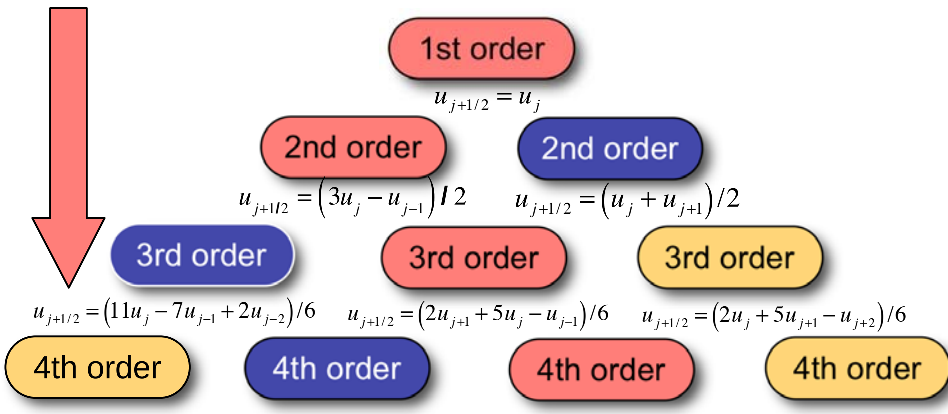
SAND2008-4223P

New Concepts for Developing Non-Oscillatory Methods



$$\mathrm{TV}\left(\mathrm{R}\left(u^n\right)\right) \leq \mathrm{TV}\left(u^n\right) + O\left(h^r\right)$$

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Sandia National Laboratories, Albuquerque

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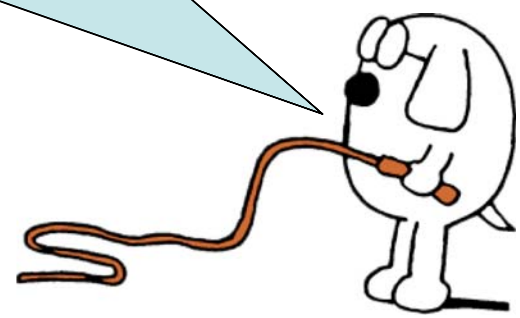
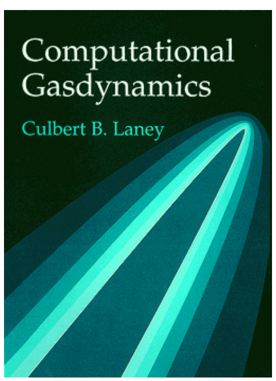
MS54, Computational Methods for Compressible Flow

Things should be made as simple as possible, but not any simpler."
- Albert Einstein

"Logically all things are created by a combination of simpler, less capable components"





- Scott Adams (as Dogbert in the Dilbert Comic Strip), noted by Culbert Laney in *Computational Gasdynamics*

Talk Outline

- Introduction: high-resolution methods
- Revisiting design concepts: what is the real total variation behavior in non-oscillatory schemes.
- Consequences
- New concepts

What is a high-resolution method? Or the role of method nonlinearity

- **The need for method nonlinearity is a consequence of Godunov's theorem:**
  - **No *linear* method can be second-order and monotone… but a *nonlinear* method can be second-order and monotone (TVD, FCT,PLM, PPM,...)!***
- **These methods hybridized the (classical) linear schemes (capitalizing on the best of each!)**
  - **To achieve higher order and physically relevant solutions e.g. LxW and upwind**
  - **This is where Dogbert's quote comes in: "*Logically all things are created by a combination of simpler, less capable components*"**

A Brief Introduction to ENO Methods

- The starting point for modern methods are 1st order monotone methods.
  – These were extended to 2nd order with TVD methods.
  – TVD methods degenerate to 1st order at extrema.
- ENO was introduced to "fix" this problem
  – UNO methods moved the TVD methods to uniform 2nd order accuracy.
  – ENO methods moved the UNO methods to arbitrarily high order.
  – WENO methods moved ENO methods toward practicality (getting rid of some of ENO's issues).

ENO Design Principles

- **Divide the methods into three main steps:**
  - **Reconstruction or spatial differencing.**
  - **Evolution (solution in the small) or Riemann solution.**
  - **Integration or time advance.**
- **The evolution step and integration are TVD (analytically at least), but the reconstruction might not be,…**
- ***… thus a focus on the reconstruction step.***
- **In monotone and TVD methods, the reconstruction is TVD, ENO methods allow variation to potentially increase, although in a controlled bounded manner.**
  - **Keeping the variation controlled can lead to convergent methods (compactness).**

Total variation is used to define methods

- **Define TV=total variation** $\mathbf{TV} = \sum \left| u_{j+1} - u_j \right|$
- **TVD means the *total variation diminishing***

$$\mathbf{TV}\left(u^{n+1}\right) \le \mathbf{TV}\left(u^{n}\right) \text{ or } \mathbf{TV}\left(\mathbf{R}\left(u^{n}\right)\right) \le \mathbf{TV}\left(u^{n}\right)$$

  - **Here R(*u*) is a reconstruction (interpolation) of *u***
  - **TVD is a manner of making nonlinear schemes monotone**
- **ENO is closely related conceptually**

$$\mathbf{TV}\left(\mathbf{R}\left(u^{n}\right)\right) \le \mathbf{TV}\left(u^{n}\right) + O\left(h^{r}\right)$$

  - **Thus ENO is almost monotone, but allows oscillations of a size** $O\left(h^{r}\right)$

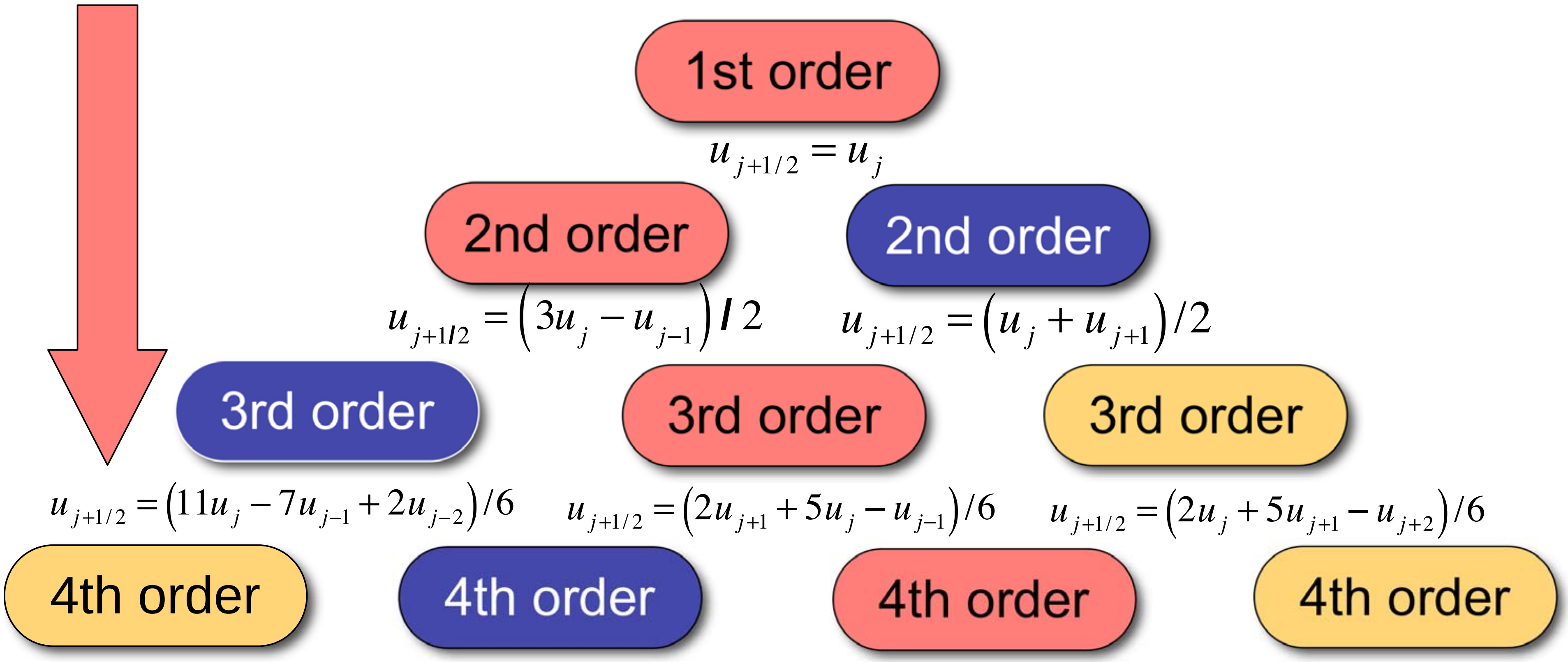
The Non-Oscillatory Concept

- The problem with TVD methods is that they reduce to 1st order at extrema (i.e. infinity norm),
  - this removes "all" oscillation creation.

$$\frac{\partial u}{\partial t} = -C_{j-1/2}\left(u_j - u_{j-1}\right) - C_{j+1/2}\left(u_{j+1} - u_j\right); C_{j\pm 1/2} \geq 0$$

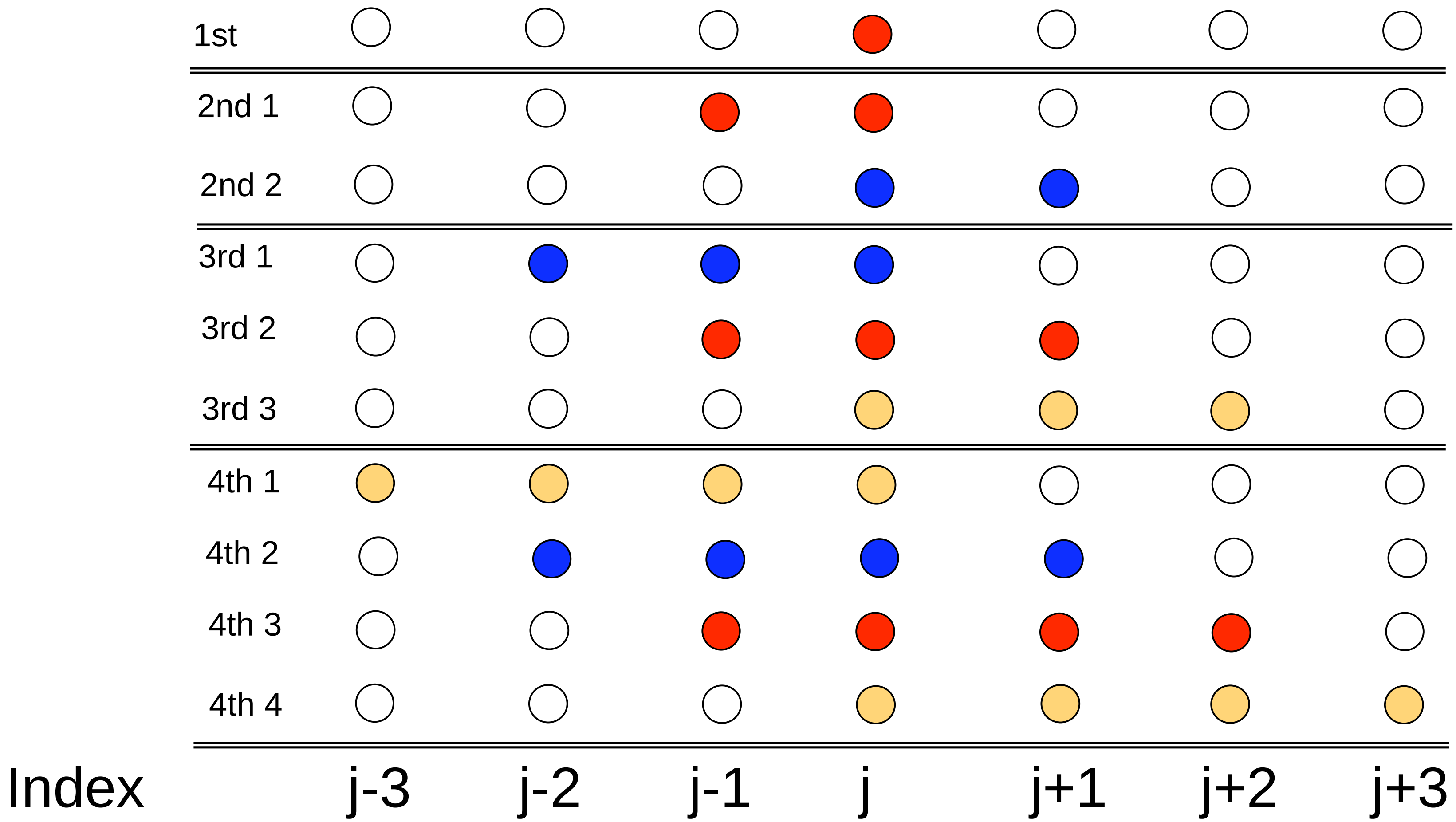
- ENO methods were developed to improve the accuracy to high-order in all norms.
- WENO was developed to fix problems observed with ENO's adaptive stencil, and allow higher order methods that more fully use the stencils.
  - Downwinding, loss of accuracy, unstable selection
  - The problem is that these methods are still plagued by a number of issues limiting their utility.

ENO Methods use smoothness to adaptively choose a stencil.

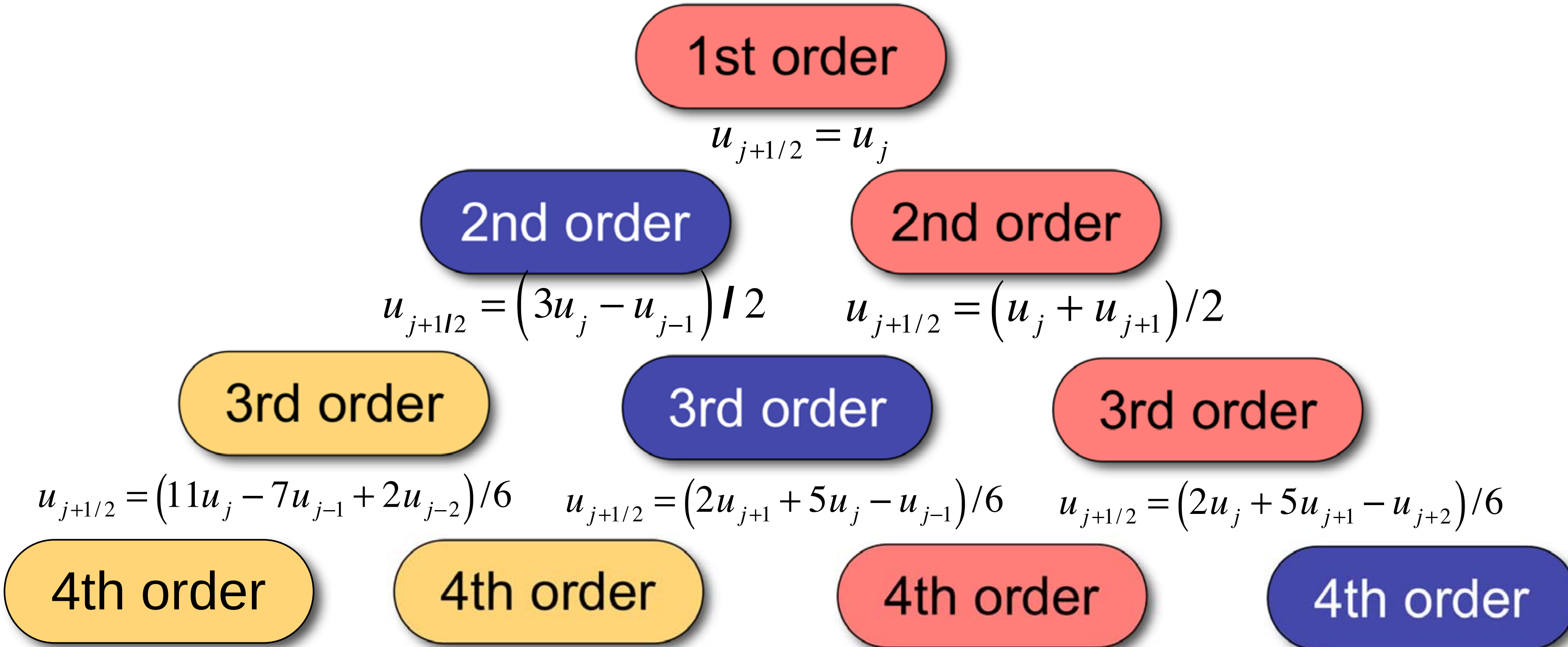


- **ENO selects stencils *adaptively* by choosing the one that is closest to the next lower order. It is hierarchical.**

These stencils can be displayed in physical space.



The same differencing may be arrived at through a different path.



- **The high-order stencils are evaluated pair-wise. This characteristic also hints at one of ENO’s pathologies.**

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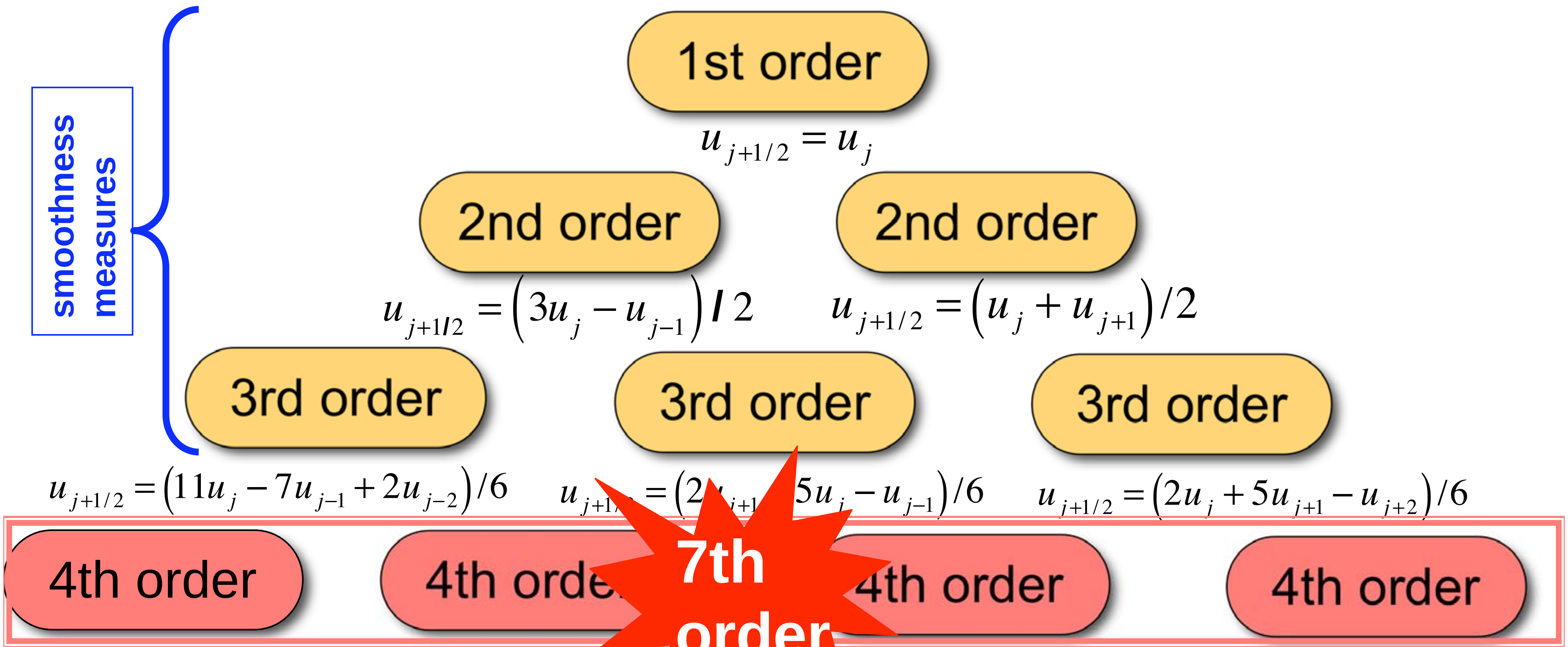
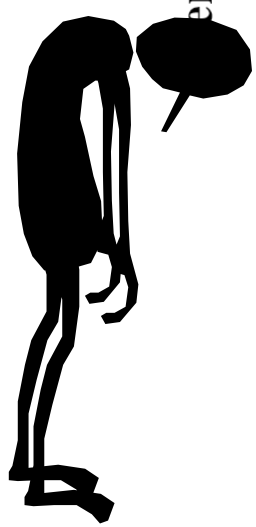
Weighted ENO methods are different in their approach, but the result is similar.

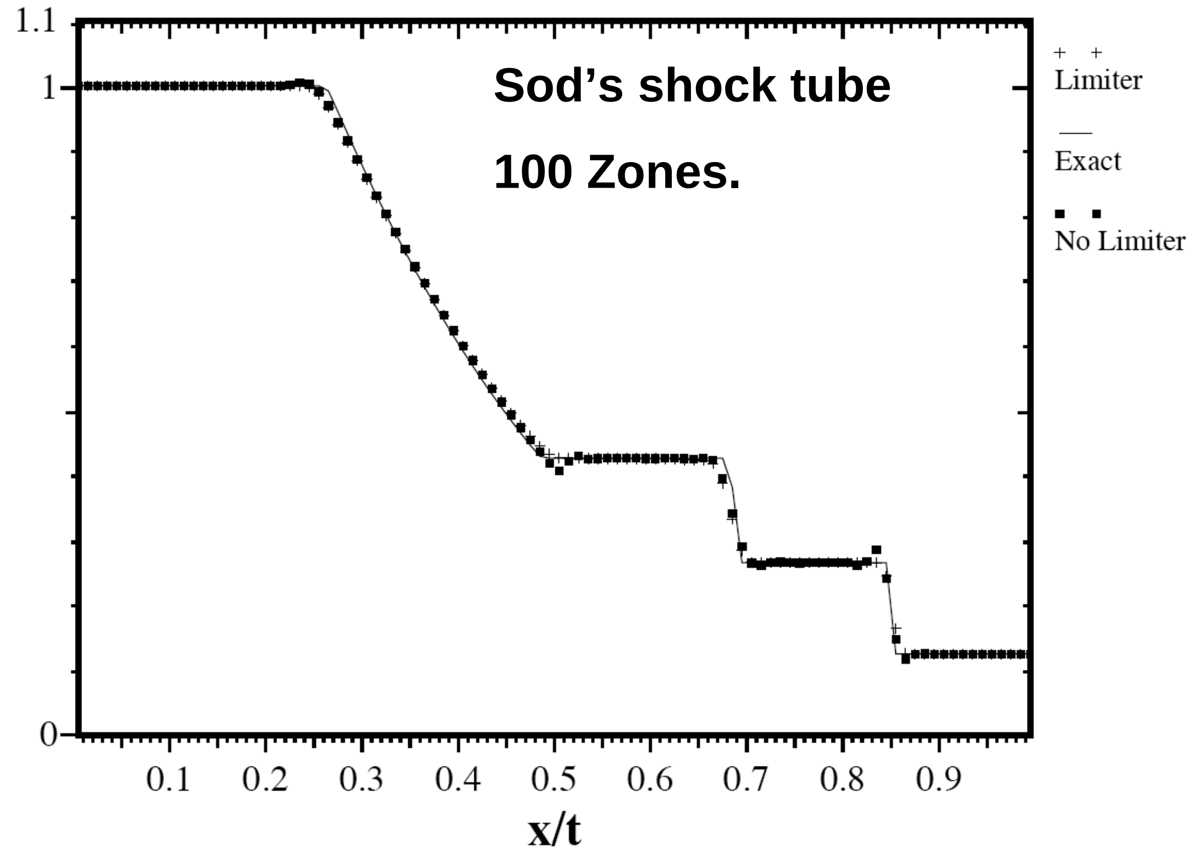


- **These methods evaluate *all* the high-order stencils and compare them (and combine them) algebraically.**
- **Weights can be chosen to achieve *2m-1* order schemes.**

Issues with Results - oscillations or limiters?

- **Sod's shock tube and 11th order WENO**
  - Oscillations w/o limiter!
  - The limiter destroys the "elegance" of the method!



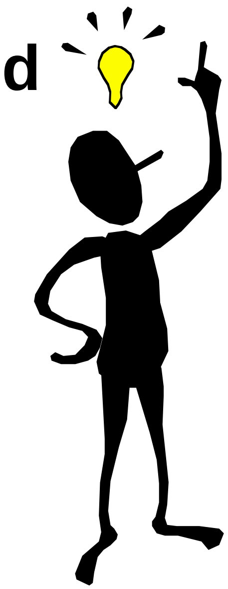


A new approach to building ENO: The Comparison Principle

- **The current ENO (and WENO) algorithms are based on the adaptive stencil approach that *recursively* finds the "smoothest" stencil.**
- **I'm proposing using a different principle than the smoothest stencil -** Using a comparison with TVD schemes to choose the high-order stencil.

$$\mathbf{TV}\left(\mathbf{R}\left(u^n\right)\right) \leq \mathbf{TV}\left(u^n\right) + O\left(h^r\right)$$

- **In other words, one would begin with a TVD stencil and choose the higher-order stencil that is closest to that TVD stencil in some sense (to be defined) .**
- **This leads to schemes similar to existing ENO and WENO schemes*, but with somewhat better properties*.**



Applying the Comparison Principle

- **The basis of the approach is the following definition for the ENO schemes,**

$$\mathrm{TV}\left(\mathrm{R}\left(u^n\right)\right) \leq \mathrm{TV}\left(u^n\right) + O\left(h^r\right)$$

- **Thus, the proposition is that choosing the stencil that is closest to the TVD method (or the comparison scheme) will satisfy this condition.**
  - **No proof (yet), but the results are very similar to existing ENO & WENO method in terms of accuracy (better), and total variation behavior (nearly identical)**
- ***The procedure resulting from the principle is flexible allowing freedom in choosing for the method's properties.***

One Might Use a Median function to accomplish the task.

- **The median(a,b,c) function chooses the function bounded by the other two and preserves accuracy**

$$\text{median}(a,b,c) = a + \text{minmod}(b-a, c-a)$$

  - **If two arguments are $O(h^m)$ then the median is $O(h^m)$**

- **An algorithm for a 4th order comparison-ENO method would look like the following (using 4-4th order fluxes as building blocks:**

$$f_a = \text{median}(f_1, f_2, f_{TVD})$$

$$f_b = \text{median}(f_3, f_4, f_{TVD})$$

$$f_{ENO} = \text{median}(f_a, f_b, f_{TVD})$$

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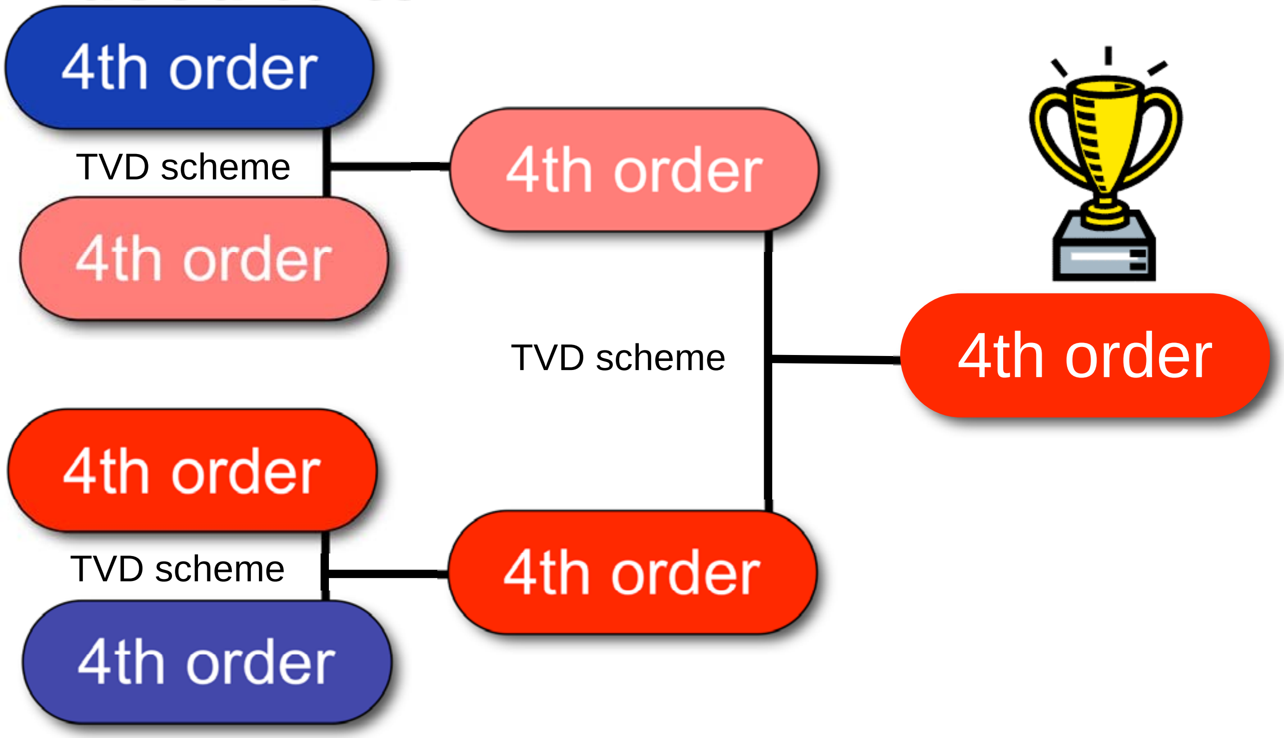
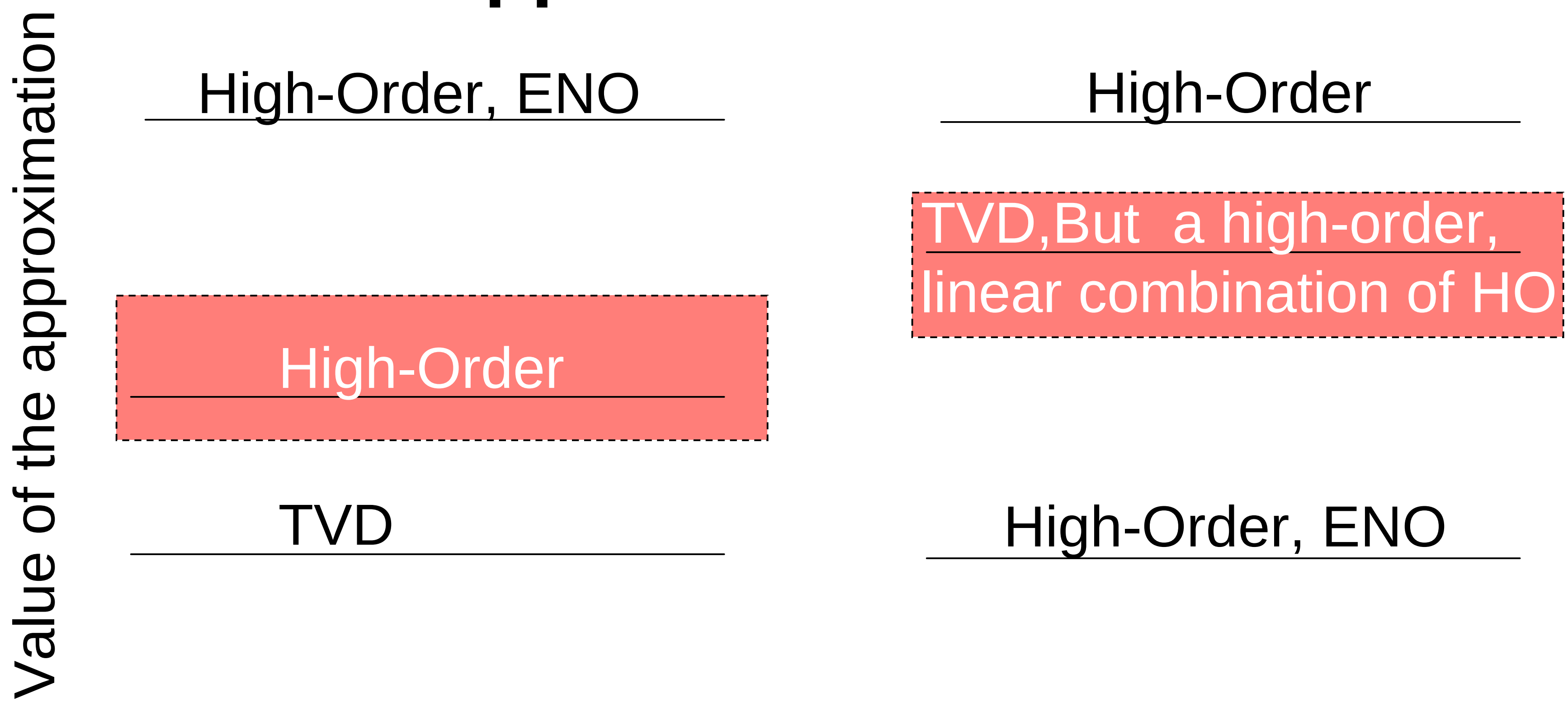
The comparison algorithm can be arranged more like "playoff"

- The TVD scheme chosen for comparison is used to test the "fitness" of each stencil.



The median function bounds the different approximations.



The median(a,b,c) returns the value bounded by the other two values. Preserves the accuracy given by at least two arguments.

The median function has some key properties.

- **One can use a median, or bounding function,** median**(*a,b,c*) that returns the middle argument of the three.**
  - **The one that is bounded by the other two**
  - ***Theorem (Huynh)*: If two arguments are $O(h^n)$ the median is too!**
  - **If one argument is $O(h^n)$ and a second is $O(h^m)$ with $m<n$, the median is $O(h^m)$**
  - ***Conjecture*: If two arguments produce a linearly stable method, the** median **will as well.**

One can emulate ENO procedures with the XMEDIAN function.

- **The median function uses the minmod function (returns the minimum magnitude argument if they have the same sign.**

$$\text{median}(a,b,c) = a + \text{minmod}(b-a, c-a)$$

- **The ENO selection procedure uses a function that returns the minimum magnitude function, mineno.**
- **The xmedian(a,b,c) would return "b" or "c" depending on which is closer to "a"**

$$\text{xmedian}(a,b,c) = a + \text{mineno}(b-a, c-a)$$

- **In this way one could make sure only the formally fourth order fluxes are chosen.**

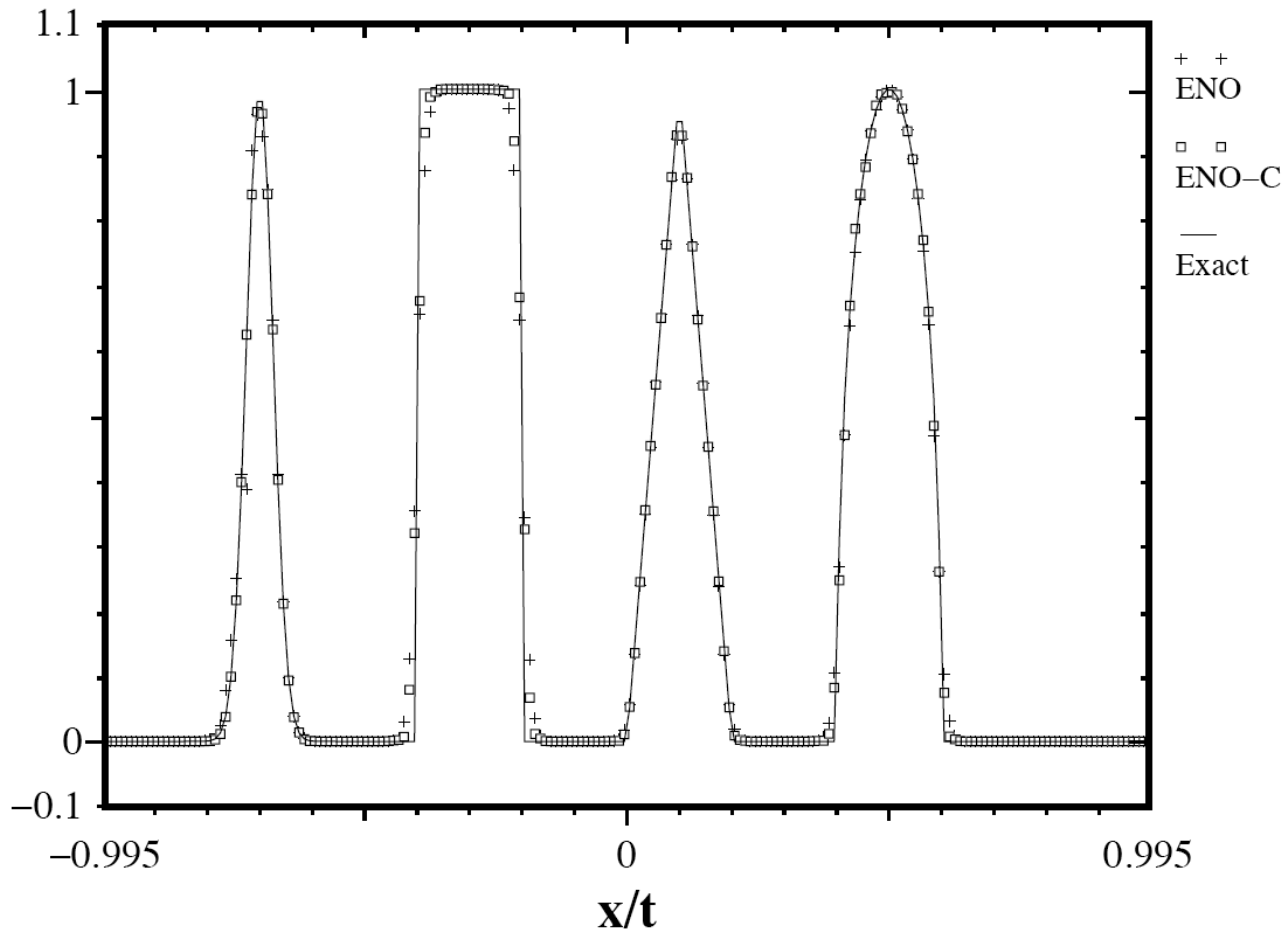
Analysis: Accuracy is unaffected.

- **The previous method would produce a formally 4th order flux. Each step produces a 4th order flux.**
- **The non-oscillatory nature comes from relation of the result to the TVD comparison method.**
  - **Each step will choose either the TVD scheme, or the 4th order flux closest to it .** $\mathrm{TV}\left(\mathrm{R}\left(u^n\right)\right) \leq \mathrm{TV}\left(u^n\right) + O\left(h^r\right)$
  - **Results indicate that this is true.**
  - ***The problem is that the median function does not contain two non-oscillatory fluxes to compare. This is a concern for nonlinear stability of the results, but no problems observed so far.***

Results: Accuracy on Scalar Waves

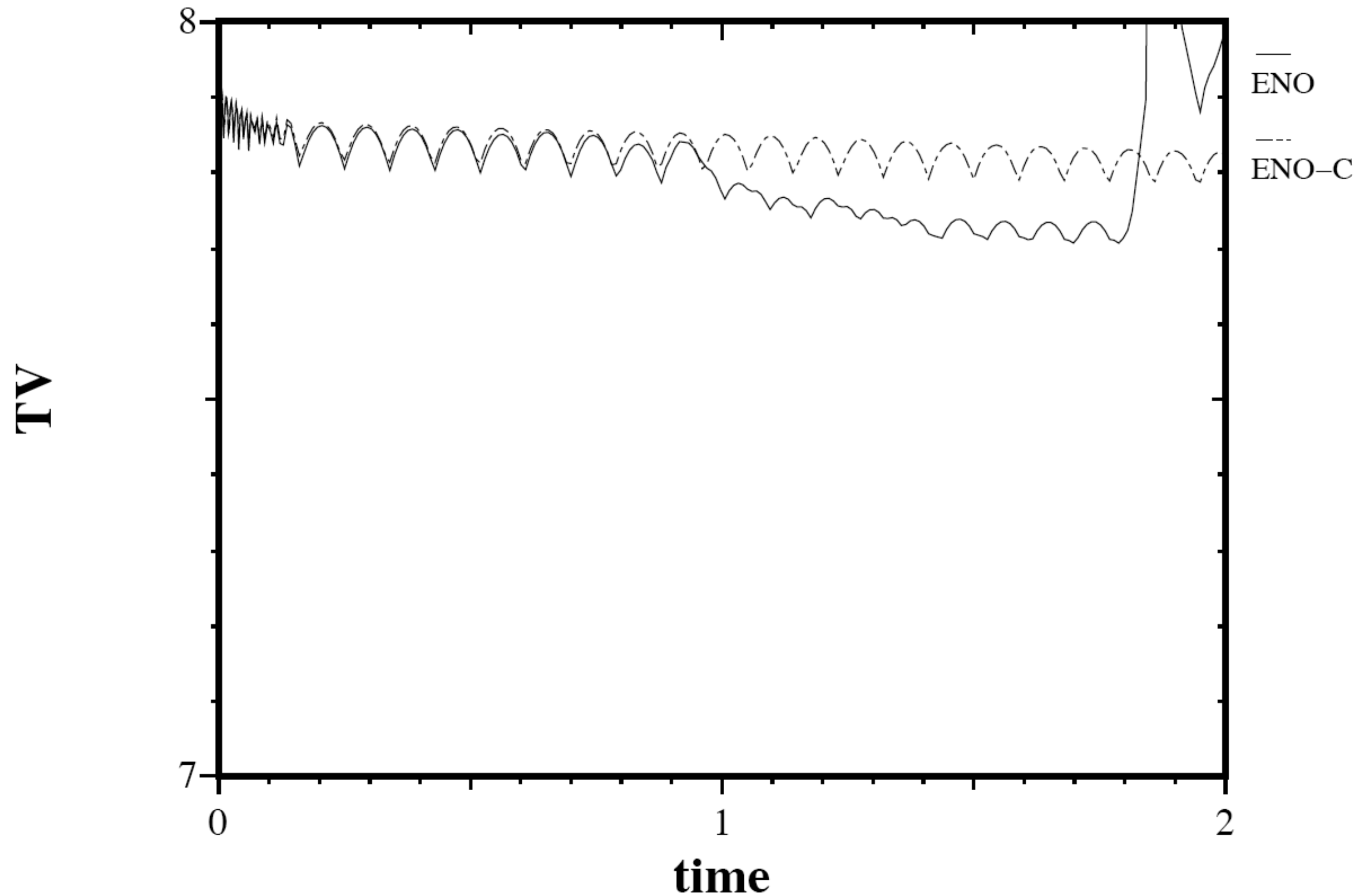
- **Compare usual ENO with a comparison ENO**

L1 Errors
ENO = 1.74e-02
ENO-C = 1.20e-02



Results: Total Variation Behavior

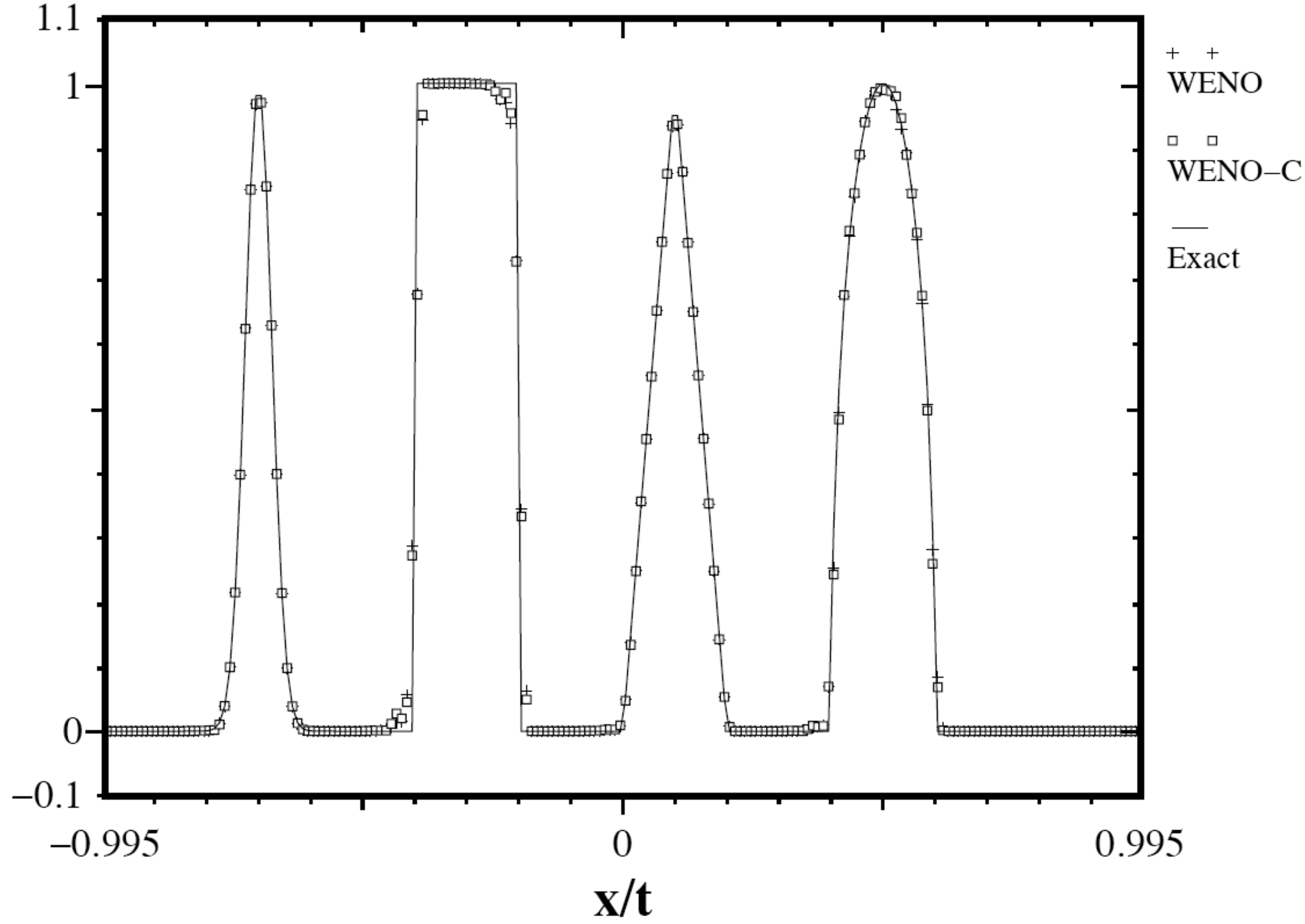
- **Look experimentally at the Total variation as a function of time.**



Results: Accuracy on Scalar Waves, 11th order WENO

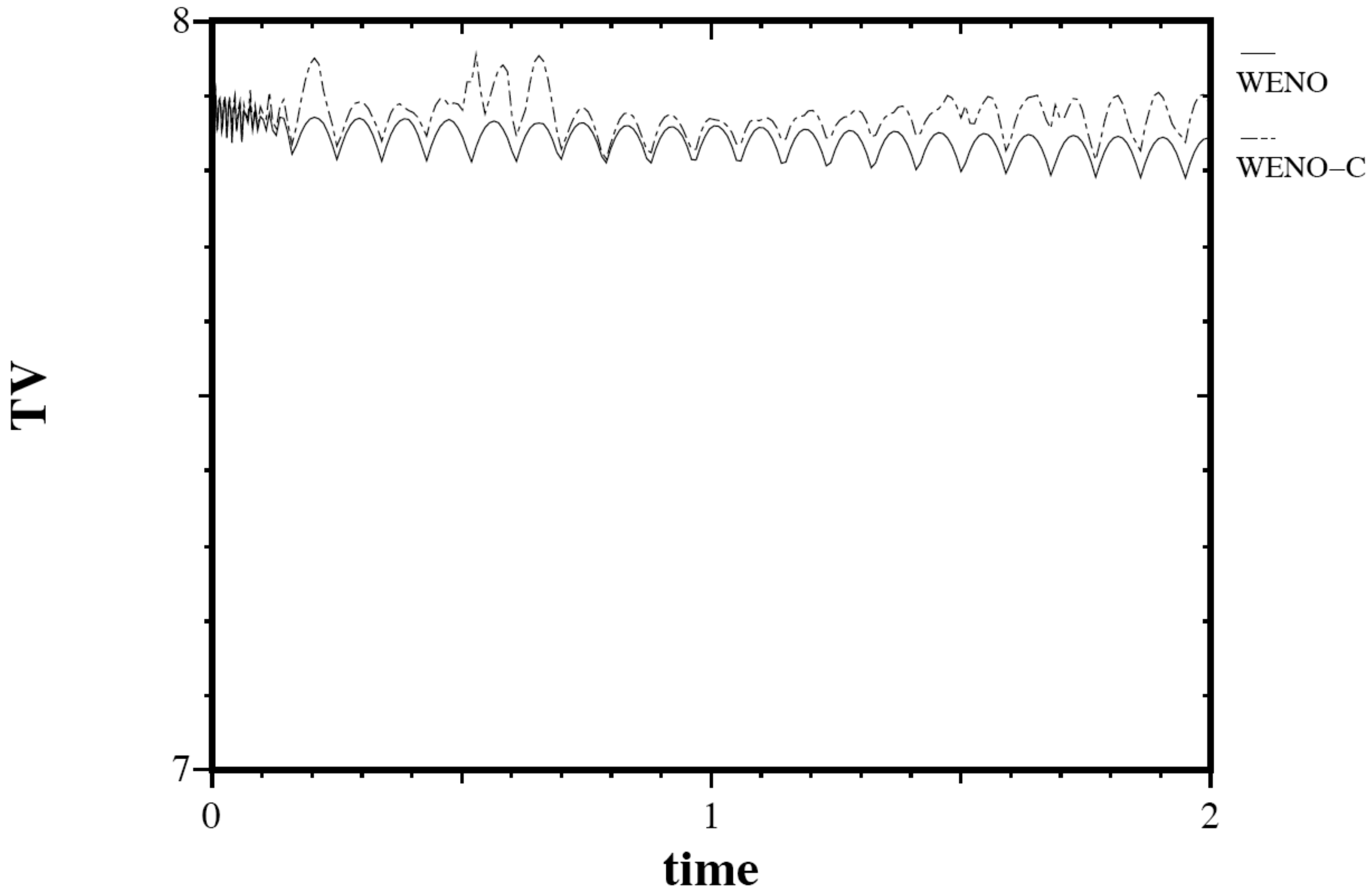
- **Compare usual WENO with a comparison WENO**

L1 Errors
WENO = 1.09e-02
WENO-C = 1.10e-02



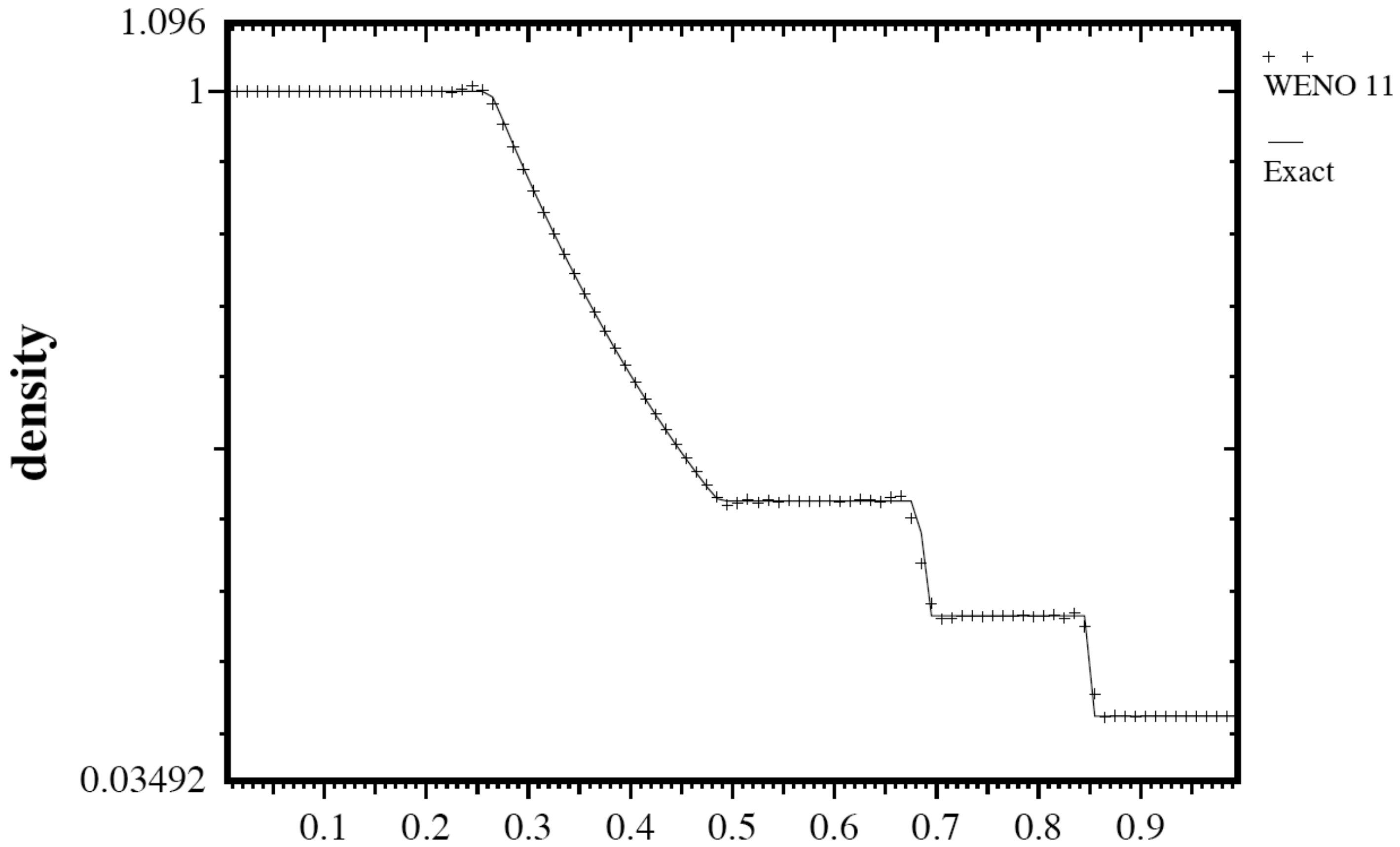
Results: Total Variation Behavior

- **Look experimentally at the Total variation as a function of time.**



Accuracy for Coupled Systems

- **Look at Sod's Shock tube again w/11th order WENO-C (comparison scheme)**
  - **about the same as the regular WENO, and with about 25% lower CPU cost.**



What would a proof of the ENO property look like?

- **Now it comes down to proving the properties of the "*Q*" function. Note that the truncation error of the first order method is well defined in terms of the total variation.**
- **The scheme can definitely be high-order and deviations in total variation will be proportional to the high-order truncation error, $O(h^r)$**
- **Q: What should the "*Q*" function look like?**
  - **A: the minmod function, just as TVD methods.**
- **Putting this together with the original method returns us to** the median function **as serving the necessary role.**
  - **Note: The second-order version of this method is TVD. A proof based on a 2nd order TVD comparison method follows the same path. Same for WENO version.**

Summary and Conclusions

- The concept of total variation is central to many high resolution schemes.
- TV converges at $O(h^2)$ for either TVD or ENO schemes, not $O(h^n)$ as originally intended.
- Thus a rigorous proof of the ENO concept is not forthcoming (they seem to be more properly TVB)
- A new means of developing ENO and WENO methods has been proposed.
  - The new methods are simpler and faster (less hierarchical, simpler to code, fewer operations).
  - The new methods produce similar (or smaller) errors.