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Time-domain boundary conditions in atmospheric acoustics

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Outline

1. Introduction.

2. Time-domain boundary conditions (TDBC) for the modified Zwikker-Kosten (ZK) impedance model.

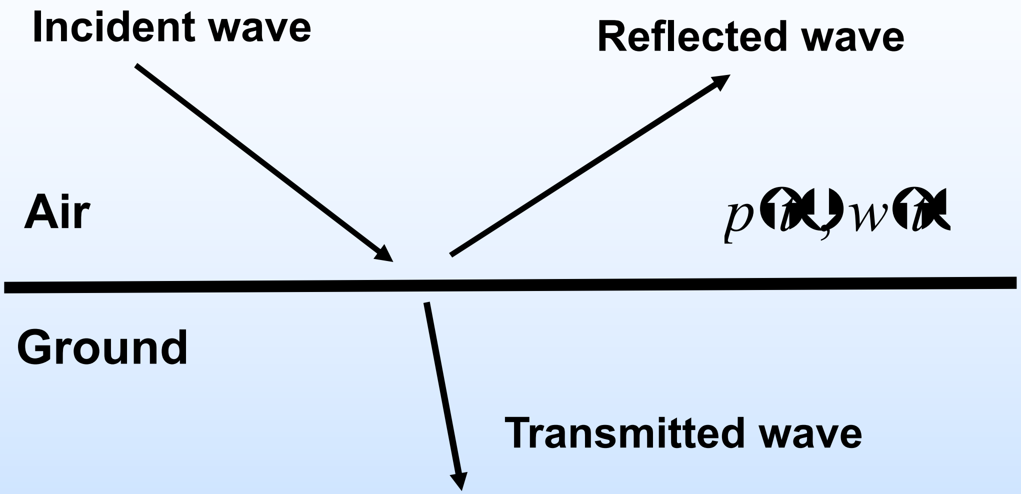
3. Causal TDBC for any impedance model.

4. Conclusions.

1. Introduction

- FDTD simulations of outdoor sound propagation is a very promising technique.
- A challenging problem in FDTD is formulation of TDBC at the ground surface.
- The goal of this paper: Development of numerically efficient and causal TDBC.

2. TDBC for modified ZK model



BC for a locally reacting surface:

$$P(\omega) = Z(\omega) W(\omega)$$

Fourier transform:

$$z(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} Z(\omega)\, d\omega$$

TDBC:

$$p(t) = \int_{-\infty}^{\infty} w(t - t') z(t')\, dt'$$

- Convolution complicates TDBC.
- Some TDBC are noncausal (e.g. Delany and Bazley model).
- The goal of the paper is to make TDBC numerically efficient and causal.

2. TDBC for modified ZK model

ZK impedance model: $Z(\omega) = Z_\infty \sqrt{\frac{1 - i\omega\tau}{-i\omega\tau}}$

Two parameters: Z_∞ and τ

Modified ZK impedance model:

$$Z_\infty = \frac{\rho c q}{\Omega}, \qquad \tau = \frac{\rho q^2 \,[\text{illegible}]}{\sigma_0 \Omega}$$

Here, tortuosity q, porosity Ω, flow resistivity σ_0, etc.

The modified ZK impedance model is almost indistinguishable from more realistic models characterized by more parameters. This comes with a price: Inside the ground the modified ZK model does not work.

2. TDBC for modified ZK model

Causal TDBC:

$$p(t) = Z_\infty\left[w(t) + \frac{1}{\tau}\int_0^t w(t-t')\, f(t'/\tau)\, dt' \right]$$

Here, the response function $f(t) = \frac{\exp(-t/2)}{2}\left[I_1(t) + I_0(t) \right]$

Slowly decaying function: $f(t) \sim t^{-1/2}$ for $t \gg 1$

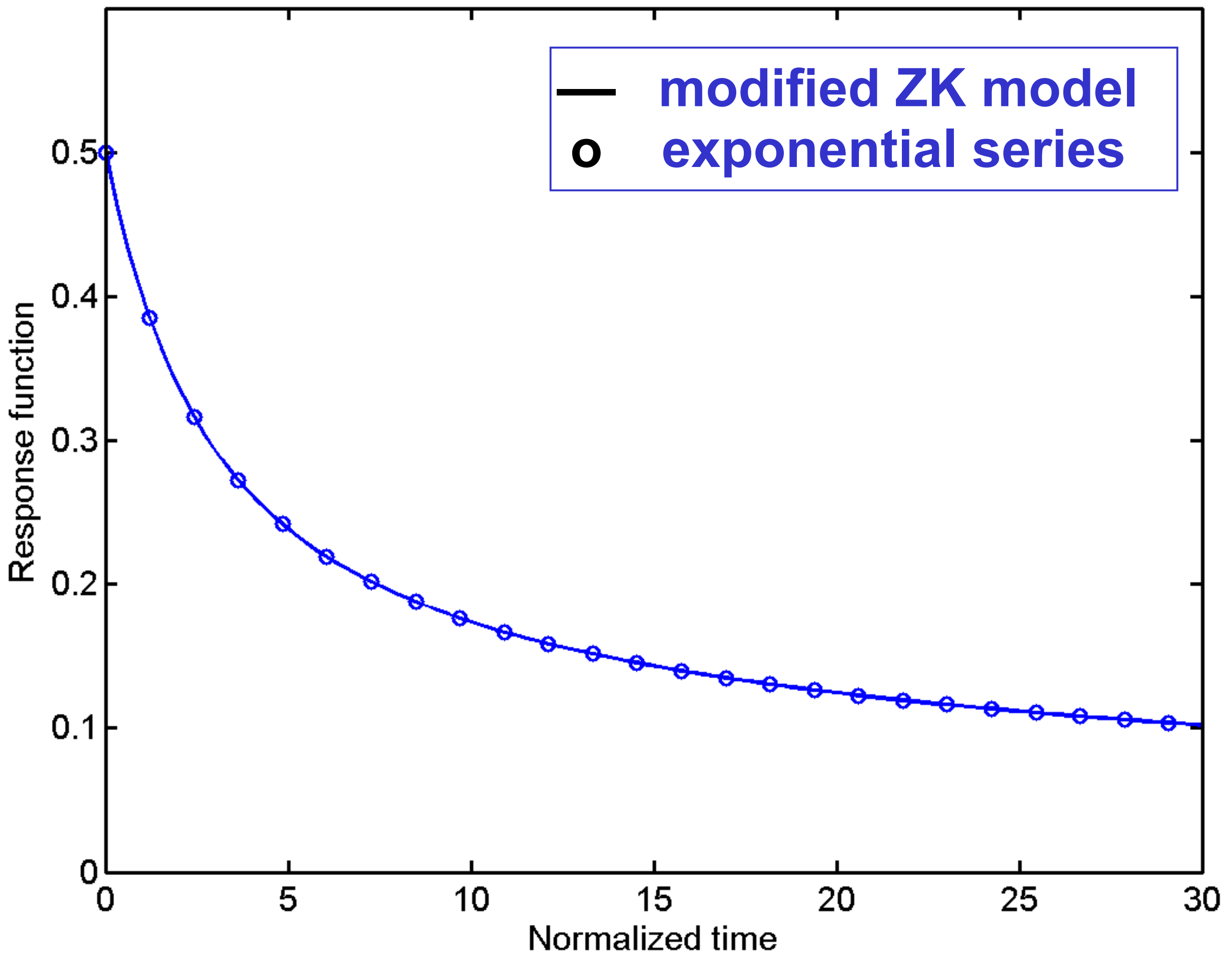
Hundreds of time steps need to be retained.

Approximation of f(t) with an exponential series:

$$f(t) \approx \sum_{k=1}^{K} a_k e^{-\lambda_k t}$$

k	a_k, ZK	λ_k, ZK
1	−0.55590	0.57559
2	0.13751	0.01157
3	0.48279	0.63507
4	0.14898	0.11357
5	0.28696	0.41958
6	−0.00037	2.63762

2. TDBC for modified ZK model



2. TDBC for modified ZK model

TDBC: $$p(t) = Z_\infty \left[w(t) + \sum_{k=1}^{K} \int_0^\infty w(t - t') A_k e^{-\lambda_k t'/\tau} \frac{dt'}{\tau} \right]$$

Discrete time steps $t = t_n = n\Delta t, \quad n = 1, 2, \ldots$.

The auxiliary variable ψ_k^n is the value of the integral.

$$\psi_k^n = e^{-\lambda_k \Delta t/\tau} \psi_k^{n-1} + A_k w(t_n) \Delta t/\tau$$

Final TDBC:

$$w(t_n) = \frac{1}{1 + \sum_{k=1}^{K} A_k \Delta \bar{t}} \left[\frac{p(t_n)}{Z_\infty} - \sum_{k=1}^{K} e^{-\lambda_k \Delta \bar{t}} \psi_k^{n-1} \right].$$

Only one time step need to be retained where 6 values of ψ_k^n are updated.

2. TDBC for modified ZK model

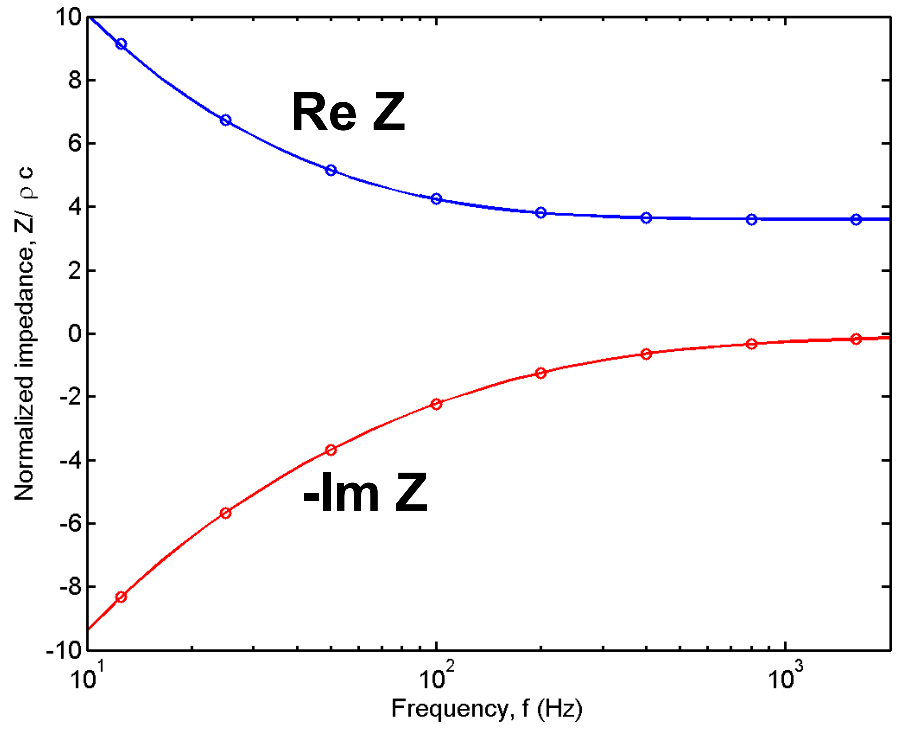
Numerical experiment:

Sound source

Air

TDBC

Ground



— modified ZK model
o numerical calculations

The porous material parameters:

$\sigma_0 = 10^4$ Pa s m^{-2}, $\Omega = 0.5$, $q = 1.8$.

The source frequency:

12.5 Hz $\le f = \omega/2\pi \le$ 8000 Hz.

$0.086 \le \omega\Delta t \le 11$

3. Causal and numerically efficient TDBC for any impedance

The main idea: Using a Pade approximation of the impedance in the frequency domain and fractional derivatives, a causal TDBC is derived.

First example: The modified ZK impedance model.

Let $x = \sqrt{-i\omega\tau}$. Then using a Pade approximation:

$$\frac{1}{Z} = \frac{x}{Z_\infty}\frac{1}{\sqrt{1+x^2}} \approx \frac{x}{Z_\infty}\frac{1+a_2x+a_3x^2}{1+b_1x+b_2x^2+b_3x^3}.$$

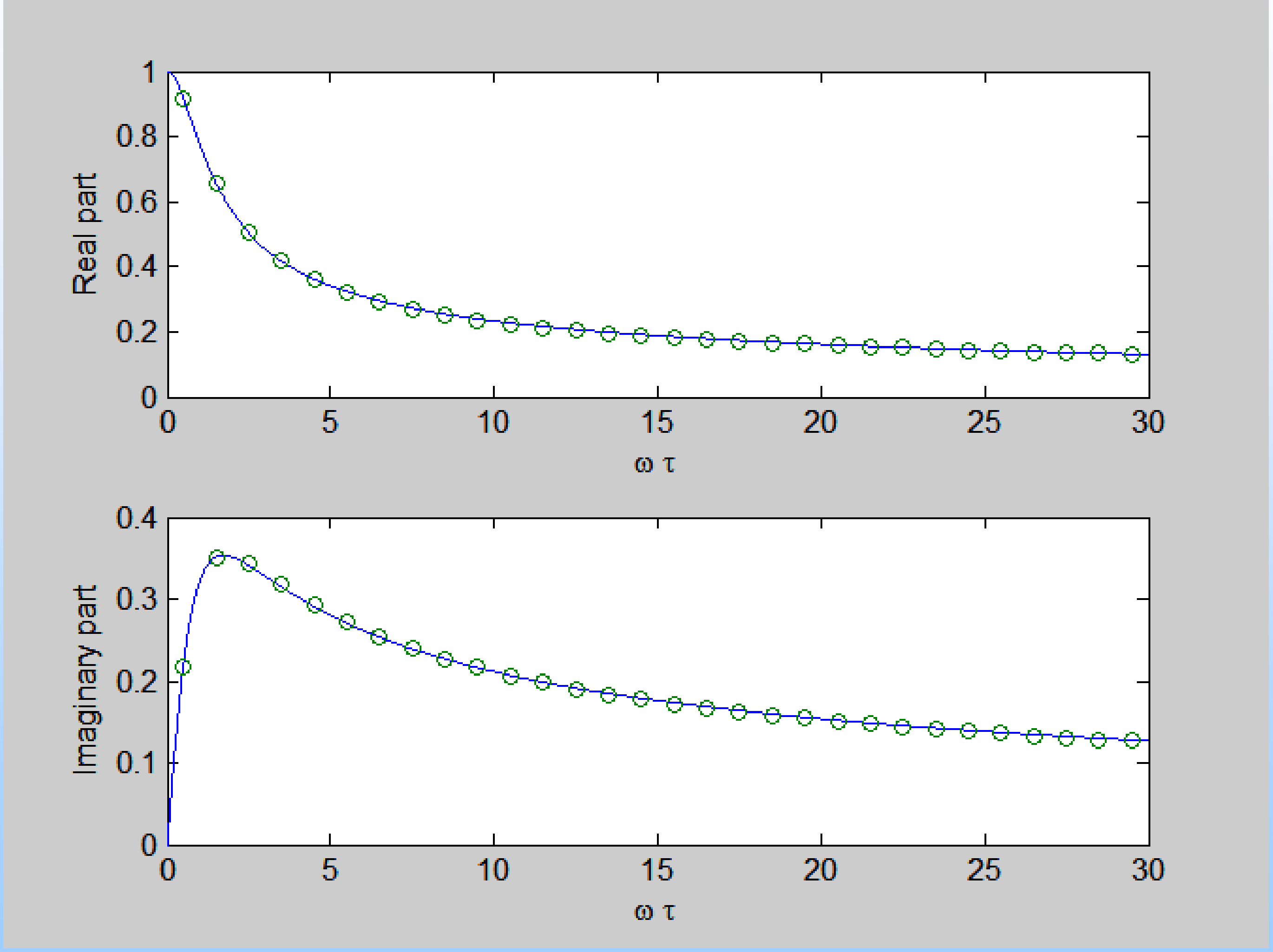
Coefficients:

$$a_2 = b_1 = 0.992, \quad a_3 = b_3 = 0.566, \; b_2 = 1.066.$$

3. Causal TDBC for any impedance

— $\frac{1}{\sqrt{1 \square x^{2}}}$

o Pade approximation



3. Causal TDBC for any impedance

BC in frequency domain: $P(\omega) = Z(\omega) W(\omega)$

Causal TDBC:

$$\frac{1}{Z_\infty}\left[1 + a_3 \tau \frac{d}{dt} + a_2 \tau^{1/2} \frac{d}{dt} D^{-1/2}\right] p(t)$$
$$= \left[b_1 + b_3 \tau \frac{d}{dt} + \left(\tau^{-1/2} + b_2 \tau^{1/2} \frac{d}{dt}\right) D^{-1/2}\right] w(t)$$

Fractional derivative: $D^{\nu} p(t) = \frac{1}{\Gamma(-\nu)} \int_{-\infty}^{t} \frac{p(t')}{(t-t')^{1+\nu}} dt'$

Grunwald formula: $D^{-1/2} p(t) = \lim_{\Delta t \to 0} (\Delta t)^{1/2} \sum_{m=0}^{\infty} \frac{\Gamma(m+1/2)}{\sqrt{\pi}\,\Gamma(m+1)} p(t - m\Delta t)$

For $m \gg 1$, $\Gamma(m+1/2)/\Gamma(m+1) \approx m^{-1/2}$ (parallels to $f(t) \approx m^{-1/2}$).

3. Causal TDBC for any impedance

Liebler et al. (2004):

$$\frac{\zeta(m+1/2)}{\zeta(m+1)} \approx \sum_{k=1}^{K} b_k e^{-\alpha_k m}.$$

Following the same procedure as earlier we define the auxiliary variable:

$$\psi_k^n = \sum_{m=0}^{\infty} b_k e^{-\alpha_k m} w(n\Delta t - m\Delta t)\sqrt{\Delta t/\delta}$$

It can be shown that $\psi_k^n = b_k w(n\Delta t)\sqrt{\Delta t/\delta} + e^{-\alpha_k}\psi_k^{n-1}.$

TDBC: $w(t_n) = F(p(t_n), \psi_k^{n-1})$

Thus, only K auxiliary functions ψ_k^n **must be stored and updated at every time step.**

Second example: The Attenborough impedance model.

Let $x = \sqrt{-i\omega\tau}$. Then using a Pade approximation:

Chandler-Wilde, Horoshenkov, JASA (1995):

$$\frac{1}{Z} = \frac{x}{Z_0}\,\frac{1 + a_2 x + a_3 x^2 + a_4 x^3}{1 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4}$$

Causal TDBC:

$$\frac{1}{Z_0}\left[1 + a_3 \tau \frac{d}{dt} + \left(a_2 \tau^{1/2}\frac{d}{dt} + a_4 \tau^{3/2}\frac{d^2}{dt^2}\right) D^{-1/2}\right] p(t)$$
$$= \left[b_1 + b_3 \tau \frac{d}{dt} + \left(\tau^{-1/2} + b_2 \tau^{1/2}\frac{d}{dt} + b_4 \tau^{3/2}\frac{d^2}{dt^2}\right) D^{-1/2}\right] w(t)$$

4. Conclusions

1. For the modified ZK impedance model, a numerically efficient algorithm for implementing TDBC is developed.

2. For any impedance model, using a Pade approximation of the impedance in the frequency domain, a causal TDBC can be derived. An efficient algorithm for numerical implementation of TDBC is then developed.