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# Domain Decomposition Solvers

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# Overview

- **Introduction**
  - why do we need iterative solvers?
  - basic domain decomposition concepts
  - types of domain decomposition preconditioners
- **Challenges**
  - constraint equations
  - very large local or global problems
  - poor convergence
- **Software Libraries**
- **References**

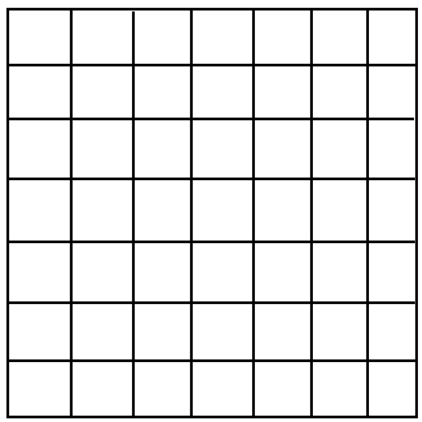
# Why do we need iterative solvers?

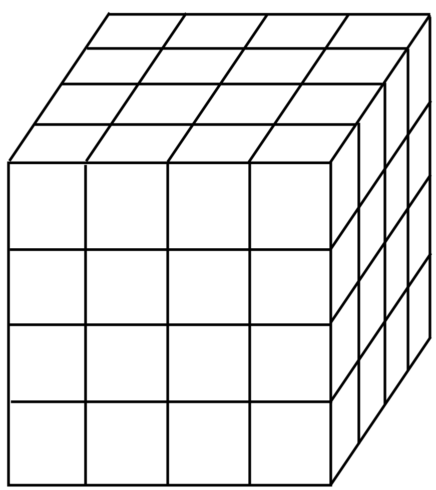
- **Direct solvers usually more robust**
  - great for solving many 2D and smaller 3D problems
  - very few, if any, knobs to adjust
  - simple black-box approach
  - requires only coefficient matrix and force vector

- **But, …**
  - memory and flops grow superlinearly with problem size
  - parallel direct solvers limited by problem size too
  - parallel direct solver speedups only possible with limited number of processors

# Direct solver complexity

- Problem: 2D or 3D finite element model with n unknowns on square or cube domain





**2D**

flops: $n^{3/2}$

memory: nlogn

**3D**

flops: $n^2$

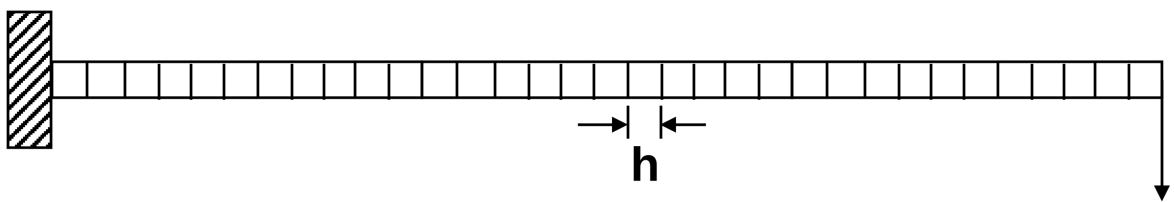
memory: $n^{4/3}$

} best possible growth

# Direct solver alternatives

- **How about conjugate gradients?**
  - flops per iteration proportional to n
  - memory requirements proportional to n
  - woo hoo, things are looking good

- **But, …**
  - number of iterations grows with problem size
  - Example: cantilevered beam of HEX elements
  - condition number grows as $1/h^2$ ($1/h^4$ for shells)



# Direct Solver Alternatives

- **How to save conjugate gradients?**
  - Instead of solving

$$Ax = b$$

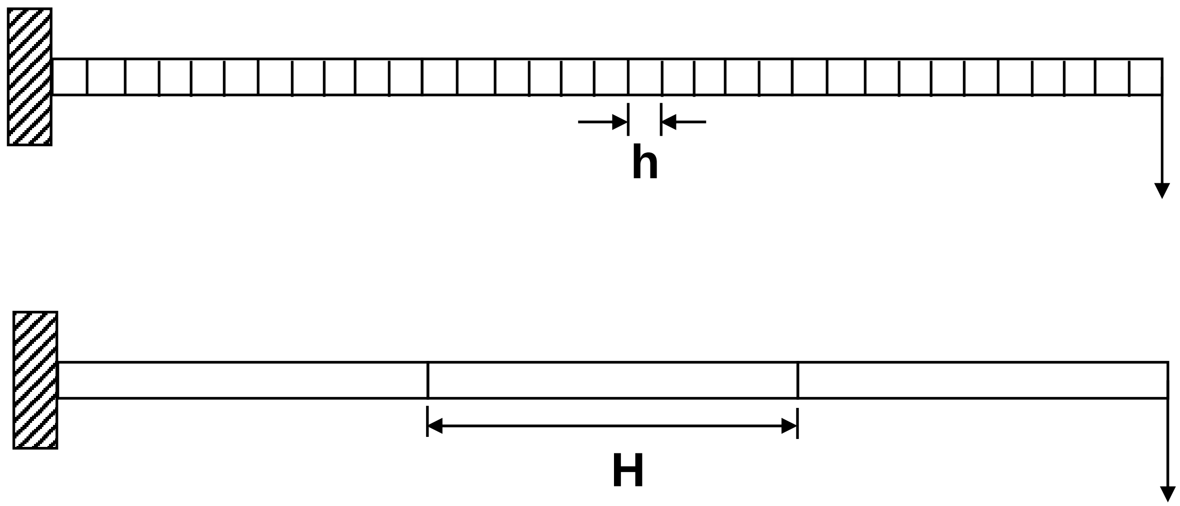
  - solve preconditioned system

preconditioner

$$M^{-1}Ax = M^{-1}b$$

  - extreme eigenvalues of $M^{-1}A$ closer together than those of A itself
  - conjugate gradients converges faster for the preconditioned system

# Preconditioner Idea

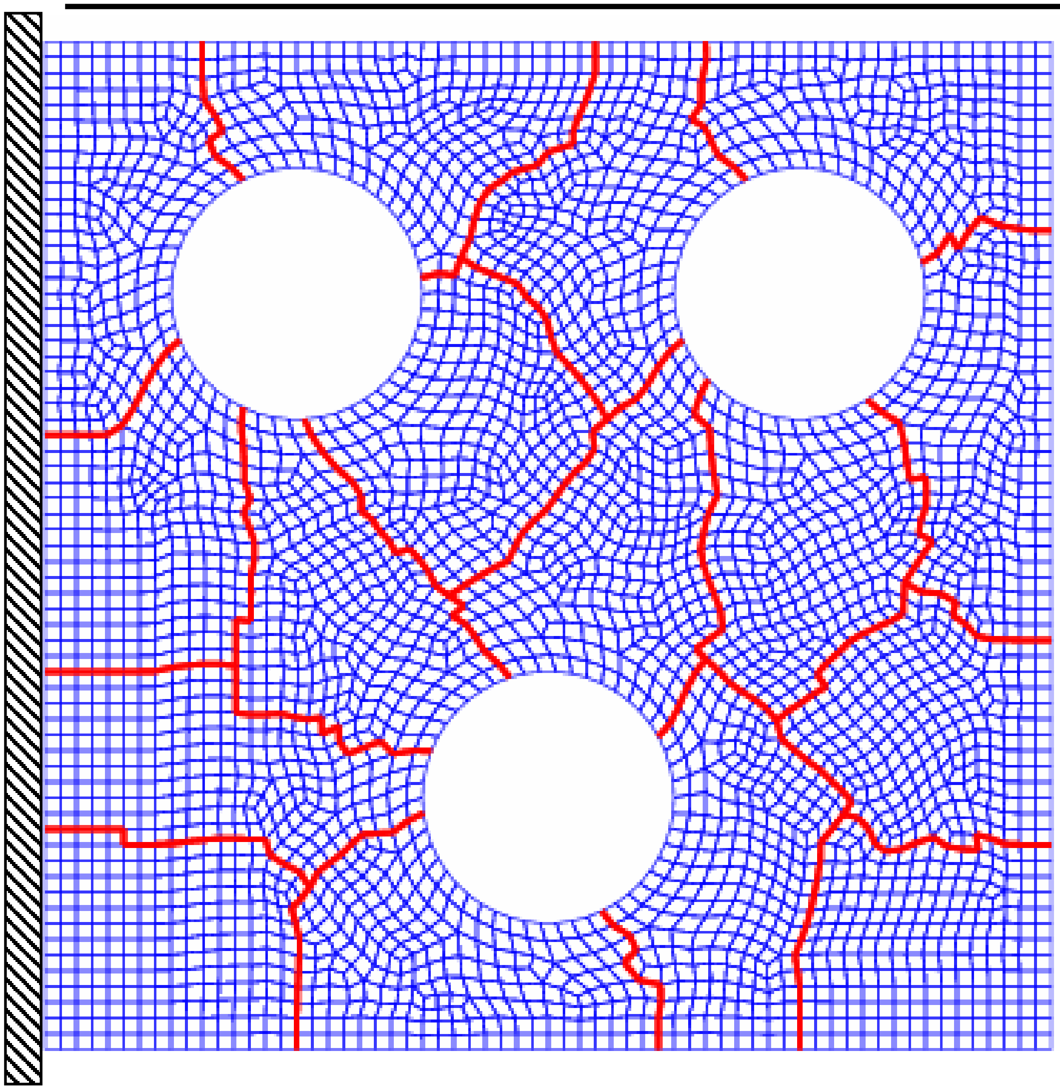



• control modes (constraint mesh) idea in Adagio/JAS

• project original problem to coarse mesh, solve, and project back to fine mesh

• how to construct coarse mesh automatically?

# Basic Domain Decomposition Concepts



- element decomposition into smaller subdomains
- each subdomain often assigned to one processor
- two-level methods have “local” subdomain solves and “global” coarse solve

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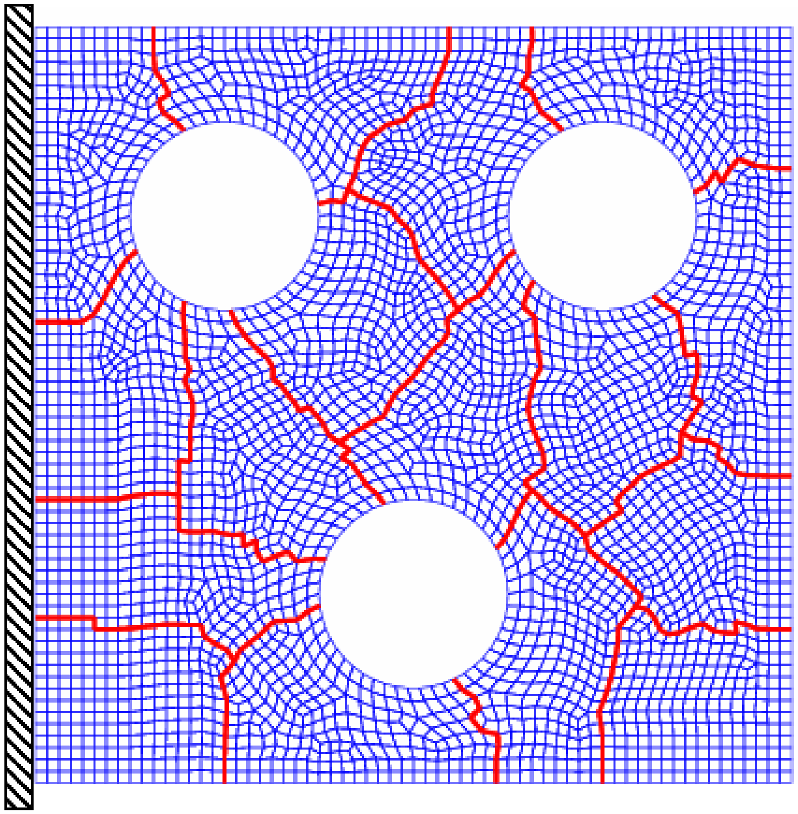
# Domain Decomposition Flavors



interface shown in red

- **Iterative Substructuring:**
  - restrict problem to interface
  - interface unknowns
    - Lagrange mulipliers, FETI
    - displacements, BDD
  - precondition interface problem
    - local “subdomain” solves
    - global “coarse” solve
  - solve using PCG

# Iterative Substructuring

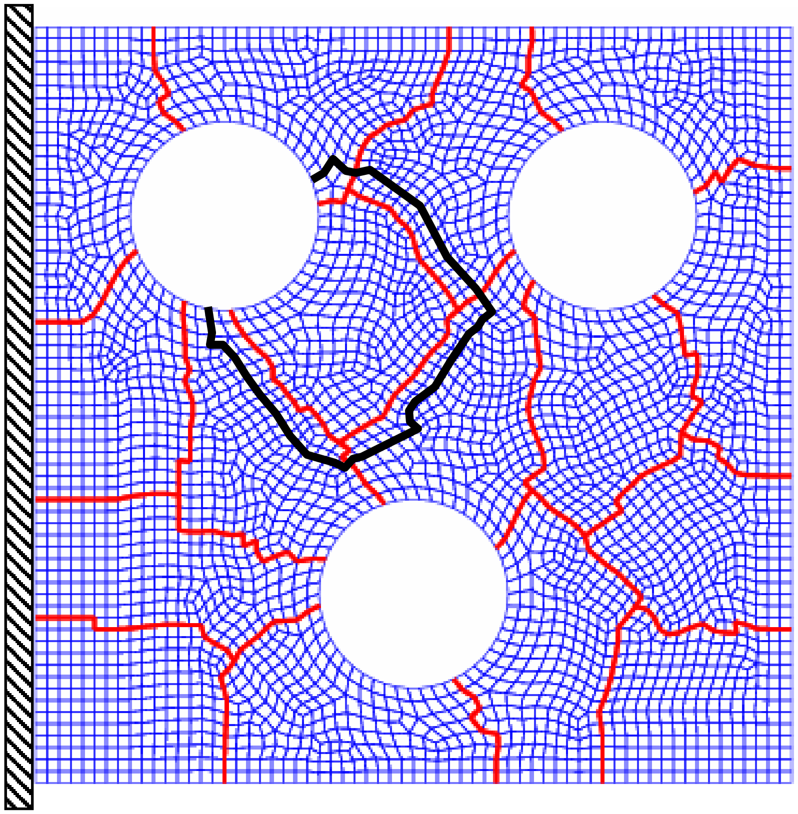
- **Condition number bounded by**

$$cond(M^{-1}A) \le C(1+\log(H/h))^2$$

**where the constant C is independent of the number of subdomains and jumps in material properties across subdomain boundaries**

- **Theory assumes shape regular elements and constant material properties in each subdomain**
- **Until recently, theory also required regular-shaped subdomains**

# Domain Decomposition Flavors



- **Overlapping Schwarz:**
  - extend each subdomain by an integer number of layers of finite elements
  - solve local problems on overlapping subdomains
  - solve a global coarse problem
  - preconditioner combines local and global solutions
  - solve using PCG

# Overlapping Schwarz

- **Condition number bounded by**

$$cond(M^{-1}A) \le C(1 + H/\delta)^{p}(1 + \log(H/h))$$

**where the constant C is independent of the number of subdomains and jumps in material properties across subdomain boundaries (δ is the overlap)**

- **Theory assumes shape regular elements and constant material properties in each subdomain**
- **Until recently, theory also required regular-shaped subdomains**

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# Some Challenges

- **Constraint equations:**

$$\begin{bmatrix} A & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

  - **indefinite saddle-point system**
  - **positive and negative eigenvalues**
  - **constraint elimination approach permits positive definite reformulation**
  - **best suited to "local" constraints, e.g. mesh tying**
  - **non-local RBE3 type constraints require greater care**

# Some More Challenges

- **Very large local or global problems:**
  - **local or global problems often solved with direct solver**
  - **if any problem gets too big, then factorization will not fit into memory**
  - **theory exists for replacing direct solves with the actions of preconditioners**
  - **idea combines best features of domain decomposition and multigrid to obtain linear complexity**

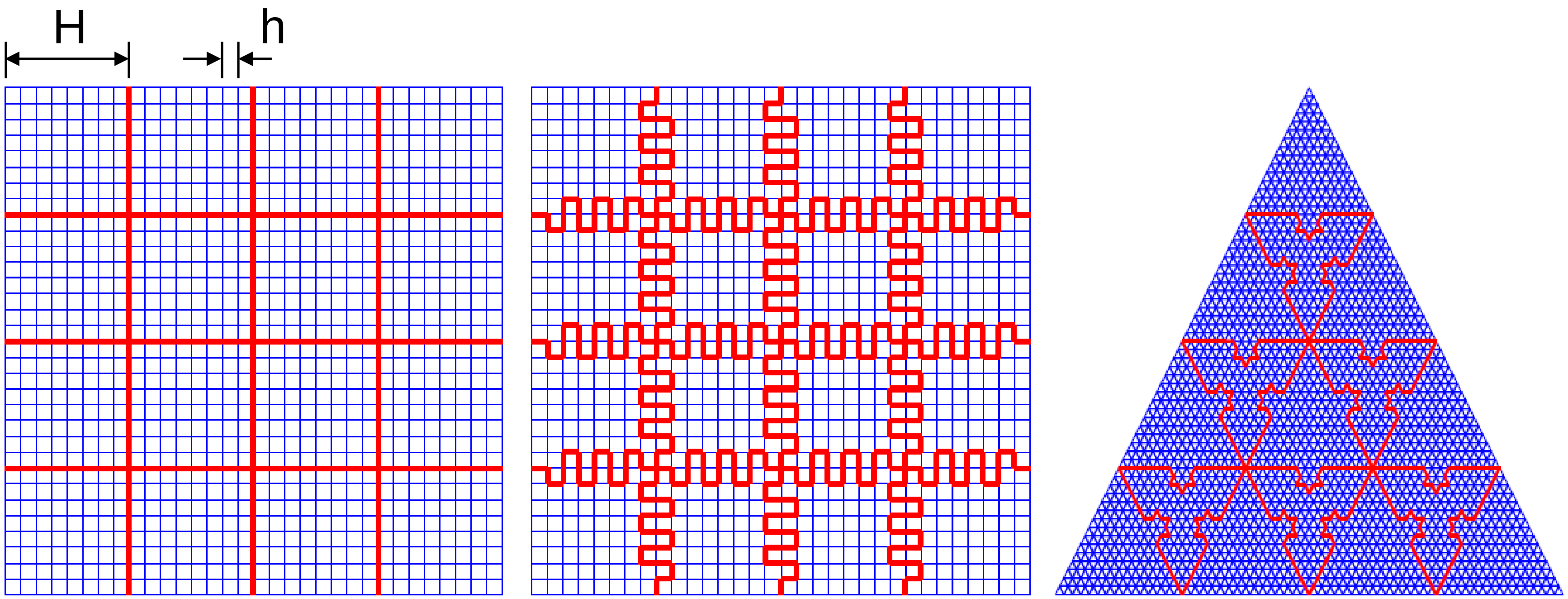
# Some More Challenges

- **Poor convergence:**
  - numbers of iterations may be unacceptably large
  - unanticipated singularities may preclude convergence
  - recent theoretical work will help guide adaptive strategies to ensure acceptable performance

# Recent Advances

- Theory for approximate solves at local and global levels (see references)
  - larger problems can be accommodated using same number of processors
  - keeps coarse problem from becoming bottleneck
- Method and theory for nearly incompressible and incompressible elasticity (Stokes)
- Theory for less-regular subdomains (see references and next page)
- Alternative approach for non-local constraints
  - under evaluation in Salinas

# Less Regular Subdomain Shapes




Type 1 Type 2 Type 3

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# Software Libraries

## CLIP and CLOP:

- **Interface preconditioner/solver CLIP**
  - **implementation of BDDC (a primal counterpart of FETI-DP)**
- **Overlapping Schwarz preconditioner/solver CLOP**
  - **coarse space based on partition of unity**
  - **well suited for problems with constraints**
- **Both CLIP and CLOP are parts of the CLAPS package in Trilinos and are available in Salinas**
- **Neither CLIP nor CLOP are being actively developed**

# Software Libraries

## GDSW: (Generalized, Dryja, Smith, Widlund)

- **Replacement for CLIP and CLOP**
  - **BDDC interface preconditioner still included**
  - **coarse space for overlapping Schwarz based on energy-minimizing harmonic extensions**
  - **includes new methods for accommodating constraints**
- **Currently under evaluation**
  - **Salinas**
  - **Aria and Kachina (incompressible fluid codes)**
- **Currently stand alone library**
  - **may or may not be included in Trilinos**

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# References

## Introductory Texts:


- B. Smith, P. Bjorstad, and W. Gropp, Domain Decomposition: Parallel Multilevel Methods for Elliptic Partial Differential Equations, Cambridge University Press, 1996.
- A. Toselli and O. Widlund, Domain Decomposition Methods: Algorithms and Theory, Springer, 2005.


## Interface Preconditioners (BDDC):


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- C. Dohrmann, "An approximate BDDC preconditioner," Numerical Linear Algebra with Applications, 14, 2007, pp. 149-168.
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## GDSW:

- M. Dryja, B. Smith, and O. Widlund, "Schwarz analysis of iterative substructuring algorithms for elliptic problems in three dimensions," SIAM Journal on Numerical Analysis, 31, 1994, pp. 1662-1694.
- C. Dohrmann, A. Klawonn, and O. Widlund, "A family of energy minimizing coarse spaces for overlapping Schwarz preconditioners, in Proceedings of the 17th International Conference on Domain Decomposition Methods in Science and Engineering, Lecture Notes in Computational Science and Engineering no. 60, 2007.

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# Discussion