$

evaluating and solving for c_2

$$c_2 = -\left(\frac{Q_0}{2k} + \frac{Q_1 L}{4k}\right)$$

results in an approximate solution

$$\overline{T}(x) = \frac{L^2}{2k}\left(Q_0 + \frac{Q_1\, L}{2}\right)\left(1 - \left(x/L\right)^2\right)$$

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Same approximate solution as the collocation method. Again, note that the *Q* terms in the approximate solution are the volumetric energy generation evaluated at the mid-point.

If there were more than one constant (say *N*) in the assumed temperature function, we would use *"N"* weighting functions that are 1 over *N* sub-domains over *0* → *L* and zero otherwise. These can be constructed by sub-dividing the domain into *N* sub-domains and using *N* heavyside functions, where $w_i=1$ over each sub-domain and zero outside of the specific sub-domain.

For example, if there were two unknown constants, then we would use two weighting functions;

•the first weighting function being $w_1=1$ over *0*→*L/2* and $w_1=0$ over *L/2*→*L* and

•the second weighting function being $w_1=0$ over *0*→*L/2* and $w_1=1$ over *L/2*→*L*.

This would result in two equations from which the two constants can be determined.

Because the weighting functions are constants over each portion of the domain, they can be factored out of the integral. The resulting equation is simply the product of a constant and the integral of the residual over the particular sub-domain. That being the case, the value of the constant (*w=1*, for our discussion) is somewhat arbitrary because we have just multiplied that integral residual equation by a constant. What is important is that the weighting functions are constants over each sub-domain and the sum of sub-domains covers the complete domain of interest in the problem.

In class, I talked about integrating over each of the *N* sub-domains with a constant. Now, I'm presenting it as integrating each weighting function over the entire domain (but the weighting function is zero over some of the domain). The resulting integrals are identical, just differing ways to "view" the method. The perspective

Galerkin Method – Assumed Quadratic Temperature Function (1)

Recall

$$\int_{\Omega} w_i(x)\, R\, dx = 0 \qquad for \quad i = 1, N$$

$$w_1(x) = f_1(x)$$

$$R = -k\frac{d^2\bar{T}}{dx^2} - Q_0 - Q_1 x$$

$$\frac{d^2\bar{T}(x)}{dx^2} = 2c_2$$

where the weighting function $w_i(x)$ has the same form as the functions used in the assumed temperature function

$$\bar{T}(x) = -c_2\left(L^2 - x^2\right) \qquad \text{and} \qquad w_1(x) = f_1(x) = -\left(L^2 - x^2\right)$$

or

$$\int_0^L \left(L^2 - x^2\right)\left(-k\frac{d^2\bar{T}}{dx^2} - Q_0 - Q_1 x\right) dx = 0$$

$$\int_0^L \left(L^2 - x^2\right)\left(-2kc_2 - Q_0 - Q_1 x\right) dx = 0$$

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If there were more than one unknown constant (say N) in the assume temperature function, then we would apply the integral using each of the functions used in the assumed temperature function. If for this problem, we had included the cubic terms, we would evaluate this integral twice; with the different weighting functions. In this case, the first instance with $w_1 = (L^2\text{-}x^2)$ and the second instance with $w_2 = (L^3\text{-}x^3)$ as the weighting function. This would result in two equations from which the two unknown constants can be determined.

NOTE: In this example, because we did not reduce the order of the residual (temperature derivatives) by employing integration-by-parts we did not employ a weak form representation. Therefore, we are solving a weighted residual integral statement, but it is not a weak form representation. You can show that in either case, the terms in the integrand will be identical. In the future, we will find more utility in using the weak form representation.

Galerkin Method – Assumed Quadratic Temperature Function (2)

By evaluating this integral

$$\int_0^L \left(L^2 - x^2\right)\left(-2k\,c_2 - Q_0 - Q_1\,x\right)dx = 0$$

and solving for the unknown constant c_2

$$c_2 = -\left(\frac{Q_0}{2k} + \frac{3Q_1L}{16k}\right)$$

results in an approximate solution

$$\bar{T}(x) = \frac{L^2}{2k}\left(Q_0 + \frac{3Q_1\,L}{8}\right)\left(1 - \left(x/L\right)^2\right)$$

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Note that the Q terms in the approximate solution are the volumetric energy generation evaluated at the 3L/8 point.

This is also the same solution as we would obtain using the variational principle. It can be shown that the Galerkin and variational principle give the same result if the problem has a classical variational statement.

Least-Squares Method – Assumed Quadratic Temperature Function (1)

Recall

$$\int_{\Omega} w_i(x)\; R\; dx = 0 \qquad for \quad i = 1, N$$

with

$$w_i(x) = \frac{\partial R}{\partial c_i}$$

$$w_1(x) = \frac{\partial R}{\partial c_2}$$

$$R = -k\frac{d^2\overline{T}}{dx^2} - Q_0 - Q_1\,x$$

$$\frac{d^2\overline{T}(x)}{dx^2} = 2c_2$$

where weighting functions are the partial derivatives with respect to the constants in the residual

The integral can be written as

$$\frac{\partial}{\partial c_i}\int_{\Omega} R^2\; dx = 0 \qquad for \quad i = 1, N$$

which is a minimization of the residual error squared

In the least-squares method, we are minimizing the error in a least-squares sense.

If there were more than one constant (say N) in the assumed temperature function, we would have to take the partial derivatives of the error integral with respect to each of the constants. The result would be N equations from which the N constants could be determined.

Least-Squares Method – Assumed Quadratic Temperature Function (2)

Because we only have one constant to evaluate, we only need to take the derivative with respect to c_2

$$\frac{d}{d\,c_2}\int_0^L \left(R^2\right) dx = \frac{d}{d\,c_2}\int_0^L \left(-k\frac{d^2\overline{T}}{dx^2} - Q_0 - Q_1\,x\right)^2 dx = 0$$

$$\frac{d}{d\,c_2}\int_0^L \left(-2\,k\,c_2 - Q_0 - Q_1\,x\right)^2\, dx = 0$$

evaluating and solving for c_2

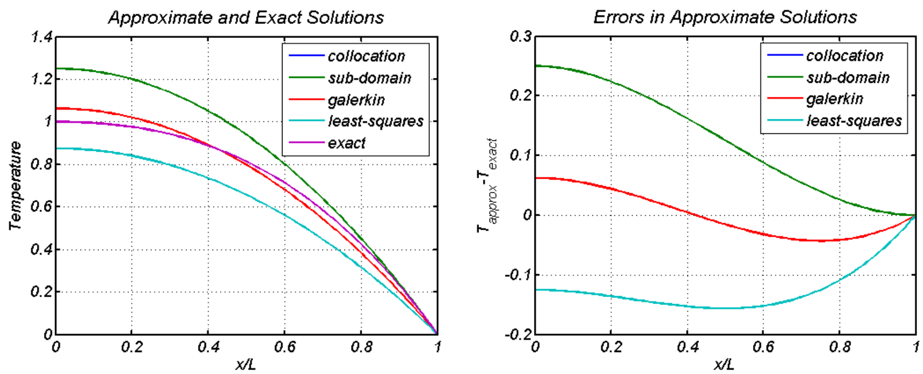
$$c_2 = -\left(\frac{Q_0}{2k} + \frac{Q_1 L}{4k}\right)$$

results in an approximate solution

$$\overline{T}(x) = \frac{L^2}{2k}\left(Q_0 + \frac{Q_1\,L}{4}\right)\left(1 - \left(x/L\right)^2\right)$$

In this case, the Q terms in the approximate solution are the volumetric energy generation evaluated at the L/4 point (weighted towards the plane of symmetry).

Comparison of MWR Results and Analytical Solution



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In each case, the main differences was where/how the volumetric energy generation was evaluated…

For collocation and sub-domain, it was at the mid-point L/2

For Galerkin, it was just inside the mid-point, at 3L/8

For least-squares, it was at the quarter point, L/4

The closer to the centerline, (smaller x), the smaller the volumetric energy generation. We see the effect in the reduced temperatures in the profiles.

All of these methods resulted in zero error at the surface (L), where the temperature was specified. We would certainly expect that since we assumed a temperature function that satisfied the boundary conditions of the problem. For this problem, only the Galerkin method had a point within the interior where the error was zero.

Summary of Methods

- Weighted integrals and weak forms permit us to consider a wider range of PDEs, when compared to minimizing functionals using the variational principle.
- Weighted integrals and weak forms reduce requirements on the assumed solutions, expand the solution space, and allow us to compute approximate solutions over domain.
- The Galerkin method produces the same solutions as variational principle (when variational functionals are available).
- It can be shown that the Galerkin method produces optimal approximate solutions in an L2 norm for certain types of equations, such as heat conduction.
- We will focus on the Galerkin FEM for heat conduction problems.

Observations for MWR

- We have assumed solution profiles over the entire domain and determined constants in the assumed solution.
- To be more general and be simpler to apply, we might consider applying these concepts to a sub-divided domain with approximate temperature profiles assumed over each domain
- This line of thinking will lead us to the Finite Element Method using the MWR Galerkin formulation for complex geometries, etc.
- Catch – before we can do that, we will have to shift our focus to the “element” concepts such as element geometry, assumed temperature profiles and gradients, evaluating the MWR integrals, etc.
- Make sure you understand these concepts for the Galerkin MWR

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Where are we going with this discussion and why?

Conceptually, we have the basic ideas here that will lead us to the Galerkin Finite Element Method.

Summary of the Strong and Weak Forms for Heat Transfer Problems

"Strong form BVP"

$$\rho C_V \frac{\partial T}{\partial t} = \frac{\partial}{\partial x_i}\left(k_{ij}\frac{\partial T}{\partial x_j}\right) + Q$$

$$T(s_i,t) = f_T(s_i,t) \quad on\ \Gamma_T$$

$$-k_{ij}\frac{\partial T}{\partial x_j} n_i = f_q(s_i,t) \quad on\ \Gamma_q$$

"Weak form BVP"

$$\int_\Omega \left[\frac{\partial w}{\partial x}\left(k_{xx}\frac{\partial T}{\partial x}\right) + \frac{\partial w}{\partial y}\left(k_{yy}\frac{\partial T}{\partial y}\right) - w\,Q \right] d\Omega - \oint_{\Gamma_q} w\, q_b\, d\Gamma = 0$$

$$T = T_{specified} \quad on\ \Gamma_T$$

$$k_{xx}\frac{\partial T}{\partial x} n_x + k_{yy}\frac{\partial T}{\partial y} n_y = q_b \quad on\ \Gamma_q$$

The weak form involves solving the PDE and boundary conditions in an average sense over the domain of interest. We will continue with that approach.

Exercise #1

Repeat the plane-wall example, but now consider the volumetric energy generation as a quadratic function of position.

$$Q(x) = Q_0 + Q_1 x^2$$

Assume: $\bar{T}(x) = -c_2\left(L^2 - x^2\right) - c_3\left(L^3 - x^3\right) - c_4\left(L^4 - x^4\right)$

Required:

- Determine the analytical solution
- Determine approximate solutions with Galerkin method by assuming 1) quadratic and 2) cubic temperature functions
- Plot analytical solution and approximate solutions
- Plot temperature error for approximate solutions
- BONUS: repeat using assume quartic temperature function