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Galerkin Reduced Order Models for Compressible Flow with Structural Interaction

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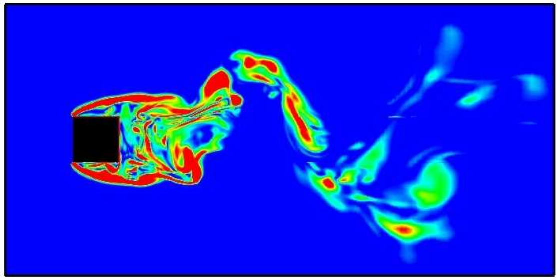
Reduced Order Models (ROMs)

- **Goal of Reduced Order Modeling**
  - Construct a surrogate numerical model that captures the essential dynamics of a full numerical model but at much cheaper expense.
- **Applications in Fluid Dynamics**
  - Predictive modeling across a parameter space, e.g. aeroelastic flutter analysis
  - System model for active flow control
  - Long-time unsteady flow analysis, e.g. fatigue of a wind turbine blade under variable wind conditions



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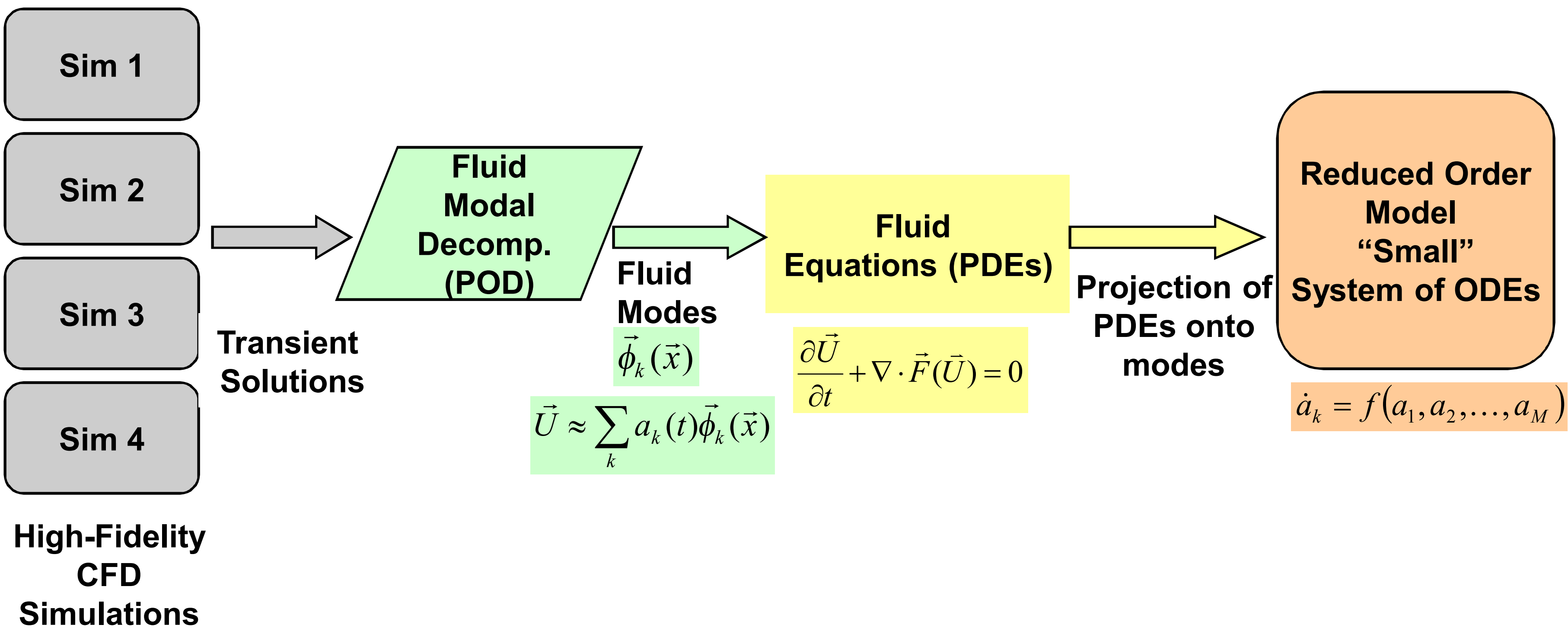
Reduced Order Modeling (ROM) Approach



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Predictive ROMs: Numerical Analysis

- The projection approach is really an alternative discretization of the governing PDEs (global Galerkin)
  - Consistency with continuous PDE: Loosely speaking, a ROM *can be* consistent with respect to the full simulations used to generate it.
  - Stability: Numerical stability is not guaranteed, in general. There are many examples of POD/Galerkin ROM instability.
  - Convergence: Consistency and stability are required.
  - Accuracy: Error estimates are often not available.

- This presentation is focused on stability of Galerkin ROMs. There are other important issues, connected with consistency, convergence, and accuracy.

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Outline

- A Stable Galerkin ROM approach
  - Background (POD, Galerkin projection)
  - Development of the stable inner product
  - Boundary conditions that preserve ROM stability
- Demonstration
  - Stability for a random basis
  - 1D acoustic pulse in a uniform flow
  - 2D cylindrical pressure pulse in a uniform flow
- Brief overview of stable coupled fluid/structure ROM method

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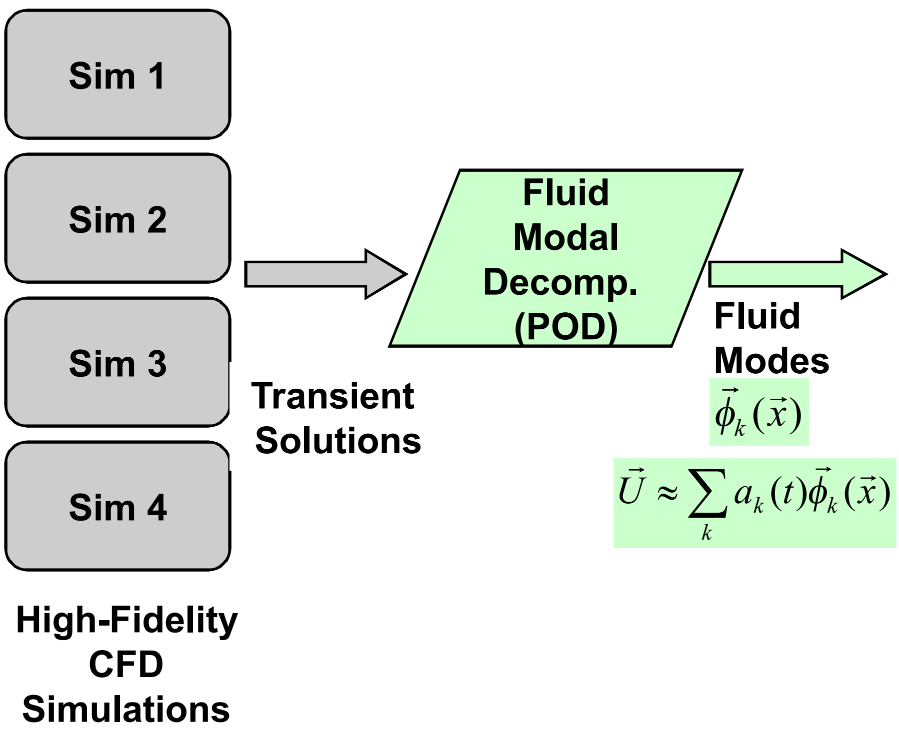
Step 1: Constructing Modes



Proper Orthogonal Decomposition (POD)

Modal Decomposition of the Solution :

$$\mathbf{u}(\mathbf{x},t) = \sum_j a_j(t)\boldsymbol{\phi}_j(\mathbf{x})$$

Ensemble of solutions from full simulation: $\{\mathbf{u}^k(\mathbf{x})\}$

Inner product, L^2 **for example :** $(u,v) = \int_\Omega u\ v\ \mathrm{d}\Omega$

Time or ensemble averaging operator : $\langle \cdot \rangle$

POD optimization problem : $\max_{\boldsymbol{\phi}\in H(\Omega)} \dfrac{\langle(\mathbf{u},\boldsymbol{\phi})^2\rangle}{\|\boldsymbol{\phi}\|^2}$

POD eigenvalue problem* :

$$\mathrm{R}\boldsymbol{\phi} = \lambda\boldsymbol{\phi}$$

(*C. W. Rowley *et al.*, *Physica D,* 2004)

$$\mathrm{R}\boldsymbol{\phi} \equiv \langle \mathbf{u}^k(\mathbf{u}^k,\boldsymbol{\phi})\rangle$$

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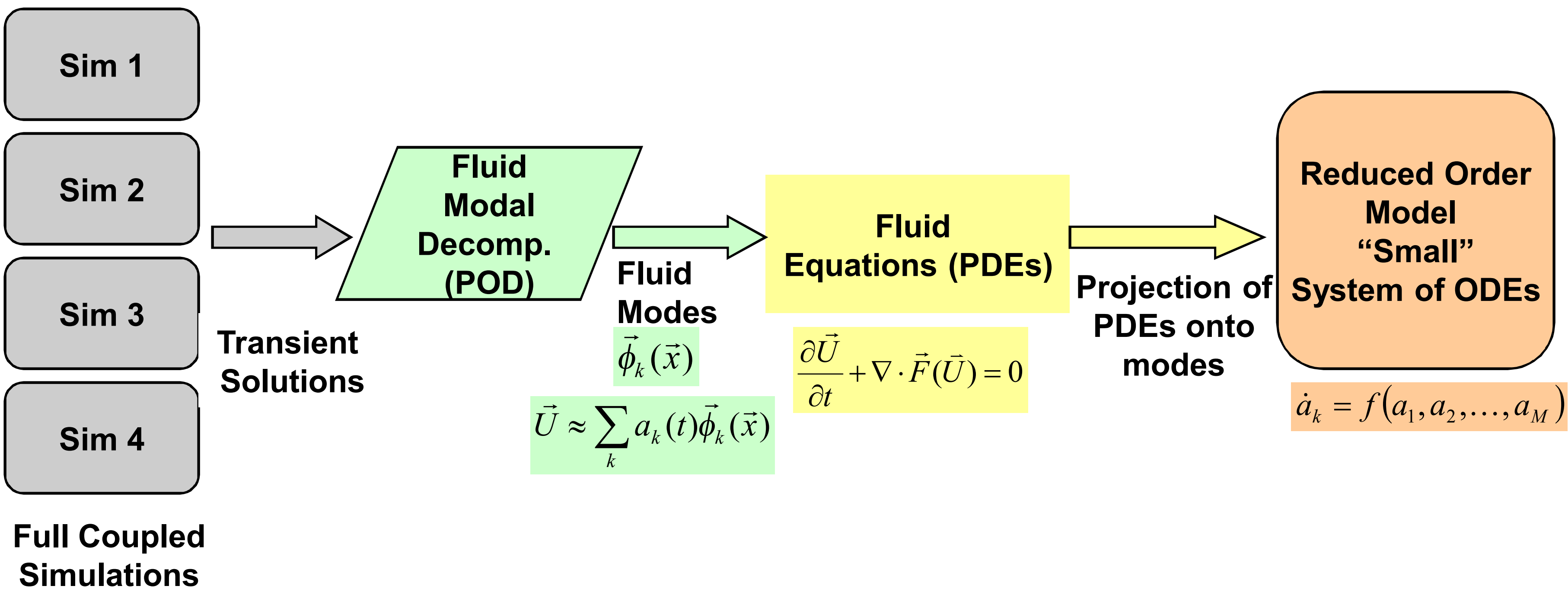
Step 2: Project the equations onto the modes



Galerkin Projection

Nonlinear PDE with linear term, quadratic and cubic nonlinearities

$$\frac{\partial \mathrm{u}}{\partial t} = \mathcal{L}\mathrm{u} + \mathcal{N}_2(\mathrm{u},\mathrm{u}) + \mathcal{N}_3(\mathrm{u},\mathrm{u},\mathrm{u})$$

Project the PDE onto a set of modes using an inner product operator

$$\left(\frac{\partial \mathrm{u}}{\partial t},\ \boldsymbol{\phi}_j\right) = (\mathcal{L}\mathrm{u},\ \boldsymbol{\phi}_j) + (\mathcal{N}_2(\mathrm{u},\mathrm{u}),\ \boldsymbol{\phi}_j) + (\mathcal{N}_3(\mathrm{u},\mathrm{u},\mathrm{u}),\ \boldsymbol{\phi}_j)$$

Substituting the modal decomposition $\mathrm{u}(\mathrm{x},t) = \sum_j a_j(t)\boldsymbol{\phi}_j(\mathrm{x})$ results in an ODE in the modal amplitudes, with pre-computable coefficients

$$\frac{\mathrm{d}a_k}{\mathrm{d}t} = \sum_l a_l(\boldsymbol{\phi}_k, \mathcal{L}(\boldsymbol{\phi}_l)) + \sum_{l,m} a_l a_m(\boldsymbol{\phi}_k, \mathcal{N}_2(\boldsymbol{\phi}_l, \boldsymbol{\phi}_m)) + \sum_{l,m,n} a_l a_m a_n(\boldsymbol{\phi}_k, \mathcal{N}_3(\boldsymbol{\phi}_l, \boldsymbol{\phi}_m, \boldsymbol{\phi}_n))$$

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Numerical Stability of a ROM

- **A practical definition**: The numerical solution does not blow up in time unexpectedly.
- **A more precise definition**: The norm of the numerical solution remains bounded in a way that is consistent with exact solutions to the governing differential equations.
- **Method of analysis**: The **energy method** uses an equation for the evolution of numerical solution "energy" to determine stability.

Linearized Euler Equations

- **Obtained by linearizing the full Euler equations about a steady base flow.**
- **Useful for aeroelasticity problems, aeroacoustics, flow instability analysis.**

$$\mathbf{q}(\mathbf{x},t) = \overline{\mathbf{q}}(\mathbf{x}) + \mathbf{q}'(\mathbf{x},t) \qquad \mathbf{q} = \begin{bmatrix} u & v & w & \zeta & p \end{bmatrix}^T$$

$$\frac{\partial \mathbf{q}'}{\partial t} + \mathbf{A}(\bar{\mathbf{q}}) \cdot \nabla \mathbf{q}' + C(\overline{\mathbf{q}})\mathbf{q}' = 0$$

where

$$\mathbf{A}(\bar{\mathbf{q}}) \equiv \left[A_x(\bar{\mathbf{q}}),\ A_y(\bar{\mathbf{q}}),\ A_z(\bar{\mathbf{q}})\right]^T,$$

$$A_x = \begin{bmatrix} \bar{u} & 0 & 0 & 0 & \bar{\zeta} \\ 0 & \bar{u} & 0 & 0 & 0 \\ 0 & 0 & \bar{u} & 0 & 0 \\ -\bar{\zeta} & 0 & 0 & \bar{u} & 0 \\ \gamma\bar{p} & 0 & 0 & 0 & \bar{u} \end{bmatrix} \quad A_y = \begin{bmatrix} \bar{v} & 0 & 0 & 0 & 0 \\ 0 & \bar{v} & 0 & 0 & \bar{\zeta} \\ 0 & 0 & \bar{v} & 0 & 0 \\ 0 & -\bar{\zeta} & 0 & \bar{v} & 0 \\ 0 & \gamma\bar{p} & 0 & 0 & \bar{v} \end{bmatrix} \quad A_z = \begin{bmatrix} \bar{w} & 0 & 0 & 0 & 0 \\ 0 & \bar{w} & 0 & 0 & 0 \\ 0 & 0 & \bar{w} & 0 & \bar{\zeta} \\ 0 & 0 & -\bar{\zeta} & \bar{w} & 0 \\ 0 & 0 & \gamma\bar{p} & 0 & \bar{w} \end{bmatrix}$$

$$C = \begin{bmatrix} \frac{\partial \bar{u}}{\partial x} & \frac{\partial \bar{u}}{\partial y} & \frac{\partial \bar{u}}{\partial z} & \frac{\partial \bar{p}}{\partial x} & 0 \\ \frac{\partial \bar{v}}{\partial x} & \frac{\partial \bar{v}}{\partial y} & \frac{\partial \bar{v}}{\partial z} & \frac{\partial \bar{p}}{\partial y} & 0 \\ \frac{\partial \bar{w}}{\partial x} & \frac{\partial \bar{w}}{\partial y} & \frac{\partial \bar{w}}{\partial z} & \frac{\partial \bar{p}}{\partial z} & 0 \\ \frac{\partial \bar{\zeta}}{\partial x} & \frac{\partial \bar{\zeta}}{\partial y} & \frac{\partial \bar{\zeta}}{\partial z} & -\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z}\right) & 0 \\ \frac{\partial \bar{p}}{\partial x} & \frac{\partial \bar{p}}{\partial y} & \frac{\partial \bar{p}}{\partial z} & 0 & \gamma\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z}\right) \end{bmatrix}$$

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Symmetrization of the Linearized Euler Equations

- The energy method can be used following "symmetrization" of the linearized Euler equations.
- Multiply the equations by the positive definite matrix:

$$H = \begin{bmatrix} \bar{\rho} & 0 & 0 & 0 & 0 \\ 0 & \bar{\rho} & 0 & 0 & 0 \\ 0 & 0 & \bar{\rho} & 0 & 0 \\ 0 & 0 & 0 & \alpha^2\gamma\bar{\rho}^2\bar{p} & \bar{\rho}\alpha^2 \\ 0 & 0 & 0 & \bar{\rho}\alpha^2 & \frac{(1+\alpha^2)}{\gamma\bar{p}} \end{bmatrix}$$

- Both H and all the HA_j are symmetric matrices

$$H\frac{\partial \mathbf{q}'}{\partial t} + HA_j\frac{\partial \mathbf{q}'}{\partial x_j} + HC\mathbf{q}' = 0$$

Stability of the Galerkin Approximation

- Define the "energy inner product" and corresponding norm:

$$(\mathbf{u}, \mathbf{v})_H \equiv \int_\Omega \mathbf{u}^T H \mathbf{v} \, d\Omega \qquad ||\mathbf{q}'||_H \equiv (\mathbf{q}', \mathbf{q}')_H^{1/2}$$

- Exact solutions to the linearized Euler equations satisfy:

$$||\mathbf{q}'(x,t)||_H \leq e^{\alpha t} ||\mathbf{q}'(x,0)||_H$$

- Introduce the approximate Galerkin solution

$$\mathbf{q}'_N = \sum_{k=1}^{M} a_k(t) \boldsymbol{\phi}_k(\mathbf{x})$$

- It turns out that the Galerkin approximation satisfies the same energy expression as for the continuous equations, i.e. it is stable.

$$||\mathbf{q}'_N(x,t)||_H \leq e^{\alpha t} ||\mathbf{q}'_N(x,0)||_H$$

- For uniform flow, the Galerkin scheme satisfies the strong stability condition:

$$||\mathbf{q}'_N(x,t)||_H \leq ||\mathbf{q}'_N(x,0)||_H$$

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Inner Product for Linearized Compressible Flow

The stability analysis dictates that we use the following "energy inner product" to both compute the POD modes and to perform the Galerkin projection.

$$(\mathbf{q}'^{(1)}, \mathbf{q}'^{(2)})_H = \int_\Omega \Big[\bar{\rho} \left(u'^{(1)} u'^{(2)} + v'^{(1)} v'^{(2)} + w'^{(1)} w'^{(2)} \right) + \alpha^2 \gamma \bar{\rho}^2 \bar{p}\, \zeta'^{(1)} \zeta'^{(2)} + \frac{1+\alpha^2}{\gamma \bar{p}}\, p'^{(1)} p'^{(2)} + \alpha^2 \bar{\rho} \left(\zeta'^{(2)} p'^{(1)} + \zeta'^{(1)} p'^{(2)} \right) \Big] \, \mathrm{d}\Omega$$

Construction of the Fluid ROM (No BC's)

- **Galerkin Projection Step**

$$\left(\boldsymbol{\phi}_j, \frac{\partial \mathbf{q}'}{\partial t}\right)_H + \left(\boldsymbol{\phi}_j, \mathbf{A}(\bar{\mathbf{q}}) \cdot \nabla \mathbf{q}'\right)_H + \left(\boldsymbol{\phi}_j, C\left(\bar{\mathbf{q}}\right) \mathbf{q}'\right)_H = 0$$

- **Reduced Order Model**

$$\dot{a}_j = -\sum_{k=1}^{M} a_k (\boldsymbol{\phi}_j, \mathbf{A}(\bar{\mathbf{q}}) \cdot \nabla \boldsymbol{\phi}_k)_H - \sum_{k=1}^{M} a_k (\boldsymbol{\phi}_j, C(\bar{\mathbf{q}}) \boldsymbol{\phi}_k)_H, \quad j = 1, \ldots, M$$

- **Matrix Form**

$$\dot{a}_j = A_{jk} a_k, \quad j = 1, \ldots, M$$

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Stable ROM Boundary Conditions

- **In Galerkin projection step, integrate the following term by parts:**

$$(\boldsymbol{\phi}_j, \mathbf{A}(\bar{\mathrm{q}}) \cdot \nabla \mathrm{q}')_H = \int_{\partial\Omega} \boldsymbol{\phi}_j^T H(\bar{\mathrm{q}}) \left(\mathbf{A}(\bar{\mathrm{q}}) \cdot \mathbf{n}\right) \mathrm{q}' \, \mathrm{d}S - \int_{\Omega} \left(\nabla \cdot \left[\boldsymbol{\phi}_j^T H(\bar{\mathrm{q}}) \mathbf{A}(\bar{\mathrm{q}})\right]\right) \mathrm{q}' \, \mathrm{d}\Omega$$

- **Boundary conditions are implemented weakly by specifying this state in the boundary integral.**
- **Characteristic boundary conditions are used.**

Energy stability is maintained if the boundary conditions are such that

$$\int_{\partial\Omega} \boldsymbol{\phi}_j^T H(\bar{\mathrm{q}}) \left(\mathbf{A}(\bar{\mathrm{q}}) \cdot \mathbf{n}\right) \mathrm{q}' \, \mathrm{d}S \geq 0$$

Numerical Implementation of Fluid ROM

- So far, all the analysis is for continuous and smooth basis functions and exact evaluation of inner product integrals.
- A discrete implementation is required that preserves stability.
- Solution:
  - Define solution snapshots and POD basis functions using a piece-wise smooth finite element representation
  - Apply Gauss quadrature rules of sufficent accuracy to exactly integrate the inner products
  - Fairly general, works for any nodal mesh that can be represented using finite elements.

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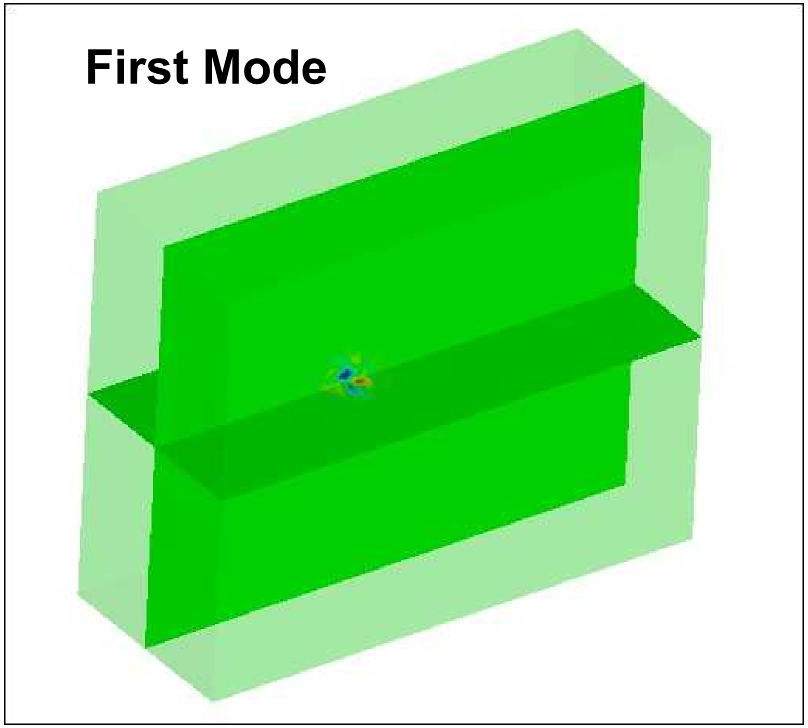
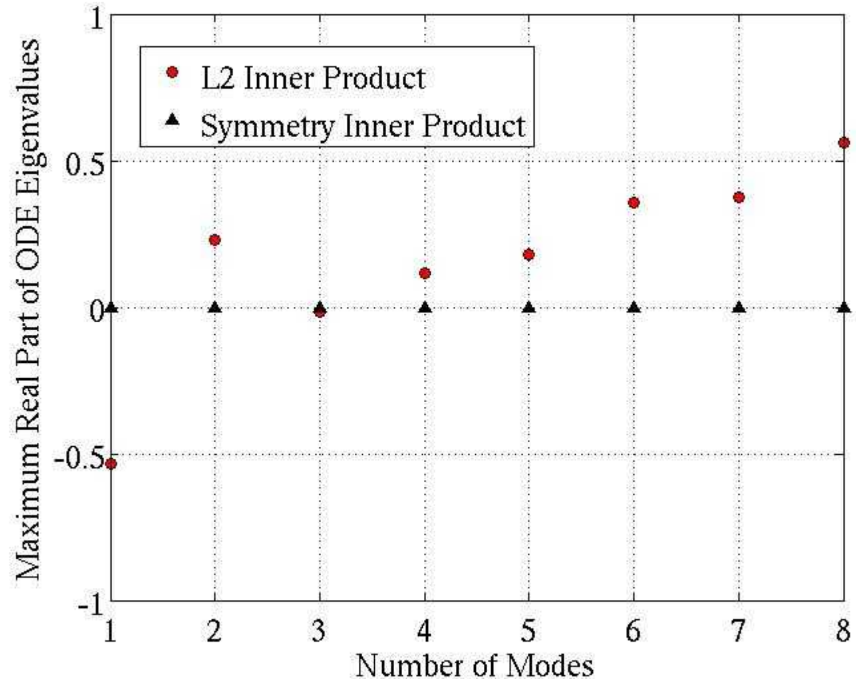
Computer Code to Generate ROMs

- For ROMs based on large CFD simulations, construction of the ROM is memory and CPU intensive
- A parallel computer code was developed to interface with a CFD code, and compute POD bases and Galerkin projection
  - Built using the parallel data structures and linear algebra routines from the Trilinos project, developed at Sandia
  - POD eigensolve performed in parallel using the RBGen Trilinos package
  - Core routines are general, with I/O interface required to handle data from different CFD codes and finite element infrastructure required to compute gradients and inner product integrals
  - Currently, only AERO-F code (C. Farhat, Stanford U.) interface is implemented, for unstructured tetrahedral meshes

Example: Purely Random Basis





- Each mode is a random disturbance field that decays to zero at the domain boundaries.
- Uniform steady base flow.
- Recall ROM form $\dot{a}_j = A_{jk} a_k$: positive real part of eigenvalues of A determine stability in time.
- Model problem for modes dominated by numerical error.

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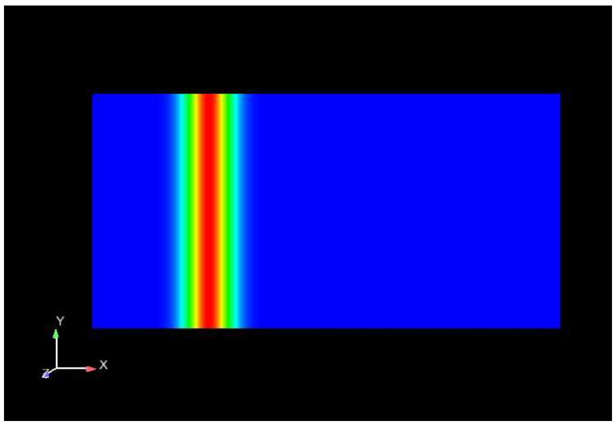
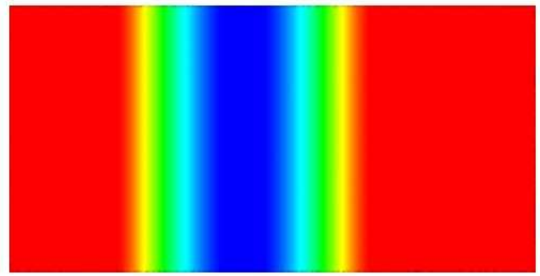
Example: 1D Acoustic Pulse

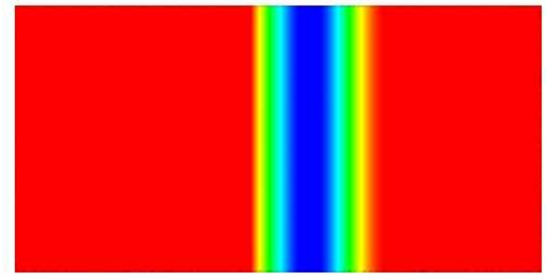
CFD animation: pressure



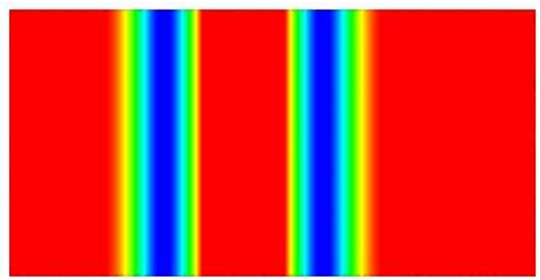
Mode 1



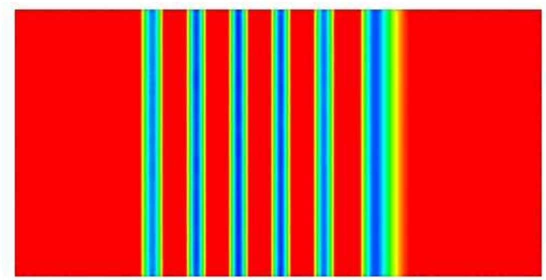
Mode 2



Mode 3



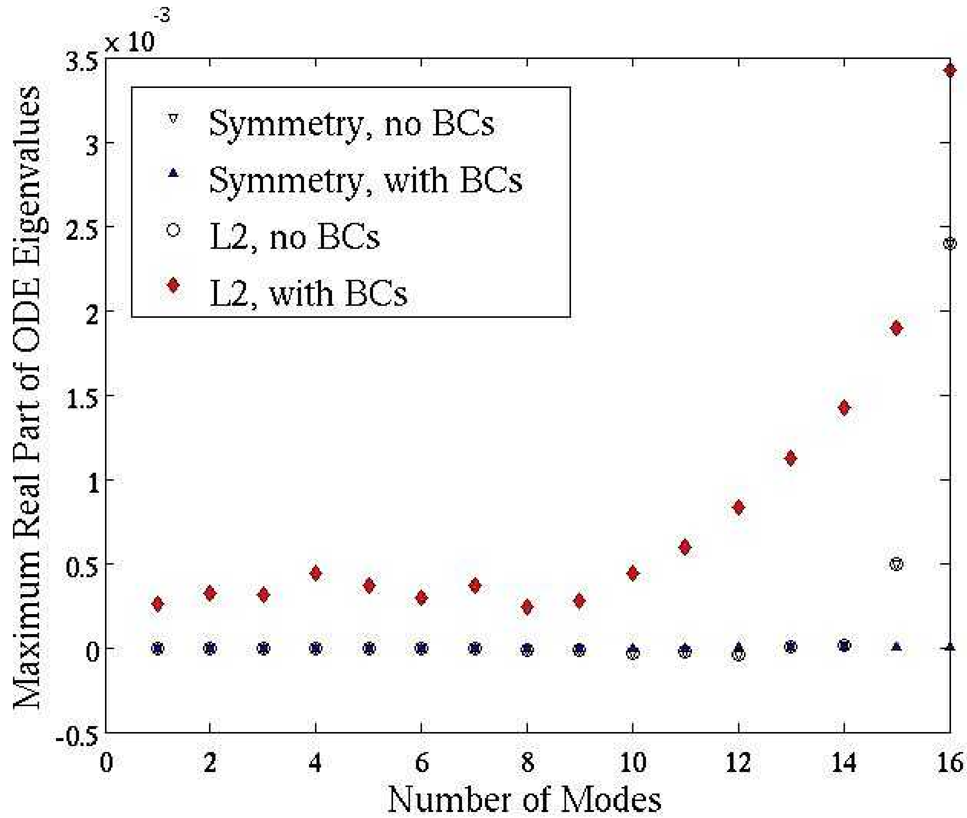
Mode 12



- **Uniform base flow with velocity *U/c* = *M* = *0.5* in the *x*-direction**
- **Acoustic pulse prescribed as an initial condition, propagates in the *x*-direction with velocity *U+c***
- **Slip wall boundary condition applied on the top, bottom, and side walls**

1D Pressure Pulse : Stability

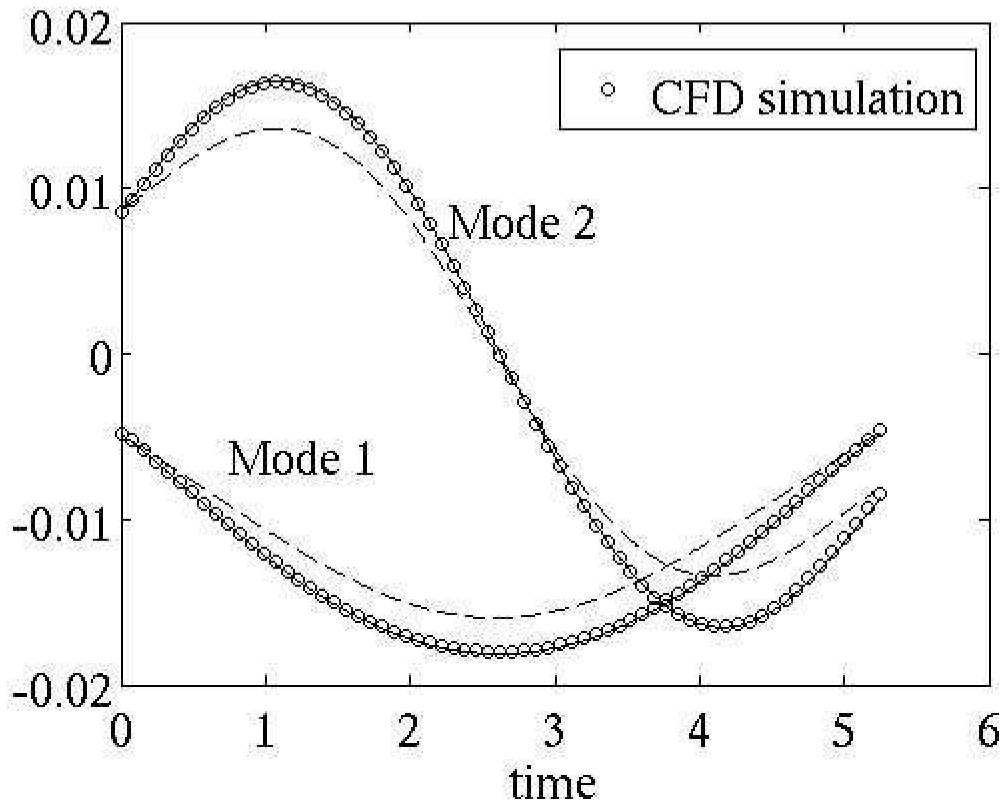
- **Four Galerkin schemes were used**
  - **Symmetry inner product, with and w/o boundary conditions**
  - **L2 inner product, with and w/o boundary conditions**
- **Only the symmetry inner product with boundary conditions remains stable**



1D Pressure Pulse Results

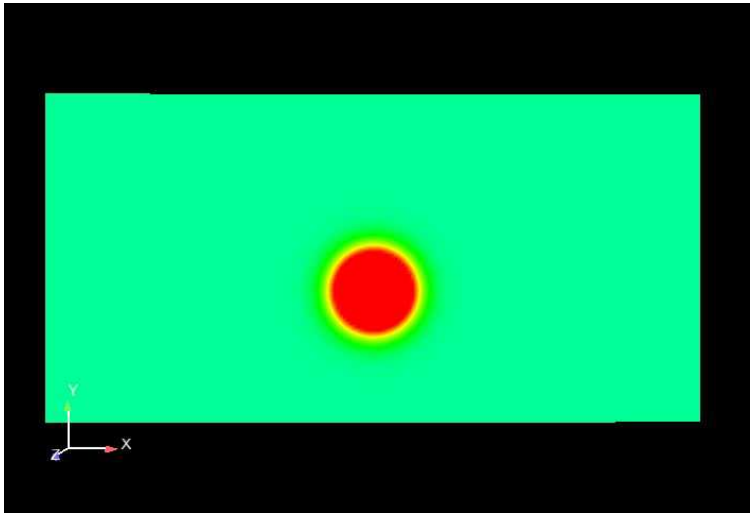
- **Comparison of the ROM result with the projection of the CFD simulation onto the POD modes**
- **This particular ROM appears to be convergent as the number of modes increases**
- **L2 ROM has comparable accuracy, since instability is weak**

Dashed line: 4 mode ROM
Solid line: 8 mode ROM

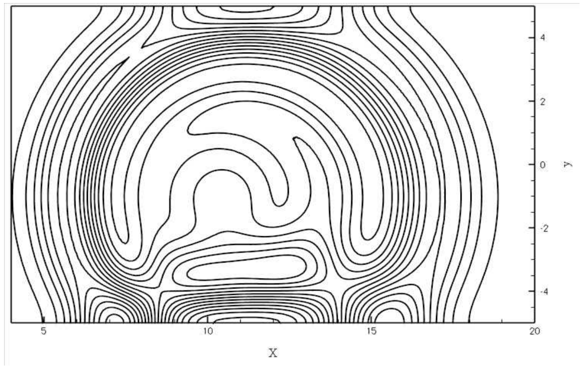


Example: 2D Pressure Pulse

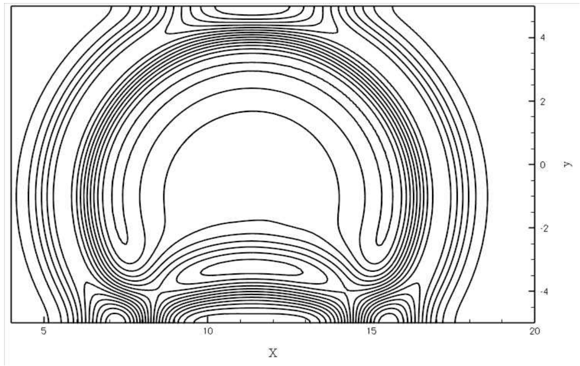
- M=0.25 uniform base flow, cylindrical Gaussian pressure pulse
- The ROM is stable
- Very good qualitative agreement with 12 mode ROM



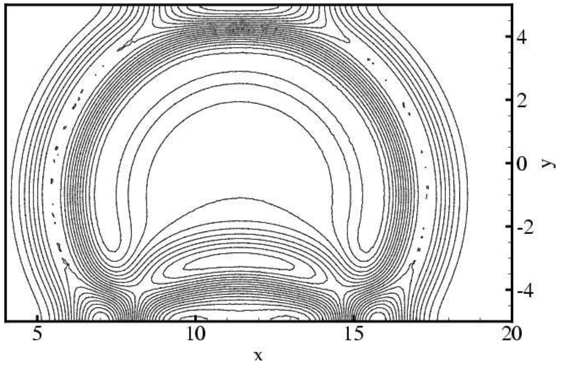
6 Mode ROM



12 Mode ROM



CFD



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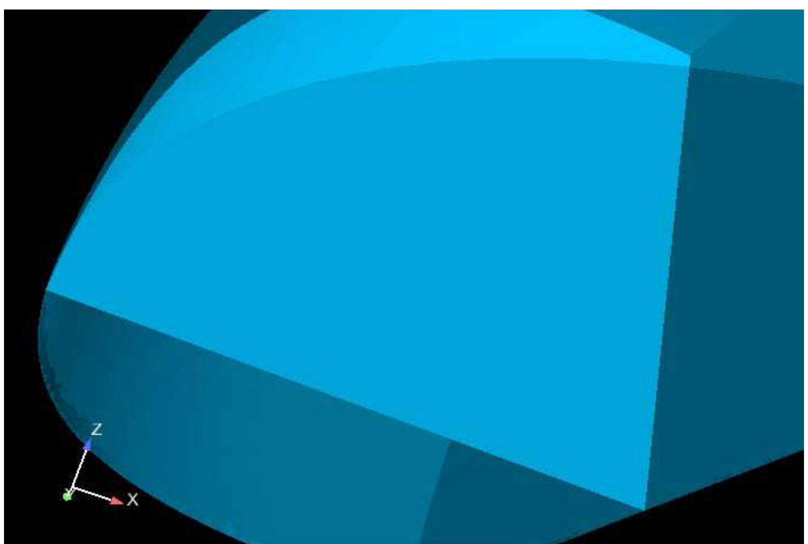
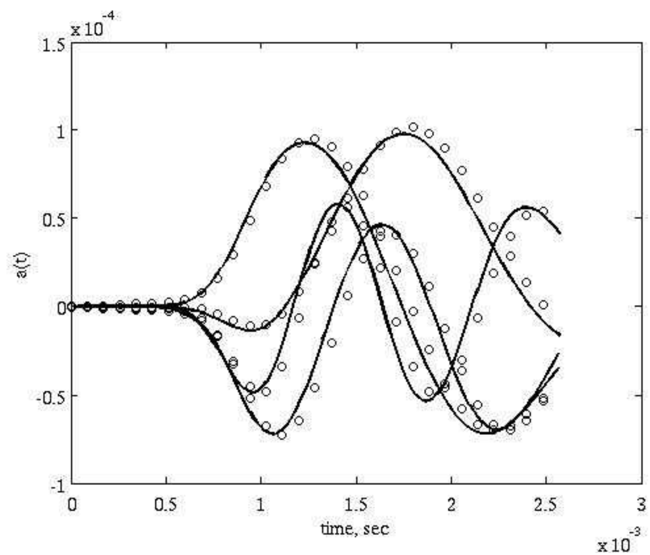
Coupled Fluid-Structure ROM

- **The fluid ROM wall boundary condition allows for coupling of a fluid ROM to a structural dynamics ROM for the case of small (linear) structural displacements.**
- **Transfer of fluid pressure loading to the structure is possible using a fluid/structure boundary integral.**

Forced Flexible Plate





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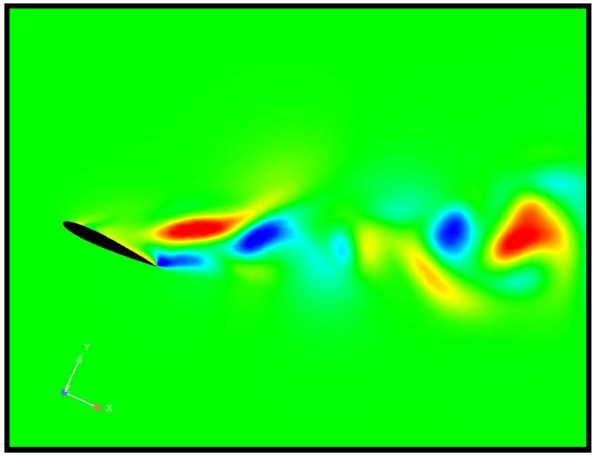
Summary

- **A Galerkin projection technique for linearized, compressible flow**
  - **Numerically stable for any choice of basis**
  - **Weak boundary conditions that preserve stability**
  - **Numerical implementation using finite elements that preserves stability**
- **ROMs using this scheme were demonstrated to be stable on several model problems**

Future Directions

- Examine ROM performance for more complicated systems.
  - Non-uniform base flow.
  - Coupled fluid-structure problem.
- Towards stable Nonlinear ROMs
  - Try out the linear symmetry inner product on the nonlinear equations.
  - The nonlinear Euler (and Navier-Stokes) equations can also be symmetrized, leading to an "entropy-stable" inner product.

Stalled Airfoil Wake



Acknowledgements

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Energy Expression for the Linearized Euler Equations

- Multiply by $\mathbf{q}'^T$, integrate by parts, and use the symmetry of H and HA_j to derive the global "energy" expression:

$$\frac{\partial}{\partial t}\int_\Omega \mathbf{q}'^T H \mathbf{q}' \,\mathrm{d}\Omega = -\int_{\partial\Omega} \mathbf{q}'^T H(A_j n_j)\mathbf{q}' \,\mathrm{d}S + \int_\Omega \mathbf{q}'^T \left(\frac{\partial}{\partial x_j}(HA_j) - HC - C^T H\right)\mathbf{q}' \,\mathrm{d}\Omega$$

- Ignore the boundary integral for now
- The above equation leads to a statement of energy behavior for exact solutions to the linearized Euler equations:

$$\frac{\partial}{\partial t}\int_\Omega \mathbf{q}'^T H \mathbf{q}' \,\mathrm{d}\Omega \leq 2\alpha \int_\Omega \mathbf{q}'^T H \mathbf{q}' \,\mathrm{d}\Omega$$

α is a constant

- Define the "energy inner product" and corresponding norm:

$$(\mathbf{u},\mathbf{v})_H \equiv \int_\Omega \mathbf{u}^T H \mathbf{v} \,\mathrm{d}\Omega \qquad ||\mathbf{q}'||_H \equiv (\mathbf{q}',\mathbf{q}')_H^{1/2}$$

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