# Coupled Conduction & Enclosure Radiation Problems

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ESP = Engineering Sciences Program

ESP100 is a course on computational solid mechanics

ESP200 is a course on digital signal processing with MATLAB

ESP300 is a course on heat transfer analysis using the finite element method

There are plans to offer additional courses in the future.

All of these courses are intended to provide a continuing education opportunity – in the spirit of the INTEC courses some years ago

## Introductory Info

### Evacuation Procedures:

- Exits are located…
- Restrooms out back

### Classification:

- Absolutely no classified discussions
- If you have a concern, let us know
- Some material may be OUO, it will be marked as such

## Summary for Coupled Conduction & Enclosure Radiation

Begin with:

- General IBVP statement for heat conduction and the radiative transfer equations for enclosures

and end with:

- Understanding of the assumptions employed and details of this class of coupled problems
- Procedure for coupling the discretized heat conduction and enclosure radiation problems
- Overview of analyst considerations

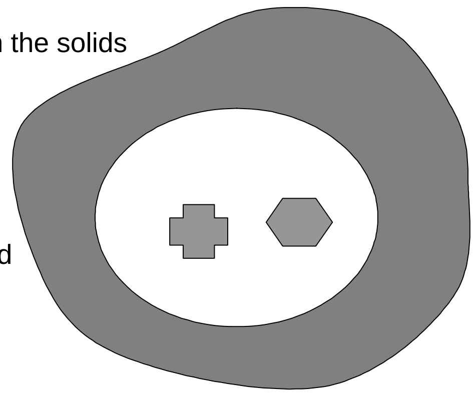
## Questions for Coupled Conduction & Enclosure Radiation

- What are the equations describing heat conduction and enclosure radiation?
- Which variables are common to and passed between the two sets of equations?
- Are there inconsistencies in the assumptions used in each set? If so, what are the inconsistencies and how do we address them?
- What assumptions are made to couple these two sets of equations?
- What approaches do we use to solve this coupled set?
- What are the advantages and disadvantages to the different coupling approaches?

# Coupled Heat Conduction and Enclosure Radiation

We need to solve the equations for

- conduction within the solids
- enclosure radiation between the solids



To do so, we will consider

- the governing equations,
- how they are related/coupled
- options for solving them

## Outline of Topics for Today

We will consider the following:

- Discretized form of the conduction equations and associated boundary conditions
- Enclosure radiation equations and associated boundary conditions
- Consistency and compatibility between the two sets of equations
- Approaches for coupling conduction and enclosure radiation
- Advantages/disadvantages of the coupling strategies
- An example problem that demonstrates some of these issues

## Initial Boundary Value Problem

The basic PDE description of heat conduction in the region $\Omega$ with boundary $\Gamma$ is

$$\rho C \frac{\partial T}{\partial t} = \frac{\partial}{\partial x_i}\left( k_{ij} \frac{\partial T}{\partial x_j} \right) + Q$$

The boundary conditions are

$$T = f^T(s_i, t) \qquad on \qquad \Gamma_T$$

With

$$\left( k_{ij} \frac{\partial T}{\partial x_j} \right) n_j + q_c + q_r + q_{enc} = f^q(s_i, t) \qquad on \qquad \Gamma_q$$

$$q_c = h_c(s_{i,} T, t)\left[ T - T_c \right]$$

$$q_r = \mathbb{F}\sigma\left[ T^4 - T_r^4 \right] = h_r(s_i, T, t, \sigma, \varepsilon \ldots)\left[ T - T_r \right]$$

$q_{enc} = f(s_{i,} \overline{T}^4, t)$ comes from the "enclosure radiation" equations



This is the standard IBVP where all of the parameters and terms were defined previously.

For our problems, the "enclosure radiation" fluxes are computed from radiative transfer equations for enclosures consisting of diffuse-gray surfaces and a non-participating (transparent) media. Cast in this form, the enclosure radiation fluxes appear as unknowns.

## Initial Boundary Value Problem (2)

You will recall that early on, we developed the equations describing the radiative transfer within an enclosure. When applied to an enclosure defined by '$N_{surf}$' discrete surfaces, this equation yields the "net heat flux" conducted away from the surface into the body.

$$\sum_{j=1}^{N_{surf}} \left[ \frac{\delta_{kj}}{\varepsilon_j} - F_{kj} \left( \frac{1-\varepsilon_j}{\varepsilon_j} \right) \right] \overline{q}_{net,j} = \sum_{j=1}^{N_{surf}} \left( \delta_{kj} - F_{kj} \right) \sigma \overline{T}_j^4$$

where $\overline{q}_{net,j} = \dfrac{Q_{net,j}}{A_j}$

These equations provide the radiative heat fluxes to completely define our IBVP. We will consider it in more detail later.



This form of the enclosure radiation equations is known as the "net-radiation" formulation. This set of equations involves both the surface temperatures and the surface radiative heat fluxes and provides the closure of the IBVP.

Note: We have used both "net" and "enc" subscripts on the radiative heat flux interchangeably. In the conduction equations, "enc" refers to the fact that these fluxes come from the enclosure radiation equations. In the enclosure radiation equations, "net" refers to the net heat flux conducted to the radiating surface. For our purposes here, both the "enc" and "net" subscripts refer to the same flux.

## FE Formulation for IBVP

Assume the standard (spatial) finite element approximation for the temperature and use the Galerkin form of the MWR

$$T(x_i,t)=\sum_{k=1}^{N}\psi_k(x_i)T_k(t)=\mathbf{\Psi}^{\mathrm{T}}(x_i)\,\mathbf{T}(t)$$

$$w(x_i)=\psi_j(x_i)=\mathbf{\Psi}(x_i)$$

Substituting these approximations into the weak form produces the finite element equations for the nodal values of temperature.

Comments:

• We assume a temperature profile that varies over each element.

• How does that fit with the assumptions for the enclosure radiative transfer?

- Uniform surface temperatures and uniform surface heat fluxes



Recall that the development of the enclosure radiation equations assumed uniform surface temperatures and surface heat fluxes.

## FE Form for IBVP

The discretized equation set is

$$\int_{\Omega} \rho C \, \mathbf{\Psi} \mathbf{\Psi}^{\mathrm{T}} \, d\Omega \frac{\partial \mathbf{T}}{\partial t} + \int_{\Omega} \frac{\partial \mathbf{\Psi}}{\partial x_i} \left( k_{ij} \frac{\partial \mathbf{\Psi}^{\mathrm{T}}}{\partial x_j} \right) d\Omega \; \mathbf{T} =$$

$$\int_{\Omega} \mathbf{\Psi} \, Q \, d\Omega + \int_{\Gamma} \mathbf{\Psi} \left( q_a - q_c - q_r - q_{enc} \right) d\Gamma$$

or in matrix form

$$\mathbf{M(T)}\dot{\mathbf{T}} + \mathbf{K(T)}\mathbf{T} = \mathbf{F}_Q\mathbf{(T)} + \mathbf{F(T)} + ????$$

We have '*N*' equations with '*N*' unknown temperatures and additional unknown radiative surface heat fluxes for the enclosure.

Where are the boundary fluxes that will come from solving the enclosure radiation equations?



If all the heat flux boundary conditions for the enclosure are known, we can compute the nodal temperature field from this set of equations.

On the next slide we will consider different approaches for handling the heat flux associated with the enclosure radiation. One approach is to leave the enclosure radiation heat flux boundary terms as unknowns in this set of equations. If we did so, we would have more unknowns than equations and would need additional equations. Those additional equations are obviously the equations describing the radiative transfer within the enclosure.

## Comments on the FE Form for IBVP

The discretized equation set is

$$\int_{\Omega} \rho C \boldsymbol{\Psi}\boldsymbol{\Psi}^{T} \, d\Omega \frac{\partial \mathbf{T}}{\partial t} + \int_{\Omega} \frac{\partial \boldsymbol{\Psi}}{\partial x_i}\left( k_{ij} \frac{\partial \boldsymbol{\Psi}^{T}}{\partial x_j} \right) d\Omega \, \mathbf{T} =$$

$$\int_{\Omega} \boldsymbol{\Psi} Q \, d\Omega + \int_{\Gamma} \boldsymbol{\Psi} (q_a - q_c - q_r - q_{enc}) \, d\Gamma$$

or in matrix form

$$\mathbf{M(T)\dot{T} + K(T)\,T = F}_Q\mathbf{(T) + F(T) - E\,q_{enc}}$$

Here the enclosure fluxes are shown as dependent variables

Can we treat them "explicitly" or "implicitly"? What difference will it make? Why one approach over the other?

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With all the flux boundary conditions known, we could compute the nodal temperature field from this set. Typically, we do not know the enclosure radiation component, so this is really a vector of unknown heat fluxes (dependent variables). As such, we could explicitly leave the enclosure radiation boundary heat flux terms in this set of equations as dependent variables. If we did so, we would have more unknowns than equations and would need additional equations. Again, we will need the equations describing the radiative transfer within the enclosure to provide closure of the equation set.

Note that this is a set of "$N_{nodes}$" equations with "$N_{nodes}$" unknown nodal temperatures and "$N_{surf}$" unknown surface radiative fluxes.

## Radiative Heat Transfer Between Surfaces Within Enclosures

Recall that radiative HT between surfaces depends on:

- radiative properties of surfaces
- geometric relationship between surfaces
- the medium between surfaces (nonparticipating in our case)

We discussed:

- radiation “view factors”
- conservation of radiative energy for enclosures
- equations describing radiative transfer in “enclosures”
  - we looked at two different forms of the equations
- coupling with equations for conduction heat transfer

In this discussion, we will get into the details of the coupling options



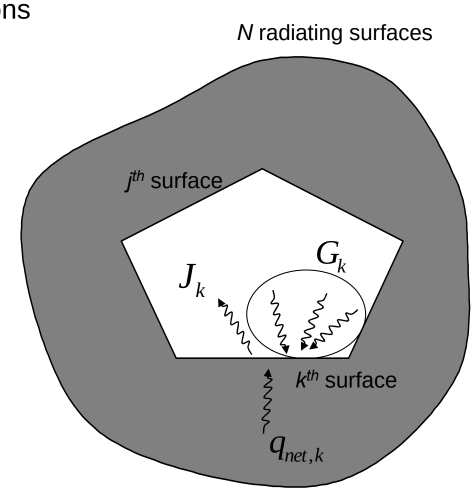
Some general comments on enclosure radiation before we get going and a little peek at what we will cover.

## Review of the Assumptions For Radiative Transfer in Enclosures

In the development of the equations for the radiative energy transfer within enclosures, the following assumptions were used:

- Non-participating media
- Diffuse-gray surfaces
- Uniform surface temperatures
- Uniform surface heat fluxes

In our discussion today, we will change our notation on the radiosity, $J_k$ , and use $\overline{q}^0$

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We started our development with an energy balance on a typical surface. With those concepts for a single surface, we can applied them to the entire enclosure, simply a collection of all the individual surfaces.

We used the following assumptions:

- •Media is non-participating (transparent)
- •Temperatures are uniform over each surface.
- •Radiative properties are independent of wavelength and direction.
- •All energy is emitted and reflected diffusely.
- •Incident and reflected energy flux is uniform over each surface.

Using these assumptions, we previously developed the “radiosity” and “net-radiation” formulations describing the radiative heat transfer within enclosures.

## Radiative Transfer for Enclosures Using the "Radiosity" Formulation

For an enclosure with *'$N_{surf}$'* surfaces with known surface temperatures

- How are the surfaces defined?
- What is an appropriate "surface" temperature?

We have a set of *'$N_{surf}$'* equations for the radiosity given by

$$\sum_{j=1}^{N_{surf}} \left( \delta_{kj} - (1-\varepsilon_k) F_{kj} \right) \overline{q}_j^0 = \varepsilon_k \sigma \overline{T}_k^4$$

Once the radiosities are computed, the net heat flux on each surface is given by

$$\overline{q}_{net,k} = \sum_{j=1}^{N_{surf}} \left( \delta_{kj} - F_{kj} \right) \overline{q}_j^0$$



In a coupled problem, we generally use temperatures computed from the heat conduction equation as known inputs to the radiosity equations. With the surface temperatures known, the radiosities can be easily computed from this set of linear algebraic equations; followed by the net heat flux calculation.

In this case, we have a set of "$N_{surf}$" equations with "2x$N_{surf}$" unknown surface radiative fluxes and surface temperatures.

## Radiative Transfer for Enclosures Using the "Net-Radiation" Formulation

The equations describing the radiative transfer in an enclosure can be written in terms of the "net heat flux" conducted away from the surface and the surface temperature.

In the "net-radiation" formulation, the set of equations becomes

$$\sum_{j=1}^{N_{surf}} \left[ \frac{\delta_{k\,j}}{\varepsilon_j} - F_{k\,j} \left( \frac{1-\varepsilon_j}{\varepsilon_j} \right) \right] \overline{q}_{net,j} = \sum_{j=1}^{N_{surf}} \left( \delta_{k\,j} - F_{k\,j} \right) \sigma \overline{T}_j^4$$

where $$\overline{q}_{net,j} = \frac{Q_{net,j}}{A_j}$$

These equations provide the second set of '$N_{surf}$' equations that relate the surface temperatures and radiative surface heat fluxes.



Again, we now have a set of "$N_{surf}$" equations, with "2x$N_{surf}$" unknowns (surface temperatures and net heat fluxes).

Knowing any "$N_{surf}$" of these, we can solve the remaining "$N_{surf}$" unknowns.

For example, given surface temperatures for all the surfaces, then the net heat flux can be computed.

## Radiative Transfer for Enclosures Using the "Net-Radiation" Formulation (2)

The "net-radiation" formulation set of equations can be expressed as

$$\sum_{j=1}^{N_{surf}} \left[ \frac{\delta_{kj}}{\varepsilon_j} - F_{kj} \left( \frac{1-\varepsilon_j}{\varepsilon_j} \right) \right] \bar{q}_{net,j} = \sum_{j=1}^{N_{surf}} \left( \delta_{kj} - F_{kj} \right) \sigma \bar{T}_j^4$$

or in matrix form as

$$\mathbf{A}(\bar{\mathbf{T}})\bar{\mathbf{q}}_{\mathbf{net}} = \mathbf{B}\bar{\mathbf{T}}^4$$

which relates the surface temperatures and radiative surface heat fluxes. There are $N_{surf}$ equations in this set.

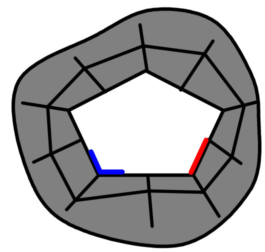


We should note here that the coefficient matrix, A(T), is an "$N_{surf}$" by "$N_{surf}$" matrix that may be densely populated (could be completely full) and asymmetric. Depending on the number of radiating surfaces, it can be very large; creating significant demands on memory. However, when the temperatures are known, it is a linear set of equations; often solved using methods such as Gauss-Seidel.

## Compatibility Issues Between Conduction and Enclosure Radiation

In general, we will assume that the surfaces making up the enclosure are defined by exposed element faces of the finite elements in the mesh.

Recall, radiative surfaces are assumed to have uniform temperatures and uniform heat fluxes



We will use an "element based" approach to define our radiation surfaces.

In some finite volume approaches, "node-based" approaches may be used to define the radiation surfaces.



An "element" based approach simplifies the computation of view factors because the radiative surfaces are simply defined as quads or triangles, etc of the element faces. If we used elements with curved surfaces, additional complexity would result.

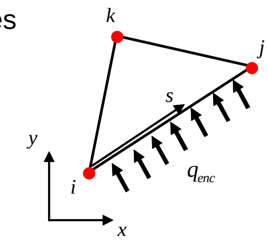
A "nodal" based approach requires the construction of a geometrically complex (polygon) around the node that would be used as a "surface" in the enclosure radiation formulation.

## Compatibility Issues Between Conduction and Enclosure Radiation (2)

Radiative fluxes applied to the element boundaries

$$\mathbf{F}^e_{q_{enc}} = \int_{\Gamma^e} \mathbf{\Psi}\, q_{enc}(x_i)\, d\Gamma$$

$$\mathbf{F}^e_{q_{enc}} = \int_{s^e} \hat{\mathbf{\Psi}}\, q_{enc}(s)\, ds$$




For elements with a linear shape function over the face

$$\mathbf{F}^e_{q_{aenc}} = \int_0^h \hat{\mathbf{\Psi}}(s)\, q_{enc}\, ds = q_{enc} \begin{Bmatrix} h/2 \\ h/2 \end{Bmatrix}$$

Recall that the rows in this element matrix correspond to the "$i^{th}$" and "$j^{th}$" rows in the global system of equations



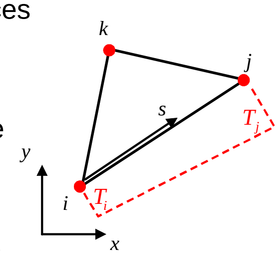
The applied heat flux from the enclosure has contributions to the global equations for nodes "i" and "j"

# Compatibility Issues Between Conduction and Enclosure Radiation (4)

In conduction equations, temperatures defined at nodes and assumed to vary over element surfaces using the shape functions (linear or bi-linear in many cases)

In enclosure radiation equations, temperatures are assumed constant over each of the radiating surfaces.

The challenge is to relate the fourth power surface temperatures in the enclosure radiation equations to the (often linear) nodal temperature field in the FEM formulation.



## Compatibility Issues Between Conduction and Enclosure Radiation (5)

There are several candidate approaches:

- Assume that the surface temperature is the temperature at the centroid of the element face. Allows the surface temperature to be expressed in terms of the nodal temperatures (important in an implicit formulation)

$$\bar{T} = \left(\hat{\boldsymbol{\psi}}\left(\xi_0, \eta_0\right)\right)^T \hat{\mathbf{T}}$$

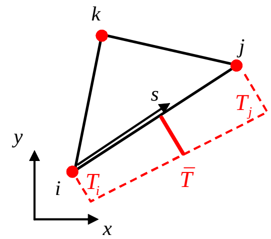
where $\xi_0, \eta_0$ are the natural coordinates of the centroid of the element face

- Integration over the element to compute an effective surface temperature that has the appropriate emissive power ($T^4$ dependence). Amenable to a staggered formulation.

## Compatibility Issues Between Conduction and Enclosure Radiation (6)

The surface temperatures over radiation surfaces can be approximated using the temperature at the centroid of the element face,

$$\bar{T}^4 \approx \bar{T}^3 \frac{(T_i + T_j)}{2} \qquad where \qquad \bar{T} = \frac{(T_i + T_j)}{2}$$




Clearly, this is an approximation for $T^4$

An advantage of this approach is that the nodal temperatures can be factored out, which allows $T^4$ terms to be included in the coefficient matrix implicitly.

Depending on the whether we use an implicit or explicit formulation, the nodal temperatures may be evaluated at either time "n" or "n+1"



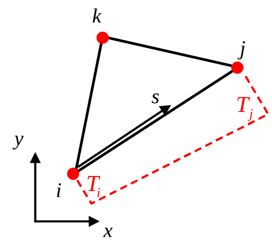
Here we have basically interpolated the temperature over the element face. We could have also interpolated T^4 (computed at the nodes) over the face. In that case, T_bar = ½ (T_i^4+T_j^4), which is clearly a different value than interpolating the temperature directly.

The differences in these two approaches can be computed analytically. The error associated with this approximation depends on the temperature gradients over the face of the element.

## Compatibility Issues Between Conduction and Enclosure Radiation (7)

A more rigorous approach for approximating radiative surface temperatures is to integrate the temperature distribution over the element face and compute an effective $T^4$ temperature with an equivalent emissive power.

$$\bar{T} = \left\{ \frac{\int_{s^e} \left(T(s)\right)^4 ds}{\int_{s^e} ds} \right\}^{0.25}$$




Clearly, this is a more rigorous approximation for the radiative temperature of the surface

One disadvantage of this approach is that the nodal temperatures can not be factored out; which prevents us from including the $T^4$ terms implicitly in the coefficient matrix.



This amounts to an area-averaged surface temperature that has the same emissive power as the radiating surface (element face).

## Three Approaches for Coupling Will Be Discussed in Detail

- "Staggered" or "cyclic" formulation
  - Solve the two sets equations separately
    - Conduction equations (N unknown temperatures)
    - Radiosity enclosure radiation equations ($N_{surf}$ unknown radiative fluxes)
  - Iterate between the two sets of equations
- "Semi-implicit" formulation
  - Solve two sets of equations similar to staggered formulation, but modify the heat fluxes to provide some implicitness (partial contributions to the stiffness matrix)
- "Fully-coupled" or "implicit" formulation
  - Form one set of nonlinear equations consisting of both
    - Conduction equations and net-radiation enclosure radiation equations
  - Solve the one set of equations for the $N$ unknown temperatures and $N_{surf}$ unknown radiative heat fluxes



Actually, we will briefly consider another fully implicit formulation, but will rule it out of further consideration due to the undesirable characteristics.

## “Staggered” or “Cyclic” Formulation

Recall we had the matrix form of the conduction equations with the enclosure radiation fluxes kept as unknowns

$$\mathbf{M}(\mathbf{T})\dot{\mathbf{T}}+\mathbf{K}(\mathbf{T})\mathbf{T}=\mathbf{F}_Q(\mathbf{T})+\mathbf{F}(\mathbf{T})-\mathbf{E}\,\mathbf{q}_{\mathbf{enc}}$$

or in final form $\hat{\mathbf{K}}(\mathbf{T})\mathbf{T}=\hat{\mathbf{F}}(\mathbf{T})-\mathbf{E}\,\mathbf{q}_{\mathbf{enc}}$

$$\sum_{j=1}^{N_{surf}}\left(\delta_{kj}-(1-\varepsilon_k)F_{kj}\right)\overline{q}_j^0=\varepsilon_k\sigma\overline{T}_k^{\,4}$$

$$q_{net,k}=\sum_{j=1}^{N_{surf}}\left(\delta_{kj}-F_{kj}\right)\overline{q}_j^0$$

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In this formulation, we cycle between the two sets of equations; solving each set independently. Individually, each equation set is not strongly nonlinear, but the T^4 nonlinear dependence does appear during the transition between the two equation sets. With known temperatures, the radiosity equation is linear; possibly with temperature dependent surface emissivities.

# Iteration Process for Staggered or Cyclic Formulation

At the beginning of this time step, at time $t^n$ with time step $\Delta t_n$

1. Predict temperature $T_p^{n+1}$ for the new time plane, $t^{n+1}$, using Adams-Bashforth explicit predictor
2. Enter iterative loop to compute end of time step fields; $T^{n+1}$ and $\overline{q}^{n+1}$

   A. Update the enclosure radiative heat flux field, $\overline{q}_c^{n+1}$, using the radiosity equations
      - For the first iteration, use the predicted temperatures, $T_p^{n+1}$
      - For subsequent iterations, use the corrected temperatures, $T_c^{n+1}$

   B. Update the corrected temperature field, $T_c^{n+1}$, using the conduction equation
      - Use radiative heat flux boundary conditions, $\overline{q}_c^{n+1}$, computed with the previously corrected temperature field, Step A.

   C. Relax corrected temperature field using corrected and previous values
      - Check for iteration convergence at this time step
      - If not converged, repeat the loop by returning to Step A

   D. Once converged, update the variables and continue to Step 3
      - Set $T^{n+1} = T_c^{n+1}$ and $\overline{q}^{n+1} = \overline{q}_c^{n+1}$

3. Check for steady state convergence using norm on temperature field
4. Estimate a new time step by comparing the predicted and corrected temperature fields

End of time step at time, $t^{n+1}$

## Comments on the "Staggered" or "Cyclic" Formulation

Advantages:

- Smaller and symmetric coefficient matrices than fully-coupled
- Smaller memory requirements allows large problems
- Amenable to transient problems, adaptive timestep approaches
- Can use multiple correction steps to improve convergence

Disadvantages:

- Can be difficult to converge steady problems, may require extensive relaxation between iterations (false transients)
- Convergence can be very slow, and quite often can diverge due to the fourth-order dependence of flux on temperature
- Can be problematic for problems with "disconnected" regions



We are solving two sets of equations by iterating between:

- Heat conduction with "$N_{nodes}$" unknown nodal temperatures
- Enclosure radiation with "$N_{surf}$" unknown heat fluxes

## "Semi-Implicit" Formulation

Recall the matrix form of the conduction equation

$$\hat{K}(T)\,T = \hat{F}(T) - E\,q_{enc}$$

Our objective is to rewrite the enclosure radiation terms more implicitly. That is, to express the radiative heat fluxes in terms of the surface (and nodal) temperatures.

By formulating these terms more implicitly, we will change the behavior of the system coupled system of equations;

- Increasing the coupling between the two sets of equations
- Improving stability and convergence characteristics



We refer to this approach as semi-implicit because we rewrite the heat fluxes in a way that a portion of those terms will end up in the global stiffness matrix, while a portion remains in the load vector on the right hand side.

Additional information can be found in the Coyote, TACO, and Calore Theory Manuals, as well as in our Reddy and Gartling textbook.

## “Semi-Implicit” Formulation (2)

To begin, we write the net radiative heat flux as the difference between the emissive power and the irradiation as

$$\overline{q}_{net,k} = \sigma \varepsilon_k \overline{T}^4 - \sum_{j=1}^{N} F_{kj}\, \overline{q}_j^0$$

Next, we approximate the heat flux at the n+1 iteration using a Taylor series expansion of this flux at the “n” iteration

$$\overline{q}_{net,k}^{n+1} = \overline{q}_{net,k}^{n} + \left.\frac{\partial \overline{q}_{net,k}}{\partial \overline{T}}\right|^{n} \left(\overline{T}^{n+1} - \overline{T}^{n}\right)$$

The partial derivative will be evaluated using the net heat flux expression above.

# "Semi-Implicit" Formulation (3)

Evaluating the partial derivative

$$\frac{\partial \bar{q}_{net,k}}{\partial \bar{T}} = \frac{\partial\left(\sigma \varepsilon_k \bar{T}^4\right)}{\partial \bar{T}} - \frac{\partial\left(\sum_{j=1}^{N} F_{k\,j}\, \bar{q}_j^0\right)}{\partial \bar{T}} \approx 4\sigma \varepsilon_k \bar{T}^3 - 0$$

and substituting into the Taylor series

$$\bar{q}_{net,k}^{n+1} = \bar{q}_{net,k}^{n} + \left.\frac{\partial \bar{q}_{net,k}}{\partial \bar{T}}\right|^{n} \left(\bar{T}^{n+1} - \bar{T}^{n}\right)$$

We obtain

$$\bar{q}_{net,k}^{n+1} = \bar{q}_{net,k}^{n} + 4\sigma \varepsilon_k \left(\bar{T}^{n}\right)^3 \left(\bar{T}^{n+1} - \bar{T}^{n}\right)$$

## "Semi-Implicit" Formulation (4)

Reordering this expression, we obtain

$$\overline{q}_{net,k}^{n+1} = \left\{ \overline{q}_{net,k}^{n} - 4\sigma\varepsilon_k \left( \overline{T}^{n} \right)^4 \right\} + \left\{ 4\sigma\varepsilon_k \left( \overline{T}^{n} \right)^3 \right\} \overline{T}^{n+1}$$

The first bracketed term is evaluated at the previous iteration and is "known." The second bracketed term involves the unknown temperature at the n+1 iteration. It will be included implicitly in the global stiffness matrix. To do that, we again have to make the mapping from the surface temperature to nodal temperatures.

Rewriting in matrix form for all surfaces in the enclosure

$$\overline{\mathbf{q}}_{net}^{n+1} = \tilde{\mathbf{F}} \left( \overline{q}_{net}, \overline{T} \right)^{n} + \tilde{\mathbf{K}}^{*} \left( \overline{T}^{3} \right)^{n} \mathbf{T}^{n+1}$$

where K* has the mapping from surface to nodal temperatures.



In this equation, the radiative flux at the iteration 'n' is computed using the temperatures at iteration 'n'.

## "Semi-Implicit" Formulation (5)

Substituting into the global system of equations, we obtain

$$\overline{\mathbf{q}}_{net}^{n+1} = \tilde{\mathbf{F}}^{n} + \tilde{\mathbf{K}}^{*}\mathbf{T}^{n+1}$$

$$\hat{\mathbf{K}}(\mathbf{T})\,\mathbf{T} = \hat{\mathbf{F}}(\mathbf{T}) - \mathbf{E}\,\mathbf{q}_{\mathbf{enc}}$$

$$\left[\hat{\mathbf{K}}(\mathbf{T}) + \mathbf{E}\tilde{\mathbf{K}}^{*}(\mathbf{T})\right]\mathbf{T} = \left\{\hat{\mathbf{F}}(\mathbf{T}) - \mathbf{E}\,\tilde{\mathbf{F}}(\mathbf{q},\mathbf{T})\right\}$$



The resulting nonlinear set of N equations is still sparse, but has additional contributions involving the temperature dependence of the radiative equations.

## Comments on the “Semi-Implicit” Formulation

Advantages:

- Smaller and symmetric coefficient matrices than fully-coupled
- Smaller memory requirements allows large problems
- Increased the coupling between the two sets of equations provides improved stability and convergence characteristics
- Amenable to both steady and transient problems
- Can use multiple correction steps to improve convergence
- Effective formulation for problems with “disconnected” regions

Disadvantages:

- Coupling may not be as robust as fully coupled formulation

# "Fully Coupled" or "Implicit" Formulation

Rewriting in matrix form, the conduction equations with the enclosure radiation heat fluxes kept as unknowns is

$$\mathbf{M}(\mathbf{T})\dot{\mathbf{T}} + \mathbf{K}(\mathbf{T})\mathbf{T} = \mathbf{F}_Q(\mathbf{T}) + \mathbf{F}(\mathbf{T}) - \mathbf{E}\,\mathbf{q}_{\mathbf{enc}}$$

or in final form

$$\hat{\mathbf{K}}(\mathbf{T})\mathbf{T} + \mathbf{E}\,\mathbf{q}_{\mathbf{enc}} = \hat{\mathbf{F}}(\mathbf{T})$$

with '*N*' equations, '*N*' unknown nodal temperatures and '$N_{surf}$' unknown surface heat fluxes

## "Fully Coupled" or "Implicit" Formulation (2)

The "net-radiation" formulation is written in terms of surface fluxes and surface (not nodal) temperatures

$$\mathbf{A}(\overline{\mathbf{T}})\overline{\mathbf{q}}_{\mathbf{net}} = \mathbf{B}\overline{\mathbf{T}}^4$$

Expressing the surface temperatures in terms of the nodal temperatures, the matrix can be rewritten as

$$\mathbf{A}(\overline{\mathbf{T}})\overline{\mathbf{q}}_{\mathbf{net}} + \mathbf{B}^*\left(\overline{\mathbf{T}}^3\right)\mathbf{T} = 0$$

Vector of nodal temperatures over enclosure surfaces

where $\mathbf{B}^*$ includes the mapping of surface temperatures to nodal temperatures. There are $N_{surf}$ equations in this set.

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We have discussed approaches for mapping between the surface temperatures and nodal temperatures earlier in this class. The details will be overlooked here.

Again, we note that the coefficient matrix, A(T), is an "$N_{surf}$" by "$N_{surf}$" matrix that may be densely populated (could be completely full). Depending on the number of radiating surfaces, it can be very large; creating significant demands on memory.

## "Fully Coupled" or "Implicit" Formulation (3)

One approach to forming a single set of equations is to analytically solve for the net heat fluxes by inverting the **A** matrix and eliminate them from the set of equations

$$\bar{\mathbf{q}}_{\mathbf{net}} = \left[\mathbf{A}(\bar{\mathbf{T}})\right]^{-1}\left[\mathbf{B}^{*}\left(\bar{\mathbf{T}}^{3}\right)\right]\mathbf{T}$$

Substituting into the conduction equation, yields a strongly nonlinear set of 'N' equations

$$\left[\hat{\mathbf{K}}(\mathbf{T}) + \mathbf{E}\left[\mathbf{A}(\bar{\mathbf{T}})\right]^{-1}\left[\mathbf{B}^{*}\left(\bar{\mathbf{T}}^{3}\right)\right]\right]\mathbf{T} = \hat{\mathbf{F}}(\mathbf{T})$$

where $\mathbf{B}^{*}$ includes the mapping of surface temperatures to nodal temperatures.

## “Fully Coupled” or “Implicit” Formulation (4)

In a general application, this approach requires inverting the coefficient matrix in the net-radiation equations. If the emissivity of any of the surfaces is temperature dependent, then we would have to invert this matrix at each iteration. This would be a serious disadvantage as the problem size increases.

An advantage of this approach is that are still only *N* unknown nodal temperatures in the system of equations. The fourth-order temperature dependence is tightly coupled in the nonlinear set of algebraic equations.

Rather than eliminating the net radiative heat fluxes from the set of equations, another approach is to keep both the nodal temperatures and the net radiative heat fluxes as unknowns in one set of equations. As we will see, this significantly increases the size of the set of algebraic equations.

## “Fully Coupled” or “Implicit” Formulation (5)

Keeping both the nodal temperatures and the net radiative heat fluxes as unknowns, the combined set of algebraic equations can be written as

$$\begin{bmatrix} \hat{\mathbf{K}} & \mathbf{E} \\ \mathbf{B}^{*}\overline{\mathbf{T}}^{3} & \mathbf{A} \end{bmatrix} \begin{Bmatrix} \mathbf{T} \\ \overline{\mathbf{q}} \end{Bmatrix} = \begin{Bmatrix} \hat{\mathbf{F}} \\ \mathbf{0} \end{Bmatrix}$$

Comments regarding this system of equations:

- Size of the coefficient matrix is increased ($N + N_{surf}$) relative to those for the staggered approach; matrix is asymmetric and nonlinear
- Strong coupling between systems improves convergence
- Amenable to iterative methods (Picard or Newton)
- Memory requirements are a serious limitation



Full matrix is larger, but provides strong coupling of nonlinear effects.

Note the sizes of the partitions in the coefficient matrix:

**K** is $N_{nodes}$ x $N_{nodes}$ (# of nodal temperatures x # of nodal temperatures)
**E** is $N_{nodes}$ x $N_{surf}$ (# of nodal temperatures x # of enclosure surfaces)

**B*** is $N_{surf}$ x $N_{nodes}$ (# of surfaces x # of nodes)
**A** is $N_{surf}$ x $N_{surf}$ (# of surfaces x # of surfaces)

## “Fully Coupled” or “Implicit” Formulation (6)

Solving this set using a Newton’s method, we rewrite the system in terms of the residuals

$$\begin{Bmatrix} \mathbf{R_T} \\ \mathbf{R_q} \end{Bmatrix} = \begin{bmatrix} \hat{\mathbf{K}} & \mathbf{E} \\ \mathbf{B}^* \overline{\mathbf{T}}^3 & \mathbf{A} \end{bmatrix} \begin{Bmatrix} \mathbf{T} \\ \overline{\mathbf{q}} \end{Bmatrix} - \begin{Bmatrix} \hat{\mathbf{F}} \\ \mathbf{0} \end{Bmatrix}$$

Now, applying Newton’s method, the system becomes

$$\begin{bmatrix} \mathbf{J}_{TT} & \mathbf{J}_{Tq} \\ \mathbf{J}_{qT} & \mathbf{J}_{qq} \end{bmatrix}^n \begin{Bmatrix} \Delta\mathbf{T} \\ \Delta\overline{\mathbf{q}} \end{Bmatrix}^{n+1} = \begin{bmatrix} \dfrac{\partial \mathbf{R}_T}{\partial \mathbf{T}} & \dfrac{\partial \mathbf{R}_T}{\partial \overline{\mathbf{q}}} \\ \dfrac{\partial \mathbf{R}_q}{\partial \mathbf{T}} & \dfrac{\partial \mathbf{R}_q}{\partial \overline{\mathbf{q}}} \end{bmatrix}^n \begin{Bmatrix} \Delta\mathbf{T} \\ \Delta\overline{\mathbf{q}} \end{Bmatrix}^{n+1} = \begin{Bmatrix} \mathbf{R}_T \\ \mathbf{R}_q \end{Bmatrix}^n$$

where ‘*n*’ and ‘*n+1*’ superscripts refer to the subsequent iterations



In general, the resulting Jacobian matrix is densely populated and asymmetric. The Jacobian is constructed to include the dominant nonlinear coupling terms with temperature dependent material properties and boundary conditions neglected. The matrix problem can be solved with an iterative method appropriate for asymmetric matrices; such as GMRES.

# Iteration Process for Fully Coupled Formulation

At the beginning of this time step, at time $t^n$ with time step $\Delta t_n$

1. Predict temperature $T_p^{n+1}$ for the new time plane, $t^{n+1}$, using Adams-Bashforth explicit predictor
2. Enter iterative loop to compute end of time step fields; $T^{n+1}$ and $\overline{q}^{n+1}$

   A. Update the both the corrected enclosure radiative heat flux field, $\overline{q}_c^{n+1}$, and the corrected temperature field, $T_c^{n+1}$, using the coupled set of conduction and net-radiation equations
   - For the first iteration, use the predicted temperatures, $T_p^{n+1}$
   - For subsequent iterations, use the corrected temperatures, $T_c^{n+1}$

   B. Relax corrected temperature field using corrected and previous values
   - Check for iteration convergence at this time step
   - If not converged, repeat the loop by returning to Step A

   C. Once converged, update the variables and continue to Step 3
   - Set $T^{n+1} = T_c^{n+1}$ and $\overline{q}^{n+1} = \overline{q}_c^{n+1}$

3. Check for steady state convergence using norm on temperature field
4. Estimate a new time step by comparing the predicted and corrected temperature fields

End of time step at time, $t^{n+1}$

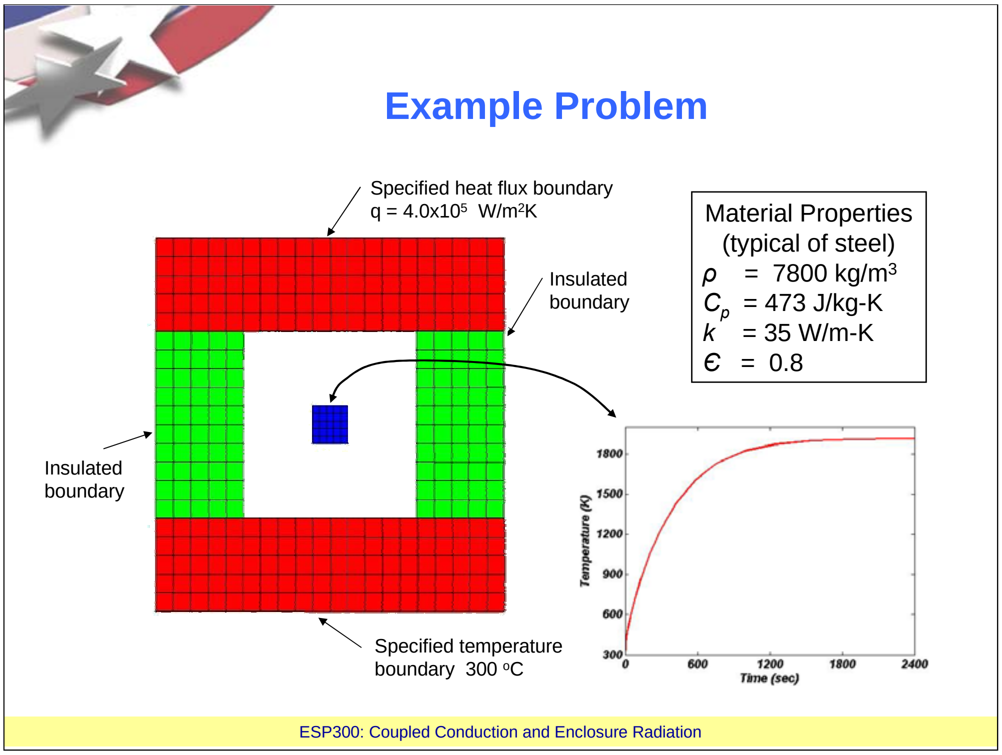
## Comments on the “Fully Coupled” or “Implicit” Formulation

Advantages:

- Strong coupling between radiation and conduction
- Improved stability and rates of convergence
- Amenable to steady problems
- Coupling effective for “disconnected” domains

Disadvantages:

- Coefficient matrix is larger with staggered approach, ‘$N + N_{surf}$’
- Coefficient is non-symmetric and fully-populated
- Memory requirements increase significantly with problem size




Note that steady-state is reached in approximately 1200 seconds. We would like the automatic time step algorithm to increase the time step to increase as we approach steady conditions.

What happens to the set of heat conduction equations for the blue domain as we approach steady-state?

## Objectives of this Example Problem

With this problem, we will investigate the staggered and the fully-coupled approaches. We will consider:

- Timestep behavior
- Number of iterations required
- Single verses multiple corrector steps
- Memory required
- CPU time required

We will end with some general observations regarding the two approaches.

## Advancing a time step involves a predictor, a corrector, and an adaptive timestep algorithm

At the beginning of this time step

1. Predict temperature for the new time plane using Adams-Bashforth explicit predictor
2. Procedure to compute temperature field at the new time plane
   - Update temperature field using either the “staggered” formulation or the “fully coupled” formulation
   - Check for convergence of new temperature field
3. Estimate a new time step by comparing the predicted and corrected temperature fields

End of time step



We have discussed this process for each of the formulations we are considering. It is included here, just as a reminder of the overall process.

## Overview of "Staggered" and "Fully Coupled" Formulations

Staggered or cyclic formulation

- Smaller & symmetric conduction matrix
- Smaller memory requirements
- Less stable, poorer convergence rates

Fully coupled formulation (w/Newton's method)

- Larger & asymmetric coefficient matrix
- Much larger memory requirements
- Better stability, better convergence rates

**Better choice depends on the specific problem**

## Review of Adaptive Time Step Algorithm

Adaptive time step algorithm maintains solution accuracy within user-defined tolerance. Timesteps are based on difference between predicted and corrected solution fields at time ‘n+1’

$$\Delta t_{n+1} = \Delta t_n \left( b \cdot \frac{\varepsilon^t}{d_{n+1}^{\Delta t}} \right)^m$$

“integration tolerance”

$$m = 1/3 \qquad b = 3\left(1 + \Delta t_{n-1}/\Delta t_n\right)$$

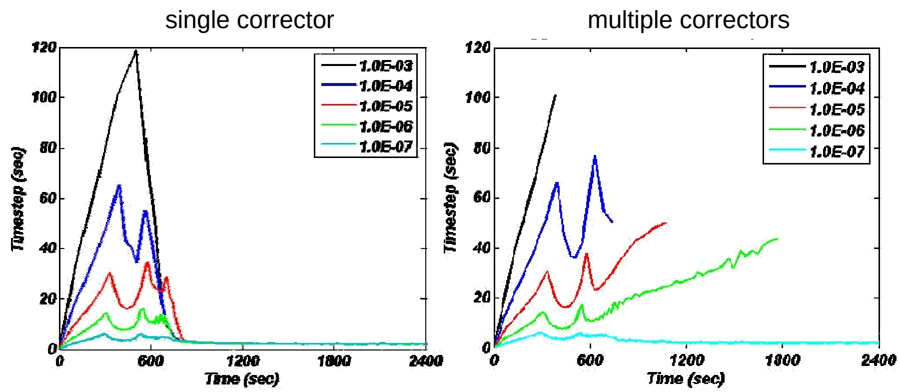
$$d_{n+1}^{\Delta t} = \left[ \frac{1}{N_{nodes} \cdot T_{\max}^2} \sum_{i=1}^{N_{nodes}} \left( T_i^{n+1} - T_{i,p}^{n+1} \right)^2 \right]^{1/2}$$



We will vary the “integration tolerance” to automatically adjust the time step to maintain a desired integration accuracy. Varying this parameter adjusts the time step to control the difference between the ‘predicted’ and ‘corrected’ temperatures.

## Staggered Formulation

Multiple correctors suffered from convergence issues

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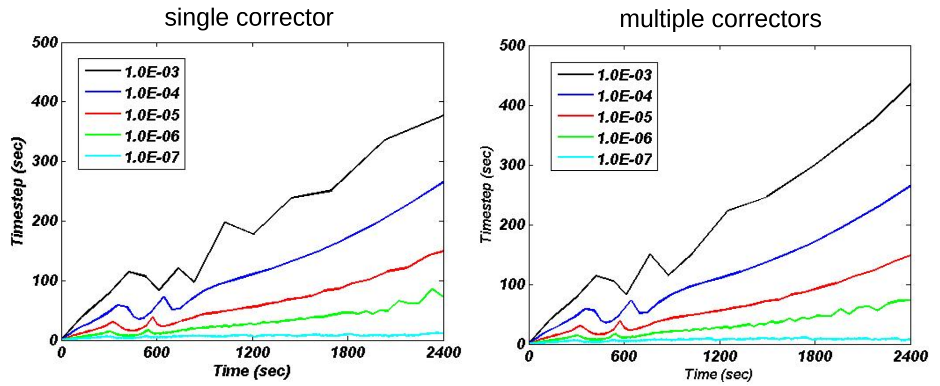
With a single corrector, the auto time step algorithm reduces the time step to maintain the integration tolerance. The solution continues to advance because the inner loop is not tightly converged with a single corrector.

With multiple correctors, the inner loop attempts to converge the conduction equations, but the matrix becomes singular as the problem approaches steady conditions and eventually the coefficient matrix essentially becomes singular. At that time, the solution will not converge. This behavior occurs because the smaller square is basically "disconnected."

## Coupled Formulation Using Newton's Method

In general, a single corrector adequately maintained integration tolerances (exception is 1.0E-03 tolerance)

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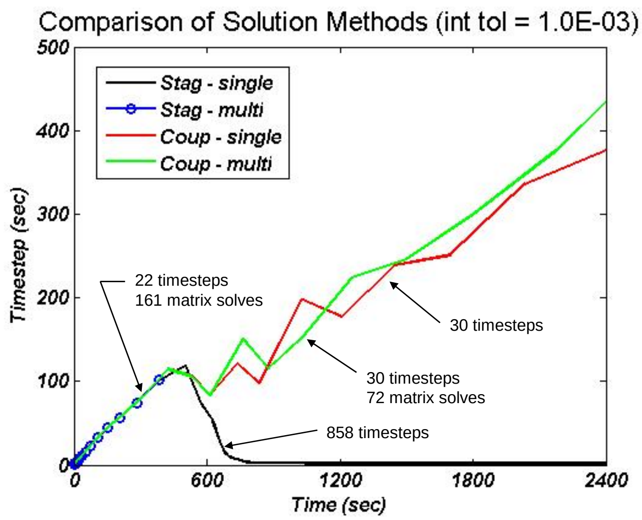
With the fully coupled formulation, the time step grows as the problem becomes more steady. It grows most for lower values of the integration tolerance.

For multiple correctors, the time step behavior is similar to the single corrector because the adaptive time step algorithm is keeping the predicted and corrected temperature fields very close. In general, the multiple corrector converged in two iterations (predictor close to the solution and not terribly nonlinear!).

## Coupled formulation provides much larger timesteps for a given integration tolerance

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As clearly shown here, the coupled formulation outperformed the staggered solution in terms of increasing the time step. We will see on the next slide that the total cpu time is also smaller for the coupled formulation for this problem.

## Summary of Performance Results

| | Staggered – single corrector | | | | | Staggered – multiple correctors | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Integration tolerance (1.0Exx) | | | | | Integration tolerance (1.0Exx) | | | | |
| | -3 | -4 | -5 | -6 | -7 | -3 | -4 | -5 | -6 | -7 |
| steps | 858 | 876 | 895 | 992 | 1176 | 22 | 43 | 89 | 200 | 1172 |
| Mtrx iters | 858 | 876 | 895 | 992 | 1176 | 161 | 330 | 418 | 558 | 1183 |
| Time (sec) | 3.5 | 3.1 | 3.8 | 3.5 | 4.3 | dnc | dnc | dnc | dnc | 4.3 |

| | Coupled – single corrector | | | | | Coupled – multiple correctors | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Integration tolerance (1.0Exx) | | | | | Integration tolerance (1.0Exx) | | | | |
| | -3 | -4 | -5 | -6 | -7 | -3 | -4 | -5 | -6 | -7 |
| steps | 30 | 52 | 99 | 200 | 547 | 30 | 52 | 99 | 200 | 545 |
| Mtrx iters | 30 | 52 | 99 | 200 | 547 | 72 | 108 | 198 | 400 | 550 |
| Time (sec) | 0.5 | 0.7 | 1.2 | 2.1 | 5.7 | 1.0 | 1.3 | 2.3 | 4.3 | 5.6 |



Because the auto time step algorithm limited large changes in the solution, the multiple correctors were really not effective. The initial correctors were reasonably close to the converged solution. The 1.0xe-03 integration tolerance was the minor exception to that observation. Because the integration tolerance was smaller and the time steps larger, on average, slightly more than two correctors were used (72 correctors for 30 time steps).

## For 'large' problems, memory issues may be a limiting factor in many cases

For the example problem used here (2-D problem with 325 elements, 396 nodes, and 60 radiating surfaces)

- For the staggered formulation (396 dof)
  - 3138 non-zero coefficients in the conduction matrix
- For the coupled/Newton formulation (454 dof)
  - 10458 non-zero coefficients in the combined matrix

For a typical practical problem (3-D problem with 431K elements, 89K nodes, and a total of 47K radiating surfaces)

- For the staggered formulation (89K dof)
  - 1.2 million non-zero coefficients in the conduction matrix
- For the coupled/Newton formulation (136K dof)
  - 943 million non-zero coefficients in the combined matrix
  - Number of dof increased by 1.5x, number of coefficients by 786x

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For our example problem, the number of unknowns increased only slightly (396 to 454 ---- by 58 unknowns) and the number of non-zero coefficients increased by a factor of 3 or so (3138 to 10458). The "real-world" problem clearly demonstrates this trend.

## "Upper bound" of memory requirements shows $N^2$ dependence for coupled formulation

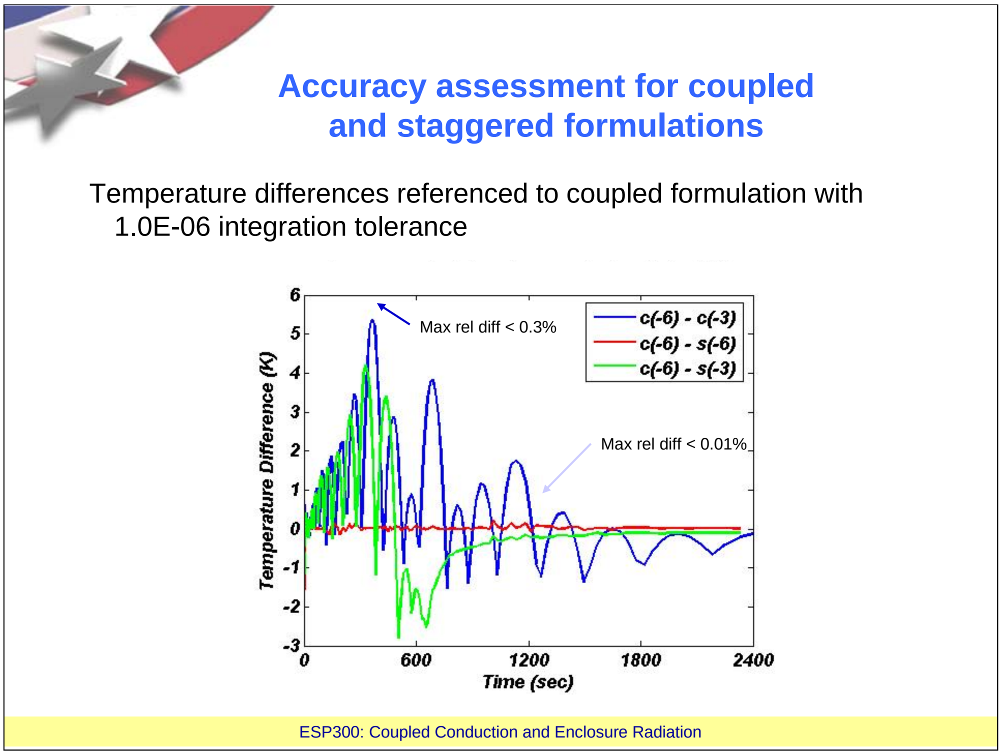
For the coupled formulation, the number of non-zero coefficients representing contributions of enclosure radiation depends on the number of radiating surfaces

The upper-bound in the ***increase*** in the number of non-zero coefficients due to enclosure radiation is

$$\Delta_{non-zeros} \leq \sum_{i}^{N_{enclosures}} 2N_i^2 + N_i \times npf_i$$

where

- $N_i$ is the number of radiating surfaces and
- $npf_i$ is the number of nodes per radiating surface (face)




Here we are considering differences in the solutions as a function of integration tolerance. The largest differences are less than 0.3%. This type of information is useful in assessing acceptable simulation parameters from a practical perspective. Recall that the solution is being compared on the face of the smaller block. Other locations may behave differently.

## Comments for the Example Problem

So, what's the "Best" approach??

Turns out "BEST" requires a problem dependent answer!

Consideration must be given to

- Accuracy requirements
- Performance requirements
- Problem size and memory limitations

Overview of these characteristics has been presented

- Benefits/cost of coupled formulation with Newton's method has been demonstrated

## Summary of Today's Class

Reviewed equations describing conduction and enclosure radiation

- Reviewed assumptions employed in each system of equations

Discussed in detail

- Variables common to both sets of equations that provide the coupling
- Inconsistencies and incompatibilities between these assumptions
- Formulations used to resolve or accommodate these inconsistencies
- Staggered and fully coupled formulations for solving these equations
- Characteristics, advantages, and disadvantages to each formulation

With our example problem, we

- Demonstrated some of the characteristics of coupling options
- Considered behavior transient response progressing to steady state

## Did We Answer our Questions for Coupled Conduction & Enclosure Radiation?

- What are the equations describing heat conduction and enclosure radiation?
- Which variables are common to and passed between the two sets or equations?
- Are there inconsistencies in the assumptions used in each set? If so, what are the inconsistencies and how do we address them?
- What assumptions are made to couple these two sets of equations?
- What approaches do we use to solve this coupled set?
- What are the advantages and disadvantages to the different approaches?