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from: D. M. Day, Org. 1542, MS-0870

subject: Comments on Verification Tests in Solid Mechanics

The purpose of this memorandum is to reproduce the uni-axial strain and bending bar problems from [3] in sufficient detail that they are readily implementable. In particular, some fundamental typographical errors are addressed.

The purpose and intent of this unorthodox document warrants further a more detailed description. The document was written on personal time by a code developer. From a well written and remarkably easy to understand article two examples are reproduced in as much detail as possible. The goal is to produce a more precise description of two verification examples, sufficiently accurate and clear to share with other code developers. Section 5 and Appendix 2 of [3] are confirmed, with revisions. But 18 step algorithm in section 7 of [3] has not been confirmed, and in my opinion is not the most useful part of [3], as it is presented.

Another purpose of this document is to serve as a starting point for conversations with my colleagues. The next step is to discuss with a Subject Matter Expert the constitutive laws available in Adagio

1 Introduction

1. There's an important typographical error in the definition of the deformation. I fixed the typo so that both a power series expansion of the deformation about $t = 0$ looks right, and also the deformation gradient is consistent.

2. Another typo is writing λ when what is meant is Λ. Basically some of the λ's are really Λ, and resolving that ambiguity correctly is required. I cannot rule out that there is some way, e.g. dimensional analysis, to tell whether or not λ is actually Λ. Other typos, e.g. equation (67) of [3] a should be changed to n, are not important.

3. For the bending bar problem to be uniquely defined, the domain must be specified, e.g. $[0,1] \times [0,8]$. All domains are rotated about the origin. For instance the domains $[0,1] \times [0,8]$ and $[1,2] \times [0,8]$ are different problems.

4. In the presentation of the bending bar problem, it is assumed that $T = 1$. The code developer needs access to T, to create fast automatic tests.

Some modifications would make the verification tests less expensive to use.

- Attempting to using the bending bar example as it is presented would be risky. It is preferable to decompose the bending bar test to simple testable components.
- Ideally an implementation in a programming language would be presented. If applicable, adding a Mathematica Notebook in an appendix would be an improvement.
- The bending bar example is presented as a black box which limits its value.

And I have a comment and a hunch. In Appendix 2 it's a little confusing to call the rotated coordinate system *polar* coordinates, with basis (e_r, e_θ). Also the parameterized (parametrized) family of bending bar problems includes the bar bent into a circular disk from [4]; neither article gives a references on the problem.

variable	description
X	material point
x	spatial point, $x = \chi(X,t)$
u	displacement $x - X$
div	div_x
t	time or traction
a	$\frac{\partial^2 x}{\partial t^2}$
$\rho(x,t)$	spatial density
ρ_0	initial material density, $\rho(X,0)$
F	deformation gradient $F_{ij} = \frac{\partial x_i}{\partial X_j}$
J	$\det F$
P	first Piola-Kirchhoff stress $P = J\sigma F^{-T}$
(E_1, E_2, E_3)	Cartesian basis
I	3×3 identity matrix
DIV	$\mathrm{DIV}_X P = \frac{\partial P_{ij}}{\partial X_j} E_i$
T	simulation stop time,
n	unit outward normal
λ	$\frac{E\nu}{(1+\nu)(1-2\nu)}$
μ	$\frac{E}{2(1+\nu)}$ Lamé modulii

Table 1. The nomenclature is standard, except P, which is not standardized.

variable	value
$\rho(x,t)$	arbitrary
T	1 second
Youngs modulus	10^6 Pa
Poissons ratio	0.25
stretch	$\Lambda > 0$
domain	brick aligned with axes

Table 2. Uni-axial Strain Problem

First sections 5,7 and Appendix 2 of [3] are nearly reproduced in order, as a kind of code review. Word processing software eliminates the cumbersome aspects of such a task. The Uni-axial Strain problem is reproduced verbatim. The next section attempts to repeat the complex algorithm for the bending bar body forces. I have not taken the time to verified the algorithm presented in [3] for the body forces. The next section revisits the derivation in Appendix 2. Several critical typographical errors are identified and fixed. Section 5 sketches how a less risky to implement user subroutine. And section 6 reviews the constitutive law, and identifies the most closely related constitutive law available in Adagio.

2 Uni-axial strain and displacement for traction boundary conditions

A verification problem satisfies the balance of momentum equation,

$$\mathrm{div}(\sigma) + \rho b = \rho a, \quad \text{spatial coordinates}, \tag{1}$$

$$\mathrm{DIV}(P) + \rho_o b = \rho_o a, \quad \text{material coordinates}. \tag{2}$$

Recall that divergence on matrices acts along the columns, or for tensors, on the last index.

The recipe is to specify a displacement field that determines the stress field. Then the body force is determined by equation (2). There are hidden constraints. Mechanics codes are designed to handle $u \in H^1(\Omega)$ (c.f. u defined by it's value at mesh nodes). Another hidden constraint is that $b \in L^2(\Omega)$, i.e. b is defined by its values on element interiors in some coordinate system. Knowledge of the interface for source terms $b(\ldots)$ is a luxury shared by few code developers.

We consider *homogeneous* uni-axial *strain* of a hyper-elastic solid. That is in the absence of translation, the mapping for a homogeneous deformation of a point X in the initial configuration to x in the deformed configuration is $x = FX$ and of course

$$u = (F - I)X.$$

Furthermore the deformation gradient F varies with time, but *not* position. Accordingly, the acceleration is $a = \ddot{F}X$. Here the reference frame is the material frame, $F(0) = I$. The choices for F are further restricted by homogeneity and uni-axiality.

The simplest non-trivial example is a deformation gradient that varies linearly through time from the initial value I to an arbitrary $F(T) = G$, $F = I(1-t)+Gt$. Thus, for this example, $\ddot{F} = 0$, and hence, the material acceleration is zero. For *uni-axial* strain corresponding to a final stretch Λ in the 1-direction, $G = \text{diag}(\Lambda, 1, 1)$. And therefore the time-varying deformation gradient is

$$F = \text{diag}(\phi(t), 1, 1), \quad \phi(t) = 1 + (\Lambda - 1)t, \quad \phi > 0 \tag{3}$$

For a *homogeneous* material, the constitutive model and its associated parameters are the same at all points in space. Accordingly, for a homogeneous deformation, the stress predicted by the constitutive model is the same at all points in space, making the divergence of stress zero. Thus, with both the acceleration and stress divergence zero, equation (2) implies that the body force must be zero. Though the body force is zero, the initial velocity field, $v = \dot{F}X$, is nonzero (so this problem offers a simple test for initializing velocity fields in a code). This problem could be solved using velocity boundary conditions. The boundary tractions, $t = \sigma n$, depend on the constitutive model. Here the Neo-Hookean constitutive model,

$$\sigma = \frac{\lambda \log J}{J} I + \frac{\mu}{J}(FF^T - I) \tag{4}$$

is adopted, for positive elastic Lamé material constants λ and μ. Substituting F from equation (3) into equation (4) gives stress as a function of stretch Λ and time t. The traction on the positive x-face, t_1, and the traction on negative x-face, t_2 are given by

$$t_1 = -t_2 = \frac{\lambda \ln \phi + \mu(\phi^2 - 1)}{\phi}.$$

The traction on the positive y-face, t_3, and the traction on the negative y-face, t_4, are

$$t_3 = -t_4 = \frac{\lambda \ln \phi}{\phi},$$

A 3D brick has traction t_5 on its positive z face and t_6 on its negative z-face. Here $t_5 = -t_6 = t_3$. For a mechanics codes not solving pure traction problems, displacements may be specified on the z faces.

3 Bending bar verification test

This problem domain is a rectangular bar with height H and base B. In this problem all material points undergo an identical deformation mode: uni-axial strain with superimposed rotation. This problem also includes a time and space-varying traction on the boundary, thus giving this problem the advantage of assessing the codes algorithms for geometrically non-linear traction boundary conditions under non-homogeneous deformations. The constitutive model is the Neo-Hookean model given by equation (4).

As explained in section 4, the following sequence of calculations evaluate the body force. Boundary conditions and initial conditions may be determined from the section 4. An algorithm for evaluating the tractions is given at the end of this section.

variable	value
initial density	$\rho_0 = 10^3 \frac{kg}{m^3}$
stop time	$T = 1s$
Youngs modulus	10^3 Pa
Poissons ratio	0.3
height	$H = 8m$
base	$B = 1m$
domain	$[X_l, X_r] \times [0, 8]$, $X_r = X_l + 1$
$v_0 = 0$	initial velocity vanishes
$\sigma_0 = 0$	initial stress vanishes

Table 3. Bending Bar Problem

1. $t = 0$, $v = 0$, $u = 0$.
2. Evaluate an amplitude function $\beta = \frac{A}{2}(1 - \cos(\frac{2\pi t}{T}))$
3. Evaluate the element rotation angle $\alpha = \beta \frac{X_2}{H}$
4. Evaluate a temporary variable

$$p_1 = (128H^3 - 8A^2HX_2^2 - 5A^3X_1X_2^2)$$
$$+4(16H^3 + A^2HX_2^2 + A^3X_1X_2^2)\cos(2\pi t)$$

5. Evaluate a temporary variable

$$p_2 = 4A^2(2H + AX_1)X_2^2\cos(4\pi t)$$
$$-4A^3X_1X_2^2\cos(6\pi t) + A^3X_1X_2^2\cos(8\pi t)$$

6. Evaluate a temporary variable

$$p_3 = -128H^3\cos(\frac{AX_2\sin(\pi t)^2}{H})$$
$$-32H^3\cos(2\pi t - \frac{AX_2\sin(\pi t)^2}{H})$$

7. Evaluate a temporary variable

$$p_4 = -32H^3\cos(2\pi t + \frac{AX_2\sin(\pi t)^2}{H})$$

8. $p_5 = \frac{A(1+\frac{AX_1 - AX_1\cos(2\pi t)}{2H})}{2H\rho_0(2H+AX_1-AX_1\cos(2\pi t))^2}$

9. Evaluate a temporary variable

$$p_6 = -8H^2\lambda + 8AH\mu X_1 + 3A^2\mu X_1^2$$
$$-4A\mu X_1(2H + AX_1)\cos(2\pi t)$$

10.

$$p_7 = A^2\mu X_1^2\cos(4\pi t) + 8H^2\lambda\log(1 + \frac{AX_1\sin(\pi t)^2}{H})$$

11. $p_8 = -12AHX_2 - 4A^2X_1X_2 + A(8H + 7AX_1)X_2\cos(2\pi t) + 4A(H - AX_1)X_2\cos(4\pi t)$
12. $p_9 = A^2X_1X_2\cos(6\pi t) + 32H^2\sin(\frac{AX_2\sin(\pi t)^2}{H})$
13.

$$p_{10} = -8H^2\sin(2\pi t - \frac{AX_2\sin(\pi t)^2}{H})$$
$$+8H^2\sin(2\pi t + \frac{AX_2\sin(\pi t)^2}{H})$$

14. Evaluate the radial component of the body force

$$b_r = \frac{\pi^2\csc(\pi t)^4(p_1 + p_2 + p_3 + p_4)}{32AH^2} + p_5(p_6 + p_7)$$

15. Evaluate the circumferential component of the body force

$$b_\theta = \frac{\pi^2\csc(\pi t)^4(p_8 + p_9 + p_{10})}{8AH}$$

16. Evaluate the the body force in Cartesian coordinates

$$b = Q(\alpha)\begin{bmatrix} b_r \\ b_\theta \end{bmatrix}$$

17. $b_z := 0$

These the algebraic expressions are not presented in a way that is easy to understand. For instance, the expressions have not been simplified. The identity $\cos(a + b) + \cos(a - b) = 2\cos(a)\cos(b)$ simplifies the expression $p_3 + p_4$. The numerator and denominator cancel in the expression for p_5. The identity $\sin(a + b) - \sin(a - b) = 2\cos(a)\sin(b)$ simplifies the expression p_{10}.

Tractions depend on t, peak amplitude A, T, bar height H, and element coordinate X, The reference element face unit outward normal N. The rotation of the bar is done using

$$Q(\alpha) = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

for $0 \le \beta \le A$ and $0 \le \alpha \le \beta$. The peak amplitude was chosen to be $A = \frac{\pi}{2}$.

1. $\beta = \frac{A}{2}(1 - \cos(\frac{2\pi t}{T}))$
2. $\Lambda = 1 + \beta\frac{X_1}{H}$
3. $U = \mathrm{diag}(1, \Lambda)$
4. $J = \det U$
5. $\overline{\sigma} = \sigma(\lambda, \mu, J, U)$
6. $\alpha = \beta\frac{X_2}{H}$
7. $Q(\alpha)$
8. $t = Q\overline{\sigma}N$

The choice of $\beta(t)$ so that $\beta(0) = 0$ and $\beta_t(0) = 0$ corresponds to trivial initial conditions.

4 Bending bar forcing functions

This is my attempt to understand the Appendix 2 from [3]. The deformation will be decomposed into a sequence of calculations. The mapping from an initial position X to a deformed position x is shown in Fig. 28 of [3]. More explanation is needed here. I think that I have corrected the algebra (in red).

$$x = \begin{bmatrix} -\frac{H}{\beta} + (X_1 + \frac{H}{\beta})\cos(\alpha) \\ (X_1 + \frac{H}{\beta})\sin(\alpha) \end{bmatrix} \tag{5}$$

This produces a consistent deformation gradient,

$$F = \begin{bmatrix} \cos(\alpha) & -(X_1 + \frac{H}{\beta})\sin(\alpha)\frac{\beta}{H} \\ \sin(\alpha) & (X_1 + \frac{H}{\beta})\cos(\alpha)\frac{\beta}{H} \end{bmatrix} = QU$$

Note that

$$\lambda \neq \Lambda = 1 + \beta\frac{X_1}{H}$$

A helpful check is to express $\chi(X, t)$ in a way that does not have singularities at $t = 0$. Noting that $\alpha = \beta\frac{X_2}{H}$, and $\mathrm{sinc}(\alpha) = \frac{\sin(\alpha)}{\alpha}$, there holds $X_2\mathrm{sinc}(\alpha) = \frac{H}{\beta}\sin(\alpha)$. A related function is represented here by its series expansion, $\frac{-1+\cos(\alpha)}{\alpha} = -\frac{\alpha}{2} + O(\alpha^3)$. The expression for the deformation without singularities is

$$x = \begin{bmatrix} \cos(\alpha) & -\frac{\alpha}{2} \\ \sin(\alpha) & \mathrm{sinc}(\alpha) \end{bmatrix} X + O(\alpha^2).$$

This is the expression to use for extremely small values of α.

F has a unique polar decomposition $F = RU$ as the product of a rotation $R = \text{diag}(Q, 1)$ and a stretch $U = \text{diag}(1, \Lambda, 1)$.

The Jacobian is Λ. It's necessary to specify the domain, specifically both X_l and X_r, as is evident from the critical points,

$$\{X : \text{det}F = 0\} = \{X : X_1 = -H/\beta\} \tag{6}$$

Here, α is the angle of rotation at the material point of interest, Λ is the amount of stretch in the 2-direction.

Then the Cauchy stress σ is computed using $\sigma = R\overline{\sigma}R^T$ where R is the rotation tensor. Turning now to the momentum equation, the indicial form of the divergence is given by

$$f_i = \frac{\partial \sigma_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}(R_{ip}\overline{\sigma}_{pq}R_{jq}).$$

By the product rule,

$$f_i = \frac{\partial R_{ip}}{\partial x_j}\overline{\sigma}_{pq}R_{jq} + R_{ip}\frac{\partial \overline{\sigma}_{pq}}{\partial x_j}R_{jq} + R_{ip}\overline{\sigma}_{pq}\frac{\partial R_{jq}}{\partial x_j}$$

Note that the polar rotation R depends only on the rotation angle α,

$$\frac{\partial R_{ip}}{\partial x_j} = \frac{dR_{ip}}{d\alpha}\frac{\partial \alpha}{\partial x_j}$$

Substituting

$$\frac{\partial \alpha}{\partial x_j} = \frac{\partial \alpha}{\partial X_n}F^{-1}_{nj},$$

introducing $\Omega = Q(\frac{\pi}{2})$, and substituting $\frac{dR}{d\alpha} = R\Omega$, produces

$$\frac{\partial R_{ip}}{\partial x_j} = R_{ik}\Omega_{kp}\frac{\partial \alpha}{\partial X_n}U^{-1}_{nm}R^{-1}_{mj}$$

The expansion of the right-hand term follows by changing the index (i, p, j) to (j, q, j).

$$R_{ip}\overline{\sigma}_{pq}\frac{\partial R_{jq}}{\partial x_j} = R_{ip}\overline{\sigma}_{pq}R_{jk}\Omega_{kq}\frac{\partial \alpha}{\partial X_n}U^{-1}_{nm}R^{-1}_{mj}$$

I like to order terms to emphasize their connection to matrix products. The left-hand term simplifies.

$$\frac{\partial R_{ip}}{\partial x_j}\overline{\sigma}_{pq}R_{jq} = \frac{\partial \alpha}{\partial X_n}U^{-1}_{nm}R^{-1}_{mj}R_{jq}\overline{\sigma}_{pq}\Omega_{kp}R_{ik} =$$

$$\frac{\partial \alpha}{\partial X_n}U^{-1}_{nq}\overline{\sigma}_{pq}\Omega_{kp}R_{ik}$$

The right-hand term is similar.

$$R_{ip}\overline{\sigma}_{pq}\frac{\partial R_{jq}}{\partial x_j} = \frac{\partial \alpha}{\partial X_n}U^{-1}_{nm}R^{-1}_{mj}R_{jk}\Omega_{kq}\overline{\sigma}_{pq}R_{ip} =$$

$$\frac{\partial \alpha}{\partial X_n}U^{-1}_{nk}\Omega_{kq}\overline{\sigma}_{pq}R_{ip}$$

We're two thirds of the way to deriving an expression for the internal force density.

$$R_{ip}\frac{\partial \overline{\sigma}_{pq}}{\partial x_j}R_{jq}.$$

Recall that $\overline{\sigma}$ depends only on Λ,

$$\frac{\partial \overline{\sigma}_{pq}}{\partial x_j} = \frac{\partial \overline{\sigma}_{pq}}{\partial \Lambda}\frac{\partial \Lambda}{\partial x_j} = \frac{\partial \overline{\sigma}_{pq}}{\partial \Lambda}\frac{\partial \Lambda}{\partial X_k}F^{-1}_{kj}$$

$F = RU$, $F^{-1}R = U^{-1}$

$$R_{ip}\frac{\partial \overline{\sigma}_{pq}}{\partial x_j}R_{jq} = \frac{\partial \Lambda}{\partial X_k}F^{-1}_{kj}R_{jq}\frac{\partial \overline{\sigma}_{pq}}{\partial \Lambda}R_{ip} =$$

$$\frac{\partial \Lambda}{\partial X_k}U^{-1}_{kq}\frac{\partial \overline{\sigma}_{pq}}{\partial \Lambda}R_{ip}$$

The divergence operator, $\nabla\cdot$, is a row vector. Let's say that f is a column vector, and write $f^T = \nabla\cdot\sigma$ to avoid any ambiguity.

$$f^T = \nabla_X\alpha U^{-1}\overline{\sigma}^T\Omega^T R^T + \nabla_X\Lambda U^{-1}\frac{\partial \overline{\sigma}^T}{\partial \Lambda}R^T +$$

$$\nabla_X\alpha U^{-1}\Omega\overline{\sigma}^T R^T$$

Here the $[,]$ notation denotes the commutator $[A,B] = AB - BA$, of a pair of square matrices. The expression for the body force in rotated coordinates is

$$\overline{f}^T = f^T R = \nabla_X\alpha U^{-1}\overline{\sigma}^T\Omega^T + \nabla_X\Lambda U^{-1}\frac{\partial \overline{\sigma}^T}{\partial \Lambda} + \nabla_X\alpha U^{-1}\Omega\overline{\sigma}^T =$$

$$= \nabla_X\alpha U^{-1}[\Omega,\overline{\sigma}^T] + \nabla_X\Lambda U^{-1}\frac{\partial \overline{\sigma}^T}{\partial \Lambda} \tag{7}$$

The purpose of the remainder of this section is to compare equations with [3]. Before the constitutive law the Lamé parameters can not yet have appeared. We know for instance

$$\nabla_X\alpha = \frac{\beta}{H}[0,1,0], \quad \nabla_X\Lambda = \frac{\beta}{H}[1,0,0]$$

I've been keeping the transpose on the stress, but of course σ is assumed to be symmetric.

$$[\Omega, \sigma] = \begin{bmatrix} -2\sigma_{21} & \sigma_{11} - \sigma_{22} & -\sigma_{23} \\ \sigma_{11} - \sigma_{22} & 2\sigma_{21} & \sigma_{13} \\ -\sigma_{32} & \sigma_{31} & 0 \end{bmatrix}$$

This means that

$$\frac{H\Lambda}{\beta}\overline{f}^T = [\sigma_{11} - \sigma_{22}, 2\sigma_{21}, \sigma_{13}] + \Lambda\sigma(1,:)_{,\Lambda}$$

The corresponding formula in [3] is complicated. Next the equation for $\overline{f}$ is written out, and that result matches the expression here.

4.1 The Neo-Hookean Case

The purpose of this section is to evaluate the body force using equation (7). Ideally this section is perfectly redundant with section 3. In verification, redundancy is wonderful. The algebra is left in here. A sanity test of the body force equation is done.

To begin, bearing in mind that $F = RU$,

$$[\Omega, \overline{\sigma}^T] = \frac{\mu}{J}[\Omega, FF^T].$$

$Y = [\Omega, FF^T]$ is symmetric, traceless. In particular

$$y_{21} = 2(\cos(\alpha)^2 - \Lambda^2 \sin(\alpha)^2), \quad y_{22} = (\Lambda^2 - 1)\cos(2\alpha)$$

In equation (7), we have evaluated the term

$$(\nabla_X \alpha) U^{-1} \mu Y = \frac{\mu\beta}{H\Lambda^2}[y_{21}, y_{22}].$$

It remains to evaluate

$$(\nabla_X \Lambda) U^{-1} \frac{\partial \overline{\sigma}^T}{\partial \Lambda} = \frac{\beta}{H}[1, 0]\frac{\partial \overline{\sigma}^T}{\partial \Lambda}$$

Differentiating equation (4),

$$\sigma = \frac{\lambda \log J}{J} I + \frac{\mu}{J}(FF^T - I),$$

involves

$$\frac{d}{dJ}\frac{\log J}{J} = \frac{1 - \log J}{J^2},$$

and also

$$F_{,J}F^T = (RU_{,J})F^T = R\mathrm{diag}(0,1)U^T R^T = R\mathrm{diag}(0,\Lambda)R^T.$$

The last step is

$$\frac{d}{dJ}\frac{FF^T - I}{J} = \frac{(F_{,J}F^T + FF^T_{,J})J - FF^T + I}{J^2} =$$

$$= R(\mathrm{diag}(0, 2J\Lambda) - (1, \Lambda^2) + (1, 1))R^T J^{-2}$$
$$= R(\mathrm{diag}(0, \Lambda^2 + 1))R^T \Lambda^{-2}$$
$$= Re_2 e_2^T R^T (1 + \Lambda^{-2}).$$

This gives the term that we needed:

$$\frac{\beta}{H}[1, 0]\frac{\partial \overline{\sigma}^T}{\partial \Lambda} =$$
$$\frac{\beta}{H}\frac{\lambda(1 - \log J)}{J^2}[1, 0] - \frac{\beta}{H}(1 + \Lambda^{-2})\mu \sin(\alpha) e_2^T R^T$$

This expression for $\overline{f}$, using intermediate variables,

$$\overline{f}^T = \frac{\lambda\beta(1 - \log\Lambda)}{H\Lambda^2}[1, 0]+$$
$$\frac{\mu\beta}{H\Lambda^2}([y_{21}, y_{22}] - (1 + \Lambda^2)\sin(\alpha) e_2^T R^T) \tag{8}$$

instead of the primitive variables, looks utterly different from section 3. As a sanity test, there should be a $\log\Lambda$ term in either p_6 or p_7 of section 3. Bearing in mind that

$$\beta = \frac{A}{2}(1 - \cos(2\frac{\pi t}{T})) = A\sin(\frac{\pi t}{T})^2,$$

the $\log\Lambda$ is there in p_7.

4.2 Acceleration and Polar Coordinates

The force density contribution from the material acceleration follows from the relation $x = \chi(X, t)$. The initial velocity $x_{,t} = x_{,\beta}\beta_{,t}$ vanishes with $\beta_{,t}$. Acceleration is just

$$\frac{\partial^2 x}{\partial t^2} = \frac{\partial}{\partial t}(x_{,\beta}\beta_{,t}) = (\frac{\partial}{\partial t}x_{,\beta})\beta_{,t} + x_{,\beta}\beta_{,tt} =$$
$$x_{,\beta\beta}\beta_{,t}^2 + x_{,\beta}\beta_{,tt} \tag{9}$$

where the derivatives with respect to β are found by differentiating equation (5). Density is $\rho = \rho_0/\Lambda$. Substituting f, ρ and $\ddot{x}$ in the momentum equation in spatial coordinates, equation (1), after choosing the material model and material constants, the total body force required for this deformation is $b = \ddot{x} - f/\rho$.

5 Interfaces

A minimal interface requires, first, the problem parameters described in Table 3. A good interface will check, using equation (6), that the deformation is well defined.

A suite of functions will be needed to determine the initial conditions, boundary conditions and body force.

One function could return $x, \dot{x}, \ddot{x}$ using equation (9), which in turn would be using an inner interface that requires X, H, β and uses equation (5). An then an outer interface would require X, H, t, T, A, compute β, using the inner interface.

A function that returns the stress σ, using equation (4), and the internal force density $f = \text{div}\sigma$ using equation (8) is called for.

6 Hyperelastic Materials

> Civilization advances by extending the number of important operations which we can perform without thinking about them. A. N. Whitehead, Introduction to Mathematics (1911)

Table 4. Nomenclature for three dimensional solid mechanics

variable	description
λ	first Lamé parameter
μ	shear modulus, second Lamé parameter
κ	bulk modulus $\kappa = \lambda + \frac{2}{3}\mu$
F	deformation gradient
J	determinant F
B	left Cauchy-Green FF^T
C	right Cauchy-Green F^TF
S	seconds Piola-Kirchoff stress
$\overline{B}$	isochoric $BJ^{-2/3}$
$\overline{I_1}$	trace $\overline{B}$
dev()	$\text{dev}(L) = L - \text{tr}(L)/3$

To use the verification problems from [3], one must thoroughly consider the constitutive law for a compressible Neo-Hookean solids. There are many Neo-Hookean materials. For Adagio, the Neo-Hookean constitutive laws do not include this one, to the best of my knowledge. This work could proceed in one of two directions: either figure out how to use the non-default hyperelastic model in Adagio, or use the default Neo-Hookean model and redo some calculations from [3] for the new material.

In this section, the material used in [3] is characterized further. The goal is to identify the closest relative in the Adagio constitutive laws, but this task cannot be done without help from a Subject Matter Expert.

6.1 The Neo-Hookean model used in the verification problems

The constitutive law in [3] is generic: it is also used in [2] and [1] without reference. In [2],

$$W = \frac{\mu}{2}(\text{trace}(C) - 3) - \mu \ln J + \frac{\lambda}{2}(\ln J)^2, \quad S = \mu(I - C^{-1}) + \lambda(\ln J)C^{-1}.$$

In [1], the same strain energy density function is used and the Kirchoff stress $\tau = \lambda \ln J I + \mu(B - I)$.

For purposes of comparison, it is standard to give the strain energy density function, which in this case is

$$W = \frac{\mu}{2}(I_1 - \ln I_3) + \frac{\lambda}{8}(\ln I_3)^2,$$

which matches and is derived in the remainder of this paragraph. The true (or Cauchy) stress σ corresponding to the hyperelastic material with strain energy density $W = W(I_1, I_2, I_3)$ [5], where in $I_1 = \text{trace}(B)$, $I_3 = \det(B) = J^2$ is

$$\sigma = \frac{2}{J}\left[\frac{\partial W}{\partial I_1}B + \frac{\partial W}{\partial I_2}\left(BI_1 - B^2\right)\right] + 2J\frac{\partial W}{\partial I_3}1.$$

By comparison with the constitutive law,

$$\sigma J = \lambda \log J + \mu(B - I)$$

reveals that $\frac{\partial W}{\partial I_2} = 0$, $\frac{\partial W}{\partial I_1} = \frac{\mu}{2}$ and

$$2J\frac{\partial W}{\partial I_3} = \frac{\lambda \ln J - \mu}{J}.$$

The latter is equivalent to

$$\frac{\partial W}{\partial I_3} = \frac{\lambda}{8}\ \frac{2 \ln I_3}{I_3} - \frac{\mu}{2}I_3^{-1},$$

and a strain energy density function follows.

6.2 A related Neo-Hookean model in Adagio

The Adagio Neo-Hookean model uses the strain energy density given in [6], chapter 9, equation 9.2.3. The stored energy function W has volumetric and deviatoric parts $U(J)$ and $\overline{W}(\overline{B})$,

$$W = U(J) + \overline{W}(\overline{B}),$$

$$U(J) = \frac{\kappa}{2}\left(\frac{1}{2}(J^2 - 1) - \ln J\right), \quad \overline{W}(\overline{B}) = \frac{\mu}{2}\text{trace}(\overline{B}).$$

The corresponding Kirchoff stress is

$$\tau = JpI + s, \quad p = \frac{\kappa}{2}(J^2 - 1)/J, \quad s = \mu\ \text{dev}(\overline{B}).$$

Initials: *D.D.*

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External Distribution:

Professor Rebecca Brannon
Department of Mechanical Engineering
University of Utah
Salt Lake City, UT
rebecca.brannon@utah.edu

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