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The Peridynamic Theory of Solid Mechanics for Modeling Material Failure and Fracture

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U.S. DEPARTMENT OF ENERGY

NNSA National Nuclear Security Administration

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Peridynamic Theory of Solid Mechanics

WHAT IS PERIDYNAMICS?

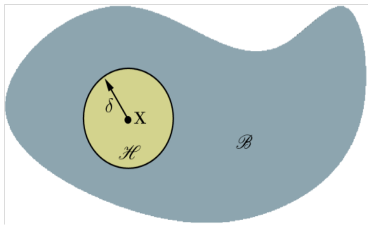
Peridynamics is a mathematical theory that unifies the mechanics of continuous media, cracks, and discrete particles

HOW DOES IT WORK?

- Peridynamics is a *nonlocal* extension of continuum mechanics
- Remains valid in presence of discontinuities, including cracks
- Balance of linear momentum is based on an *integral equation*:

$$\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t) = \underbrace{\int_{\mathscr{B}} \left\{ \underline{\mathbf{T}}[\mathbf{x},t]\langle\mathbf{x}'-\mathbf{x}\rangle - \underline{\mathbf{T}}'[\mathbf{x}',t]\langle\mathbf{x}-\mathbf{x}'\rangle \right\} \, \mathrm{d}V_{\mathbf{x}'}}_{\text{Divergence of stress replaced with integral of nonlocal forces.}} + \mathbf{b}(\mathbf{x},t)$$

The point X interacts directly with all points within its horizon



S.A. Silling. Reformulation of elasticity theory for discontinuities and long-range forces. *Journal of the Mechanics and Physics of Solids*, 48:175-209, 2000.

Silling, S.A. and Lehoucq, R. B. Peridynamic Theory of Solid Mechanics. *Advances in Applied Mechanics* 44:73-168, 2010.

Peridynamic Theory of Solid Mechanics

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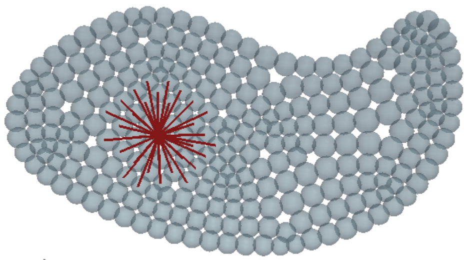
CONSTITUTIVE LAWS IN PERIDYNAMICS

- Peridynamic *bonds* connect any two material points that interact directly
- Peridynamic forces are determined by *force states* acting on bonds

$$\underbrace{\underline{\mathbf{T}}[\mathbf{x},t]}_{\text{Force State}} \; \underbrace{\langle \mathbf{x}'_i - \mathbf{x} \rangle}_{\text{Bond}}$$

- Force states are determined by constitutive laws and are functions of the deformations of all points within a neighborhood
- *Material failure* is modeled through the breaking of peridynamic bonds
  - Example: critical stretch bond breaking law

DISCRETIZATION OF A PERIDYNAMIC BODY



Direct discretization of the strong form of the balance of linear momentum [1]

$$\rho(\mathbf{x})\ddot{\mathbf{u}}_h(\mathbf{x},t) = \sum_{i=0}^{N} \left\{ \underline{\mathbf{T}}[\mathbf{x},t] \langle \mathbf{x}'_i - \mathbf{x} \rangle - \underline{\mathbf{T}}'[\mathbf{x}'_i,t] \langle \mathbf{x} - \mathbf{x}'_i \rangle \right\} \Delta V_{\mathbf{x}'_i} + \mathbf{b}(\mathbf{x},t)$$

[1] S.A. Silling and E. Askari. A meshfree method based on the peridynamic model of solid mechanics. *Computers and Structures*, 83:1526-1535, 2005.

Constitutive Models for Peridynamics

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PERIDYNAMIC FORCE STATES MAP BONDS TO PAIRWISE FORCE DENSITIES

- Peridynamic constitutive laws can be grouped into two categories
  - *Bond-based*: bond forces depend only on a single pair of material points
  - *State-based*: bond forces depend on deformations of all neighboring material points

Microelastic Material [1]

- Bond-based constitutive model
- Pairwise forces are a function of bond stretch

$$s = \frac{y - x}{x}$$

- Magnitude of pairwise force density given by

$$\underline{t} = \frac{18k}{\pi\delta^4} s$$

Linear Peridynamic Solid [2]

- State-based constitutive model
- Deformation decomposed into deviatoric and dilatational components

$$\theta = \frac{3}{m} \int_{\mathcal{H}} (\underline{\omega}\, \underline{x}) \cdot \underline{e}\, dV \qquad \underline{e}^d = \underline{e} - \frac{\theta\, \underline{x}}{3}$$

- Magnitude of pairwise force density given by

$$\underline{t} = \frac{3k\theta}{m}\, \underline{\omega}\, \underline{\mathrm{x}} + \frac{15\mu}{m}\, \underline{\omega}\, \underline{e}^{\mathrm{d}}$$

Definitions

$\underline{x}$	bond vector
x	initial bond length
y	deformed bond length
s	bond stretch
$\underline{e}$	bond extension
$\underline{e}^d$	deviatoric bond extension
$\underline{\omega}$	influence function
V	volume
$\mathcal{H}$	neighborhod
m	weighted volume
θ	dilatation
δ	horizon
k	bulk modulus
μ	shear modulus
$\underline{t}$	pairwise force density

1. S.A. Silling. Reformulation of elasticity theory for discontinuities and long-range forces. *Journal of the Mechanics and Physics of Solids*, 48:175-209, 2000.
2. S.A. Silling, M. Epton, O. Weckner, J. Xu, and E. Askari, Peridynamic states and constitutive modeling, *Journal of Elasticity*, 88, 2007.

Classical Material Models Can be Applied in Peridynamics

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NON-ORDINARY STATE-BASED APPROACH [1]

1. Compute an approximate deformation gradient based on the initial and current locations of material points in nonlocal neighborhood

<u>Approximate Deformation Gradient</u>

$$\bar{\mathbf{F}} = \left(\sum_{i=0}^{N} \underline{\omega}_i \, \underline{\mathbf{Y}}_i \otimes \underline{\mathbf{X}}_i \, \Delta V_{\mathbf{x}_i} \right) \mathbf{K}^{-1}$$

<u>Shape Tensor</u>

$$\mathbf{K} = \sum_{i=0}^{N} \underline{\omega}_i \, \underline{\mathbf{X}}_i \otimes \underline{\mathbf{X}}_i \, \Delta V_{\mathbf{x}_i}$$

2. Kinematic data passed to classical material model
3. Classical material model computes stress
4. Stress converted to pairwise forces

$$\underline{\mathbf{T}} \left\langle \mathbf{x}' - \mathbf{x} \right\rangle = \underline{\omega} \, \sigma \mathbf{K}^{-1} \left\langle \mathbf{x}' - \mathbf{x} \right\rangle$$

5. Apply stabilization term to suppress low-energy modes (optional)

[1] S. Silling, M. Epton, O. Weckner, J. Xu, and E. Askari. Peridynamic states and constitutive modeling. Journal of Elasticity, 88:151-184, 2007.

Material Failure Is Controlled by a Bond-Failure Law

THE CRITICAL-STRETCH MODEL IS THE SIMPLEST BOND-FAILURE LAW [1]

- A bonds fails when its extension exceeds a critical value
- Bond failure is irreversible

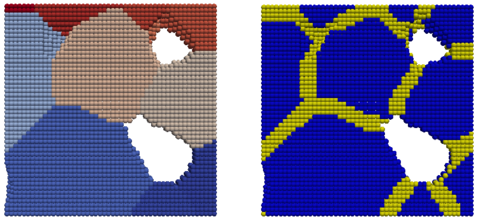
$$s_{\mathrm{max}} = \frac{||\underline{e}||_{\mathrm{max}}}{||\underline{x}||}$$

$$\phi = \begin{cases} 0 & \text{if} \quad s_{\mathrm{max}} < s_{\mathrm{crit}} \\ 1 & \text{if} \quad s_{\mathrm{max}} \geq s_{\mathrm{crit}} \end{cases}$$

- Damage results from the accumulation of broken bonds
- Critical stretch parameter is tied to the energy release rate (experimentally measureable)

Example: Modified critical-stretch law for polycrystalline materials [2]

- Modified critical-stretch law for failure of polycrystalline material



- Bond failure law favors material damage along grain boundaries
- Contact algorithm controls material interactions after bonds are broken

1. Silling, S.A. and Askari, E. A meshfree method based on the peridynamic model of solid mechanics. *Computers and Structures* 83:1526-1535, 2005.

2. D. Littlewood, V. Tikare, and J. Bignell. Informing Macroscale Constitutive Laws through Modeling of Grain-Scale Mechanisms in Plutonium Oxide. Workshop on Nonlocal Damage and Failure: Peridynamics and Other Nonlocal Models, San Antonio, Texas, March 11-12 2013.

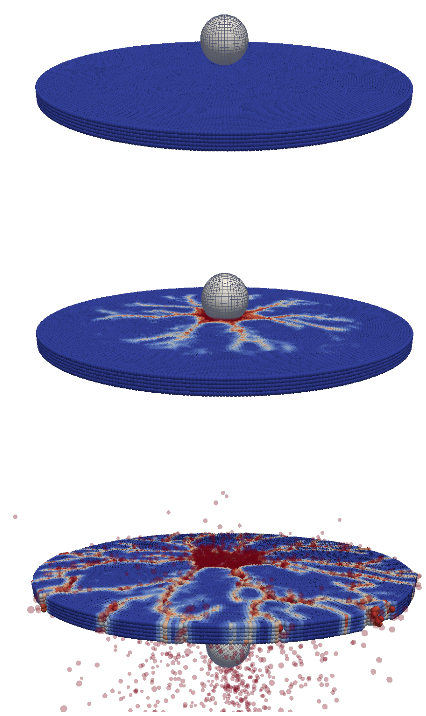
Contact in Peridynamic Simulations

- A *short-range force* approach has been used in the majority of peridynamic simulations to date [1]
  - Spring-like repulsive force
  - Active when relative distance, *r*, is below contact radius, r_c

$$f_{\mathrm{c}} = \begin{cases} C\,(r_{\mathrm{c}} - r)\,\Delta V_1 \Delta V_2 & \text{if} \quad r \leq r_{\mathrm{c}} \\ 0 & \text{if} \quad r > r_{\mathrm{c}} \end{cases}$$

  - Does not require explicit definition of contact surfaces
  - Friction may be incorporated by decomposing relative motion into normal and tangential components

- More sophisticated contact models are possible
  - Example: iterative penalty enforcement to drive the contact gap to zero [2]



Simulation of brittle fracture

1. Silling, S.A. and Askari, E. A meshfree method based on the peridynamic model of solid mechanics. *Computers and Structures* 83:1526-1535, 2005.
2. SIERRA Solid Mechanics Team, Sierra/SolidMechanics 4.22 user's guide, SAND Report 2011-7597, Sandia National Laboratories, Albuquerque, NM and Livermore, CA, 2011.

Relationship between Classical and Peridynamic Theories

PERIDYNAMIC OPERATORS ARE ANALOGUES OF THE CLASSICAL THEORY

Relation	**Peridynamic Theory**	**Standard Theory**
Kinematics	$\underline{\mathbf{Y}}\langle \mathbf{x}' - \mathbf{x}\rangle = \mathbf{y}(\mathbf{x}') - \mathbf{y}(\mathbf{x})$	$\mathbf{F} = \frac{\partial \mathbf{y}}{\partial \mathbf{x}}(\mathbf{x})$
Linear Momentum Balance	$\rho\ddot{\mathbf{u}}(\mathbf{x}) = \int_{\mathcal{H}_{\mathbf{x}}} \{\underline{\mathbf{T}}[\mathbf{x},t]\langle \mathbf{x}' - \mathbf{x}\rangle - \underline{\mathbf{T}}[\mathbf{x}',t]\langle \mathbf{x} - \mathbf{x}'\rangle\}\, dV_{\mathbf{x}'} + \mathbf{b}(\mathbf{x})$	$\rho\ddot{\mathbf{u}}(\mathbf{x}) = \nabla \cdot \boldsymbol{\sigma}(\mathbf{x}) + \mathbf{b}(\mathbf{x})$
Constitutive Model	$\underline{\mathbf{T}} = \hat{\underline{\mathbf{T}}}(\underline{\mathbf{Y}})$	$\boldsymbol{\sigma} = \widehat{\boldsymbol{\sigma}}(\mathbf{F})$
Angular Momentum Balance	$\int_{\mathcal{H}_{\mathbf{x}}} \{\underline{\mathbf{Y}}\langle \mathbf{x}' - \mathbf{x}\rangle \times \underline{\mathbf{T}}\langle \mathbf{x}' - \mathbf{x}\rangle\}\, dV_{\mathbf{x}'} = 0$	$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$

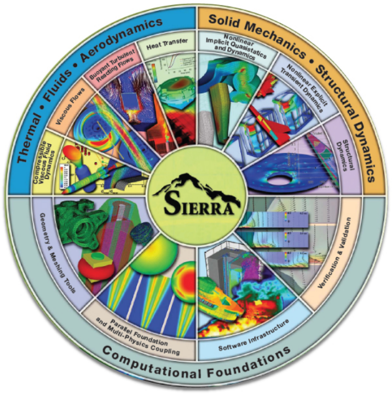
Peridynamic Codes

PERIDIGM

- Computing Research Center's open-source computational peridynamics code
- Built on *Trilinos* software toolset

SIERRA/SOLIDMECHANICS

- Engineering mechanics simulation code suite supporting the nation's nuclear weapons mission as well as other customers
- Advanced Simulation and Computing (ASC) code



OTHERS

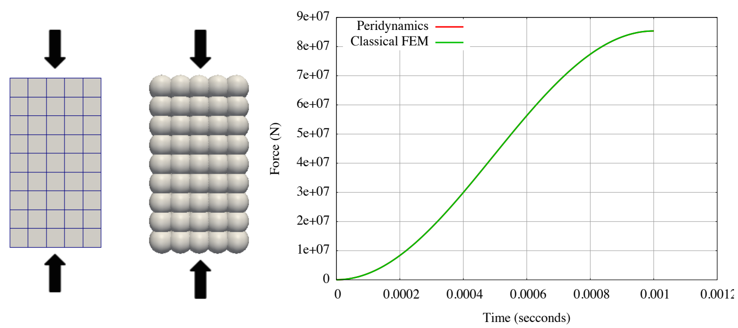
- EMU, LAMMPS

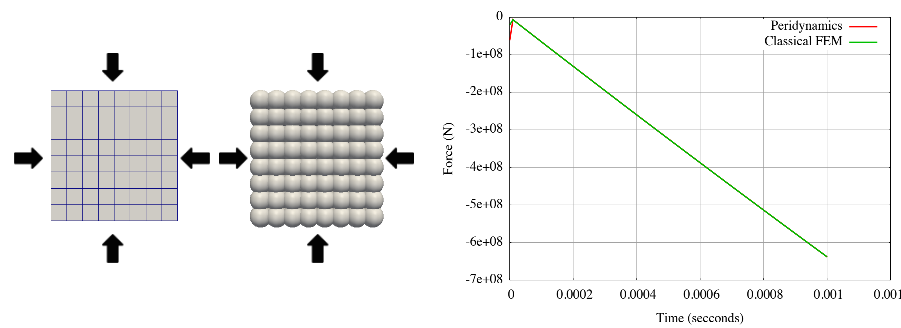
1. Parks, M.L., Littlewood, D.J., Mitchell, J.A., and Silling, S.A. Peridigm users' guide v1.0.0. Sandia Report SAND-2012-7800, 2012.
2. SIERRA Solid Mechanics Team. Sierra/SolidMechanics 4.32 user's guide, 2014.

Examples of Simple Test Problems

Uniaxial and hydrostatic compression

- Tests constructed such that peridynamics and classical FEM should yield same result
- Simulation results verified for numerous material models

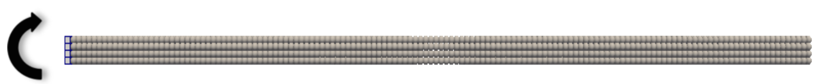




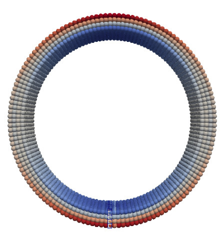
Beam bending

- Test peridynamics with neo-Hookean material model against classical beam bending theory
- Simulation gives expected bending response and stress distribution

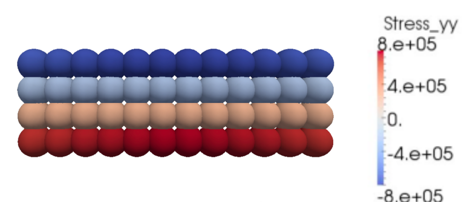
Applied rotation



Fixed support



Increased pure bending eventually produces circle



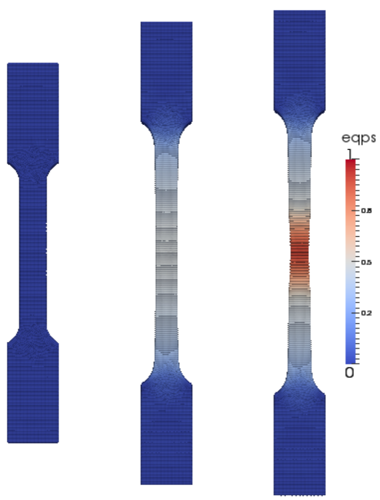
Linear stress distribution through cross section

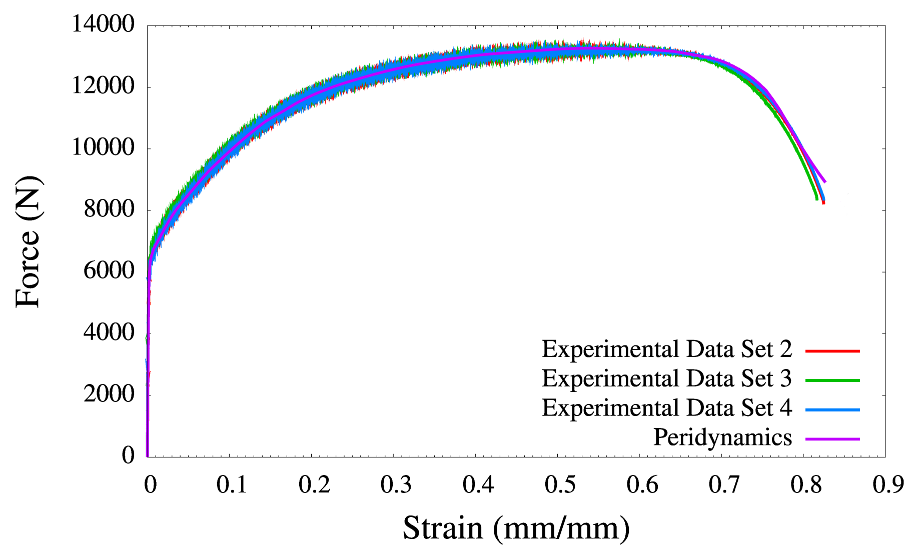
Dogbone Tensile Test

CONSTITUTIVE MODEL CALIBRATINO AGAINST EXPERIMENTAL DATA

- Dogbone specimen
  - 304L stainless steel (very ductile)
  - Quasi-static loading conditions
- Peridynamic model
  - Non-ordinary state-based peridynamic
  - Elastic-plastic material constitutive model

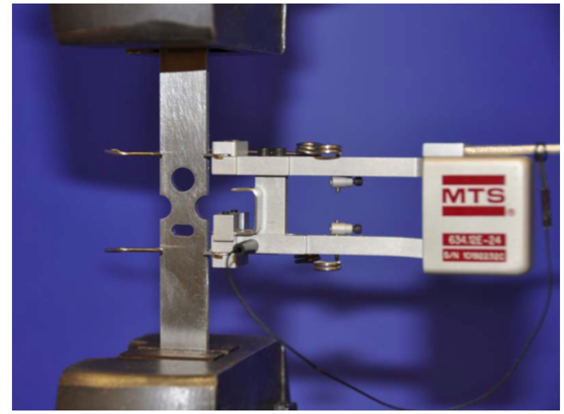
Young's Modulus	199.95e3 MPa
Poisson's Ratio	0.285
Yield Stress	220.0 MPa
Piecewise linear hardening curve	



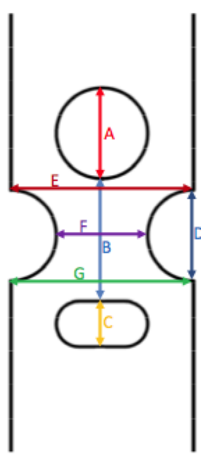


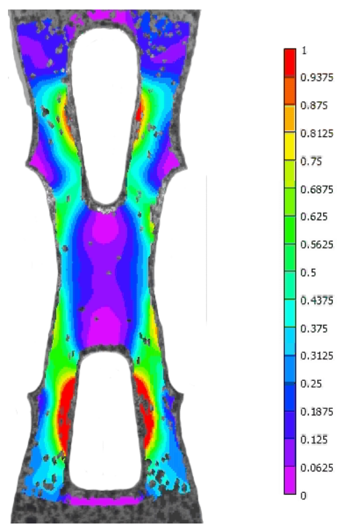
Necking Experiment

- Modified dogbone specimen
  - 304L stainless steel (very ductile)
  - Quasi-static loading conditions
- Peridynamic model
  - Non-ordinary state-based peridynamic
  - Elastic-plastic material constitutive model

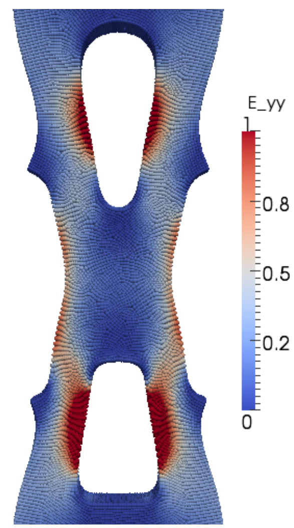


[Boyce]

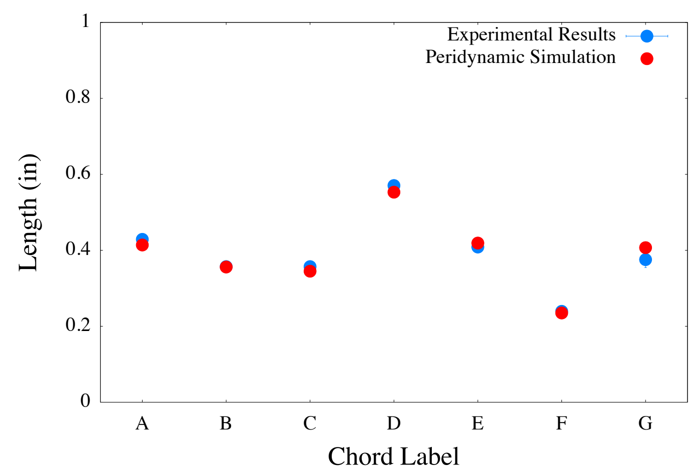




Experimental DIC image [Boyce]



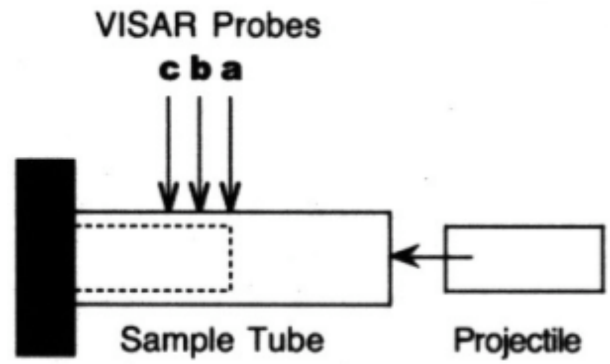
Post-test simulation result



Expanding Tube Experiment

- Experimental setup:
  - Tube expansion via collision of Lexan projectile and plug within AerMet tube
  - Accurate recording of velocity and displacement on tube surface

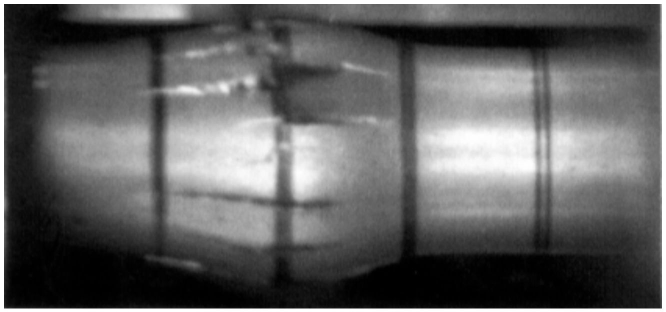
- Modeling approach:
  - AerMet tube modeled with peridynamics, elastic-plastic material model with linear hardening
  - Lexan plugs modeled with traditional FEM, EOS-enabled Johnson-Cook material model



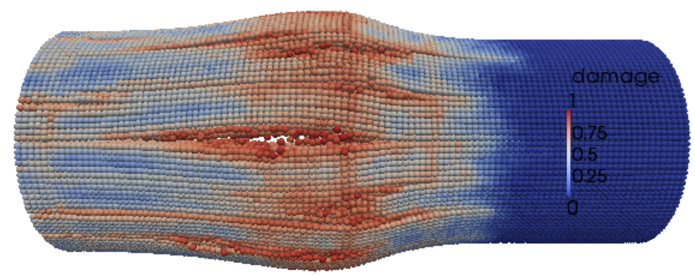
[Vogler et. al]

Vogler, T.J., Thornhill, T.F., Reinhart, W.D., Chhabidas, L.C., Grady, D.E., Wilson, L.T., Hurricane, O.A., and Sunwoo, A. Fragmentation of materials in expanding tube experiments. *International Journal of Impact Engineering*, 29:735-746, 2003.

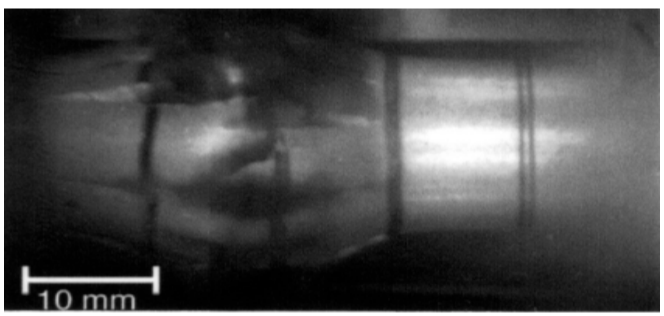
Expanding Tube Experiment



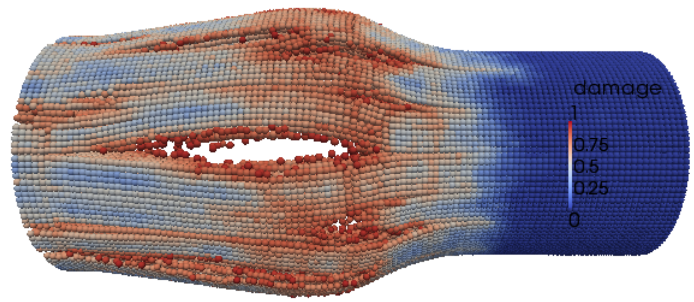
Experimental image at 15.4 microseconds [Vogler et. al]



Simulation at 15.4 microseconds



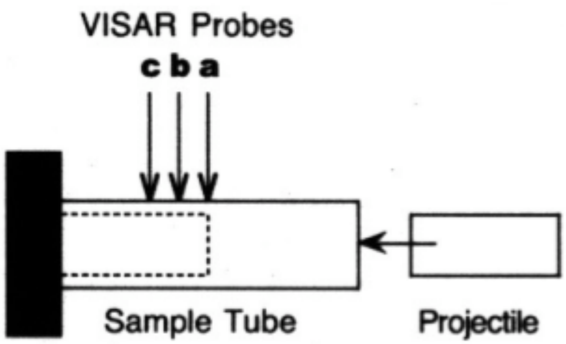
Experimental image at 23.4 microseconds [Vogler et. al]

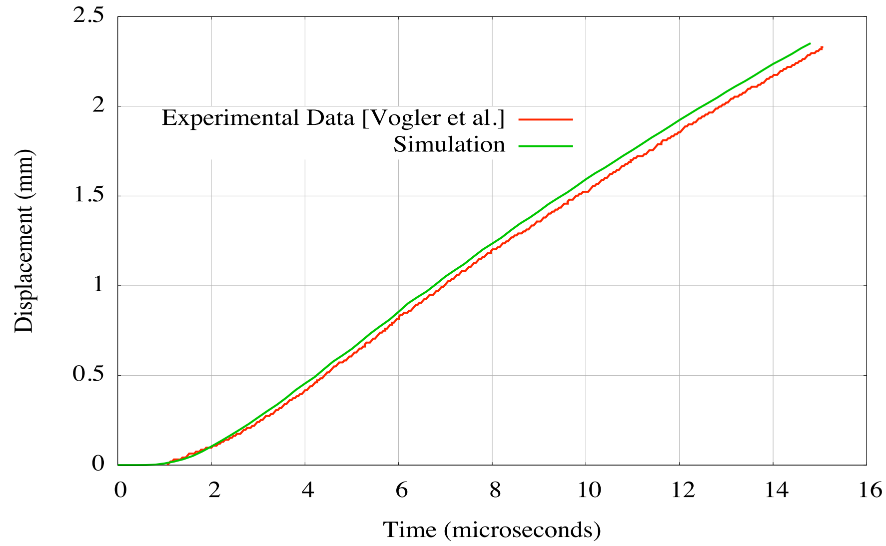


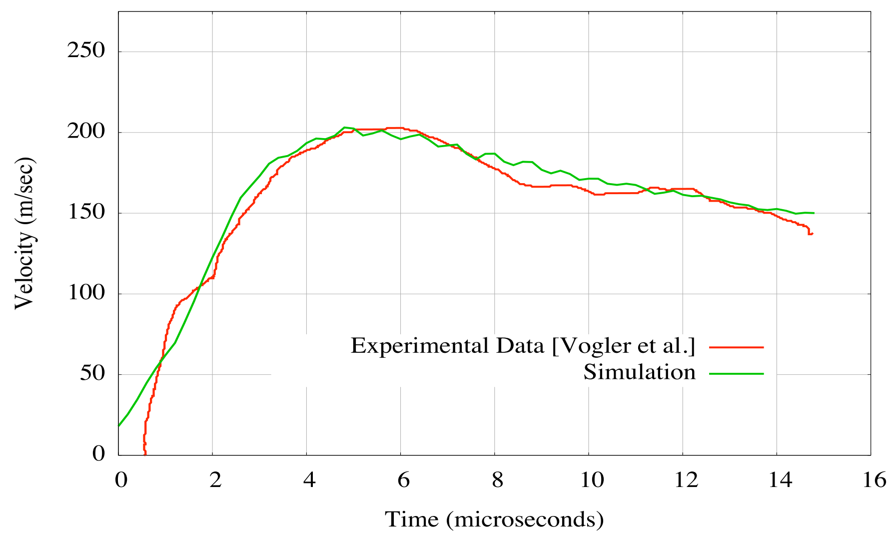
Simulation at 23.4 microseconds

Predicted Displacement and Velocity on Tube Surface

Displacement and velocity
on tube surface
at probe position A



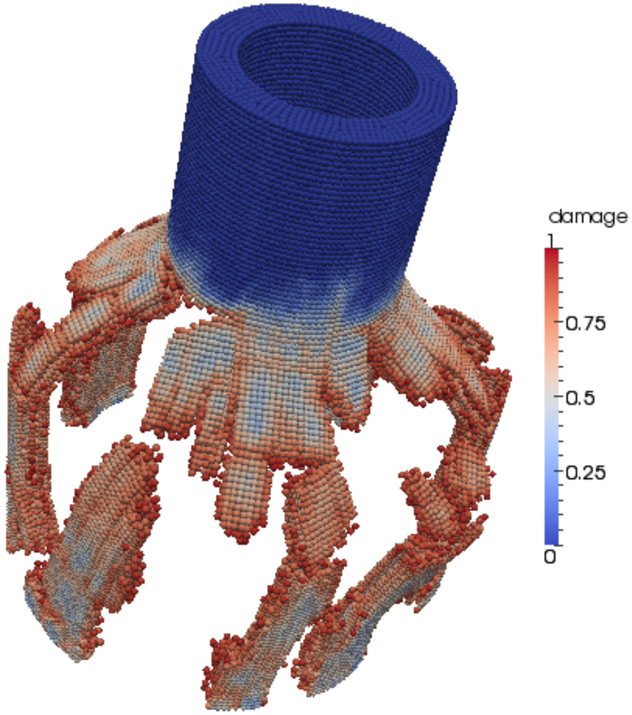




Fragmentation Pattern

QUALITATIVE COMPARISON OF FRAGMENTATION RESULTS

- Vogler et. al reported significant uncertainty in results at late time
- Approximately half the tube remained intact
- Vogler et. al recovered 14 fragments with mass greater than one gram



Simulation at 84.8 microseconds

Ongoing R&D Efforts in Peridynamics

- ★ Local-nonlocal coupling for integrated fracture modeling
  - Peridynamic partial stress formulation to enable a variable horizon
  - Blending-based coupling approaches
- Position-aware constitutive laws
  - Address behavior of a class of peridynamic constitutive laws at free surfaces
- Improvements to meshfree discretization
- Coarse graining approaches for multiscale modeling
- Multi-physics models
  - Mechanics, thermal, fluid flow, etc.
- Validation of peridynamics for specific Sandia applications
  - Problems of direct relevance to national security
  - Comparison against experimental data

Local-Nonlocal Coupling

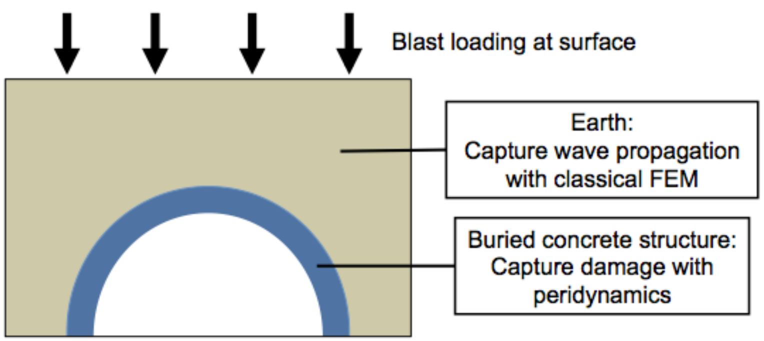
APPLICATION OF PERIDYNAMICS TO SANDIA MISSIONS REQUIRES INTEGRATION WITH CLASSICAL MODELS

- Standard finite element codes based on classical continuum mechanics provide a robust and mature technology for a broad set of applications
- Peridynamics offers a framework for modeling material failure
- Goal: Unify the strengths of peridynamics and classical continuum mechanics

KEY CHALLENGES

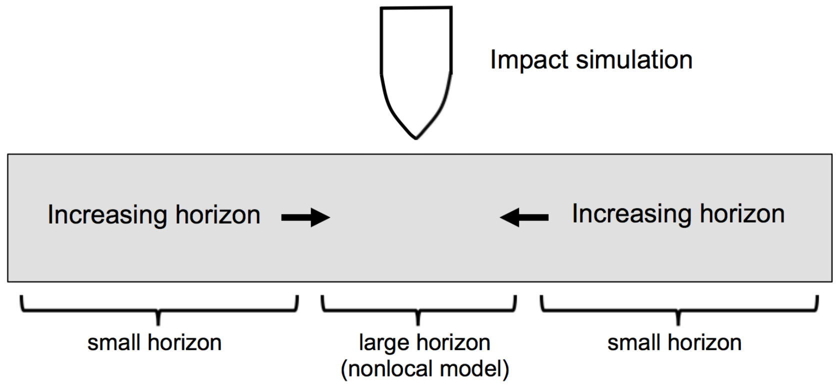
- The nonlocal governing equations of peridynamics differ inherently from those in the classical (local) theory
- Coupling strategies must avoid nonphysical artifacts at the interface of local and nonlocal models

Vision
Apply peridynamics only in regions susceptible to material failure



New Research Focus: Variable Nonlocal Length Scale

A variable nonlocal length scale facilitates transitions between peridynamics and classical continuum mechanics



EXISTING FORMULATIONS MUST BE EXPANDED TO SUPPORT A VARIABLE HORIZON

- Current peridynamic constitutive models do not support a variable horizon
- Goal: develop an alternative formulation that mitigates spurious artifacts in the presence of a variable nonlocal length scale
- Approach: target one-dimensional patch tests (expose spurious artifacts, if any)
  - Linear displacement field must be equilibrated
  - Quadratic displacement field must produce constant acceleration

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Proposed Solution

PARTIAL STRESS FORMULATION FOR PERIDYNAMIC CONSTITUTIVE LAWS [1]

$$\nu_o(\mathbf{x}) := \int_{\mathcal{H}} \underline{\mathbf{T}}[\mathbf{x}]\langle\xi\rangle \otimes \xi \, dV_{\mathbf{x}'}$$

- Guaranteed to pass the linear patch test (even with a varying horizon)
- Partial stress and full peridynamic stress[2] are equal if the force state T[x] is independent of x
  - Example: homogeneous body under homogeneous deformation
  - Result suggests that partial stress is a good approximation of the full peridynamic stress under smooth deformation
- Partial stress formulation is not a good candidate for modeling material failure
- Provides a natural transition between the full peridynamic formulation and a classical stress-strain formulation (hybrid approach)

[1] Silling, S., and Seleson, P., Variable Length Scale in a Peridynamic Body, SIAM Conference on Mathematical Aspects of Materials Science, Philadelphia, PA, June 12, 2013.

[2] Lehoucq, R.B., and Silling, S.A. Force flux and the peridynamic stress tensor, Journal of the Mechanics and Physics of Solids, 56:1566-1577, 2008.

Application of Partial Stress within Peridynamics Framework

INTERNAL FORCE CALCULATION REQUIRES DIVERGENCE OPERATOR

- Internal force evaluated as divergence of partial stress

$$\mathbf{L}(\mathbf{x}) = \nabla \cdot \nu(\mathbf{x}) = \mathrm{Tr}(\nabla \nu(\mathbf{x}))$$

$$\nabla \nu(\mathbf{x}) = \int_{\mathcal{H}} \underline{\omega} \langle \xi \rangle \{\nu(\mathbf{x}') - \nu(\mathbf{x})\} \otimes \xi \, dV_{\mathbf{x}'} \; \mathbf{K}^{-1}$$

- The partial stress can be applied within the meshless approach of Silling and Askari [1]

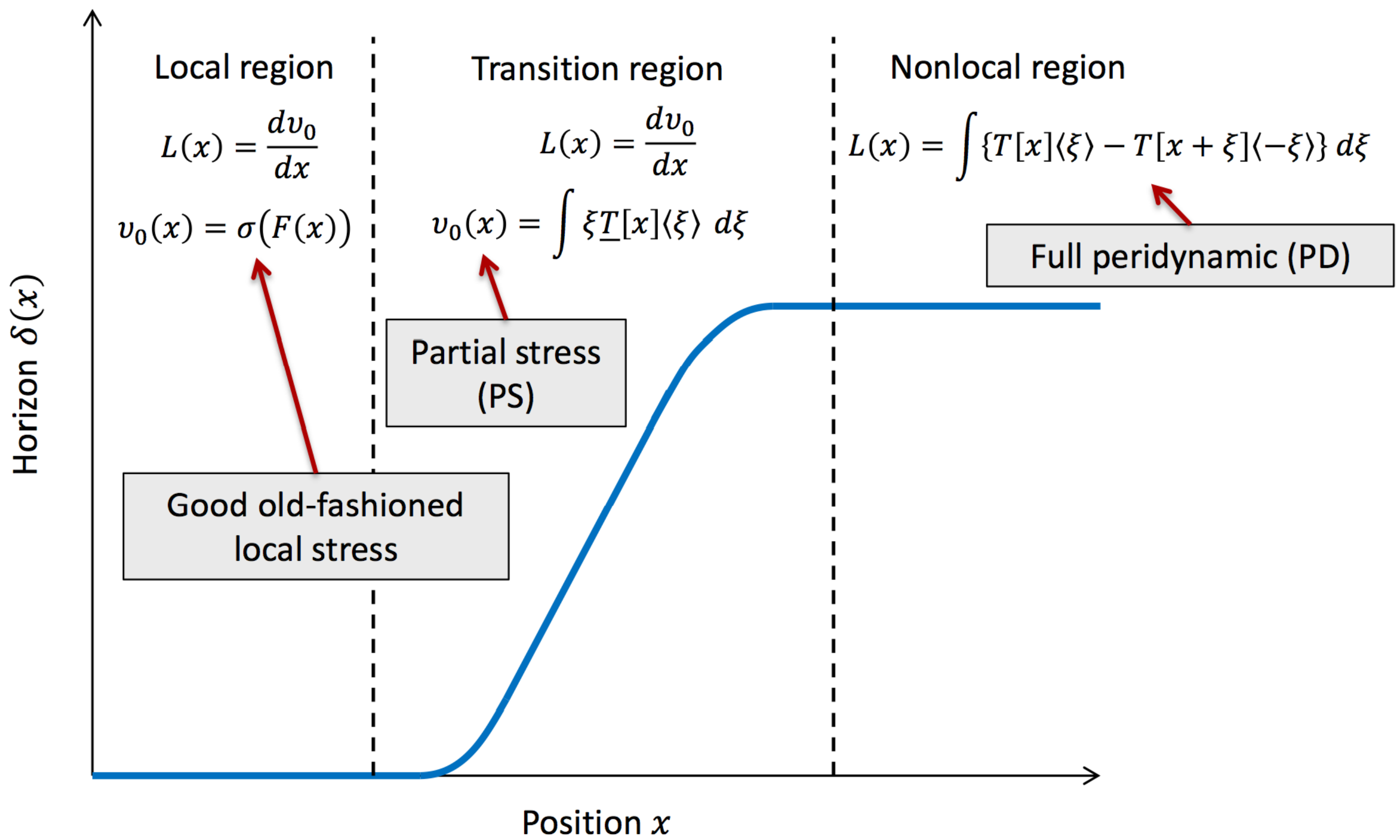
$$\nabla \cdot \nu(\mathbf{x}) = \mathrm{Tr}\left(\left(\sum_{n=1}^{N} \underline{\omega} \langle \xi^n \rangle \{\nu(\mathbf{x}^n) - \nu(\mathbf{x})\} \otimes \xi^n \, \Delta V^n \right) \mathbf{K}^{-1} \right)$$

- The partial stress can also be applied within a standard finite-element scheme

[1] S.A. Silling and E. Askari. A meshfree method based on the peridynamic model of solid mechanics. *Computers and Structures*, 83:1526-1535, 2005.

Utilize the Partial Stress Formulation in a Transition Region

ALTER THE PERIDYNAMIC HORIZON WITHIN A BODY TO APPLY NONLOCALITY ONLY WHERE NEEDED



[Stewart Silling]

Patch Tests for Partial Stress Formulation

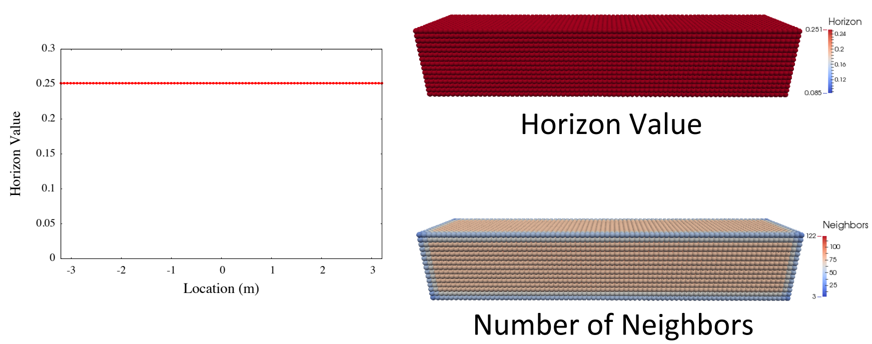
SUBJECT RECTANGULAR BAR TO PRESCRIBED DISPLACEMENT FIELDS

- Examine response under linear and quadratic displacement fields
- Investigate standard formulation with both constant and varying peridynamic horizon
- Investigate partial stress formulation with both constant and varying peridynamic horizon

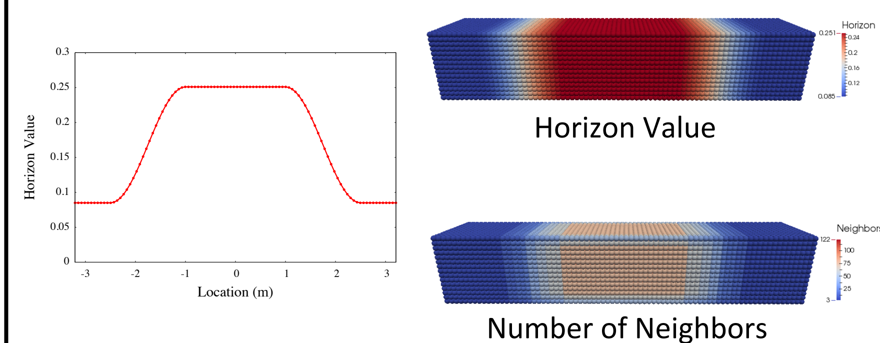
Elastic Correspondence Material Model

Density	7.8 g/cm^3
Young's Modulus	200.0 GPa
Poisson's Ratio	0.0
Stability Coefficient	0.0

Constant Horizon



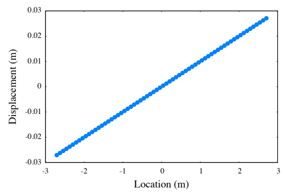
Varying Horizon



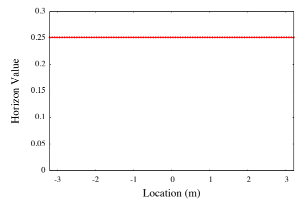
Patch Test: Prescribed Linear Displacement

<u>Test set-up</u>

Prescribe linear displacement field



Constant horizon throughout bar

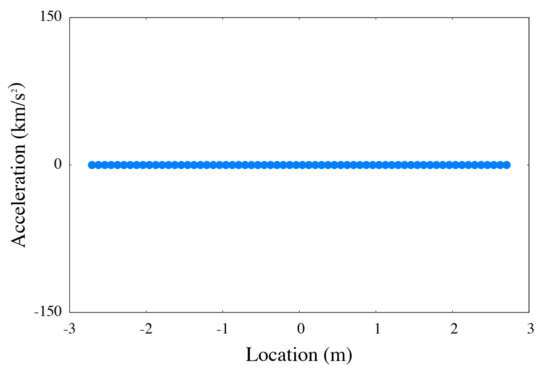


Can the standard model and the partial-stress model recover the expected zero acceleration?

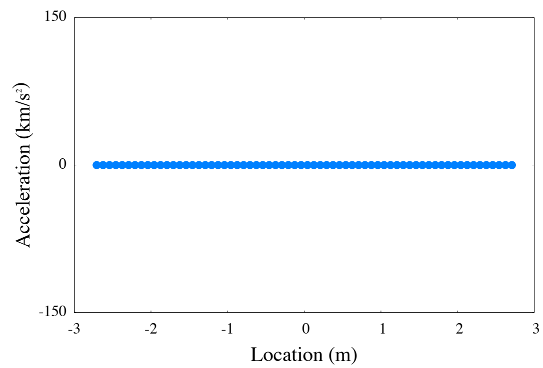
Both models produce the expected result when the horizon is **constant**

Test Results: Acceleration over the length of the bar

Standard material model



Partial-stress formulation

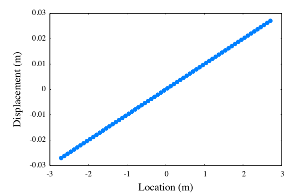


Note: nodes near ends of bar excluded from plots

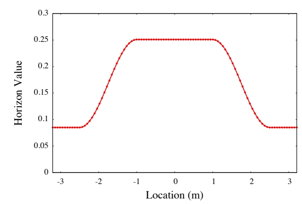
Patch Test: Prescribed Linear Displacement

Test set-up

Prescribe linear displacement field



Variable horizon

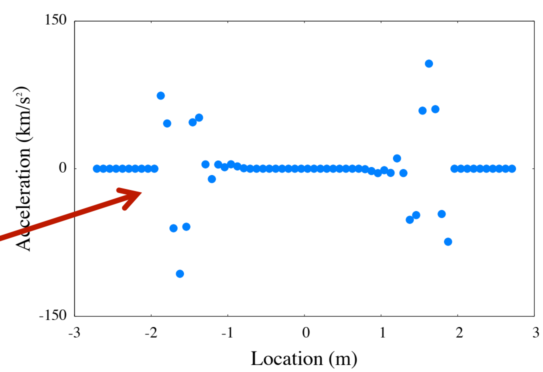


Can the standard model and the partial-stress model recover the expected zero acceleration?

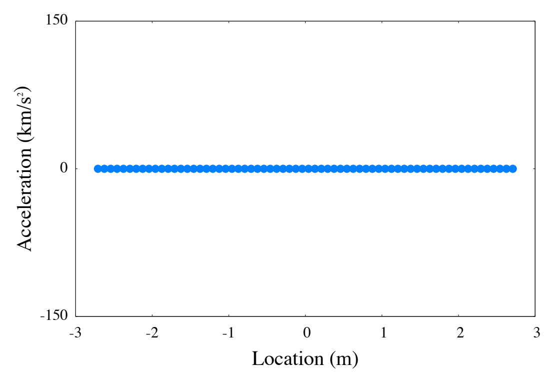
Only the **partial stress** formulation produce the expected result when the horizon is **varying**

Test Results: Acceleration over the length of the bar

Standard material model



Partial-stress formulation

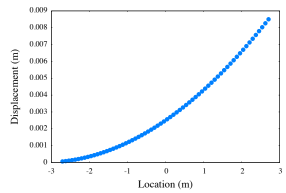


Spurious "ghost forces" present in standard formulation

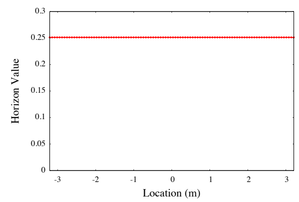
Patch Test: Prescribed Quadratic Displacement

Test set-up

Prescribe quadratic displacement field



Constant horizon throughout bar

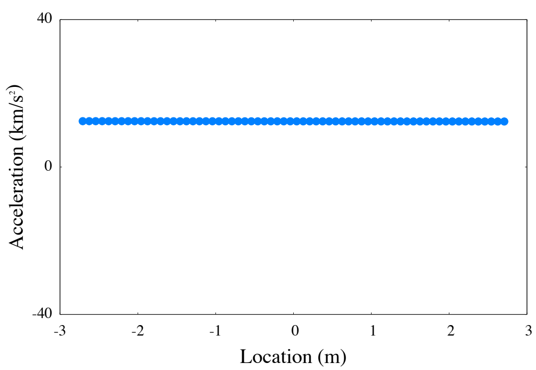


Can the standard model and the partial-stress model recover the expected constant acceleration profile?

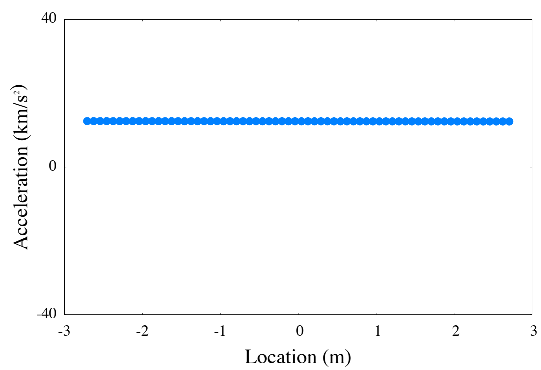
Both models produce the expected result when the horizon is **constant**

Test Results: Acceleration over the length of the bar

Standard material model



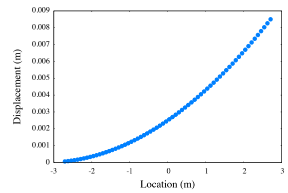
Partial-stress formulation



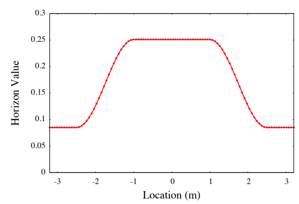
Patch Test: Prescribed Quadratic Displacement

Test set-up

Prescribe quadratic displacement field



Variable horizon

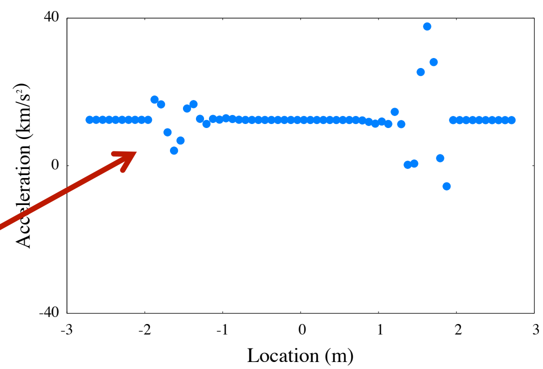


Can the standard model and the partial-stress model recover the expected constant acceleration?

Only the **partial stress** formulation produce the expected result when the horizon is **varying**

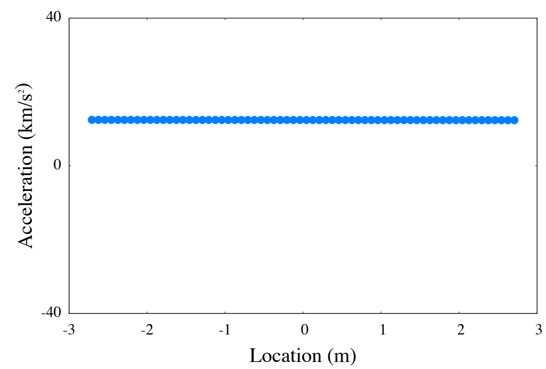
Test Results: Acceleration over the length of the bar

Standard material model



Spurious "ghost forces" present in standard formulation

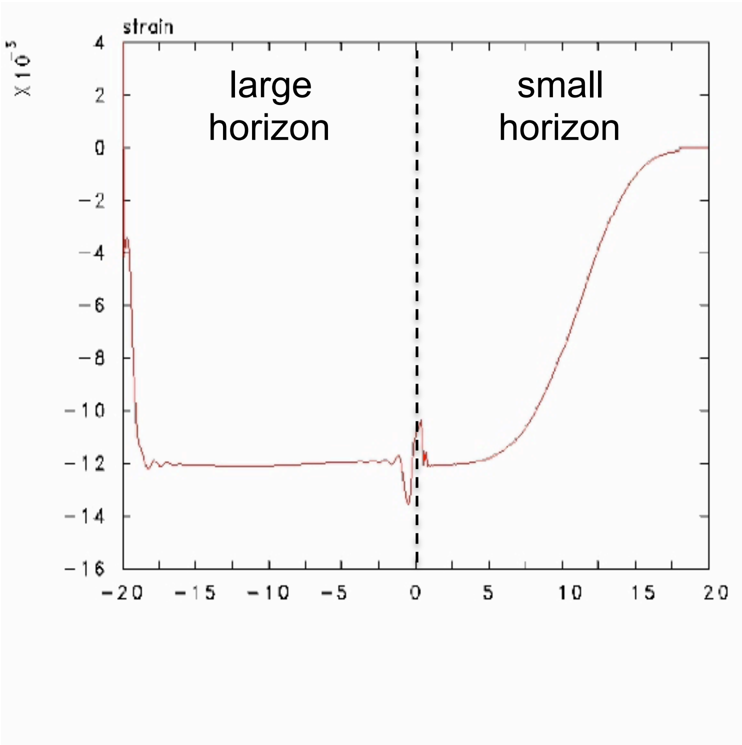
Partial-stress formulation



Wave Propagation through Region of Varying Horizon

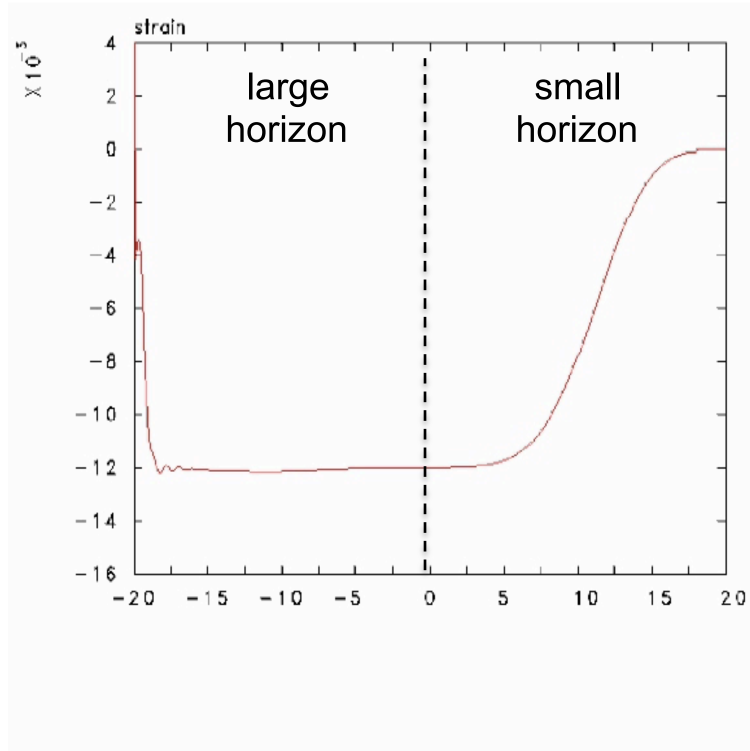
<u>Standard peridynamic model</u>

Numerical artifacts present at transition from large horizon to small horizon



<u>Partial-stress approach</u>

Greatly reduces artifacts, enables smooth transition between large and small horizons



[1] Silling, S., and Seleson, P., Variable Length Scale in a Peridynamic Body, SIAM Conference on Mathematical Aspects of Materials Science, Philadelphia, PA, June 12, 2013.

What about Performance?

USE OF A VARIABLE HORIZON IMPACTS PERFORMANCE IN SEVERAL WAYS

- Use of a variable horizon can reduce neighborhood size
  - Less computational cost per internal force evaluation
  - Reduces number of unknowns in stiffness matrix for implicit time integration
- Use of a variable horizon can reduce the critical time step
  - Critical time step is strongly dependent on the horizon [1, 2]
  - Smaller time step results in more total steps to solution for explicit transient dynamic simulations
  - Important note: the critical time step for analyses combining peridynamics and classical finite analysis is generally determine by the classical finite elements

Total Number of Bonds
(equal to number of nonzeros in stiffness matrix)

Constant Horizon	92.6 million
Varying Horizon	46.5 million

Stable Time Step [1, 2]
(explicit transient dynamics)

Constant Horizon	2.03e-5 sec.
Varying Horizon	7.15e-6 sec.

[1] S.A. Silling and E. Askari. A meshfree method based on the peridynamic model of solid mechanics. *Computers and Structures*, 83:1526-1535, 2005.

[2] Littlewood, D.J, Thomas, J.D., and Shelton, T.R. Estimation of the Critical Time Step for Peridynamic Models. SIAM Conference on the Mathematical Aspects of Material Science, Philadelphia, Pennsylvania, June 9-12, 2013.

Questions?

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Sandia National Laboratories

Multiscale Science (Org. 1444)

Back-Up Slides

Suppression of Low-Energy Modes

Penalize deformation that deviates from regularized deformation gradient [1]

Predicted location of neighbor

$$\mathbf{x}_n'^{\star} = \mathbf{x}_n + \bar{\mathbf{F}}_n \, (\mathbf{x}_o' - \mathbf{x}_o)$$

Hourglass vector

$$\mathbf{\Gamma}_{\text{hg}} = \mathbf{x}_n'^{\star} - \mathbf{x}_n'$$

Hourglass vector projected onto bond

$$\gamma_{\text{hg}} = \mathbf{\Gamma}_{\text{hg}} . \, (\mathbf{x}_n' - \mathbf{x}_n)$$

Stabilization force

$$\mathbf{f}_{\text{hg}} = -\mathrm{C}_{\text{hg}} \underbrace{\left(\frac{18k}{\pi\delta^4}\right)}_{\substack{\text{micro-}\\\text{modulus}}} \underbrace{\frac{\gamma_{\text{hg}}}{\|\mathbf{x}_o' - \mathbf{x}_o\|}}_{\substack{\text{hourglass}\\\text{stretch}}} \underbrace{\frac{\mathbf{x}_n' - \mathbf{x}_n}{\|\mathbf{x}_n' - \mathbf{x}_n\|}}_{\substack{\text{bond unit}\\\text{vector}}} \Delta V_{\mathrm{x}} \, \Delta V_{\mathrm{x}'}$$

[1] D. Littlewood, K. Mish, and K. Pierson. Peridynamic simulation of damage evolution for structural health monitoring. Proceedings of ASME 2012 International Mechanical Engineering Congress and Exposition (IMECE2012), Houston, TX, November 9-15, 2012.

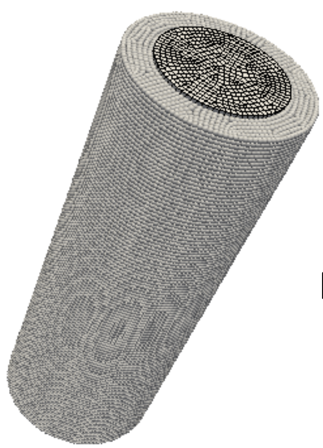
Combined Peridynamics / Classical FEM Model

- Lexan projectile and plug
  - Modeled with classical FEM
  - Johnson-Cook constitutive model

Parameter	Value
Density	1.19 g/cm^3
Young's Modulus	2.54 GPa
Poisson's Ratio	0.344
Yield Stress	75.8 MPa
Hardening Constant *B*	68.9 MPa
Rate Constant C	0.0
Hardening Exponent *N*	1.0
Thermal Exponent *M*	1.85
Reference Temperature	70.0 °F
Melting Temperature	500.0 °F

- AerMet tube
  - Modeled with peridynamics
  - Elastic-plastic constitutive model

Parameter	Value
Density	7.87 g/cm^3
Young's Modulus	194.4 GPa
Poisson's Ratio	0.3
Yield Stress	1.72 GPa
Hardening Modulus	1.94 GPa
Critical Stretch	0.02



Model Discretization

Peridynamics and Higher-Order Gradient Methods

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- Local models contain no length scale

$$\ddot{u}(x) = au''(x)$$

- Higher-order gradients introduce length scale in a weak sense

$$\ddot{u}(x) = au''(x) + bu''''(x)$$

➡ Dimensional analysis shows that sqrt(b/a) has units of length

- Peridynamics is a (strongly) nonlocal model

Peridynamic model (nonlocal)

$$\rho\ddot{u}(x,t) = \int_{-\delta}^{\delta} \frac{c}{|\epsilon|}\left[u(x+\epsilon,t) - u(x,t)\right] d\epsilon$$

Taylor Series ↓

Higher-order gradient model (weakly nonlocal)

$$\rho\ddot{u}(x,t) = \mathrm{K}_a\left[\frac{\partial^2 u}{\partial x^2} + \frac{\delta^2}{24}\frac{\partial^4 u}{\partial x^4} + \frac{\delta^4}{1080}\frac{\partial^6 u}{\partial x^6} + \ldots\right]$$

$\lim\ \delta \to 0$ ←

Local model

$$\rho\ddot{u}(x,t) = \mathrm{K}_a\frac{\partial^2 u}{\partial x^2}$$

S.A. Silling and R.B. Lehoucq, Convergence of peridynamics to classical elasticity theory, *Journal of Elasticity*, 93(1), 2008.

Pablo Seleson, Michael L. Parks, Max Gunzburger, and Richard B. Lehoucq. Peridynamics as an upscaling of molecular dynamics. *Multiscale Modeling and Simulation*, 8(1), 2009.