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Enriched Finite Element Methods in ARIA and GOMA for Thermal/Fluids Applications

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Outline

- Motivation
  - Why interface capturing instead of moving mesh (ALE)
  - Why enriched finite elements
  - Relevant applications
- Description
  - eXtended Finite Element Method (XFEM)
- Case Study: XFEM investigation of laser welding
- Implementation issue: quadrature for discontinuous integrands
- Beyond XFEM: Application independent dynamic interfaces using Conformal Decomposition Finite Element (CDFEM)

How Level Set, Enriched Finite Element Methods Can Help us do Our Job Better

Fundamental Motivations
- Conformal mesh generation is typically more expensive than analysis (approximately 80% of total effort for complex models)
- Arbitrary Lagrangian Eulerian (ALE) is extremely powerful but cannot capture topological change
  - Precludes interfacial breakup and merging

Level Set Methods Have Promise
- To help us simulate complex geometries without conformal meshes
  - Do the complex simulations we do now with faster turnaround time and with less analyst time
- To enable us to simulate physics we cannot currently address
  - We cannot use ALE to simulate the merging and breakup that occur in laser welding and foam decomposition

Problem Class: Dynamic Interface Problems
- Typical application area for level set methods
- Examples: multiphase flow and phase change problems like laser welding, drop dynamics, mold filling, and foam decomposition
- Benefits
  - Difficult, if not impossible, to address using ALE

Problem Class: Topologically Complex, but Stationary Interfaces
- A less obvious application area
- Examples: conduction in composite materials, single phase flow in porous media
- Benefits
  - Avoid conformal mesh generation
  - Avoid contact between disparate meshes

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Finite Element Methods for Interfaces in Fluid/Thermal Applications

Boundary Fitted Meshes
- Supports wide variety of interfacial conditions accurately
- Requires boundary fitted mesh generation
- Not feasible for arbitrary topological evolution (ALE)
  - Mesh quality degrades with evolution, phase breakup and merging are precluded.

eXtended Finite Element Methods (XFEM)
- Dolbow et al. (2000), Belytchko et al. (2001)
- Successfully applied to numerous problems ranging from crack propagation to phase change to multiphase flow
- Supports weak conditions accurately, mixed and Dirichlet conditions are actively researched (Dolbow et al.)
- Avoids boundary fitted mesh generation
- Supports general topological evolution (subject to resolution requirements)
- Requires modified matrix structure and element assembly including interpolation and integration
  - Modified quadrature rules being actively researched

Generalized Finite Element Methods (GFEM)
- Strouboulis et al. (2000)
- Combination of standard finite element and partition of unity enrichment

Immersed Finite Element Methods
- Li et al. (2003)
- Supports selected jumps across material boundaries (discontinuous gradient or value)

Conformal Decomposition Finite Element Method (CDFEM)
- Enrichment by adding nodes along interfaces

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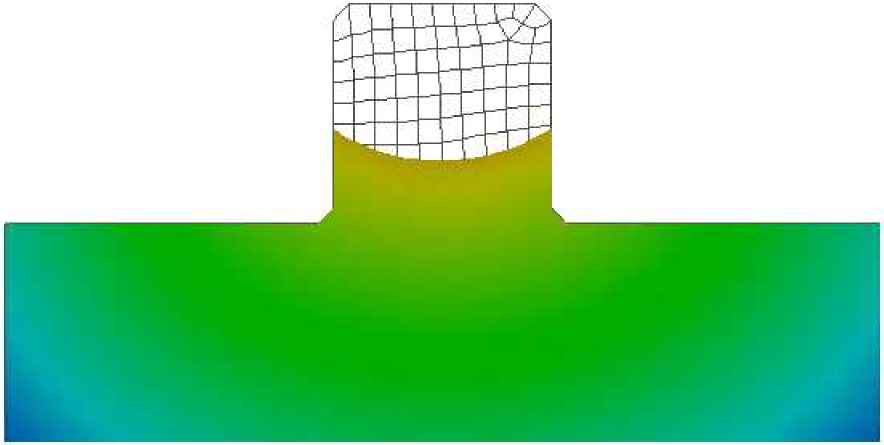
Dynamic Interface Example: Flow in a Microchannel

Low capillary number flow into a microchannel (Ca=0.01)

- Complex interfacial physics
  - Surface tension dominated flow
  - Wetting model plays critical role

- ALE simulations require frequent remeshing
  - Expensive analyst time, introduces inaccuracy
- Level set simulation performed in one simulation

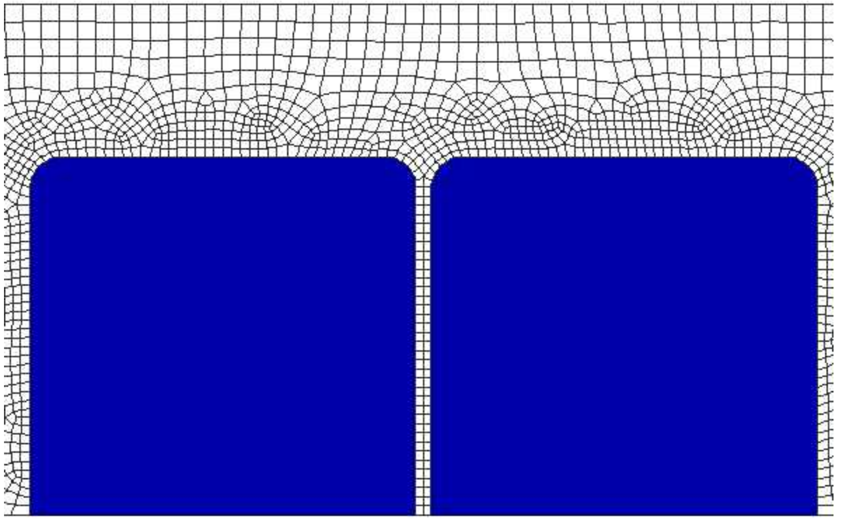


Dynamic Interface Example: Laser Welding

Material joining by intense localized heating

- Extremely complex interfacial physics
  - Radially distributed laser heating
  - Vapor recoil pressure
  - Vaporization heat loss
  - Radiation and convection heat loss
  - Critical role of surface tension

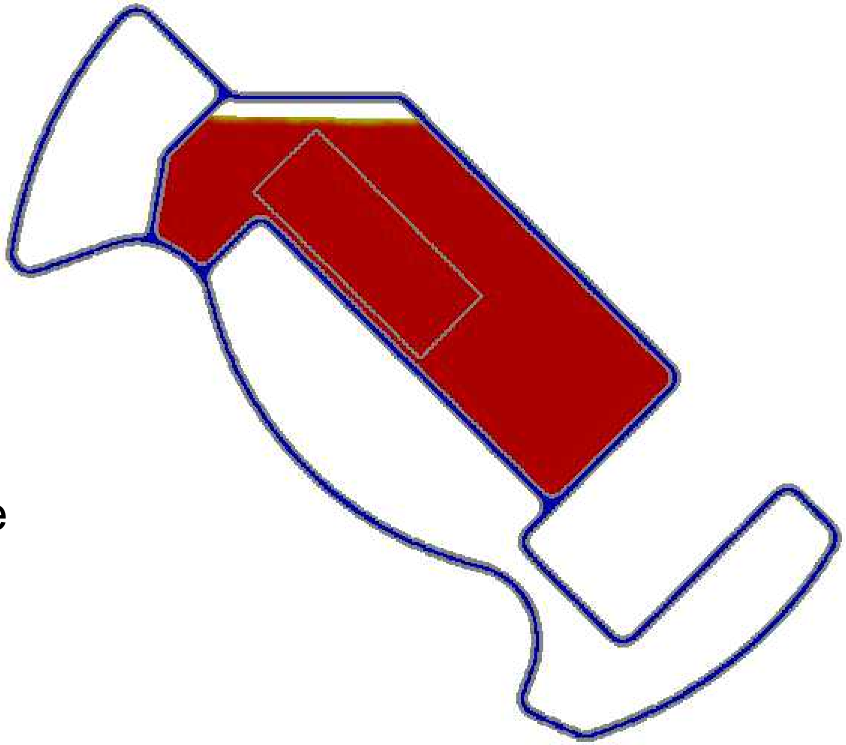
- ALE simulations require frequent remeshing
  - Expensive analyst time, introduces inaccuracy
- Level set - XFEM simulations capture topological change
  - Complete set of interfacial conditions applied along level set interface cutting through elements



Dynamic Interface Example: REF Modeling

Foam Removed by Surface Reaction and Flow

- Complex interfacial physics
  - First-order surface reaction
  - Foam modeled as viscous liquid
  - Surface velocity include flow and reaction components
- ALE simulations not feasible
  - Despite relatively slow interfacial motion, changing topology makes remeshing difficult, if not impossible
- Level set - XFEM simulations capture topological change
  - Complete set of interfacial conditions applied along level set interface cutting through elements



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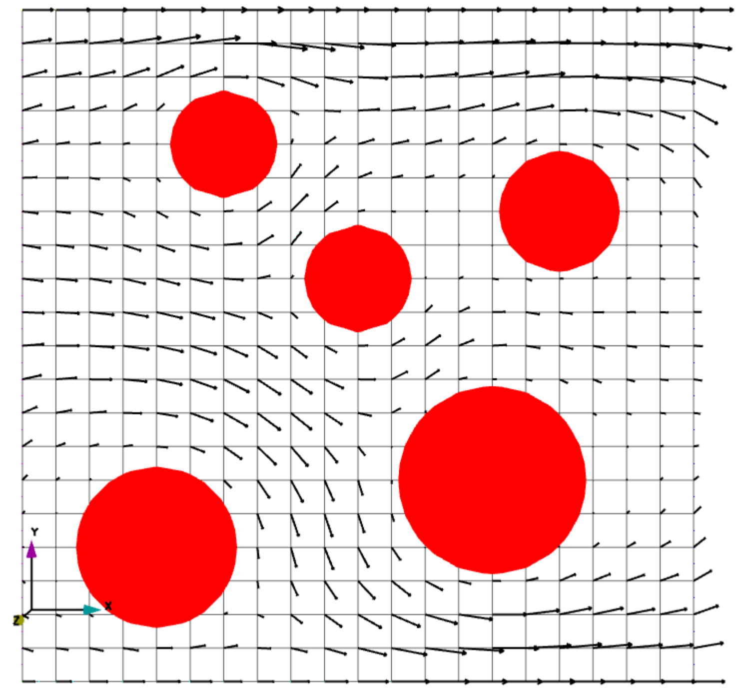
Topologically Complex Interfaces Example: Flow in Random Media

Conformal mesh approach

- Current method of choice whenever feasible
- Requires generating conformal mesh
  - Expensive in terms of analyst time
  - Sometimes impossible for quadrilateral or hexahedral meshes
  - May require separate mesh and contact bc's if modeling physics in solids



Level set method

- Accurate, cost effective approach that should be pursued
- Non-conforming mesh more easily generated
  - Geometry description used to generate level set function rather than conformal mesh
  - Allows much faster prototyping and parameter studies including geometry modification

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Topologically Complex Interfaces Example: Conduction in Composites

Conformal mesh approach

- Current method of choice whenever feasible
- Requires generating conformal mesh
  - Expensive in terms of analyst time
  - Sometimes impossible for quadrilateral or hexahedral meshes
  - May require separate mesh and contact bc's if modeling physics in solids

Level set method

- Accurate, cost effective approach that should be pursued
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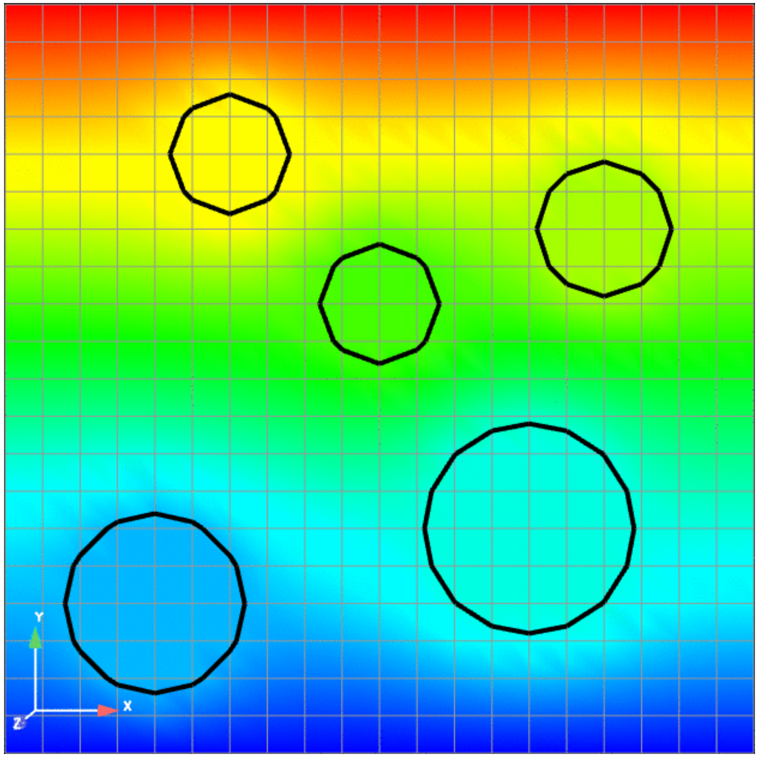

Cylindrical inclusions with 1000x higher conductivity

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Technical Approach: Interfacial Motion Modeling

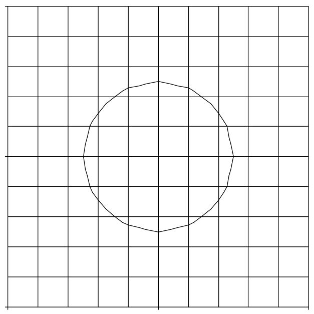
Solve for signed distance from interface

$$\frac{\partial \phi}{\partial t} + u \cdot \nabla \phi = 0$$

Interface normal and curvature computed

$$\vec{n} = \nabla \phi, \kappa = \nabla \cdot \nabla \phi$$

Phase Boundary



Interfacial discontinuities accounted for by modifying stress tensor, heat flux, species flux for elements along interface

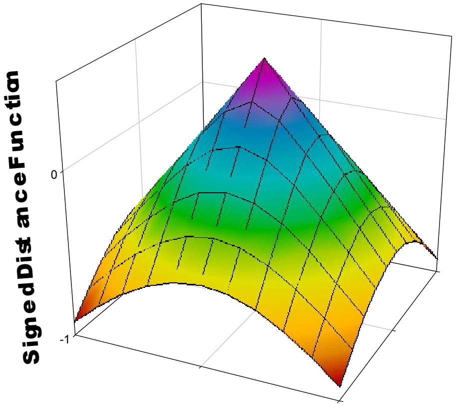
- Stress Tensor term

  $$\tau_{ij} = \sigma \delta(\phi)\left[\delta_{ij} - n_i n_j\right]$$

  – Delta function may be sharp or diffuse function of distance

- Similar for Heat, Species Flux

Level Set Representation



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Level Sets in Finite Elements: Extended Finite Element Method

Extended Finite Element: Finite Element Method for Embedded Interfacial Jumps

- Dolbow et al (2000)

Enrich elements containing discontinuities

- Add extra degrees of freedom, a_i

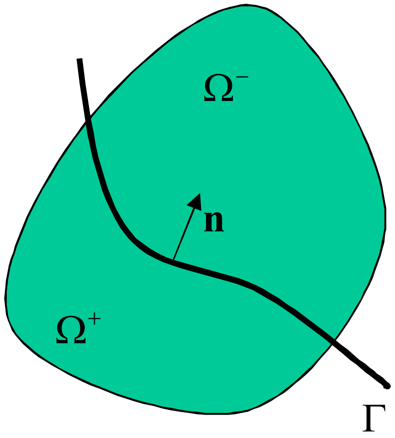
$$T = \sum_i N_i T_i + \sum_i N_i g_i a_i$$

- Basis functions for extended dofs have two parts
- Standard continuous variation within element, N_i
- Discontinuous extending function, g_i
- Typical form for discontinuous value

$$g_i(x) = H(\phi(x)) - H(\phi_i), \;\; \phi_i \equiv \phi(x_i)$$

- Typical form for discontinuous gradient

$$g_i(x) = |\phi(x)| - |\phi_i|$$



Extended Finite Element Method

Features

- Enforces continuity across element faces
  - Enrichment is nodal
- Element contributions are discontinuous

  Element contribution to residual

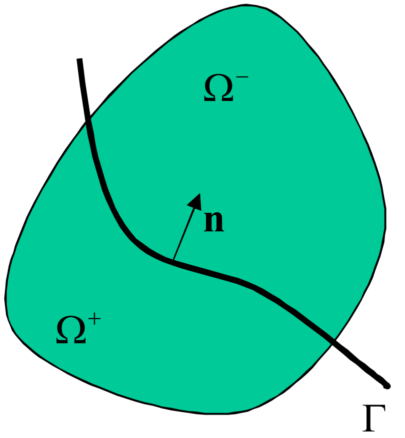
$$R_i = -\int_{\Omega} \nabla N_i \cdot k \nabla T \, d\Omega$$

becomes

$$R_i = -\int_{\Omega^-} \nabla N_i \cdot k \nabla T \, d\Omega - \int_{\Omega^+} \nabla N_i \cdot k \nabla T \, d\Omega$$

$$-\int_{\Gamma} N_i^+ \mathbf{n} \cdot \mathbf{Q}^+ \, d\Omega + \int_{\Gamma} N_i^- \mathbf{n} \cdot \mathbf{Q}^- \, d\Omega$$

  - Weight functions are discontinuous
  - Gradients are discontinuous
  - Requires conformal integration



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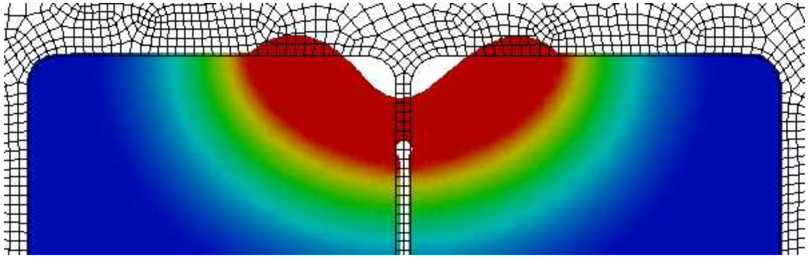
Case Study: XFEM Study of Laser Welding in GOMA

Process Characteristics

- Topological change
  - Weld pool formation and deformation
  - Weld pool merging and undesirable void formation
- Surface dominated physics
  - Surface laser heat flux
  - Surface heat loss by conduction and radiation
  - Strong surface tension effects (including Marangoni)

Resulting Simulation Requirements

- Eulerian method required to capture discontinuities as the move across fixed grid
- Finite element method desirable for material property variation and natural description of interfacial fluxes
- eXtended Finite Element Method (XFEM) required to account for interfacial discontinuities



Implementation – Applying XFEM to Laser Welding

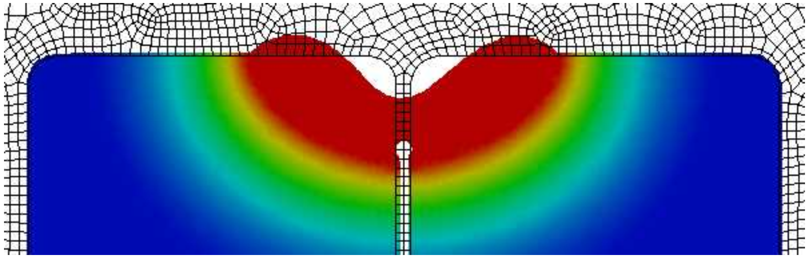
Problem Discretization
- Fixed unstructured mesh
- Solid-liquid interface described by enthalpy method
  - Specific heat is temperature dependent to account for latent heat
  - Viscosity sharp function of temperature around between solidus and liquidus
- Liquid-vapor interface described by level set method

Variable Enrichment
- Variables allowed to be discontinuous across liquid-vapor interface

Subelement Integration
- Required to integrate discontinuous quantities resulting from discontinuous variables and trial functions

Interfacial conditions
- XFEM approach produces natural mechanism for applying interfacial fluxes
- Several options discussed in literature for handling surface tension

Coupling
- Implemented in code designed for fully coupled, Newton's method
- Choice of surface tension application made this impossible
  - Final algorithm involves loosely coupling the level set evolution to the mass, momentum, and energy evolution



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Implementation – Enriched Quantities

Discontinuous quantities

- Surface tension produces sharply discontinuous pressure across the interface
  - Pressure is enriched with Heaviside function

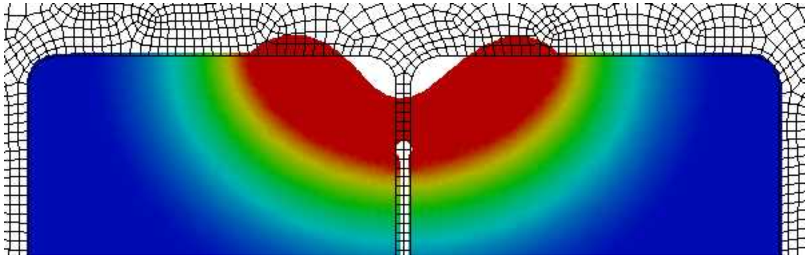
$$P(\mathbf{x}) = \sum_i P_i N_i(\mathbf{x}) + \sum_i a_i N_i(\mathbf{x}) g_i(\mathbf{x})$$

$$g_i(\mathbf{x}) = H(\phi(\mathbf{x})) - H(\phi_i), \ \ \phi_i \equiv \phi(\mathbf{x}_i)$$

- Velocity gradient is discontinuous due to jump in viscosity
  - The is a secondary effect compared to pressure discontinuity
  - Experiments with gradient-type enrichment have not yielded significant differences
  - Currently, velocity is not enriched

One-sided quantities

- Temperature in vapor is irrelevant since surface heat transfer is better described using laser heat input and radiative boundary conditions
  - Variable itself is not enriched, but trial function is multiplied by Heaviside function
  - This truncation of the integration domain produces boundary integral due to integration by part of diffusive terms



Implementation Issue: XFEM Integration

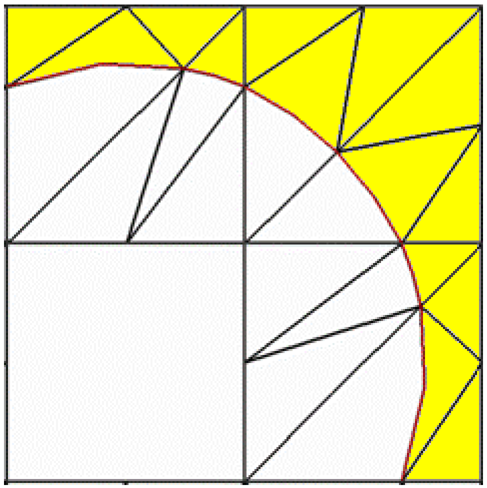
Modified Element Quadrature

- Basis and trial functions are now discontinuous

$$R_i = -\int_{\Omega} \nabla N_i \cdot k \nabla T \, d\Omega$$

$$R_i = -\int_{\Omega^-} \nabla N_i^- \cdot k^- \nabla T^- \, d\Omega - \int_{\Omega^+} \nabla N_i^+ \cdot k^+ \nabla T^+ \, d\Omega - \int_{\Gamma} N_i^+ \mathbf{n} \cdot \mathbf{Q}^+ \, d\Omega + \int_{\Gamma} N_i^- \mathbf{n} \cdot \mathbf{Q}^- \, d\Omega$$

- For XFEM-Level set methods, functions become generalized functions of the level set variable
  - Heaviside and Dirac delta functions



Moderately Invasive Feature in XFEM Codes

- Quadrature rule depends on level set variable
- Coupling issues, time derivative evaluation

Several Solutions

- Diffuse integration
  - Smoothed generalized functions
- Subelement integration
  - Subdivide elements into conformal subelements
  - Implementation issues
- Develop new integration rules for generalized functions
  - Derive new integration rules that account for generalized functions

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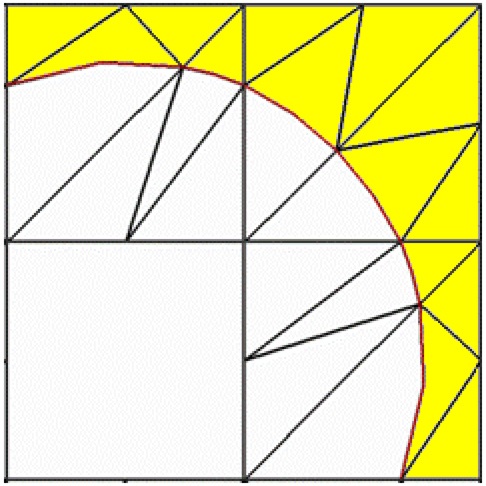
XFEM Subelement Integration - Issues



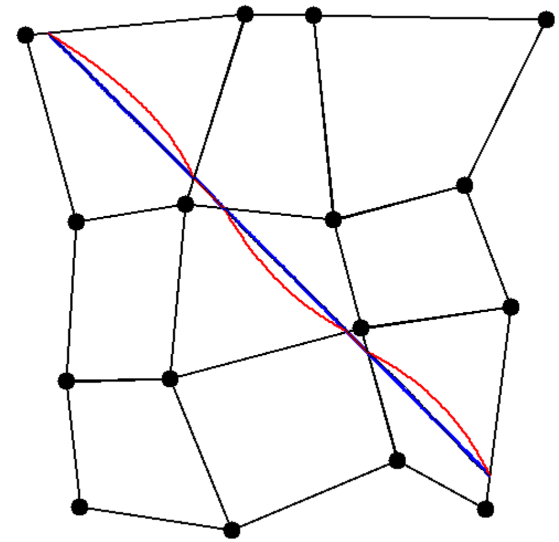
Basics

- Decompose non-conformal element into conformal subelements
- Perform standard Gauss integration over subelements

Important Implementation Details

- What is definition of subelements?
  - Option 1: Coordinates of subelements are parametric coordinates for owning element
  - Option 2: Coordinates of subelements are real coordinates
- What order are the subelements?
  - Are linear sides sufficient for obtaining optimal rates of convergence?

XFEM Subelement Integration - Issues

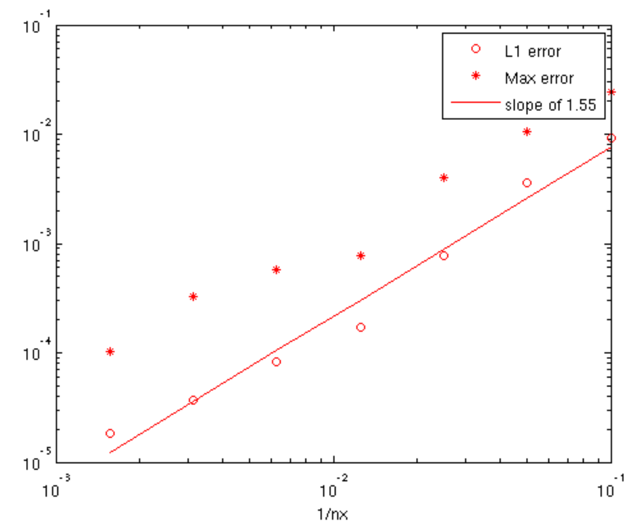


Subelements in Parametric Coordinates

- Low order subelements with linear sides
  - Figure: desired surface vs. actual surface
  - Suboptimal accuracy
- Higher order subelements with parabolic sides
  - Figure: actual surface (thin black curve nearly coincident with desired surface)
  - More costly quadrature
  - Must use root finder for internal nodes
  - Achieves optimal convergence rate

Subelements in Real Coordinates

- Low order subelements with linear sides
  - Must solve simple but nonlinear system for every quadrature point, every time step
  - Achieves optimal convergence rate
  - 3D: How are non-planar hex faces handled?



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Integration Rules for Elements with Generalized Functions - Motivation

Philosophical

- Integration rules designed to exactly integrate finite element functions
  - Enriched functions need modified quadrature rules

Pragmatic When Compared with Alternatives

- Diffuse methods
  - Simple but inaccurate, inconsistent
- Subelement methods
  - Must be carefully implemented
  - Must specifically account for degenerate cases

Allows Advanced Capabilities

- Provides analytical Jacobian information
  - Required by full Newton codes
  - Make interfacial optimization possible

Possible Disadvantages

- Possibly increases number of quadrature points for same element
- Difficult, if not impossible to derive for higher order elements

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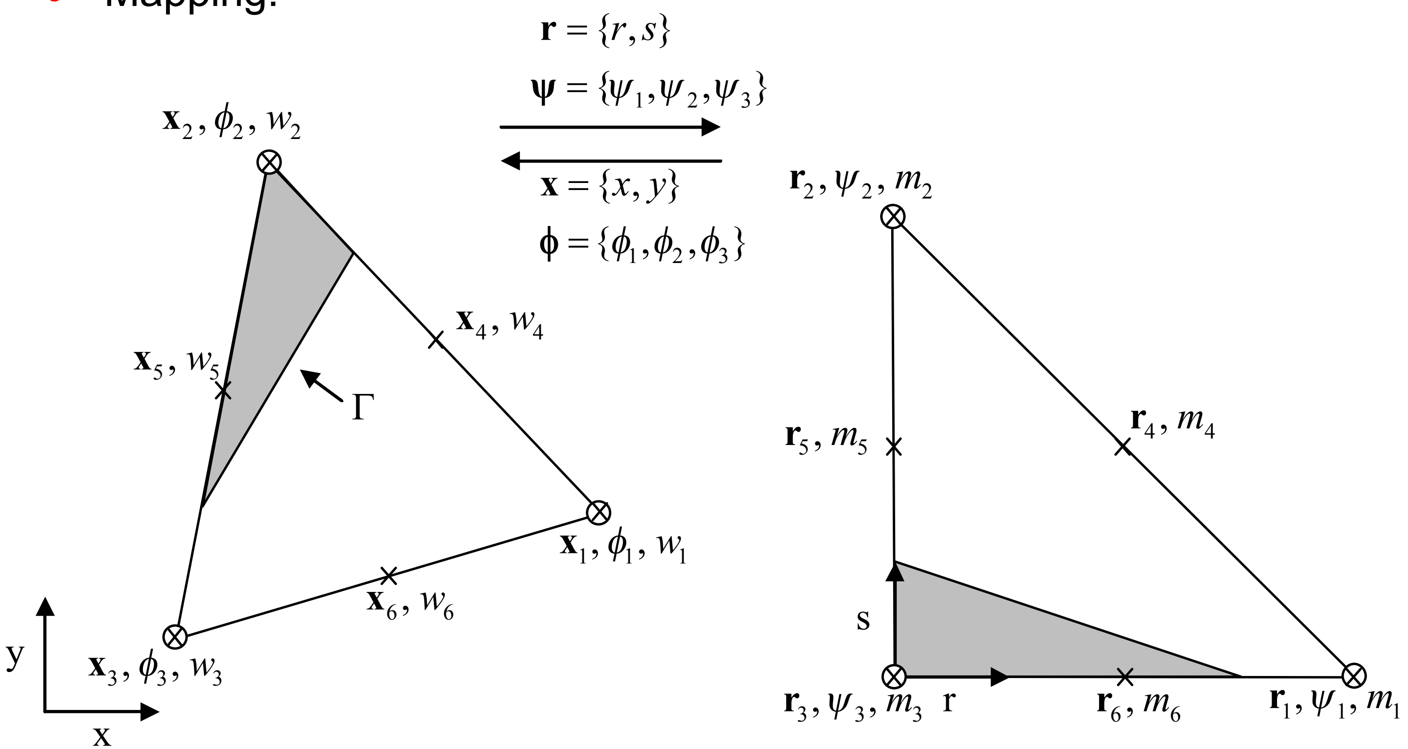
Generalized Function Quadrature - Method

Approach

- Develop quadrature rules capable of exactly integrating finite element functions including a generalized function of the level set variable
  - Piecewise polynomial times Heaviside or Dirac delta function
- Form:

$$\int_{\Omega^+} g(\mathbf{x})\, d\Omega_{\mathbf{x}} = \sum_{i=1}^{6} w_i^{+}(\boldsymbol{\phi})\, g(\mathbf{x}_i)\, J(\mathbf{x}_i) \qquad \int_{\Gamma} g\, d\Gamma_{\mathbf{x}} = \sum_{i=1}^{6} w_i^{\Gamma}(\boldsymbol{\phi}) \left|\nabla \phi(\mathbf{x}_i)\right| g(\mathbf{x}_i) J(\mathbf{x}_i)$$

- Mapping:



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Generalized Function Quadrature - Method

- Form linear system for weights

$$I_f^{\Delta}(\psi) \equiv \int_{\Delta} f(\mathbf{r})\, d\Omega_{\mathbf{r}} = \sum_{i=1}^{6} m_i^{\Delta}(\phi) f(\mathbf{r}_i)$$

$$\mathbf{A}\,\mathbf{m}^{\Delta}(\psi) = \mathbf{I}^{\Delta}(\psi)$$

  - Require all monomials in a quadratic function be exactly integrated

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ r_1 & r_2 & r_3 & r_4 & r_5 & r_6 \\ s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ r_1 s_1 & r_2 s_2 & r_3 s_3 & r_4 s_4 & r_5 s_5 & r_6 s_6 \\ r_1^2 & r_2^2 & r_3^2 & r_4^2 & r_5^2 & r_6^2 \\ s_1^2 & s_2^2 & s_3^2 & s_4^2 & s_5^2 & s_6^2 \end{bmatrix} \begin{bmatrix} m_1^{\Delta}(\psi) \\ m_2^{\Delta}(\psi) \\ m_3^{\Delta}(\psi) \\ m_4^{\Delta}(\psi) \\ m_5^{\Delta}(\psi) \\ m_6^{\Delta}(\psi) \end{bmatrix} = \begin{bmatrix} I_1^{\Delta}(\psi) \\ I_r^{\Delta}(\psi) \\ I_s^{\Delta}(\psi) \\ I_{rs}^{\Delta}(\psi) \\ I_{r^2}^{\Delta}(\psi) \\ I_{s^2}^{\Delta}(\psi) \end{bmatrix}$$

  - Select quadrature point locations
    - Valid quadrature rules yield nonsingular matrix,
    - Normally quadrature point locations considered unknowns that are selected such that integration achieves desired order with minimal number of points
    - Arbitrary interface location makes fortuitous point selection impossible
    - Simplest valid quadrature rules involve points on the nodes and edges

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Generalized Function Quadrature - Method

- Form linear system for weights, cont'd
  - Analytically evaluate integrals as function of nodal level set values

$$I_1^\Delta(\psi) = \frac{\psi_3^2}{2\,\Delta_{31}\,\Delta_{32}} \qquad I_{rs}^\Delta(\psi) = \frac{\psi_3^4}{24\,\Delta_{31}^2\,\Delta_{32}^2}$$

$$I_r^\Delta(\psi) = \frac{\psi_3^3}{6\,\Delta_{31}^2\,\Delta_{32}} \qquad I_{r^2}^\Delta(\psi) = \frac{\psi_3^4}{12\,\Delta_{31}^3\,\Delta_{32}}$$

$$I_s^\Delta(\psi) = \frac{\psi_3^3}{6\,\Delta_{31}^2\,\Delta_{32}} \qquad I_{s^2}^\Delta(\psi) = \frac{\psi_3^4}{12\,\Delta_{31}\,\Delta_{32}^3}$$

$$\Delta_{31} \equiv \psi_3 - \psi_1$$

$$\Delta_{32} \equiv \psi_3 - \psi_2$$

- Solve for weights as functions of nodal level set values

$m_i^\Delta(\psi)$	functional form
$m_1^\Delta(\psi)$	$-I_r^\Delta(\psi) + 2I_{r^2}^\Delta(\psi)$
$m_2^\Delta(\psi)$	$-I_s^\Delta(\psi) + 2I_{s^2}^\Delta(\psi)$
$m_3^\Delta(\psi)$	$I_1^\Delta(\psi) - 3I_r^\Delta(\psi) - 3I_s^\Delta(\psi) + 4I_{rs}^\Delta(\psi) + 2I_{r^2}^\Delta(\psi) + 2I_{s^2}^\Delta(\psi)$
$m_4^\Delta(\psi)$	$4I_{rs}^\Delta(\psi)$
$m_5^\Delta(\psi)$	$4(I_s^\Delta(\psi) - I_{rs}^\Delta(\psi) - I_{s^2}^\Delta(\psi))$
$m_6^\Delta(\psi)$	$4(I_r^\Delta(\psi) - I_{rs}^\Delta(\psi) - I_{r^2}^\Delta(\psi))$

Results

- Weights are continuous functions of nodal level set values
  - Allows analytical Jacobian formation
  - All degenerate cases handled without special consideration
- Weights are not positive definite

Generalized Function Quadrature – Test Problem

Conduction in Annulus and Spherical Shell

- Poisson equation, $k = 1$, $q = 1$

  $$\nabla \cdot k \nabla T + q = 0$$

- Boundary conditions
  - Insulated inner surface
  - Robin-type output surface, $h = 10$

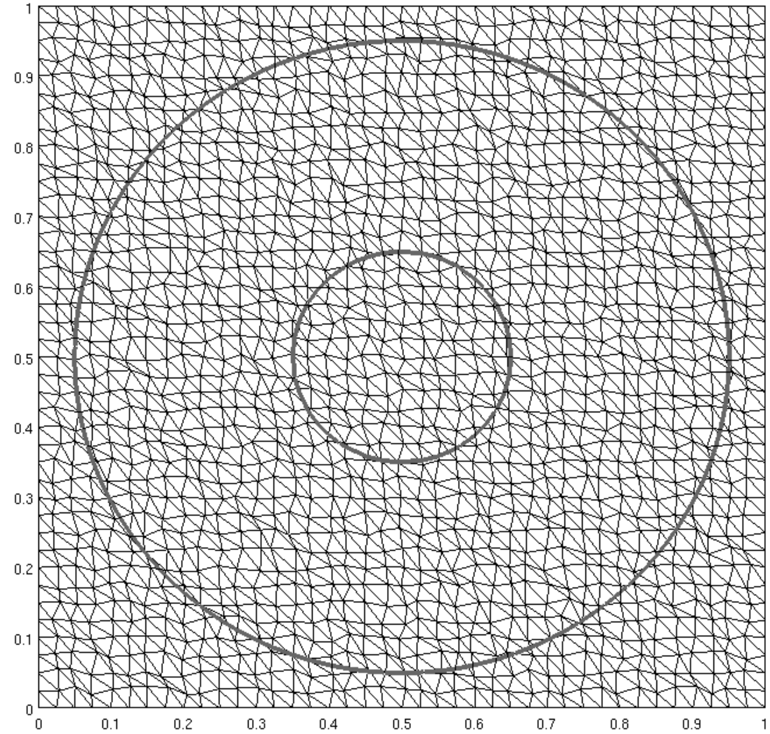
  $$-\mathbf{n}_{outer} \cdot k \nabla T = h\left(T - 0\right)$$

Discretization

- Linear triangle and tetrahedral elements, linear temperature, linear level set function
- Randomly perturbed nodes of structured mesh
  - Rigorous test for deformed meshes

Validation

- Compare against exact solutions



$$T^{2D}(r) = \frac{q}{4k}\left(R_o^2 - r^2\right) + \frac{q}{2hR_o}\left(R_o^2 - R_i^2\right) - \frac{qR_i^2}{2k}\left(\log\left(R_o\right) - \log\left(r\right)\right)$$

$$T^{3D}(r) = \frac{q}{3hR_o^2}\left(R_o^3 - R_i^3\right) - \frac{q}{6kr}\left(r^3 + 2R_i^3\right) + \frac{q}{6kR_o}\left(R_o^3 + 2R_i^3\right)$$

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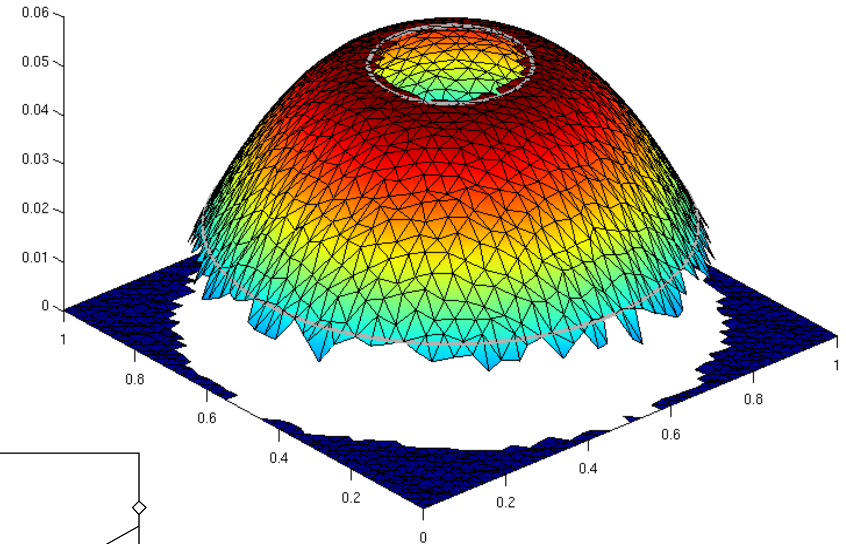
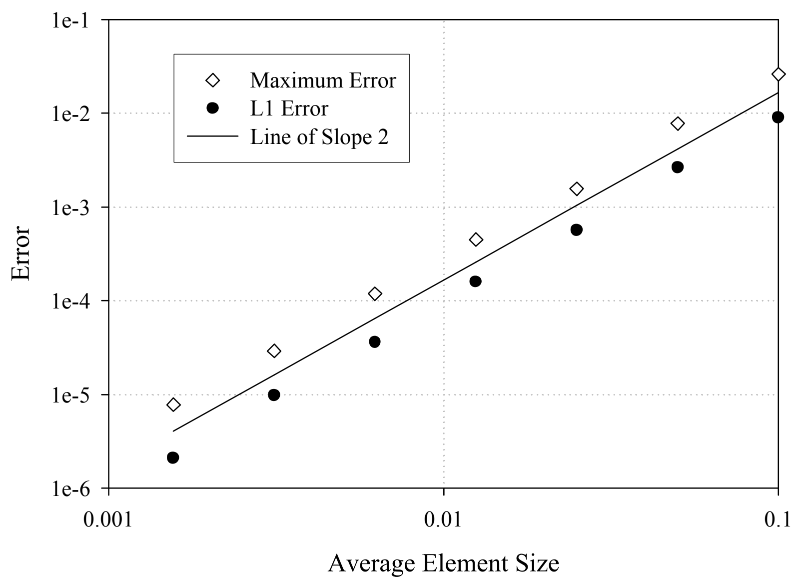
Generalized Function Quadrature – 2D Test

Results

- Visualization - Elements that use ghost nodes and exterior nodes are removed
- Sharp discontinuities captured along inner and outer surfaces
- 2nd order accuracy demonstrated over multiple decades





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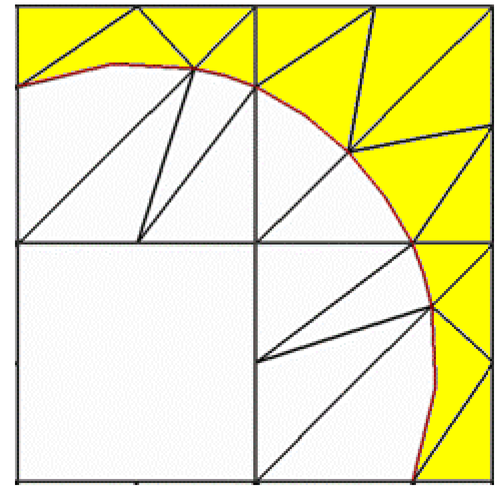
Beyond XFEM: Conformal Decomposition Finite Element Methods (CDFEM)

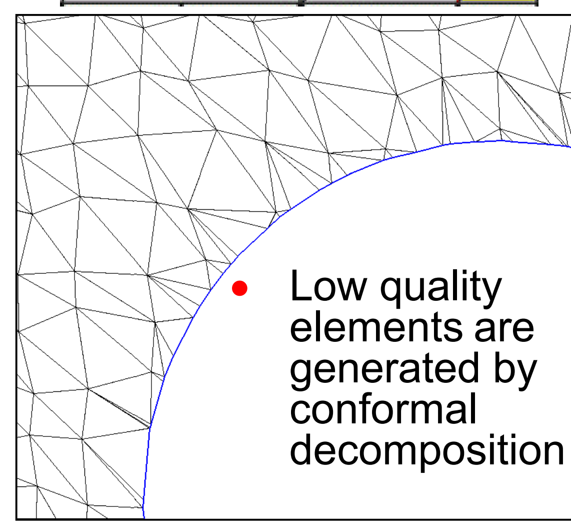
Simple Concept
- Decompose non-conformal elements into conformal ones
- Obtain solution on conformal elements

Related Work
- Li et al. (2003) FEMCGAN: FEM on Cartesian Grid with Added Nodes
  - Focus on Cartesian Grid. Considered undesirable because it lost original matrix structure.
- Mathematical works: Chen and Zhou (1998), Riviere and Girault (2006)
- Others?

Properties
- Supports wide variety of interfacial conditions accurately (identical to boundary fitted mesh)
- Avoids boundary fitted mesh generation
- Supports general topological evolution (subject to resolution requirements)
- Requires modified matrix structure (additional elements)
  - Similar to finite element adaptivity
- Uses standard finite element assembly including data structures, interpolation, and quadrature

Questions
- Accuracy? Conformal elements can have vanishing quality.
- Relationship to XFEM?





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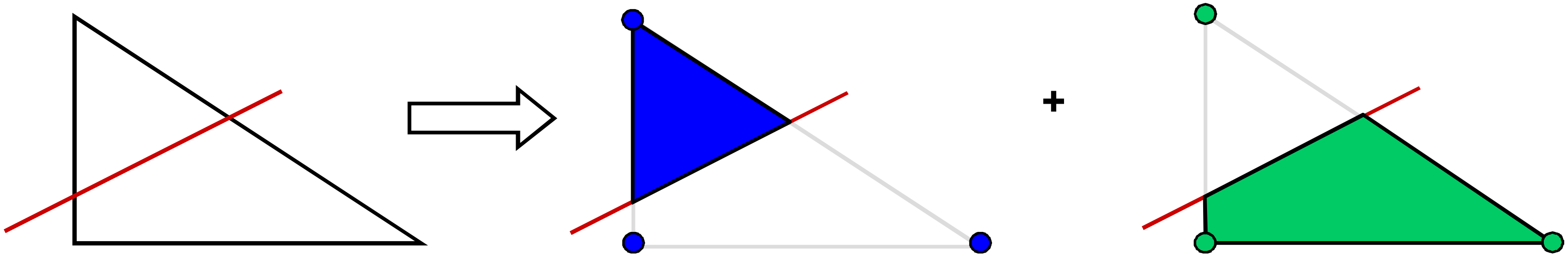
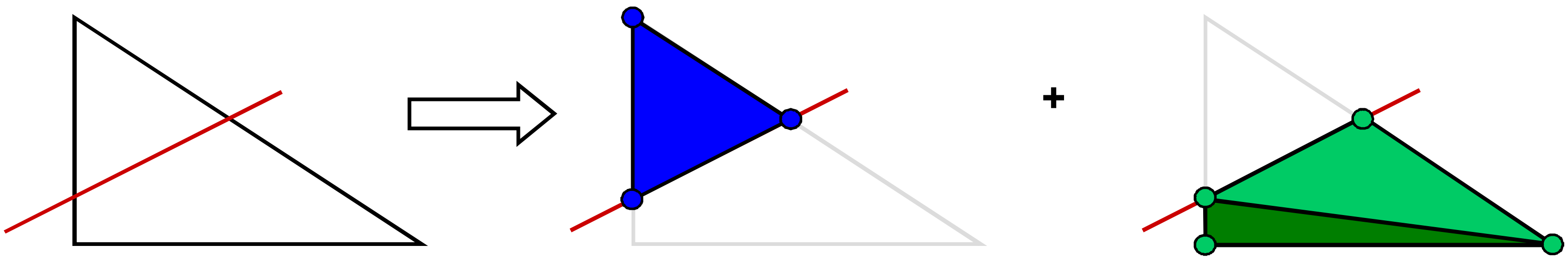
XFEM – CDFEM Comparison

XFEM Approximation



CDFEM Approximation



- Identical IFF interfacial nodes in CDFEM are constrained to match XFEM values at nodal locations
- CDFEM space contains XFEM space

XFEM – CDFEM Comparison, cont'd

Approximation

- CDFEM space contains XFEM space
  - Accuracy of CDFEM no less than XFEM? Li et al. (2003)
  - CDFEM can recover XFEM solution by constraining interfacial nodes
    - Separate linear algebra step outside of element assembly routines

Boundary Conditions

- CDFEM readily handles interfacial Dirichlet conditions
  - Simply apply Dirichlet conditions to interfacial nodes
- Gives another view of difficulty with Dirichlet conditions in XFEM
  - CDFEM recovers XFEM when interfacial nodes constrained to XFEM space
  - CDFEM provides optimal solution for Dirichlet problem when interfacial nodes are given by Dirichlet conditions
  - Attempting to satisfy both sets of constraints simultaneously over-constrains the problem

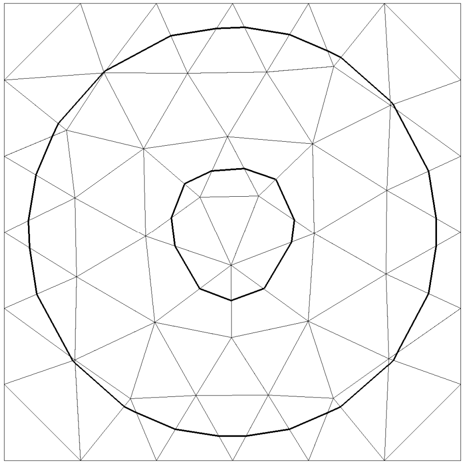
Implementation

- Conformal decomposition can be performed external to all assembly routines
  - For stationary interfaces the decomposition can be performed once on the input mesh

CDFEM Implementation

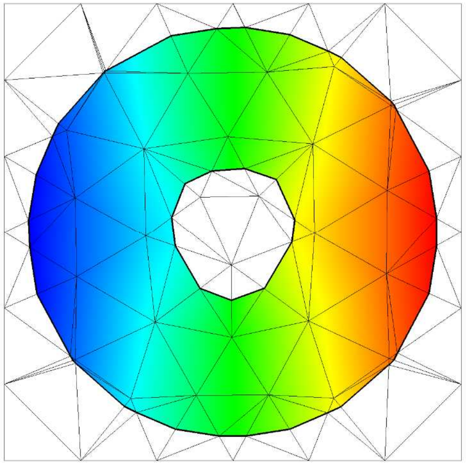
For Steady State Problems

- Stationary Interfaces
  - Conformal decomposition can be performed once
  - Non-conformal mesh input, KRINO performs conformal decomposition, ARIA solves transport
  - Provides test of accuracy, performance, and implementation



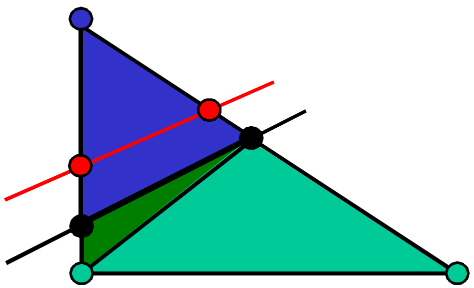
For Transient Problems

- Must perform decomposition based on current interface location
  - Level set provides convenient description
- Similar requirements to adaptive refinement
  - Dynamic data structures, matrix graph
  - Prolongation of solution to new nodes
- Transparent to physics code (Element assembly)

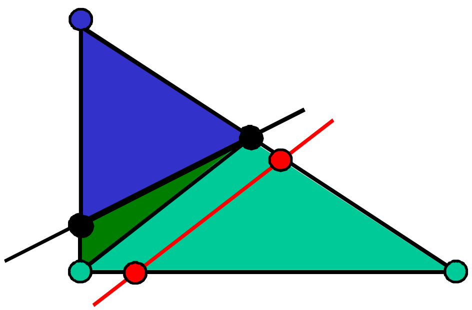


Moving CDFEM Goals

- How do we handle the moving interface?



- What do we do when nodes change sign?



- Goals
  - Try to recover moving mesh case for moving interface
  - Try to preserve minima, maxima
- Proposal
  - Prolongation: Set “old” value to value of nearest point on interface
  - Dynamics: Use ALE style (u-dxdt) for advection term

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CDFEM Verification

- Two-Dimensional Potential Flow About a Cylinder (static)
  - Analytical solution provides quantitative measure of accuracy
    - Accuracy of velocity potential and its gradient computed in volume and on interface
  - Allows experiments with various boundary conditions
- Three-Dimensional Potential Flow About a Sphere (static)
  - Analytical solution provides quantitative measure of accuracy
    - Accuracy of velocity potential and its gradient computed in volume and on interface
  - Allows experiments with various boundary conditions
- Two-Dimensional Viscous, Incompressible Couette Flow (static)
  - Analytical solution provides quantitative measure of accuracy
  - Test of conformal decomposition for viscous, incompressible flow
- Three-Dimensional Viscous Flow about a Periodic Array of Spheres (static)
  - Comparison with Boundary Element results
  - Examines behavior of decomposition up to sphere overlap
- Advection of Weak Discontinuity (dynamic)
  - Shows ability to capture discontinuities
  - Analytical solution provides quantitative measure of accuracy
- Solidification of 1-D Bar (dynamic)
  - Shows ability to capture discontinuities
  - Analytical solution provides quantitative measure of accuracy

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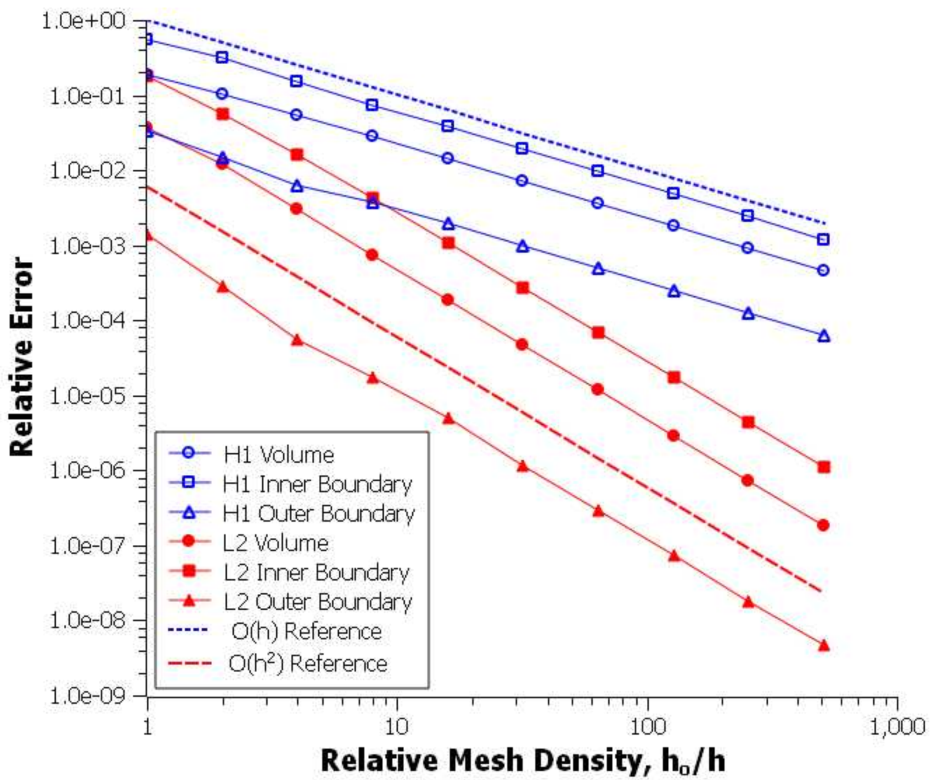
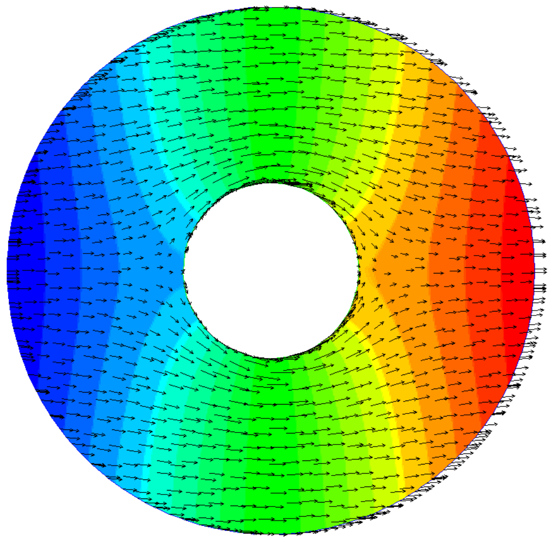
CDFEM Simulation of Steady, Potential Flow about a Circular Cylinder

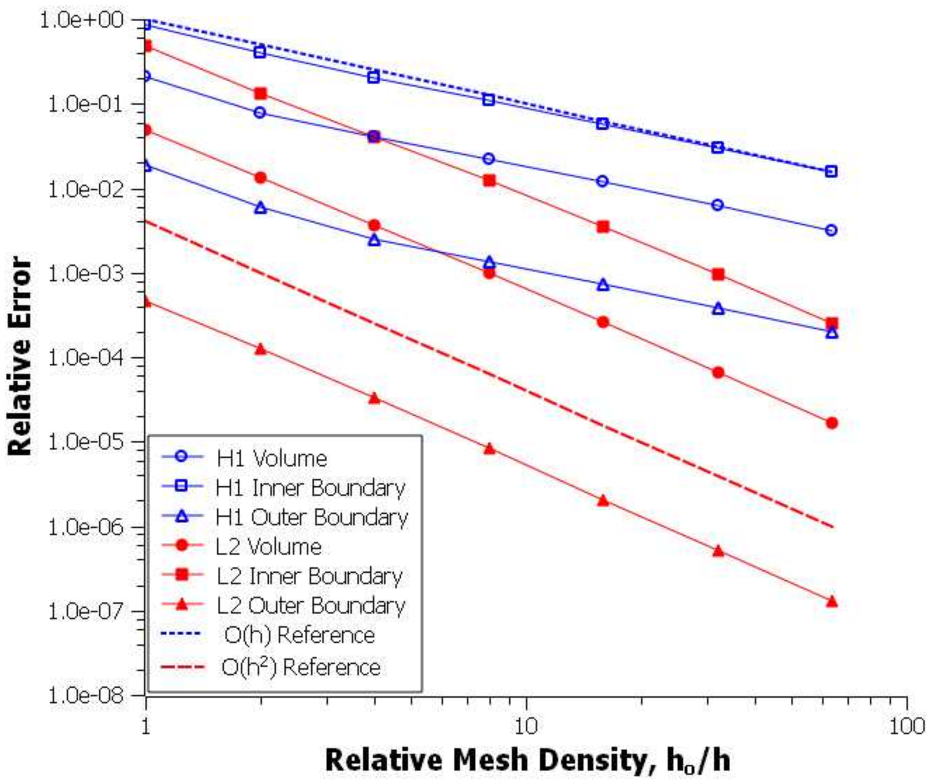
- Embedded curved boundaries
- Dirichlet BC on outer surface, Natural BC on inner surface
- Optimal convergence rates for solution and gradient both on volume and boundaries

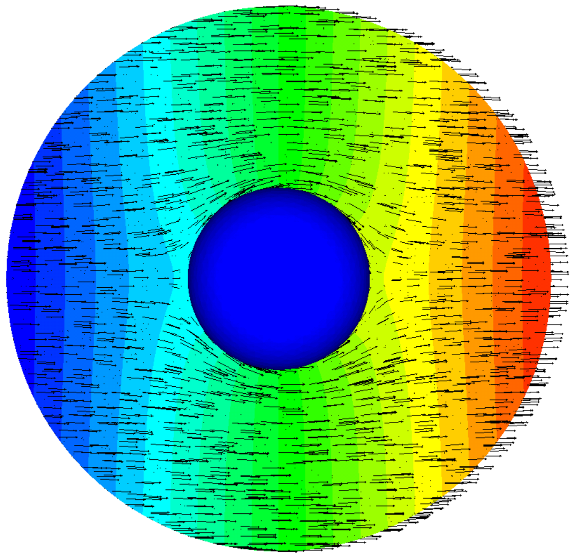




CDFEM Simulation of Steady, Potential Flow about a Sphere

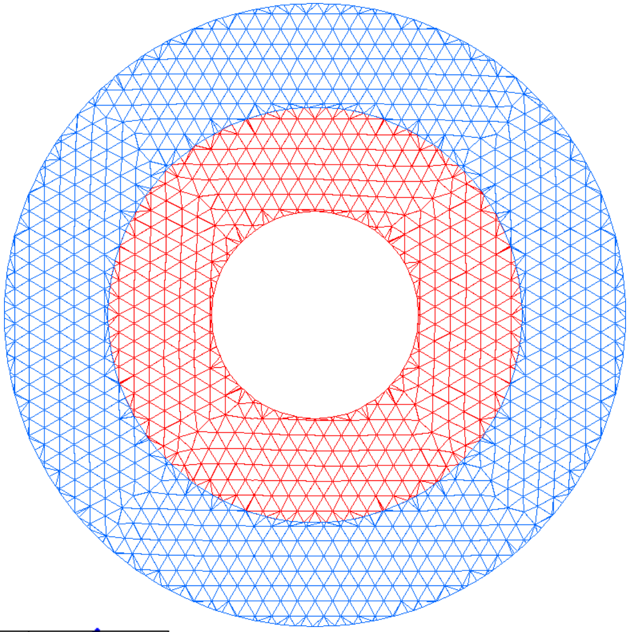
- Embedded curved boundaries
- Dirichlet BC on outer surface, Natural BC on inner surface
- Optimal convergence rates for solution and gradient both on volume and boundaries

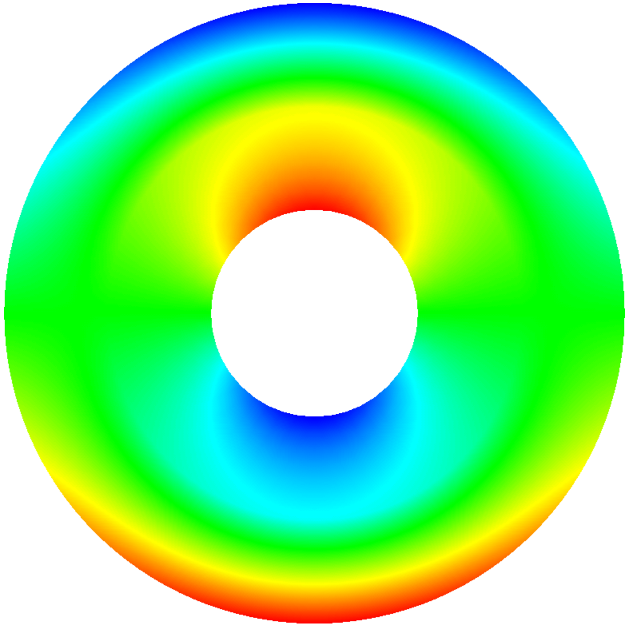


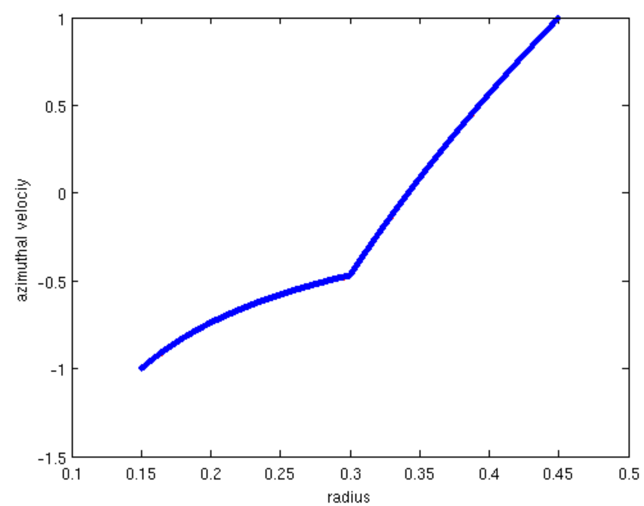


CDFEM Simulation of Steady, Fluid-Fluid Interface Problem: Couette Flow

- Two-Phase Flow between concentric cylinders
  - Counter-rotating cylinders
  - 4:1 viscosity ratio
  - No surface tension
- Dirichlet conditions on inner and outer surfaces, weak discontinuity along interface
- Cut regular, unstructured mesh along outer, inner, and interfacial radii







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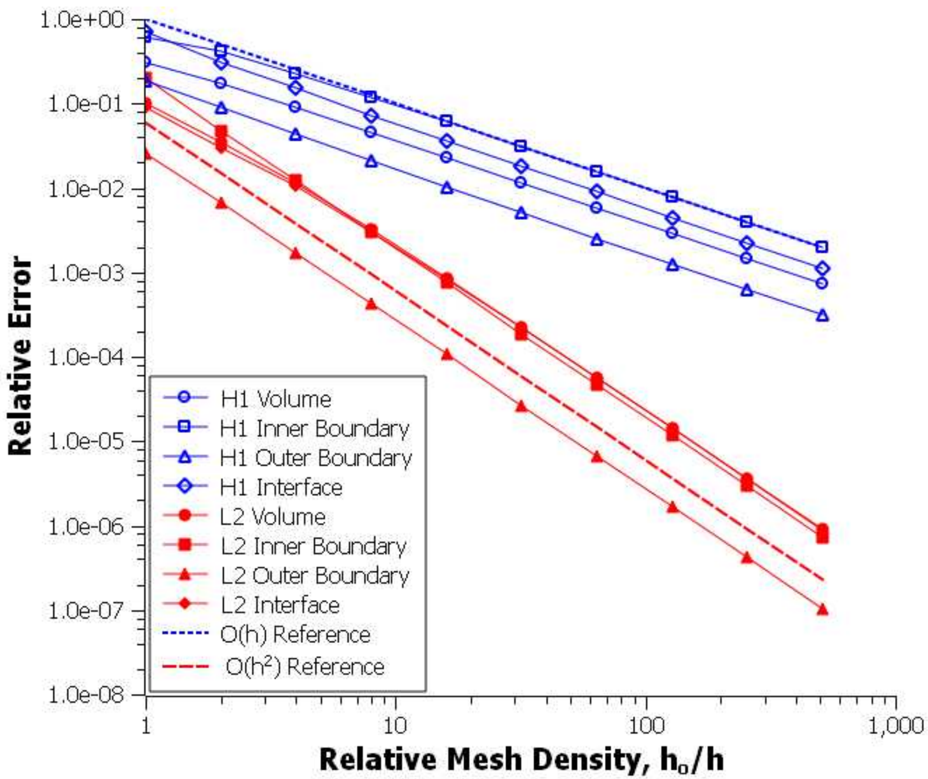
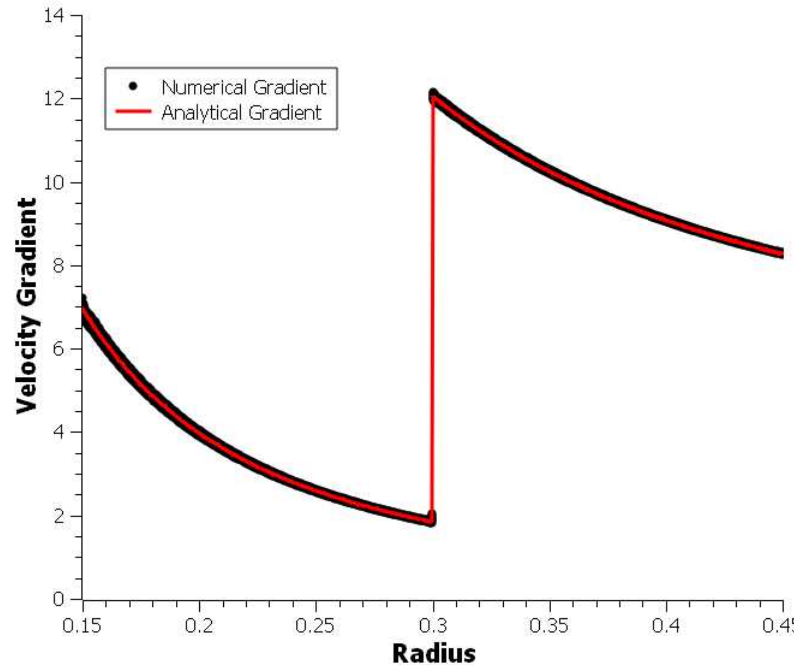
CDFEM Simulation of Steady, Fluid-Fluid Interface Problem: Couette Flow

- Embedded curved boundaries
- Dirichlet BC on inner and outer surface
- Weak discontinuity in velocity captured sharply and accurately
- Optimal convergence rates for solution and gradient both on volume and boundaries





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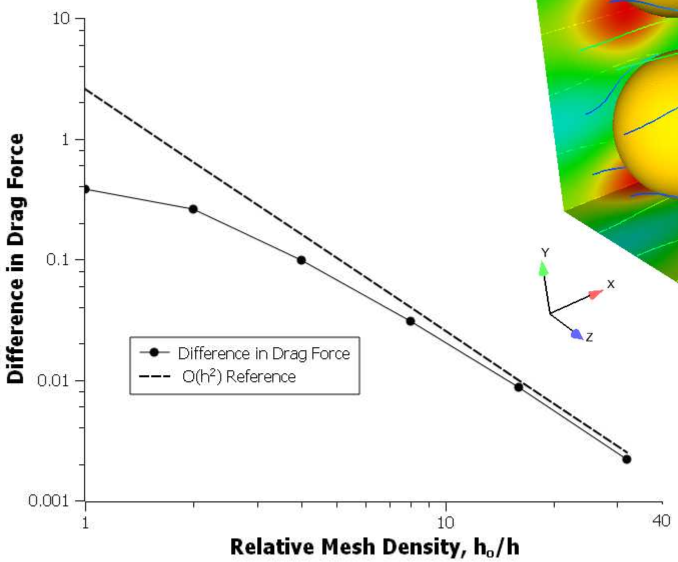
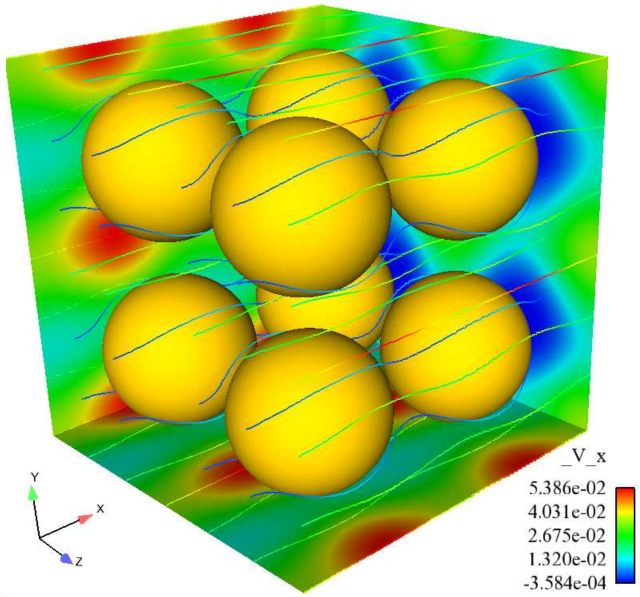
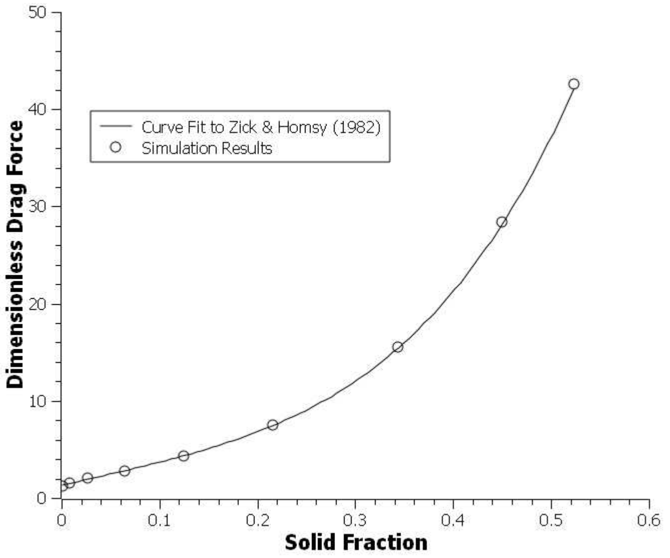
CDFEM Simulation of Steady, Viscous Flow about a Periodic Array of Spheres

- Embedded curved boundaries
- Dirichlet BC on sphere surface
- Accurate results right up to close packing limit
- Sum of nodal residuals provides accurate/convergent measure of drag force







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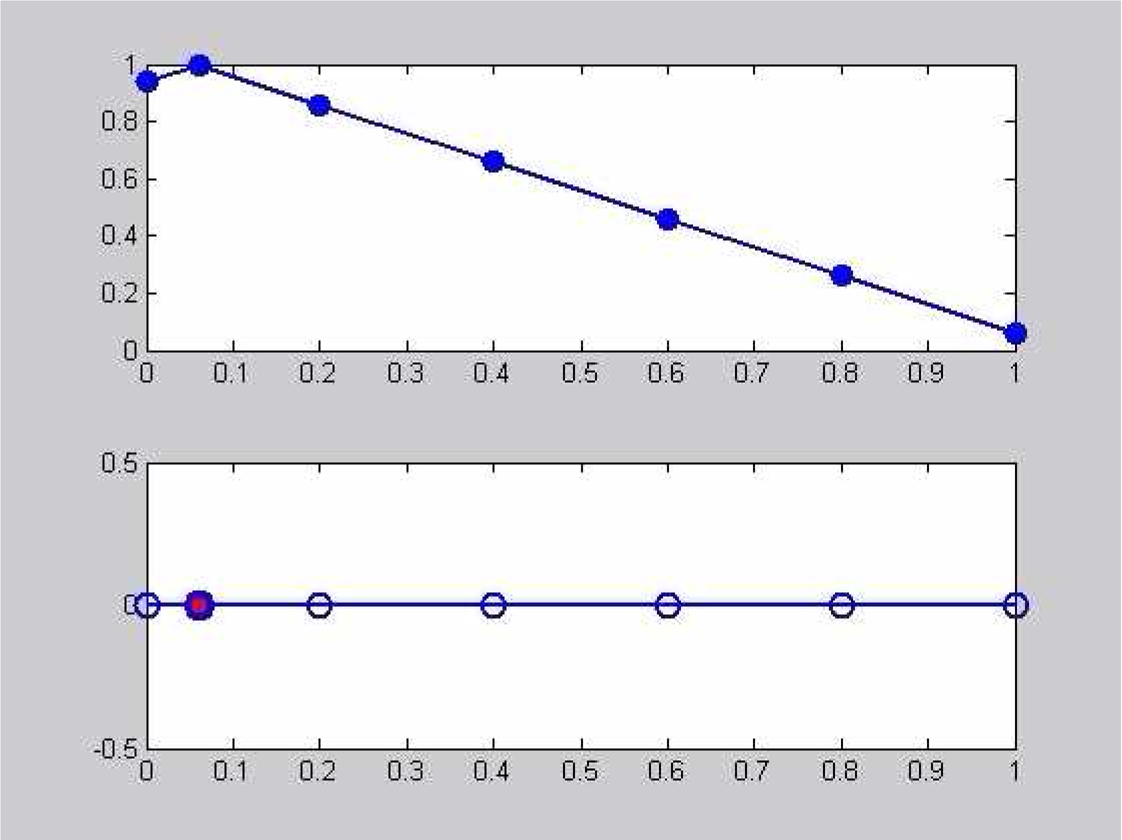
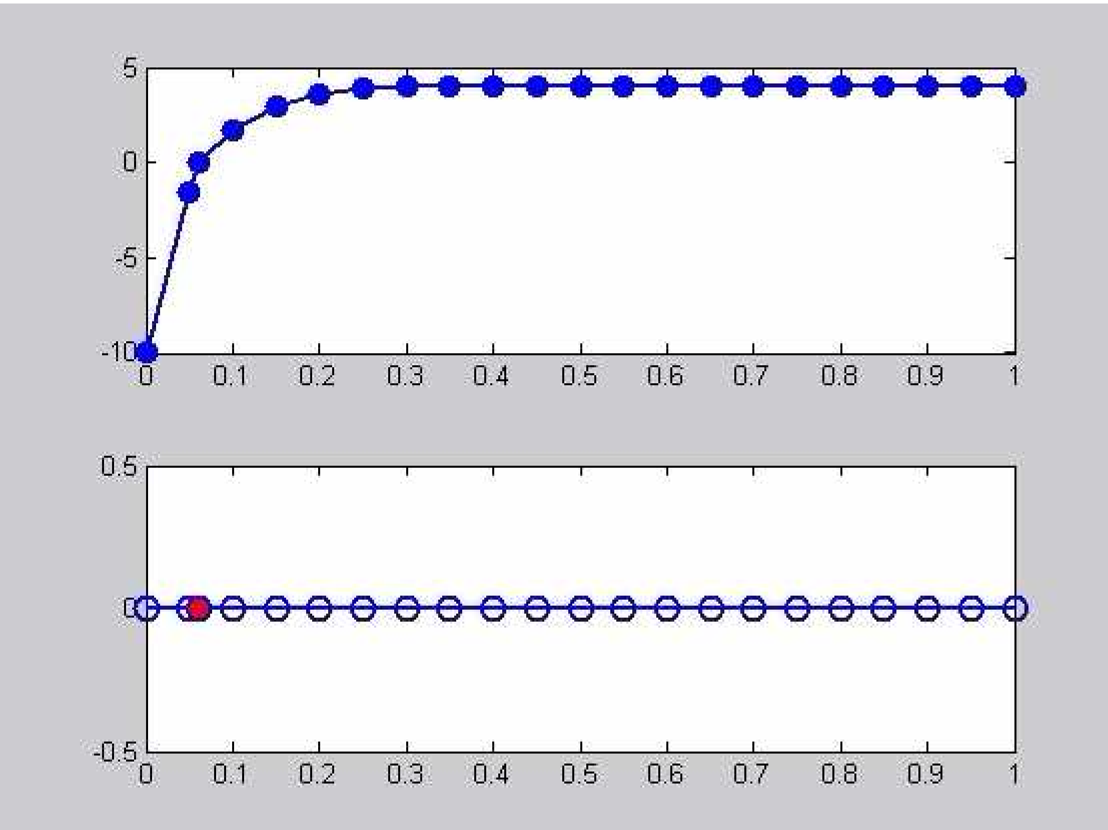
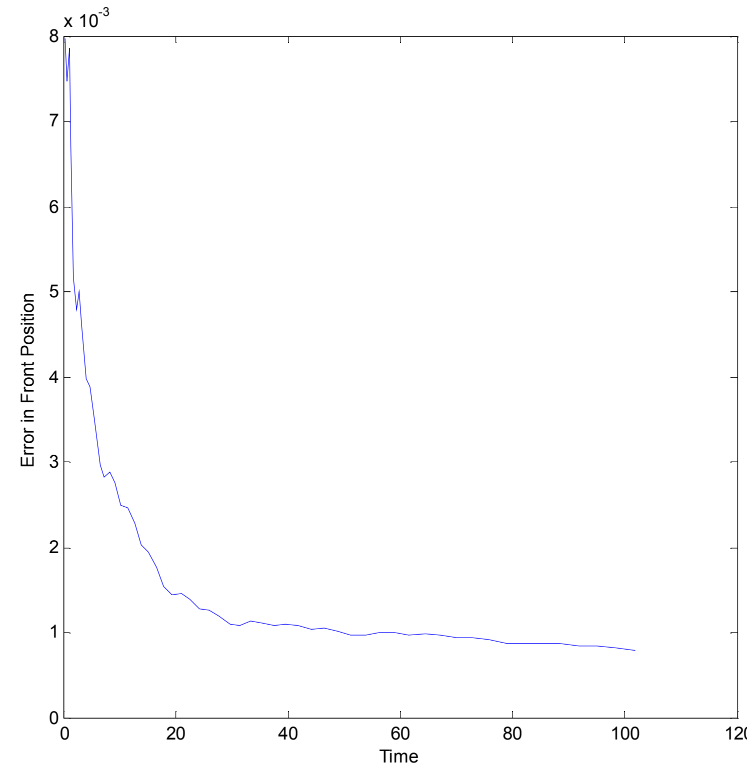
Dynamic CDFEM: 1-D Advection of a Piecewise Linear Field



- Exact preservation of linear field
- Does not pollute Max-Min

Dynamic CDFEM: 1-D Phase Change





- Great agreement with exact solution

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Summary and Conclusions

XFEM is Powerful Tool for Multiphase Dynamics

- Simulations in realistic geometries reveal important physics not seen in ALE simulations
- Combination of discontinuous variables and one-sided variables make powerful technique
- Wide variety of weak integrated conditions implemented on level set surface

Care Must be Taken When Using Subelement Integration

- Definition of subelements – Parametric coordinates?
  - Accuracy: Low order subelements can lead to suboptimal convergence
  - Performance: Higher order subelements involve over-integration and root finding for internal nodes
- Definition of subelements – Real coordinates?
  - Performance: Quadrature point location inversion
  - Accuracy: Element face conformity for hexes in 3D?

Analytic Integration for Generalized Functions

- Can be used to formulate fixed point integration rules with weights that depend continuously on nodal level set values
- Provides analytic Jacobian information
- Handles degenerate cases smoothly without special consideration
- Higher order elements not practical

CDFEM

- Simple method for handling arbitrary interfacial discontinuities
  - Transparent to underlying finite element assembly
- Recovers XFEM when added nodes are constrained to lie in XFEM space
- Demonstrates optimal rates of convergence for both Neumann and Dirichlet BC on curved surfaces