

ON STABILIZED FINITE ELEMENT METHODS FOR THE STOKES PROBLEM IN THE SMALL TIME STEP LIMIT

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Joint work with

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WHY IS THE SMALL TIME STEP LIMIT INTERESTING?

- Reacting flow simulations

- the accurate temporal resolution of fast chemical reactions requires small time steps, even for implicit time stepping schemes
- in fact, the time step often needs to be much smaller than that required for the accurate resolution of diffusion and convection phenomena
- thus, the time step used is disparately small compared to the spatial grid size

- Low-order time stepping schemes used in conjunction with high-order spatial discretizations
 - for example, such a combination can yield errors proportional to
$$C_1\Delta t + C_2h^\beta \quad \text{for } \beta > 1$$
 - to equilibrate the temporal and spatial contributions to the error, one has to take a small time step
- So, in these and other settings, the need to use small time steps is driven by questions of accuracy
 - normally, one does not expect any issue with stability if small enough time steps are used

WHY ARE STABILIZED METHOD INTERESTING?

- Stabilized methods for effecting spatial discretizations are in very common use in engineering practice
 - they allow for the use of **equal-order interpolation**
 - the same degree polynomials defined with respect to the same grid can be used for all variables
- mixed-Galerkin finite element methods do not allow for equal-order interpolation

- equal-order interpolation is very useful in **large-scale multi-physics codes**
 - the codes deal with several dozen independent variables, e.g., several chemical species, velocity components, etc.
 - these codes require the transfer of information between its different components
 - the tasks of keeping track of all the variables and of information transfer are both greatly **facilitated when one can use equal-order interpolation**
- Thus, many large multi-physics codes use
 - consistently stabilized finite element methods for spatial discretization
 - implicit finite difference schemes for temporal discretization
 - projection schemes are also used

- Consistently stabilized finite element methods for the Stokes problem were developed in the [steady-state setting](#); three well-known methods in this class are

- [pressure-Poisson stabilization](#)

T. HUGHES, L. FRANCA, AND M. BALESTRA; A new finite element formulation for computational fluid dynamics: Circumventing the Babuska-Brezzi condition: A stable Petrov-Galerkin formulation of the Stokes problem accommodating equal-order interpolations, *Comp. Meth. Appl. Mech. Engrg.* **59** 1986, 85-99

- [Galerkin least-squares stabilization](#)

T. HUGHES AND L. FRANCA; A new finite element formulation for computational fluid dynamics: VII. The Stokes problem with various well-posed boundary conditions: symmetric formulations that converge for all velocity pressure spaces, *Comp. Meth. Appl. Mech. Engrg.* **65** 1987, 85-96

- [Douglas-Wang stabilization](#)

J. DOUGLAS AND J. WANG; An absolutely stabilized finite element method for the Stokes problem, *Math. Comp.* **52** 1989, 495-508

- In the steady-state setting, these methods have proven to be very useful
- These methods were then used in **time-dependent settings** where they also have enjoyed much success
- However, when
 consistently stabilized finite element methods and
 implicit time integrators are used
to effect spatial and temporal discretization
 instabilities in the pressure approximations are observed
when **small time steps are used**

- The poor approximation of the pressure when using consistently stabilized finite element methods with small time steps was
 - was first pointed out to us by our colleague John Shadid of the Sandia National Laboratories
 - was subsequently studied in some recent papers
 - P. BOCHEV, M. GUNZBURGER, AND J. SHADID; On stabilized finite element methods for transient problems with varying time scales, *Comp. Meth. Appl. Mech. Engrg.* **193** 2004, 1471-1489
 - P. BOCHEV, M. GUNZBURGER, AND R. LEHOUCQ; On stabilized finite element methods for transient problems with varying time scales, *Proc. ECOMASS 2004, Jyväskylä, Finland, 2004*
 - these papers focused on
 - the **fully discrete** in space and time **equations**
 - demonstrating the **sufficiency of a lower bound** on the time step

- Here, we focus on
 - the semi-discrete in space equations
 - demonstrating the necessity of a lower bound on the time step
- Along the way, we show that
 - the semi-discrete pressure operator is unstable
 - this identifies the source of the instabilities in the pressure approximations in the small time step limit
 - a sufficiently large time step stabilizes the fully discrete pressure equation

- So, in the time-dependent case, consistently stabilized finite element methods find themselves in a curious and unusual situation
 - the semi-discrete in space equations are ill-posed
 - the fully-discrete in space and time equations are well-posed, provided the time step is large enough
 - thus, it is not surprising that something bad happens in the small time step limit if the spatial grid size does not also tend to zero sufficiently fast
- Because the cause of the instability is the semi-discrete pressure operator
 - the instability has nothing to do with reaction or convection terms terms
 - thus, although fast reaction terms provide a motivation for using small time steps, those terms do not cause the problems
- thus, it suffices (very fortunately for us) to simply consider the Stokes equations

MIXED-GALERKIN FINITE ELEMENT METHODS FOR THE STOKES EQUATIONS

- Consider the Stokes system

$$\left\{ \begin{array}{lll} \frac{\partial \mathbf{u}}{\partial t} - \Delta \mathbf{u} + \nabla p = \mathbf{f} & \text{in} & \Omega \times (0, T) \\ \nabla \cdot \mathbf{u} = 0 & \text{in} & \Omega \times (0, T) \\ \mathbf{u} = \mathbf{0} & \text{on} & \Gamma \times (0, T) \\ \mathbf{u}|_{t=0} = \mathbf{u}_0 & \text{in} & \Omega \end{array} \right.$$

- Function spaces $\mathcal{V} = H_0^1(\Omega) \quad \mathcal{P} = L_0^2(\Omega)$
- Bilinear forms: with $(\cdot, \cdot)$ denoting the $L^2(\Omega)$ inner product
$$a(\mathbf{u}, \mathbf{v}) = (\nabla \mathbf{v}, \nabla \mathbf{u}) \quad b(\mathbf{v}, p) = -(p, \nabla \cdot \mathbf{v})$$

- Weak formulation: seek $\mathbf{u} \in \mathcal{V}$ and $p \in \mathcal{P}$ such that for almost all $t \in (0, T]$

$$\left\{ \begin{array}{l} (\mathbf{u}_t, \mathbf{v}) + a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathcal{V} \\ b(\mathbf{u}, q) = 0 \quad \forall q \in \mathcal{P} \end{array} \right.$$

- Semi-discretization in space

– choose conforming finite-dimensional subspaces $\mathcal{V}^h \subset \mathcal{V}$ and $\mathcal{P}^h \subset \mathcal{P}$

– then, seek $\mathbf{u}^h \in \mathcal{V}^h$ and $p^h \in \mathcal{P}^h$ such that for all $t \in (0, T]$

$$\left\{ \begin{array}{l} (\mathbf{u}_t^h, \mathbf{v}^h) + a(\mathbf{u}^h, \mathbf{v}^h) + b(\mathbf{v}^h, p^h) = (\mathbf{f}, \mathbf{v}^h) \quad \forall \mathbf{v}^h \in \mathcal{V}^h \\ b(\mathbf{u}^h, q^h) = 0 \quad \forall q^h \in \mathcal{P}^h \end{array} \right.$$

– the discretized initial data is determined from

$$(\mathbf{u}^h|_{t=0}, \mathbf{v}^h) = (\mathbf{u}_0, \mathbf{v}^h) \quad \forall \mathbf{v}^h \in \mathcal{V}^h$$

- Let

$\{\boldsymbol{\xi}_i^h\}_{i=1}^N$ denote a basis for $\mathcal{V}^h$ $\{\chi_k^h\}_{k=1}^M$ denote a basis for $\mathcal{P}^h$

so that

$$\mathbf{u}^h(\mathbf{x}, t) = \sum_{j=1}^N U_j(t) \boldsymbol{\xi}_j^h(\mathbf{x}) \qquad p^h = \sum_{m=1}^M P_m(t) \chi_m^h(\mathbf{x})$$

for some functions $\{U_j(t)\}_{j=1}^N$ and $\{P_m(t)\}_{m=1}^M$

- Then, the semi-discrete equations are equivalent to

$$\begin{pmatrix} \mathbb{M} \dot{\vec{U}} \\ \vec{0} \end{pmatrix} + \begin{pmatrix} \mathbb{A} & \mathbb{B}^\top \\ \mathbb{B} & 0 \end{pmatrix} \begin{pmatrix} \vec{U} \\ \vec{P} \end{pmatrix} = \begin{pmatrix} \vec{F} \\ \vec{0} \end{pmatrix} \qquad \text{and} \qquad \mathbb{M} \vec{U}|_{t=0} = \vec{U}_0$$

where

$$\mathbb{A}_{ij} = a(\boldsymbol{\xi}_i^h, \boldsymbol{\xi}_j^h) = (\nabla \boldsymbol{\xi}_i^h, \nabla \boldsymbol{\xi}_j^h) \qquad \mathbb{B}_{im} = b(\chi_m^h, \boldsymbol{\xi}_i^h) = -(\chi_m^h, \nabla \cdot \boldsymbol{\xi}_j^h)$$

$$\mathbb{M}_{ij} = (\boldsymbol{\xi}_i^h, \boldsymbol{\xi}_j^h) \qquad (\vec{F})_i = (\mathbf{f}, \boldsymbol{\xi}_i^h) \qquad (\vec{U}_0)_j = (\mathbf{u}_0, \boldsymbol{\xi}_j^h)$$

- The conformity $\mathcal{V}^h \subset \mathcal{V}$ and $\mathcal{P}^h \subset \mathcal{P}$ of the approximation spaces is not sufficient to guarantee the stability of mixed-Galerkin finite element methods for the Stokes problem
- In addition, these spaces are required to satisfy the LBB or inf-sup compatibility condition

$$\inf_{q^h \in \mathcal{P}^h, q^h \neq 0} \sup_{\mathbf{v}^h \in \mathcal{V}^h, \mathbf{v}^h \neq 0} \frac{(q^h, \nabla \cdot \mathbf{v}^h)}{\|\mathbf{v}^h\|_1 \|q^h\|_0} = \kappa_h \geq \kappa_h^{\min} > 0$$

- The inf-sup constant κ_h has the matrix characterization

$$\kappa_h^2 = \min_{\vec{Z} \in \mathbb{R}^M} \frac{\vec{Z}^\top \mathbb{B} \mathbb{A}^{-1} \mathbb{B}^\top \vec{Z}}{\vec{Z}^\top \mathbb{M}_p \vec{Z}} \quad \text{where} \quad (\mathbb{M}_p)_{ij} = (\chi_i^h, \chi_j^h)$$

- The LBB condition rules out the use of equal-order interpolation with respect to the same mesh for both $\mathcal{V}^h$ and $\mathcal{P}^h$

- Note that the semi-discrete mixed-Galerkin finite element equations can be written in the form

$$(\mathbf{u}_t^h, \mathbf{v}^h) + a(\mathbf{u}^h, \mathbf{v}^h) + b(\mathbf{v}^h, p^h) - b(\mathbf{u}^h, q^h) - (\mathbf{f}, \mathbf{v}^h) = 0$$

$$\forall \{\mathbf{v}^h, q^h\} \in \mathcal{V}^h \times \mathcal{P}^h, \quad \forall t \in (0, T]$$

with initial data determined from

$$(\mathbf{u}^h|_{t=0}, \mathbf{v}^h) = (\mathbf{u}_0, \mathbf{v}^h) \quad \forall \mathbf{v}^h \in \mathcal{V}^h$$

CONSISTENTLY STABILIZED FINITE ELEMENT METHODS FOR THE STOKES EQUATIONS

- Let $\mathcal{T}_h$ denote a subdivision of the domain Ω into finite elements
 - then $\mathcal{K}$ denotes a finite element in $\mathcal{T}_h$
 - then $(\cdot, \cdot)_{\mathcal{K}}$ denotes the $L^2(\mathcal{K})$ inner product
- The discretized initial data is again determined from

$$(\mathbf{u}^h|_{t=0}, \mathbf{v}^h) = (\mathbf{u}_0, \mathbf{v}^h) \quad \forall \mathbf{v}^h \in \mathcal{V}^h$$

- Then, consistently stabilized methods for the Stokes problem take the form

$$\begin{array}{lcl}
 (\mathbf{u}_t^h, \mathbf{v}^h) + a(\mathbf{u}^h, \mathbf{v}^h) + b(\mathbf{v}^h, p^h) - b(\mathbf{u}^h, q^h) - (\mathbf{f}, \mathbf{v}^h) & \begin{array}{c} \text{mixed} \\ \Leftarrow \\ \text{Galerkin} \\ \text{terms} \end{array} & \\
 - \sum_{\mathcal{K} \in \mathcal{T}_h} \tau \left(\mathbf{u}_t^h - \Delta \mathbf{u}^h + \nabla p^h - \mathbf{f}, -\gamma \Delta \mathbf{v}^h + \nabla q^h \right)_{\mathcal{K}} & \begin{array}{c} \text{stabilization} \\ \Leftarrow \\ \text{terms} \end{array} &
 \end{array}$$

$$= 0 \quad \forall \{ \mathbf{v}^h, q^h \} \in \mathcal{V}^h \times \mathcal{P}^h, \quad \forall t \in (0, T]$$

where

τ is a stabilization parameter

and γ is used to define the three most popular stabilized methods

$\gamma = 0$	pressure-Poisson stabilization
$\gamma = 1$	Galerkin least-squares stabilization
$\gamma = -1$	Douglas-Wang stabilization

- Let's dissect the stabilization terms

$$\underbrace{\sum_{\mathcal{K} \in \mathcal{T}_h}}_{\substack{\text{sum} \\ \text{over} \\ \text{elements}}} \underbrace{\tau}_{\substack{\text{stabi-} \\ \text{lization} \\ \text{parameter}}} \underbrace{\left(\mathbf{u}_t^h - \Delta \mathbf{u}^h + \nabla p^h - \mathbf{f} \right)}_{\substack{\text{residual of} \\ \text{momentum} \\ \text{equation}}} \underbrace{\left(-\gamma \Delta \mathbf{v}^h + \nabla q^h \right)}_{\substack{\text{test} \\ \text{function} \\ \text{terms}}} \Bigg)_{\mathcal{K}}$$

- The crucial term that provides stabilization is the pressure-Poisson term

$$- \sum_{\mathcal{K} \in \mathcal{T}_h} \tau \left(\mathbf{u}_t^h - \Delta \mathbf{u}^h + \nabla p^h - \mathbf{f} \right) \underbrace{\left(-\gamma \Delta \mathbf{v}^h + \nabla q^h \right)}_{\mathcal{K}}$$

- this accounts for the minus sign in front of the stabilization term

- The stabilization terms vanish for the exact solution of the Stokes equations

$$-\sum_{\mathcal{K} \in \mathcal{T}_h} \tau \left(\mathbf{u}_t - \Delta \mathbf{u} + \nabla p - \mathbf{f}, -\gamma \Delta \mathbf{v}^h + \nabla q^h \right)_{\mathcal{K}} = 0$$

- the exact solution of the Stokes equations also satisfies the mixed-Galerkin formulation

$$(\mathbf{u}_t, \mathbf{v}^h) + a(\mathbf{u}, \mathbf{v}^h) + b(\mathbf{v}^h, p) - b(\mathbf{u}, q^h) - (\mathbf{f}, \mathbf{v}^h) = 0$$

- as a result, the exact solution of the Stokes equations satisfies the stabilized equations

- for this reason, this class of methods is referred to as consistently stabilized methods

- The parameter γ in the test function terms serves to define different methods

- pressure-Poisson stabilization $\gamma = 0$

$$- \sum_{\mathcal{K} \in \mathcal{T}_h} \tau \left(\mathbf{u}_t^h - \Delta \mathbf{u}^h + \nabla p^h - \mathbf{f}, + \nabla q^h \right)_{\mathcal{K}}$$

- provides the simplest means for effecting stabilization and consistency
- in the stationary case, it is **seemingly unconditionally stable**
i.e., it is stable for all values for the stability parameter

- Galerkin least-squares stabilization $\gamma = 1$

$$- \sum_{\mathcal{K} \in \mathcal{T}_h} \tau \left(\mathbf{u}_t^h - \Delta \mathbf{u}^h + \nabla p^h - \mathbf{f}, - \Delta \mathbf{v}^h + \nabla q^h \right)_{\mathcal{K}}$$

- in the stationary case, it provides a **symmetric** stabilization term
- in the stationary case, it is only **conditionally stable** i.e., it is stable for only a limited range of values for the stability parameter

– Douglas-Wang stabilization $\gamma = -1$

$$\sum_{\mathcal{K} \in \mathcal{T}_h} \tau \left(\mathbf{u}_t^h - \Delta \mathbf{u}^h + \nabla p^h - \mathbf{f}, -\Delta \mathbf{v}^h - \nabla q^h \right)_{\mathcal{K}}$$

- in the stationary case, the test function terms involve the **adjoint** of the momentum equation operator
- in the stationary case, it is **provably unconditionally stable**, i.e., it is stable for all values for the stability parameter

- Why have different consistently stabilized finite element methods been introduced?
 - pressure-Poisson stabilization was the first consistently stabilized finite element method
 - Galerkin least-squares stabilization was introduced because of its symmetry property
 - Douglas-Wang stabilization was introduced because it can be rigorously proved to be unconditionally stable
 - no such proof exists for pressure-Poisson stabilization
 - however, computational evidence indicates that pressure-Poisson stabilization does lead to unconditional stability
 - the Galerkin least-squares method is only conditionally stable

- an unconditional stability proof does exist for an optimally convergent, modified (inconsistent) pressure-Poisson stabilization method

P. BOCHEV AND M. GUNZBURGER; An absolutely stable pressure-Poisson stabilized finite element method for the Stokes equations; *SIAM J. Numer. Anal.* **42** 2004, 1189-1207

- a detailed study of consistently stabilized finite element methods for the stationary Stokes equations can be found in

T. BARTH, P. BOCHEV, M. GUNZBURGER, AND J. SHADID; A Taxonomy of consistently stabilized finite element methods for the Stokes problem, *SIAM J. Sci. Comput.* **25** 2004, 1585–1607

- If one is going to use a consistent stabilization method, there is no compelling justification for [using anything but pressure-Poisson stabilization methods](#)
 - thus, for the time being, we focus on that method
 - we return to the Galerkin least-squares and Douglas-Wang methods later
- Since a main motivation for using stabilized finite element methods is that they allow for [equal-order interpolation](#), we will use them in that manner

- In the paper

P. BOCHEV, M. GUNZBURGER, AND J. SHADID; On stabilized finite element methods for transient problems with varying time scales, *Comp. Meth. Appl. Mech. Engrg.* **193** 2004, 1471-1489

the stability parameter τ was chosen as

$$\tau = \delta h^2$$

for some constant δ and the sufficient condition for stability

$$\Delta t > \delta h^2$$

was demonstrated

— computational evidence indicates that the optimal choice is $\delta = 0.05$; see

T. BARTH, P. BOCHEV, M. G., AND J. SHADID; A Taxonomy of consistently stabilized finite element methods for the Stokes problem, *SIAM J. Sci. Comput.* **25** 2004, 1585–1607

— we also choose $\tau = \delta h^2$

— we will also discuss another choice for τ

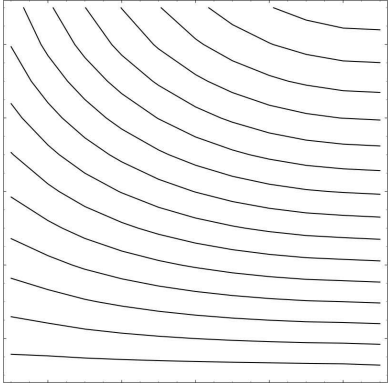
SOME MOTIVATIONAL COMPUTATIONAL EXAMPLES

- A uniform grid of 200 triangles on the unit square is used $h = 0.1$
- The smooth solenoidal exact solution is chosen as

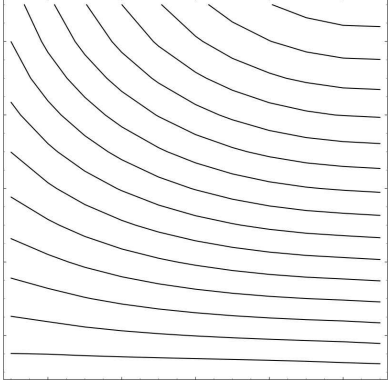
$$\mathbf{u} = \begin{pmatrix} \frac{\partial \psi}{\partial y} \\ -\frac{\partial \psi}{\partial x} \end{pmatrix} \quad \text{with} \quad \psi(x, y) = x^2(1-x)^2 \sin^2(\pi y)$$

$$p(x, y) = \sin(x) \cos(y) + (\cos(1) - 1) \sin(1)$$

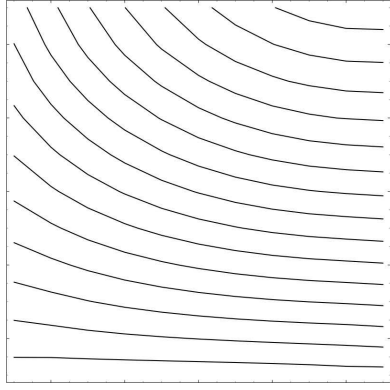
- We first look at pressure plots determined using a **mixed-Galerkin** finite element method
 - of course, we cannot use equal-order interpolation
 - so we use the Taylor-Hood element pair
 - continuous piecewise **quadratic** velocity approximation
 - continuous piecewise **linear** pressure approximation
 - with respect to the same grid
 - this element pair is known to define a stable mixed-Galerkin finite element method



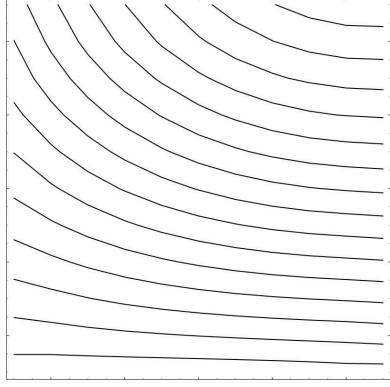
$$\Delta t = 0.001$$



$$\Delta t = 0.0001$$



$$\Delta t = 0.00001$$

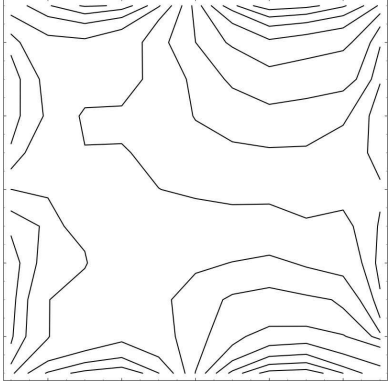


$$\Delta t = 0.000001$$

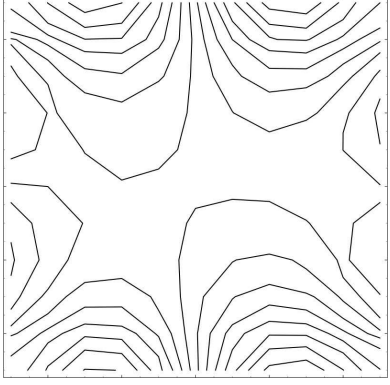
Approximate pressures determined using the Taylor-Hood element pair for different time steps

- This is an illustration of the fact that mixed Galerkin finite element methods do not encounter any difficulties, e.g., pressure instabilities, in the small time step limit
-

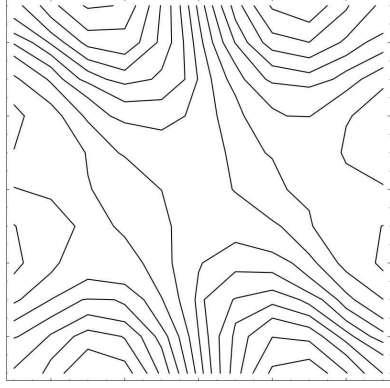
- We next look at pressure plots determined using a consistently stabilized finite element method
 - pressure-Poisson stabilization is used $\gamma = 0$
 - we still have $h = 0.1$ so that $\tau = \delta h^2 = \delta/100$
 - standard continuous piecewise
 - linear,
 - quadratic, and
 - cubic
- nodal finite elements spaces are used for both the velocity and pressure



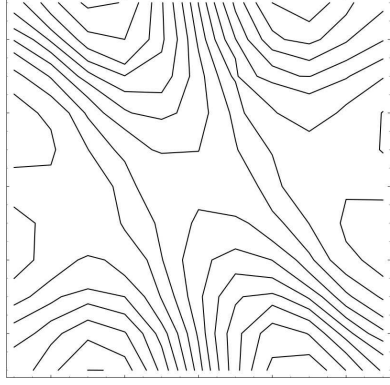
$$\Delta t = 0.001$$



$$\Delta t = 0.0001$$

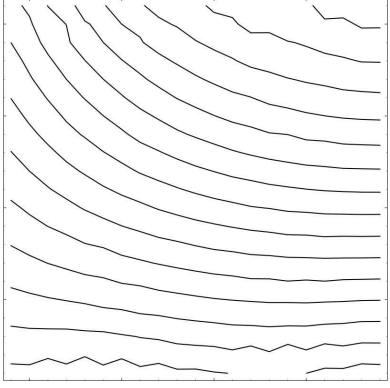


$$\Delta t = 0.00001$$

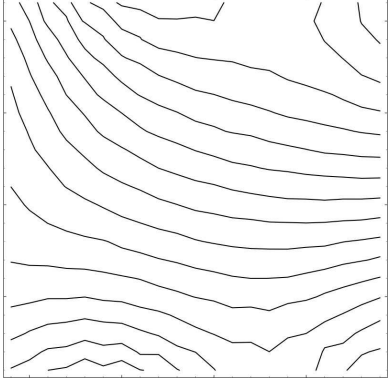


$$\Delta t = 0.000001$$

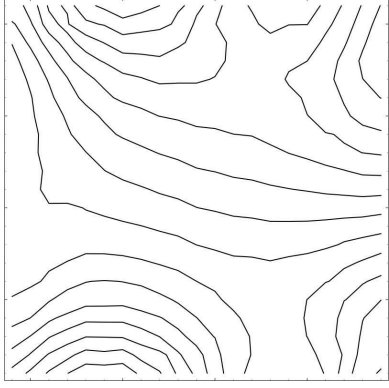
Approximate pressures determined using pressure-Poisson stabilization with $\delta = 0.05$ for different time steps and for equal-order piecewise **linear** interpolation



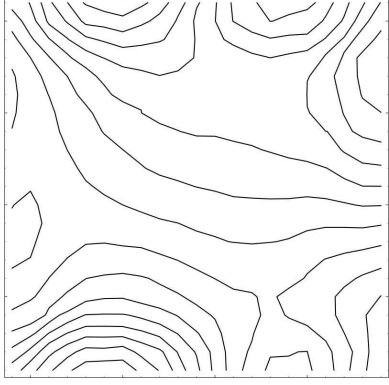
$$\Delta t = 0.001$$



$$\Delta t = 0.0001$$

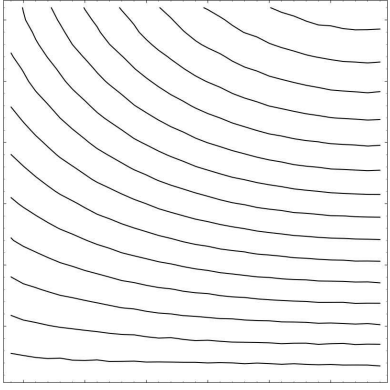


$$\Delta t = 0.00001$$

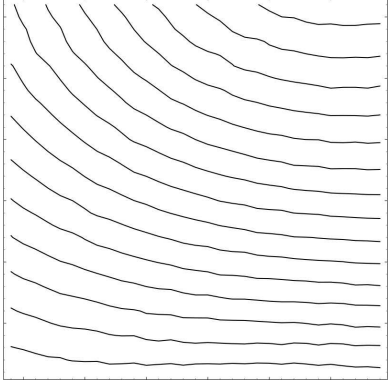


$$\Delta t = 0.000001$$

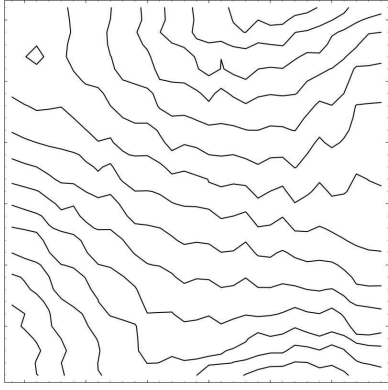
Approximate pressures determined using pressure-Poisson stabilization with $\delta = 0.05$ for different time steps and for equal-order piecewise **quadratic** interpolation



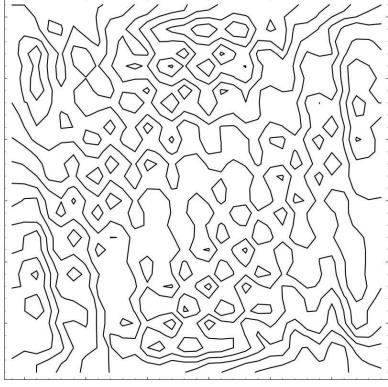
$$\Delta t = 0.001$$



$$\Delta t = 0.0001$$



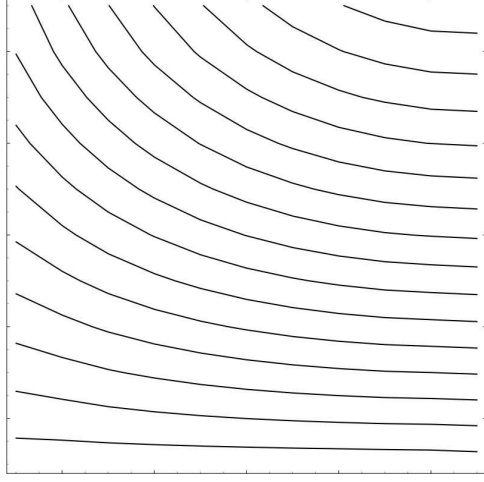
$$\Delta t = 0.00001$$



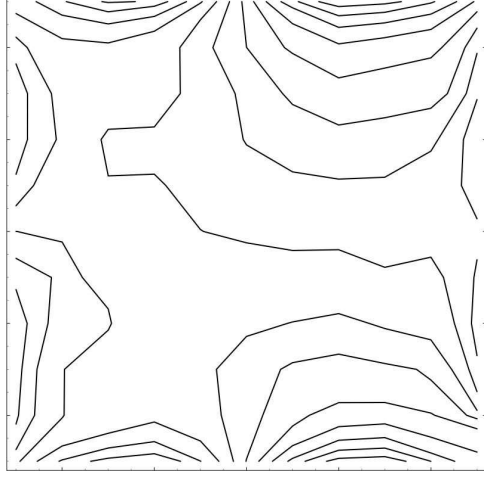
$$\Delta t = 0.000001$$

Approximate pressures determined using pressure-Poisson stabilization with $\delta = 0.05$ for different time steps and for equal-order piecewise **cubic** interpolation

- Why are the piecewise **linear** results wrong even for large time steps?



Taylor
Hood



PP
linear

$$\Delta t = 0.001 \quad \delta = 0.05$$

— for piecewise linear elements, the stabilization term

$$- \sum_{K \in \mathcal{T}_h} \tau \left(\mathbf{u}_t^h - \Delta \mathbf{u}^h + \nabla p^h - \mathbf{f}, \nabla q^h \right)_K$$

reduces to

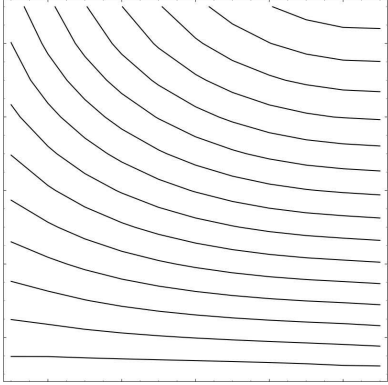
$$- \sum_{K \in \mathcal{T}_h} \tau \left(\mathbf{u}_t^h + \nabla p^h - \mathbf{f}, \nabla q^h \right)_K$$

which, as it happens, induces an incorrect boundary condition on the pressure

— but note the smooth apparent “convergence” (to the wrong answer) of the pressure approximation

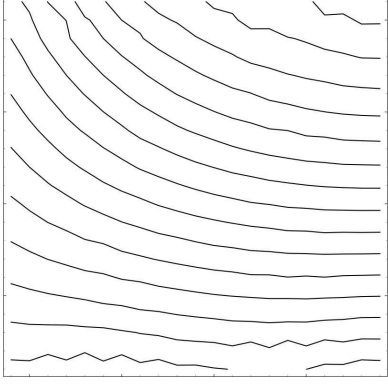
- it would be difficult to detect that something is wrong with the pressure approximation

- What happens to the piecewise **quadratic** results as Δt decreases?

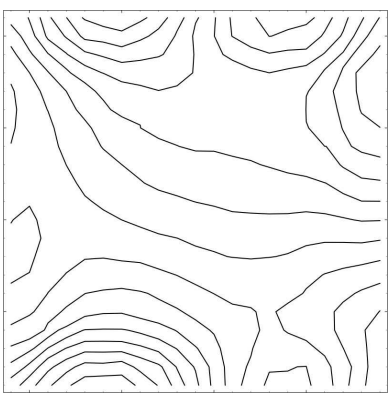


Taylor-Hood

PP $\Delta t = 0.001$

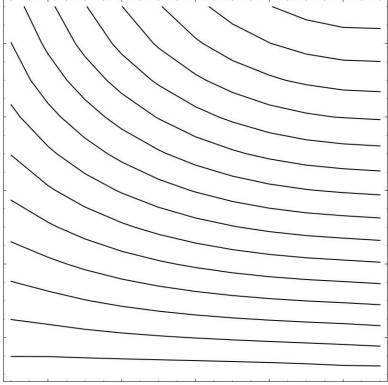


PP $\Delta t = 0.000001$



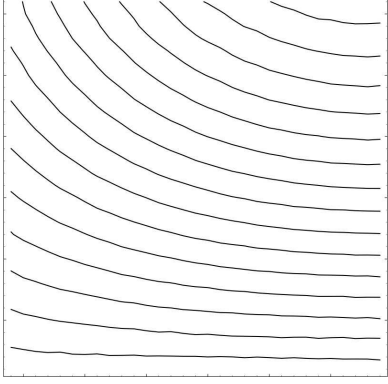
- for “large” Δt , the results appear to be sort of correct
- but as Δt gets small, the pressure approximation deviates from the correct solution
- this transition seem to occur smoothly
 - again, it would be difficult to detect that something is wrong with the pressure approximation

- What happens to the piecewise **cubic** results as Δt decreases is even stranger

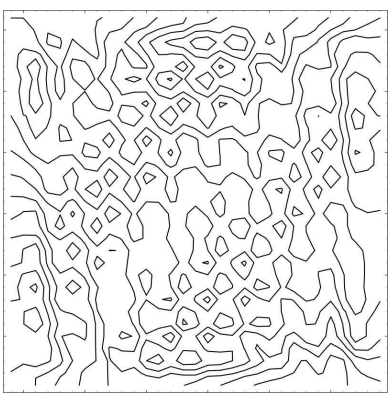


Taylor-Hood

PP $\Delta t = 0.001$

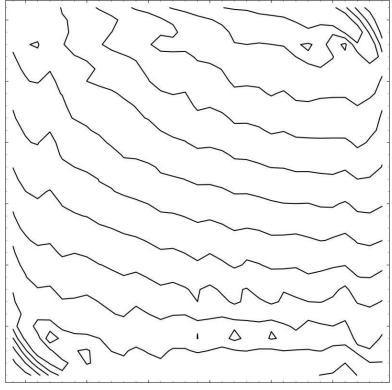


PP $\Delta t = 0.000001$

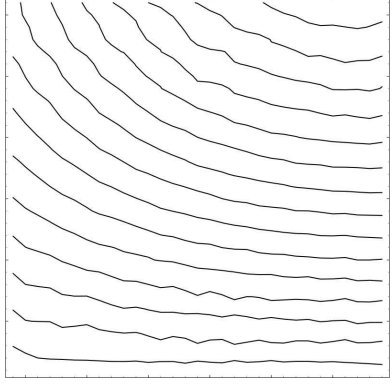


- now as Δt gets small, **spurious oscillations appear** in the pressure approximation
- the oscillations get worse as Δt decreases further

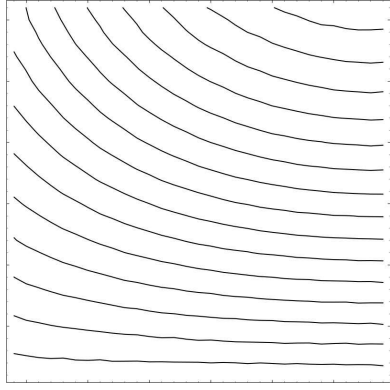
- Let's investigate the cubic case some more
 - we examine what happens as we vary δ



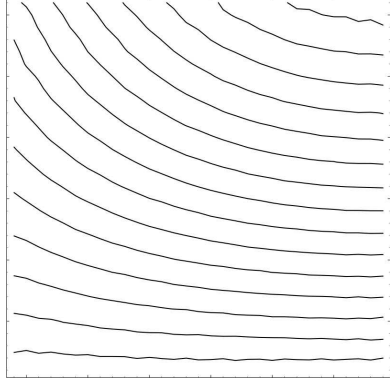
$$\delta = 5$$



$$\delta = 0.5$$



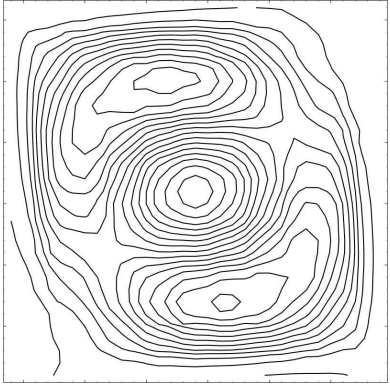
$$\delta = 0.05$$



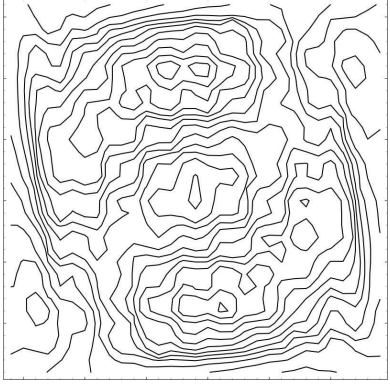
$$\delta = 0.005$$

$$\Delta t = 0.1 \quad \Leftarrow \quad \text{large time step}$$

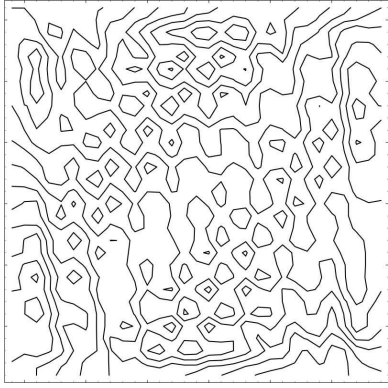
Good answers for a wide range of δ



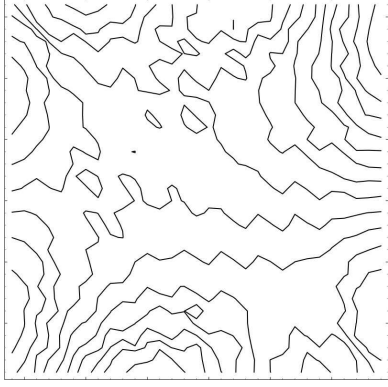
$$\delta = 5$$



$$\delta = 0.5$$



$$\delta = 0.05$$



$$\delta = 0.005$$

$\Delta t = 0.000001$ $\Leftarrow$ **small time step**

Garbage for all values of δ

ANALYSIS OF SEMI-DISCRETE EQUATIONS

- For the pressure-Poisson stabilized case, the semi-discrete equations are given by

$$\begin{pmatrix} \mathbb{M} \dot{\vec{U}} \\ \tau \mathbb{B} \dot{\vec{U}} \end{pmatrix} + \begin{pmatrix} \mathbb{A} & \mathbb{B}^\top \\ -\mathbb{B} & \tau \mathbb{S} \end{pmatrix} \begin{pmatrix} \vec{U} \\ \vec{P} \end{pmatrix} = \begin{pmatrix} \vec{F} \\ \tau \vec{G} \end{pmatrix}$$

where

$$\mathbb{K}_{km} = (\nabla \chi_m^h, \nabla \chi_k^h) \quad \mathbb{S}_{kj} = (\Delta \xi_j^h, \nabla \chi_k^h) \quad (\vec{G})_k = (\mathbf{f}, \nabla \xi_k^h)$$

- Eliminating $\dot{\vec{U}}$, one obtains the semi-discrete pressure equation

$$(\mathbb{K} - \mathbb{B} \mathbb{M}^{-1} \mathbb{B}^\top) \vec{P} = \vec{G} - \mathbb{B} \mathbb{M}^{-1} \vec{F} + \left(\frac{1}{\tau} \mathbb{B} + \mathbb{S} + \mathbb{B} \mathbb{M}^{-1} \mathbb{A} \right) \vec{U}$$

- Clearly, the stability of the pressure approximation depends on the properties of the semi-discrete pressure matrix

$$\mathbb{K} - \mathbb{B}\mathbb{M}^{-1}\mathbb{B}^\top$$

- this matrix is the difference of two discrete Laplacians
 - the pressure-Poisson matrix $\mathbb{K}$
 - the composed matrix $\mathbb{B}\mathbb{M}^{-1}\mathbb{B}^\top$ involving discrete divergence and gradient matrices $\mathbb{B}$ and $\mathbb{B}^\top$, respectively

- In the paper

R. CODINA AND M. VÁZQUEZ, AND O. ZIENKIEWICZ; A general algorithm for compressible and incompressible flows. Part III: the semi-implicit form, *Int. J. Numer. Meth. Fluids.* **27** 1998, 13-32

it was shown that the semi-discrete pressure matrix is symmetric and positive semi-definite but no estimates for the smallest eigenvalue were provided

- We have derived several refinements of this result; in particular, we have derived grid dependent bounds for the smallest eigenvalue

PROPOSITION

- Let

$$\mu_{max}^2 \equiv \max_{\vec{Q} \in \mathfrak{R}^M} \frac{\vec{Q}^\top \mathbf{B} \mathbf{M}^{-1} \mathbf{B}^\top \vec{Q}}{\vec{Q}^\top \mathbf{K} \vec{Q}}$$

- Then $\mu_{max}^2 \leq 1$ and

$$(1 - \mu_{max}^2) \vec{Q}^\top \mathbf{K} \vec{Q} \leq \vec{Q}^\top (\mathbf{K} - \mathbf{B} \mathbf{M}^{-1} \mathbf{B}^\top) \vec{Q} \quad \forall \vec{Q} \in \mathfrak{R}^M$$

- Moreover, there exists a $\vec{Q}_{max}$ such that **equality** holds
-

- We also have obtained an upper bound, but that is not of interest to our discussion
- We next give a variational characterization of μ_{max}

PROPOSITION

- Let

$$\|\vec{Q}\|_{\mathbb{K}}^2 = \vec{Q}^\top \mathbb{K} \vec{Q} \quad \text{and} \quad \|\vec{V}\|_{\mathbb{M}}^2 = \vec{V}^\top \mathbb{M} \vec{V}$$

- Then

$$\mu_{max} = \sup_{q^h \in \mathcal{P}^h, q^h \neq 0} \sup_{\mathbf{v}^h \in \mathcal{V}^h, \mathbf{v}^h \neq \mathbf{0}} \frac{(q^h, \nabla \cdot \mathbf{v}^h)}{|q^h|_{\mathbf{1}} \|\mathbf{v}^h\|_{\mathbf{0}}} = \max_{\vec{Q} \in \mathfrak{R}^M} \max_{\vec{V} \in \mathfrak{R}^N} \frac{\vec{Q}^\top \mathbb{B} \vec{V}}{\|\vec{V}\|_{\mathbb{M}} \|\vec{Q}\|_{\mathbb{K}}}$$

- Note that the norms are “reversed” with respect to the inf-sup stability condition associated with Stokes equations!

$$\inf_{q^h \in \mathcal{P}^h, q^h \neq 0} \sup_{\mathbf{v}^h \in \mathcal{V}^h, \mathbf{v}^h \neq \mathbf{0}} \frac{(q^h, \nabla \cdot \mathbf{v}^h)}{\|\mathbf{v}^h\|_{\mathbf{1}} \|q^h\|_{\mathbf{0}}} = \kappa_h > 0$$

- To answer (in the negative) the question

does there exist a $\hat{\mu}_{max} < 1$ such that $\mu_{max} \leq \hat{\mu}_{max}$ uniformly in $h > 0$?

we specialize to the unit square and to equal-order bilinear, biquadratic, and bicubic interpolation

THEOREM

- Let h denote a characteristic mesh size

- Then,

$$1 - \omega h + O(h^2) \leq \mu_{\max} \leq 1$$

where

$$\omega = \begin{cases} 2 & \text{for bilinear elements} \\ \frac{2}{3} & \\ 4 & \text{for biquadratic elements} \\ \frac{4}{15} & \\ 16 & \text{for bicubic elements} \\ \frac{16}{105} & \end{cases}$$

-
- The results obtained lead to the first result we are looking for

COROLLARY

- There exists $\vec{Q} \in \mathfrak{R}^M$ such that

$$0 \leq \vec{Q}^\top (\mathbb{K} - \mathbb{B}\mathbb{M}^{-1}\mathbb{B}^\top) \vec{Q} \leq (2\omega h + O(h^2)) \vec{Q}^\top \mathbb{K} \vec{Q}$$

- Thus, the semi-discrete pressure matrix $\mathbb{K} - \mathbb{B}\mathbb{M}^{-1}\mathbb{B}^\top$ is not uniformly invertible
- Note that $\mathbb{K} - \mathbb{B}\mathbb{M}^{-1}\mathbb{B}^\top$ does not depend on the stabilization parameter τ
 - thus the non-uniform invertibility of semi-discrete pressure matrix is unconditional
 - for all values of the stabilization parameter τ , the semi-discrete pressure equation is ill posed

THE STABILIZING ROLE OF TIME DISCRETIZATION

- Curiously, time discretization has a [stabilizing](#) effect on the ill-posed semi-discrete equations
- The fully discrete equations obtained using the backward Euler method are given by

$$\begin{pmatrix} \mathbb{M} + \Delta t \mathbb{A} & \Delta t \mathbb{B}^\top \\ (\tau - \Delta t) \mathbb{B} - \tau \Delta t \mathbb{S} & \tau \Delta t \mathbb{K} \end{pmatrix} \begin{pmatrix} \vec{U}^{(n+1)} \\ \vec{P}^{(n+1)} \end{pmatrix} = \begin{pmatrix} \mathbb{M} \vec{U}^{(n)} \\ \tau \mathbb{B} \vec{U}^{(n)} \end{pmatrix} + \Delta t \begin{pmatrix} \vec{F}^{(n+1)} \\ \tau \vec{G}^{(n+1)} \end{pmatrix}$$

where

$$\vec{U}^{(0)} = \vec{U}_0$$

and

$(\cdot)^{(n)}$ denotes the approximation at time level $t_n = n\Delta t$

- Eliminating $\vec{U}^{(n+1)}$, one obtains

$$(\mathbf{K} + \mathbf{NB}^\top) \vec{P}^{(n+1)} = \frac{1}{\Delta t} (\mathbf{NM} + \mathbf{B}) \vec{U}^{(k)} + \mathbf{N} \vec{F}^{(n+1)} + \vec{G}^{(n+1)}$$

where

$$\mathbf{N} = -\mathbf{B}(\mathbf{M} + \Delta t \mathbf{A})^{-1} + \Delta t \left(\mathbf{S} + \frac{1}{\tau} \mathbf{B} \right) (\mathbf{M} + \Delta t \mathbf{A})^{-1}$$

THEOREM

- As $\Delta t \rightarrow 0$, we have that

$$\mathbb{K} + \mathbf{N}\mathbf{B}^\top = \left(\mathbb{K} - \mathbf{B}\mathbf{M}^{-1}\mathbf{B}^\top \right) + \frac{\Delta t}{\tau} \left(\mathbf{B}\mathbf{M}^{-1}\mathbf{B}^\top + O(\Delta t) \right) + O(\Delta t)$$

- If

$$\frac{\Delta t}{\tau} \geq \alpha > 0$$

for any $\alpha > 0$, then

$$\vec{Q}^\top (\mathbb{K} + \mathbf{N}\mathbf{B}^\top) \vec{Q} \geq \frac{1}{2} \min\{1, \alpha\} \|\vec{Q}\|_{\mathbb{K}}^2$$

-
- A consequence of this theorem, we reach our ultimate goal: for fixed τ , showing the **necessity** of a uniform lower bound on the time step

COROLLARY

- If

$$\frac{\Delta t}{\tau} \geq \alpha > 0$$

where the value of $\alpha > 0$ is independent of h and Δt , then the fully discrete pressure matrix $\mathbb{K} + \mathbb{N}\mathbb{B}^\top$ is uniformly positive definite

- On the other hand, if

$$\frac{\Delta t}{\tau} \rightarrow 0 \quad \text{as} \quad \Delta t \rightarrow 0$$

$$\mathbb{K} + \mathbb{N}\mathbb{B}^\top \rightarrow \mathbb{K} - \mathbb{B}\mathbb{M}^{-1}\mathbb{B}^\top$$

i.e., the fully-discrete pressure matrix reduces to the unstable semi-discrete pressure matrix $\mathbb{K} - \mathbb{B}\mathbb{M}^{-1}\mathbb{B}^\top$

- In other words, we see from

$$\mathbb{K} + \mathbb{N}\mathbb{B}^\top = \mathbb{K} - \textcolor{red}{\mathbb{B}\mathbb{M}^{-1}\mathbb{B}^\top} + \frac{\Delta t}{\tau} \left(\textcolor{green}{\mathbb{B}\mathbb{M}^{-1}\mathbb{B}^\top} + O(\Delta t) \right) + O(\Delta t)$$

that implicit time discretization contributes the stabilizing term

$$\frac{\Delta t}{\tau} \textcolor{green}{\mathbb{B}\mathbb{M}^{-1}\mathbb{B}^\top}$$

- if $\Delta t/\tau > \alpha$ with α fixed, this term is sufficient to overcome the destabilizing term

$$-\textcolor{red}{\mathbb{B}\mathbb{M}^{-1}\mathbb{B}^\top}$$

- however, if $\Delta t/\tau \rightarrow 0$, the stabilization term disappears and we are left with the unstable semi-discrete pressure matrix $\mathbb{K} - \mathbb{B}\mathbb{M}^{-1}\mathbb{B}^\top$

CLEANING UP SOME LOSE ENDS

- What about other choices for the stabilization parameter τ ?

- we have chosen to use the **spatial** $\tau = \delta h^2$

- another possibility is the **transient** choice

$$\tau = \frac{\Delta t}{2} \left(1 + \left(\frac{\Delta t}{\delta h^2} \right)^2 \right)^{-1/2}$$

- the **good** news: h fixed and $\Delta t \rightarrow 0 \Rightarrow \Delta t/\tau > 2$

- the **bad** news: for h fixed and $\Delta t \rightarrow 0 \Rightarrow \tau \rightarrow 0$
 $\Rightarrow$ stabilization is lost!

- therefore, we see that the small time-step limit definition of τ is subject to two conflicting constraints
 - to stabilize, τ must scale as $O(h^2)$
 - to ensure that $\Delta t/\tau > \alpha$, it must scale as $O(\Delta t)$
- these constraints on τ are impossible to satisfy if Δt and h are allowed to vary independently
 - in particular, in the small time-step limit,
 - the spatial discretization step h must **necessarily** decrease as $\Delta t \rightarrow 0$

- What about the velocity approximation?

- it can be shown that (using backward Euler time stepping)

$$U^{(n+1)} = U^{(n)} + O(\Delta t)$$

- independently of the stabilization parameter

- hence, there are no stability problems for velocity approximations

- What about Galerkin least-squares and Douglas-Wang stabilization?

— for any $\gamma \in \{-1, 0, 1\}$, the semi-discrete equations are given by

$$\begin{pmatrix} (\mathbf{M} + \gamma \tau \mathbf{C}) \dot{\vec{U}} \\ \tau \mathbf{B} \dot{\vec{U}} \end{pmatrix} + \begin{pmatrix} \mathbf{A} - \gamma \tau \mathbf{D} & \mathbf{B}^\top + \gamma \tau \mathbf{S}^\top \\ -\mathbf{B} - \tau \mathbf{S} & \tau \mathbf{K} \end{pmatrix} \begin{pmatrix} \vec{U} \\ P \end{pmatrix} = \begin{pmatrix} \vec{F} + \gamma \tau \vec{H} \\ \tau \vec{G} \end{pmatrix}$$

where

$$\mathbf{C}_{ij} = \sum_{\mathcal{K} \in T_h} (\boldsymbol{\xi}_j^h, \Delta \boldsymbol{\xi}_i^h)_{\mathcal{K}} \quad \mathbb{D}_{ij} = \sum_{\mathcal{K} \in T_h} (\Delta \boldsymbol{\xi}_j^h, \Delta \boldsymbol{\xi}_i^h)_{\mathcal{K}}$$

$$(\vec{H})_i = \sum_{\mathcal{K} \in T_h} (\mathbf{f}, \Delta \boldsymbol{\xi}_i^h)_{\mathcal{K}}$$

— the associated semi-discrete pressure matrix is given by

$$\mathbb{K} - (\mathbb{B} + \gamma\tau\mathbb{S})(\mathbb{M} + \gamma\tau\mathbb{C})^{-1}\mathbb{B}^\top$$

— the invertibility of the matrix $\mathbb{M} + \gamma\tau\mathbb{C}$ may place a further restriction on the stabilization parameter τ

— in any case, with $\tau = \delta h^2$, we have that

$$\mathbb{K} - (\mathbb{B} + \gamma\tau\mathbb{S})(\mathbb{M} + \gamma\tau\mathbb{C})^{-1}\mathbb{B}^\top \sim \mathbb{K} - \mathbb{B}\mathbb{M}^{-1}\mathbb{B}$$

i.e., for $\gamma = -1$ or $\gamma = 1$, **the semi-discrete pressure matrix is spectrally equivalent** to the corresponding matrix for $\gamma = 0$

- we have made this notion precise by providing the comparability constants

— thus, **for $\gamma = -1$ and $\gamma = 1$** , the semi-discrete pressure matrix is again not uniformly invertible

$\Rightarrow$ **end up with the same conclusions as for $\gamma = 0$**

- What does this all mean in practice?

- we have $\tau = \delta h^2$ and we need to have

$$\Delta t > \alpha h^2 \quad \text{for fixed } \alpha$$

- since

- the time step may be governed by features not under our control, e.g., fast reactions,

satisfaction of the lower bound requires that

- the spatial grid size be sufficiently small, i.e.,

$$h < \sqrt{\frac{\Delta t}{\alpha}}$$

- in this case,

- the spatial grid size is not determined by spatial accuracy considerations

so that

- the spatial grid size may end up being prohibitively small

- What about non-uniform grids?

— for non-uniform grids, it may be desirable to choose the stabilization term in the form

$$- \sum_{\mathcal{K} \in \mathcal{T}_h} \tau_{\mathcal{K}} \left(\mathbf{u}_t^h - \Delta \mathbf{u}^h + \nabla p^h - \mathbf{f}, -\gamma \Delta \mathbf{v}^h + \nabla q^h \right)_{\mathcal{K}}$$

where

$$\tau_{\mathcal{K}} = \delta h_{\mathcal{K}}^2$$

and $h_{\mathcal{K}}$ is the diameter of the element $\mathcal{K}$

- What's the connection with pressure-projection methods?

- interestingly, the operator $\mathbb{K} - \mathbb{B}\mathbb{M}^{-1}\mathbb{B}^T$ also arises in pressure-projection methods
- in a pressure-projection method, this matrix effectively relaxes the discretized continuity equation to

$$-\mathbb{B}\vec{U} + \Delta t(\mathbb{K} - \mathbb{B}\mathbb{M}^{-1}\mathbb{B}^T)\vec{P} = 0$$

- here, all that is needed for stabilization is that the matrix $\mathbb{K} - \mathbb{B}\mathbb{M}^{-1}\mathbb{B}^T$ be positive semi-definite
 - it does not have to be uniformly positive definite

- a pressure-projection method implemented with equal-order finite element spaces will also become unstable when Δt is small relative to h^2
- the cause of this instability is fundamentally different from the one in stabilized methods
- in a pressure-projection method, the instability arises from the insufficient amount of stabilization provided by $\Delta t(\mathbb{K} - \mathbb{B}\mathbb{M}^{-1}\mathbb{B}^T)$ for small Δt

THE MORALS OF THE STORY

- A lower bound on the time step is necessary if one uses a fully-discrete, consistent stabilization method
 - if one has to use small time steps, instabilities arise
 - however, if one does not have to use small time steps, consistently stabilized finite element methods are good methods to use
- If small time steps are needed, one should use mixed Galerkin finite element methods, if possible
 - in fact, mixed Galerkin finite element methods are always a good choice, unless they are ruled out by practical considerations

- If practical issues, e.g., complex, multiphysics problems, prevent the use of standard mixed Galerkin finite element methods, possible alternatives to consistently stabilized finite element methods include
 - coupled, space-time discretizations
 - inconsistently stabilized finite element methods
 - least-squares finite element methods

DETAILS

- Details about today's talk can be found in the paper

P. BOCHEV, M. GUNZBURGER, AND R. LEHOUCQ; On stabilized finite element for the Stokes problem in the small time step limit, to appear in *Int. J. Numer. Meth. Fluids*