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On stabilized finite element for the Stokes problem in the small time step limit

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Method of lines

A popular approach for the discretization of the unsteady incompressible Navier-Stokes equations is

1. Consistently stabilized finite elements for the spatial discretization (Hughes et. al.)
2. Finite difference in time (backward Euler, Crank-Nicolson)

Small time step issues

Recent papers indicate that small time steps (relative to the square of the mesh size) give rise to poor pressure approximations

1. *On inf-sup stabilized finite element methods for transient problems*, Bochev, Gunzburger and Shadid, CMAME 2004

2. *On stabilized finite element methods for transient problems with varying time scales*, Bochev, Gunzburger, and Lehoucq, Proceedings ECOMASS 2004

Summary of new work

- Previous two papers focused on the fully discrete equations
  1. Sufficiency of a lower bound on the time step
- Present work focuses on the semi-discrete equations
  1. Source of the problem is that the semi-discrete pressure operator is unstable
  2. Lower bound on the time step is also necessary
  3. Sufficiently large time step stabilizes the fully discrete pressure equation

Why small time steps?

- Multi-physics computations. For example, in reacting flow simulations (MPSalsa, Shadid et al) chemistry demands small time steps.
- Low order finite difference in time combined with quadratic and higher order finite element methods
- Because the pressure operator instability is intrinsic, incompressible Stokes flow is all we need consider

Overview

- Consistently stabilized methods
- Motivating examples
- Semi-discrete pressure operator and analysis
- fully discrete pressure operator and analysis
- conclusion

Unsteady Stokes

$$
\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} - \triangle \mathbf{u} + \nabla p &= \mathbf{f} && \text{in} \quad \Omega \times (0,T) \\
\nabla \cdot \mathbf{u} &= 0 && \text{in} \quad \Omega \times (0,T) \\
\mathbf{u} &= \mathbf{0} && \text{on} \quad \Gamma \times (0,T) \\
\mathbf{u}\big|_{t=0} &= \mathbf{u}_0 && \text{in} \quad \Omega
\end{aligned}
$$

Semi-discretization

$$\mathbf{u}_h(\mathbf{x},t) = \sum_{j=1}^{N} U_j(t)\boldsymbol{\xi}_j^h(\mathbf{x}), \qquad p_h(\mathbf{x},t) = \sum_{j=1}^{M} P_j(t)\chi_j^h(\mathbf{x}),$$

$$\mathrm{Span}\{\boldsymbol{\xi}_i^h\}_{i=1}^N \equiv \mathcal{V}_h \subset \mathbf{H}_0^1(\Omega), \quad \mathrm{Span}\{\chi_i^h\}_{i=1}^N \equiv \mathcal{P}_h \subset L_0^2(\Omega)$$

$$\mathbf{u}_h(\cdot,t) \in \mathcal{V}_h, \quad p_h(\cdot,t) \in \mathcal{P}_h$$

$$\begin{aligned}
(\dot{\mathbf{u}}_h(\cdot,t),\mathbf{v}_h) + G\big(\{\mathbf{u}_h(\cdot,t),p_h(\cdot,t)\},\{\mathbf{v}_h,q_h\}\big) &= (\mathbf{f}(\cdot,t),\mathbf{v}_h) \\
(\mathbf{u}_h(\cdot,0),\mathbf{v}_h) &= (\mathbf{u}_0,\mathbf{v}_h)
\end{aligned}$$

$$\forall(\mathbf{v}_h,q_h) \in \mathcal{V}_h \times \mathcal{P}_h, \quad \forall t \in (0,T)$$

$$G\big(\{\mathbf{u}_h,p_h\},\{\mathbf{v}_h,q_h\}\big) = (\nabla\mathbf{u}_h,\nabla\mathbf{v}_h) - (p_h,\nabla\cdot\mathbf{v}_h) - (q_h,\nabla\cdot\mathbf{u}_h)$$

Semi-discretization

$$\mathbf{u}_h(\mathbf{x},t) = \sum_{j=1}^{N} U_j(t)\boldsymbol{\xi}_j^h(\mathbf{x}), \qquad p_h(\mathbf{x},t) = \sum_{j=1}^{M} P_j(t)\chi_j^h(\mathbf{x}),$$

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$$\mathbf{u}_h(\cdot,t) \in \mathcal{V}_h, \quad p_h(\cdot,t) \in \mathcal{P}_h$$

$$\begin{aligned}
(\dot{\mathbf{u}}_h(\cdot,t),\mathbf{v}_h) + G\big(\{\mathbf{u}_h(\cdot,t),p_h(\cdot,t)\},\{\mathbf{v}_h,q_h\}\big) &= (\mathbf{f}(\cdot,t),\mathbf{v}_h)\\
(\mathbf{u}_h(\cdot,0),\mathbf{v}_h) &= (\mathbf{u}_0,\mathbf{v}_h)
\end{aligned}$$

$$\forall(\mathbf{v}_h,q_h) \in \mathcal{V}_h \times \mathcal{P}_h, \quad \forall t \in (0,T)$$

$$G\big(\{\mathbf{u}_h,p_h\},\{\mathbf{v}_h,q_h\}\big) = (\nabla\mathbf{u}_h,\nabla\mathbf{v}_h) - (p_h,\nabla\cdot\mathbf{v}_h) - (q_h,\nabla\cdot\mathbf{u}_h)$$

Index 2 differential algebraic equation

$$\begin{pmatrix} \mathbb{M}\dot{U} \\ 0 \end{pmatrix} + \begin{pmatrix} \mathbb{A} & \mathbb{B}^T \\ \mathbb{B} & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}$$

$$\mathbb{A}_{ij} = (\nabla \boldsymbol{\xi}_i^h, \nabla \boldsymbol{\xi}_j^h), \quad \mathbb{B}_{ij} = -(\chi_i^h, \nabla \cdot \boldsymbol{\xi}_j^h)$$

$$\mathbb{M}_{ij} = (\boldsymbol{\xi}_i^h, \boldsymbol{\xi}_j^h), \quad F_i = (\mathbf{f}, \boldsymbol{\xi}_i^h)$$

inf-sup compatibility

$$\inf_{q_h \in \mathcal{P}_h,\, q_h \neq 0} \; \sup_{\mathbf{v}_h \in \mathcal{V}_h,\, \mathbf{v}_h \neq \mathbf{0}} \frac{(q_h, \nabla \cdot \mathbf{v}^h)}{\|\mathbf{v}_h\|_1 \|q_h\|_0} = \kappa_h \geq \kappa_h^{\min} > 0\,,$$

inf-sup compatibility

$$\inf_{q_h \in \mathcal{P}_h,\, q_h \neq 0} \sup_{\mathbf{v}_h \in \mathcal{V}_h,\, \mathbf{v}_h \neq \mathbf{0}} \frac{(q_h, \nabla \cdot \mathbf{v}^h)}{\|\mathbf{v}_h\|_1 \|q_h\|_0} = \kappa_h \geq \kappa_h^{\min} > 0\,,$$

$$\kappa_h^2 = \min_Z \frac{Z^T \mathbb{B} \mathbb{A}^{-1} \mathbb{B}^T Z}{Z^T \mathbb{M}_p Z}, \quad (\mathbb{M}_p)_{ij} = (\boldsymbol{\chi}_i^h, \boldsymbol{\chi}_j^h)$$

inf-sup compatibility

$$\inf_{q_h \in \mathcal{P}_h,\, q_h \neq 0} \; \sup_{\mathbf{v}_h \in \mathcal{V}_h,\, \mathbf{v}_h \neq \mathbf{0}} \frac{(q_h, \nabla \cdot \mathbf{v}^h)}{\|\mathbf{v}_h\|_1 \|q_h\|_0} = \kappa_h \geq \kappa_h^{\min} > 0\,,$$

$$\kappa_h^2 = \min_Z \frac{Z^T \mathbb{B} \mathbb{A}^{-1} \mathbb{B}^T Z}{Z^T \mathbb{M}_p Z}, \quad (\mathbb{M}_p)_{ij} = (\boldsymbol{\chi}_i^h, \boldsymbol{\chi}_j^h)$$

rules out equal order interpolation on the same mesh

Consistently stabilized methods

Residual stabilization enables equal order polynomial interpolation for velocity and pressure

$$
\begin{aligned}
&(\dot{\mathbf{u}}_h(\cdot,t),\mathbf{v}_h) + G\big(\{\mathbf{u}_h(\cdot,t),p_h(\cdot,t)\},\{\mathbf{v}_h,q_h\}\big) - (\mathbf{f}(\cdot,t),\mathbf{v}_h)\\
&\quad - \sum_{\mathcal{K}\in\mathcal{T}_h} \tau_{\mathcal{K}}\,\big(\dot{\mathbf{u}}_h(\cdot,t) - \triangle\mathbf{u}_h(\cdot,t) + \nabla p_h(\cdot,t) - \mathbf{f}(\cdot,t), -\gamma\triangle\mathbf{v}_h + \nabla q_h\big)_{\mathcal{K}} = 0
\end{aligned}
$$

stabilization parameter

test function

momentum residual

$\gamma = 0$ pressure-Poisson

$\gamma = -1$ Galerkin least-squares

$\gamma = 1$ Douglas-Wang

Recall the sufficient condition

On inf-sup stabilized finite element methods for transient problems, Bochev, Gunzburger and Shadid, CMAME 2004 demonstrated that a sufficient condition was

$$\Delta t > \delta h^2 \qquad \tau = \delta h^2$$ stabilization parameter

Again, we'll show that this is also necessary and so there is a lower bound on the time step

2D pressure plots

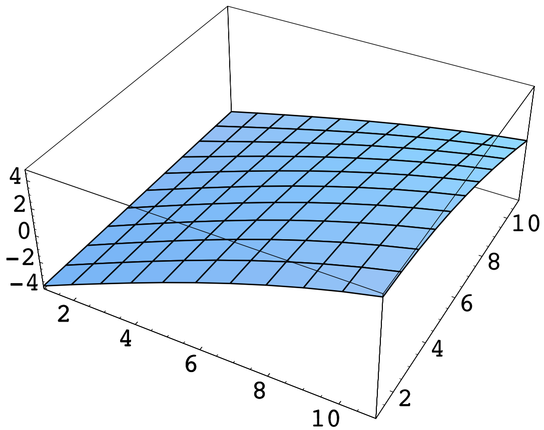
- Uniform grid on the unit square
- 200 triangles (121 vertices)
- Cubic nodal elements for the velocity and pressure

$$\mathbf{u} = \begin{pmatrix} \frac{\partial \psi}{\partial x} \\ -\frac{\partial \psi}{\partial y} \end{pmatrix} \qquad \psi(x, y) = x^2(1-x)^2 \sin^2(\pi y)$$

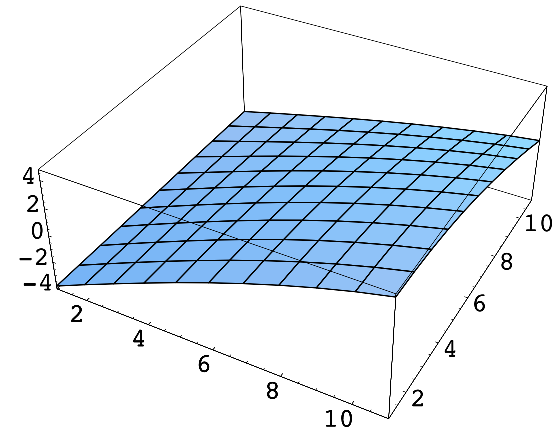
$$p(x, y) = \sin(x)\cos(y) + (\cos(1) - 1)\sin(1)$$

Taylor-Hood $\Delta t = 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}$

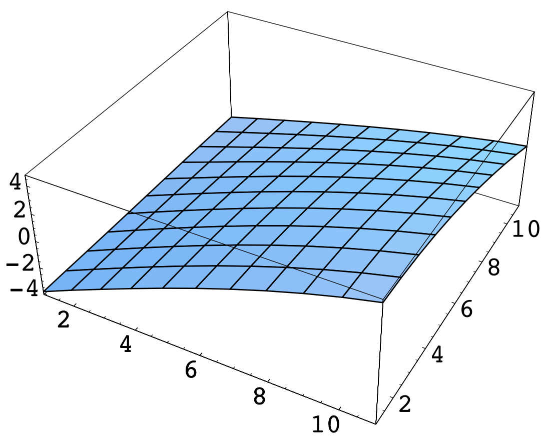
Taylor-Hood; dt=0.001



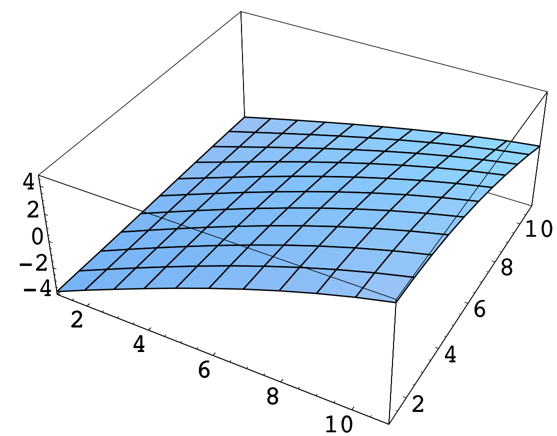
Taylor-Hood; dt=0.0001



Taylor-Hood; dt=0.00001

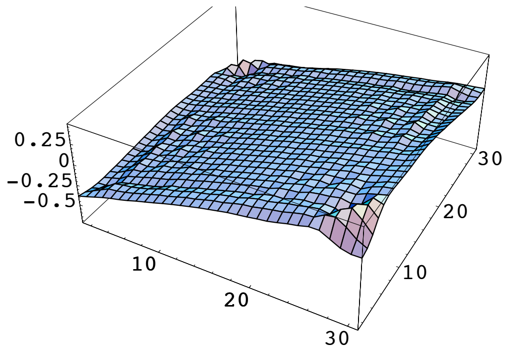


Taylor-Hood; dt=0.000001

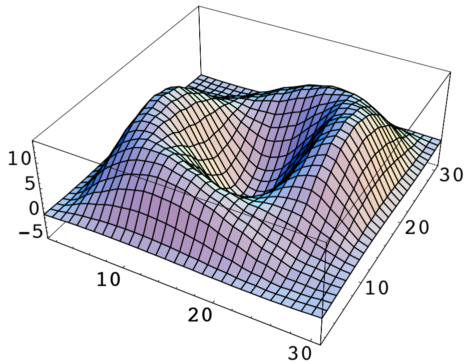


Presssure-Poisson stabilized

$\Delta t = 10^{-1}$



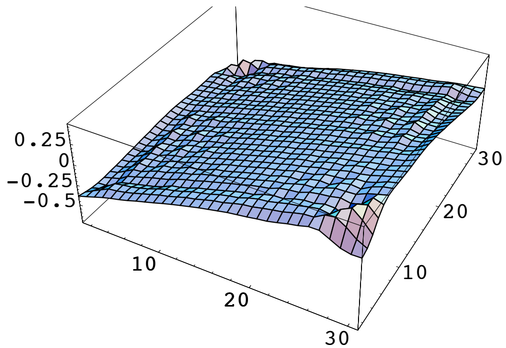
$\Delta t = 10^{-6}$

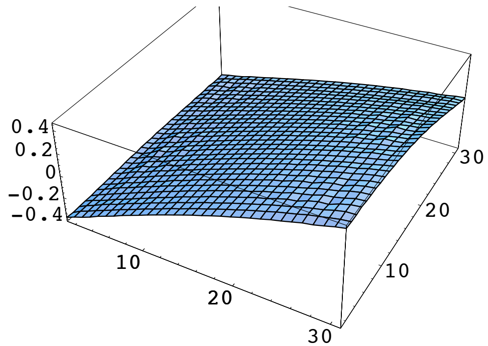


$\delta = 5$

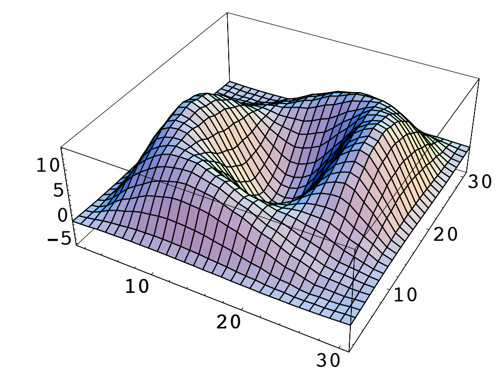
Presssure-Poisson stabilized

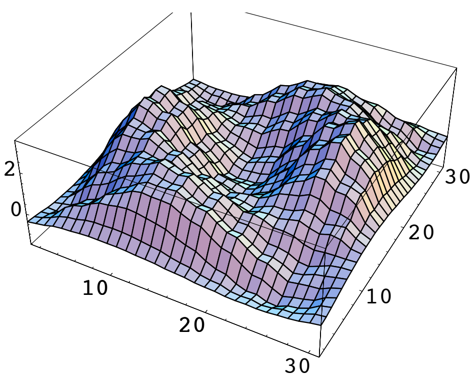
$\Delta t = 10^{-1}$ $\Delta t = 10^{-1}$





$\Delta t = 10^{-6}$ $\Delta t = 10^{-6}$

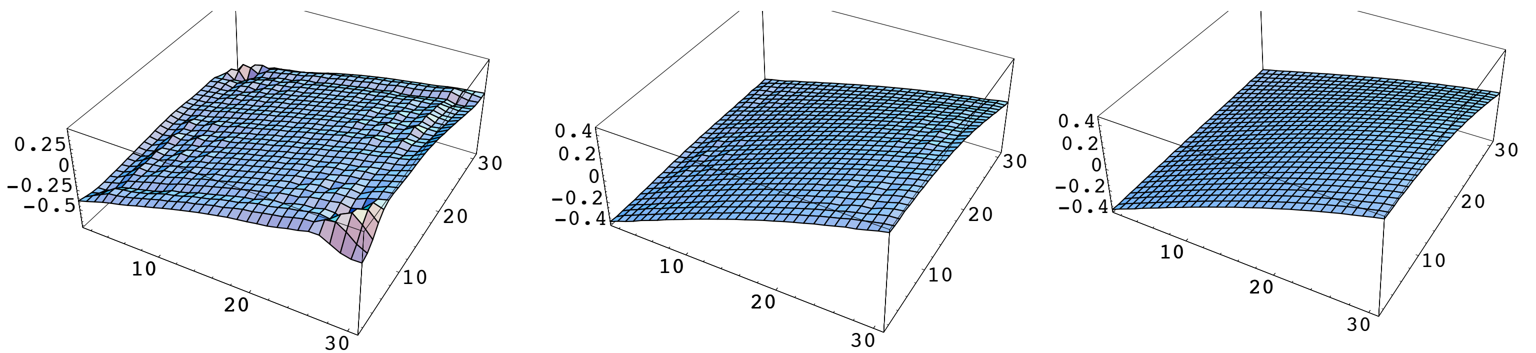




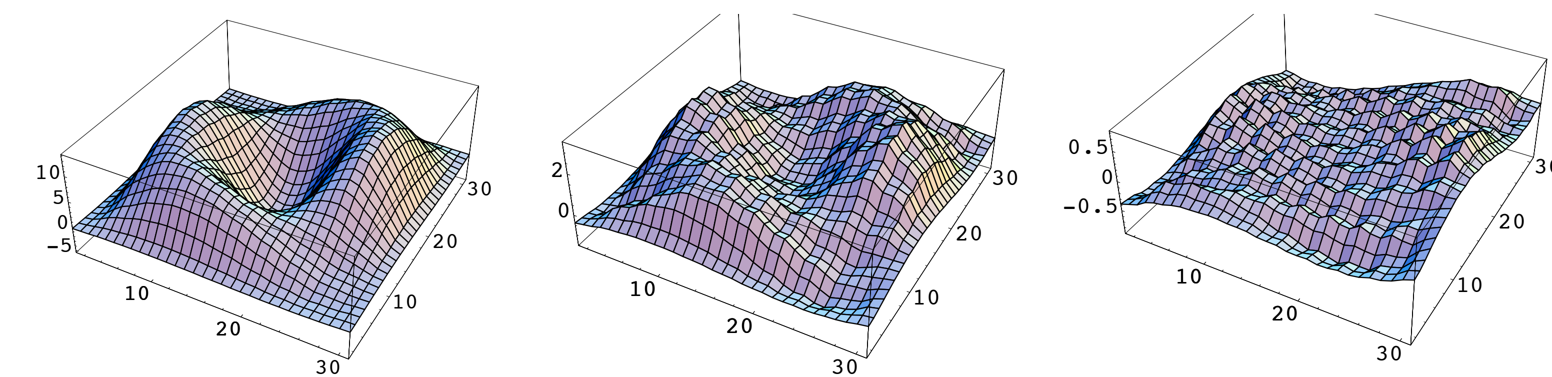
$\delta = 5$ $\delta = 5 \cdot 10^{-1}$

Presssure-Poisson stabilized

$\Delta t = 10^{-1}$ $\Delta t = 10^{-1}$ $\Delta t = 10^{-1}$



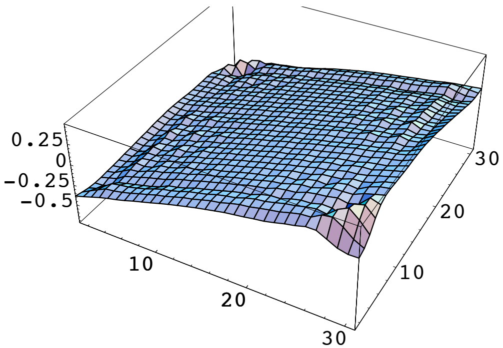
$\Delta t = 10^{-6}$ $\Delta t = 10^{-6}$ $\Delta t = 10^{-6}$



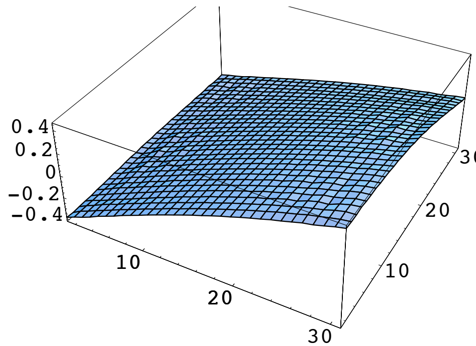
$\delta = 5$ $\delta = 5 \cdot 10^{-1}$ $\delta = 5 \cdot 10^{-2}$

Presssure-Poisson stabilized

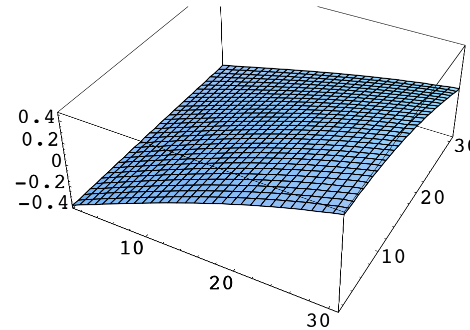
$\Delta t = 10^{-1}$



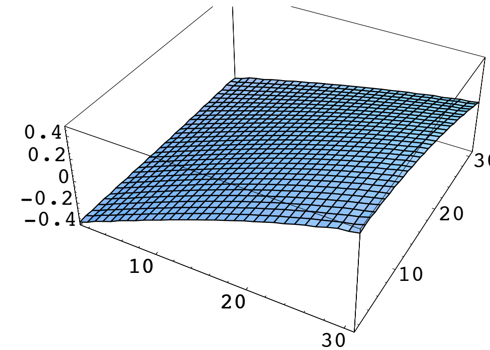
$\Delta t = 10^{-1}$



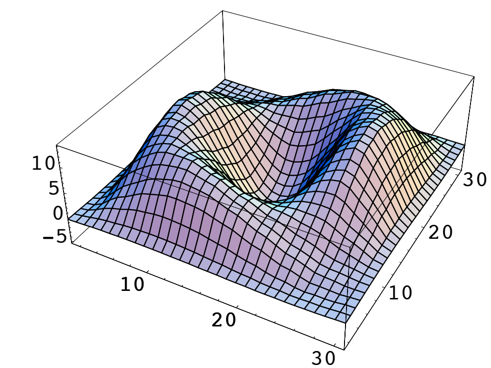
$\Delta t = 10^{-1}$



$\Delta t = 10^{-1}$

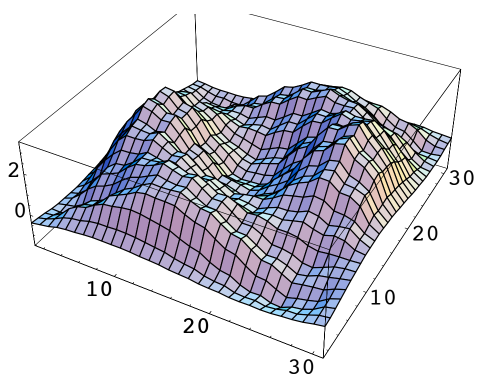


$\Delta t = 10^{-6}$



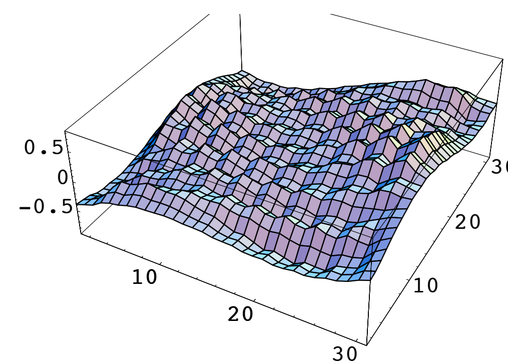
$\delta = 5$

$\Delta t = 10^{-6}$



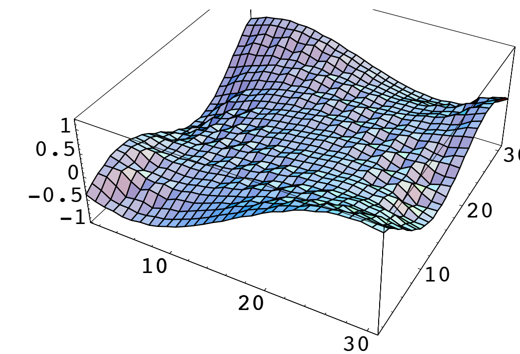
$\delta = 5 \cdot 10^{-1}$

$\Delta t = 10^{-6}$



$\delta = 5 \cdot 10^{-2}$

$\Delta t = 10^{-6}$



$\delta = 5 \cdot 10^{-3}$

Index 1 differential algebraic equation

Consider the pressure-Poisson case $\tau = \delta h^2$

$$\begin{pmatrix} \mathbb{M}\dot{U} \\ \tau\mathbb{B}\dot{U} \end{pmatrix} + \begin{pmatrix} \mathbb{A} & \mathbb{B}^T \\ -\mathbb{B} - \tau\mathbb{S} & \tau\mathbb{K} \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ \tau G \end{pmatrix}$$

$$\mathbb{K}_{ij} = (\nabla\chi_j^h, \nabla\chi_i^h), \quad \mathbb{S}_{ij} = (\Delta\boldsymbol{\xi}_j^h, \nabla\chi_i^h), \quad G_i = (\mathbf{f}, \nabla\xi_i^h).$$

Semi-discrete pressure operator

$$\left(\mathbb{K} - \mathbb{B}\mathbb{M}^{-1}\mathbb{B}^T\right) P = G - \mathbb{B}\mathbb{M}^{-1}F + \left(\frac{1}{\tau}\mathbb{B} + \mathbb{S} + \mathbb{B}\mathbb{M}^{-1}\mathbb{A}\right) U.$$

Theorem

$$(1-\mu_{max}^2)Q^T\mathbb{K}Q \leq Q^T\left(\mathbb{K}-\mathbb{B}\mathbb{M}^{-1}\mathbb{B}^T\right)Q \qquad \forall\, Q \in \mathbb{R}^M$$

$$\mu_{max}^2 \equiv \max_Q \frac{Q^T\mathbb{B}\mathbb{M}^{-1}\mathbb{B}^TQ}{Q^T\mathbb{K}Q}$$

We also have an upper bound (see paper) but let's keep things simple

Previous work and our contribution

- Codina, Vazquez, and Zienkiewicz, IJNMF, 1998 demonstrate that the semi-discrete pressure operator is positive semi-definite
- We refine this result to show that the eigenvalues (Rayleigh-quotients) are within the unit interval and provide a variational characterization

Variational characterization of the smallest and largest singular

$$\mu_{max} = \sup_{q_h \in \mathcal{P}_h,\, q_h \neq 0} \ \sup_{\mathbf{v}_h \in \mathcal{V}_h,\, \mathbf{v}_h \neq \mathbf{0}} \frac{(\nabla q_h, \mathbf{v}_h)}{|q_h|_1 \|\mathbf{v}_h\|_0} = \max_{Q \in \mathbf{R}^M} \ \max_{V \in \mathbf{R}^N} \frac{Q^T \mathbb{B} V}{\|V\|_{\mathbb{M}} \|Q\|_{\mathbb{K}}}$$

$$\|Q\|^2_{\mathbb{K}} = Q^T \mathbb{K} Q, \quad \|V\|^2_{\mathbb{M}} = V^T \mathbb{M} V$$

Variational characterization of the smallest and largest singular

$$\mu_{max} = \sup_{q_h \in \mathcal{P}_h,\, q_h \neq 0} \; \sup_{\mathbf{v}_h \in \mathcal{V}_h,\, \mathbf{v}_h \neq \mathbf{0}} \frac{(\nabla q_h, \mathbf{v}_h)}{|q_h|_1 \|\mathbf{v}_h\|_0} = \max_{Q \in \mathbf{R}^M} \; \max_{V \in \mathbf{R}^N} \frac{Q^T \mathbb{B} V}{\|V\|_{\mathbb{M}} \|Q\|_{\mathbb{K}}}$$

$$\|Q\|_{\mathbb{K}}^2 = Q^T \mathbb{K} Q, \quad \|V\|_{\mathbb{M}}^2 = V^T \mathbb{M} V$$

The norms are reversed with respect to the inf-sup stability constant associated with Stokes!

$$\frac{(q_h, \nabla \cdot \mathbf{v}^h)}{\|\mathbf{v}_h\|_1 \|q_h\|_0}$$

Sharp estimates on the unit square

$$1 - \omega h + O(h^2) \leq \mu_{max} \leq 1$$

$$\omega = \begin{cases} \frac{2}{3} & Q_1 - Q_1 \\ \frac{4}{15} & Q_2 - Q_2 \\ \frac{16}{105} & Q_3 - Q_3 \end{cases}$$

linear element

quadratic element

cubic element

$$\exists Q \quad \ni \quad Q^T \left(\mathbb{K} - \mathbb{B} \mathbb{M}^{-1} \mathbb{B}^T \right) Q \leq \left(2\omega h + O(h^2) \right) Q^T \mathbb{K} Q$$

Recall the semi-discrete pressure operator (pressure-Poisson case)

$$\left(\mathbb{K} - \mathbb{B}\mathbb{M}^{-1}\mathbb{B}^T\right) P = G - \mathbb{B}\mathbb{M}^{-1} F + \left(\frac{1}{\tau}\mathbb{B} + \mathbb{S} + \mathbb{B}\mathbb{M}^{-1}\mathbb{A}\right) U.$$

Unstable! The smallest eigenvalue goes to zero as the mesh size does!

$$\tau = \delta h^2$$

Recall the semi-discrete pressure operator (pressure-Poisson case)

$$\boxed{\left(\mathbb{K} - \mathbb{B}\mathbb{M}^{-1}\mathbb{B}^T\right)} P = G - \mathbb{B}\mathbb{M}^{-1} F + \left(\frac{1}{\tau}\mathbb{B} + \mathbb{S} + \mathbb{B}\mathbb{M}^{-1}\mathbb{A}\right) U.$$

Unstable! The smallest eigenvalue goes to zero as the mesh size does!

$$\tau = \delta h^2$$

origin of the time step restriction

$$\Delta t > \alpha h^2$$

Fully discrete equations (backward Euler)

$$\begin{pmatrix} \mathbb{M} + \Delta t \mathbb{A} & \Delta t \mathbb{B}^T \\ (\tau - \Delta t)\mathbb{B} - \tau \Delta t \mathbb{S} & \tau \Delta t \mathbb{K} \end{pmatrix} \begin{pmatrix} U^{k+1} \\ P^{k+1} \end{pmatrix} = \begin{pmatrix} \mathbb{M} & 0 \\ \tau \mathbb{B} & 0 \end{pmatrix} \begin{pmatrix} U^{k} \\ P^{k} \end{pmatrix} + \Delta t \begin{pmatrix} F^{k+1} \\ \tau G^{k+1} \end{pmatrix}$$

Fully discrete equations (backward Euler)

$$\begin{pmatrix} \mathbb{M} + \Delta t \mathbb{A} & \Delta t \mathbb{B}^T \\ (\tau - \Delta t)\mathbb{B} - \tau \Delta t \mathbb{S} & \tau \Delta t \mathbb{K} \end{pmatrix} \begin{pmatrix} U^{k+1} \\ P^{k+1} \end{pmatrix} = \begin{pmatrix} \mathbb{M} & 0 \\ \tau \mathbb{B} & 0 \end{pmatrix} \begin{pmatrix} U^{k} \\ P^{k} \end{pmatrix} + \Delta t \begin{pmatrix} F^{k+1} \\ \tau G^{k+1} \end{pmatrix}$$

$$(\mathbb{K} + \mathbb{N}\mathbb{B}^T) P^{k+1} = \left[\frac{1}{\Delta t} \Big(\mathbb{N}\mathbb{M} + \mathbb{B} \Big) U^k + \mathbb{N} F^{k+1} + G^{k+1} \right]$$

$$\mathbb{N}\mathbb{B}^T = \left(\Delta t \mathbb{S} + (\frac{\Delta t}{\tau} - 1)\mathbb{B} \right) (\mathbb{M} + \Delta t \mathbb{A})^{-1} \mathbb{B}^T$$

Theorem

$$Q^T\left(\mathbb{K}+\mathbb{N}\mathbb{B}^T\right)Q = Q^T\left(\mathbb{K}+\left(\frac{\Delta t}{\tau}-1\right)\left(\mathbb{B}\mathbb{M}^{-1}\mathbb{B}^T+O(\Delta t)\right)\right)Q+O(\Delta t)$$

$$Q^T\left(\mathbb{K}+\mathbb{N}\mathbb{B}^T\right)Q \geq \frac{1}{2}\min\{1,\alpha\}\|Q\|^2_{\mathbb{K}}, \qquad \frac{\Delta t}{\tau} \geq \alpha > 0$$

$$\mathbb{K}+\mathbb{N}\mathbb{B}^T \longrightarrow \mathbb{K}-\mathbb{B}\mathbb{M}^{-1}\mathbb{B}^T \qquad \frac{\Delta t}{\tau} \to 0, \quad \Delta t \to 0$$

Velocity approximation when using backward Euler

$$U^{k+1} = U^k + O(\Delta t)$$

Hence, no problems over one time step

Differential algebraic equations associated with the other approaches

$$\begin{pmatrix} (\mathbb{M} + \gamma\tau\mathbb{C})\,\dot{U} \\ \tau\mathbb{B}\,\dot{U} \end{pmatrix} + \begin{pmatrix} \mathbb{A} - \gamma\tau\mathbb{D} & \mathbb{B}^T + \gamma\tau\mathbb{S}^T \\ -\mathbb{B} - \tau\mathbb{S} & \tau\mathbb{K} \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F + \gamma\tau H \\ \tau G \end{pmatrix}$$

$$\mathbb{C}_{ij} = \tau \sum_{\mathcal{K}\in\mathcal{T}_h} \left(\boldsymbol{\xi}_j^h, \Delta\boldsymbol{\xi}_i^h\right)_{\mathcal{K}} \qquad \mathbb{D}_{ij} = \tau \sum_{\mathcal{K}\in\mathcal{T}_h} \left(\Delta\boldsymbol{\xi}_j^h, \Delta\boldsymbol{\xi}_i^h\right)_{\mathcal{K}} \qquad (H)_i = \tau \sum_{\mathcal{K}\in\mathcal{T}_h} \left(\mathbf{f}, \triangle\boldsymbol{\xi}_i^h\right)_{\mathcal{K}}$$

$\gamma = 0$ pressure-Poisson

$\gamma = -1$ Galerkin least-squares $\qquad \tau = \delta h^2$

$\gamma = 1$ Douglas-Wang

Theorems

$$\mathbb{K} - (\mathbb{B} + \gamma\tau\mathbb{S})(\mathbb{M} + \gamma\tau\mathbb{C})^{-1}\mathbb{B}^T$$

general semi-discrete pressure operator

$$\tau = \delta h^2, \quad \gamma = 0, \pm 1$$

$$\mathbb{K} - (\mathbb{B} + \gamma\tau\mathbb{S})(\mathbb{M} + \gamma\tau\mathbb{C})^{-1}\mathbb{B}^T \sim \mathbb{K} - \mathbb{B}\mathbb{M}^{-1}\mathbb{B}$$

- Galerkin least-squares and Douglas-Wang are equivalent to Pressure-Poisson stabilization
- We have explicit equivalency constants (see paper)

Selecting the time step in practice

$$\Delta t > \alpha h^2 \qquad \tau = \delta h^2$$

$$c_1 h^r \approx c_2 (\Delta t)^s \quad \Longrightarrow \quad h^r \approx \frac{c_2}{c_1} (\frac{\alpha}{\delta})^s (\delta h^2)^s$$

Roughly, the order of the time integrator needs to be twice the order of the finite elements

Conclusions

- Lower bound on the time step is necessary
- If this is an issue
  1. use space-time formulation
  2. stable element pairing (mixed order interpolation)
  3. non residual based stabilization method
- Connection with pressure projection methods (didn't discuss)

For more details, see *On stabilized finite element for the Stokes problem in the small time step limit*, under revision for IJNMF

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