SAND2013-7521P

# Both lazy and efficient: Compressed sensing and applications


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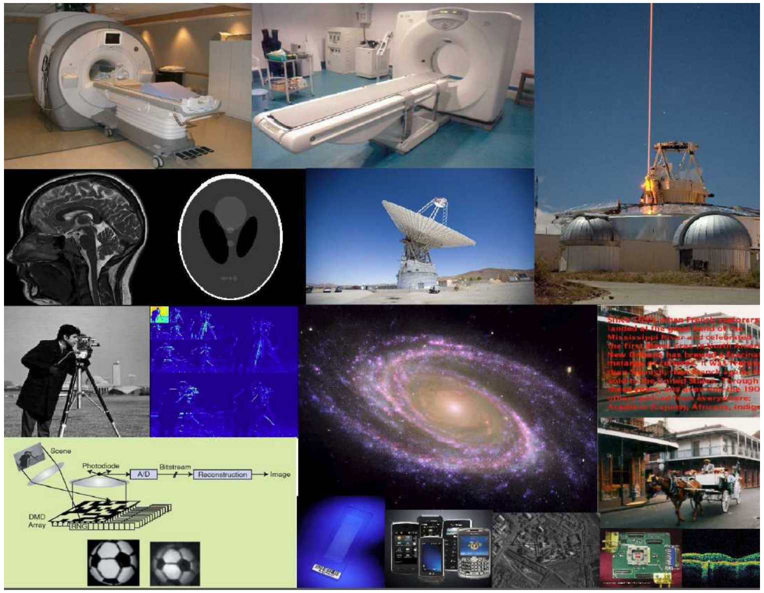


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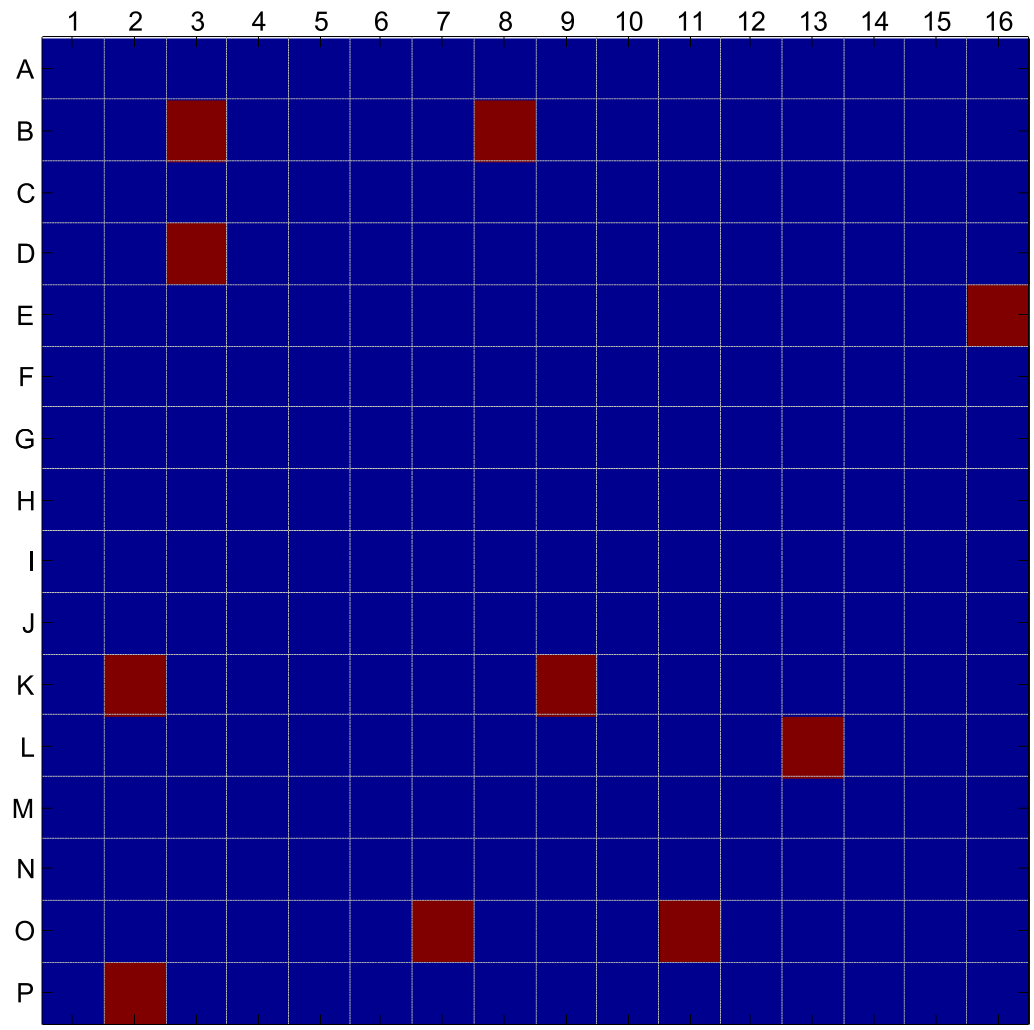
# Outline

- Battleship: a surprising fact
- Intro to compressed sensing theory
- Examples of compressed sensing
  - Early: single-pixel camera, MRI
  - Recent @ Sandia: electron microscopy, SAR

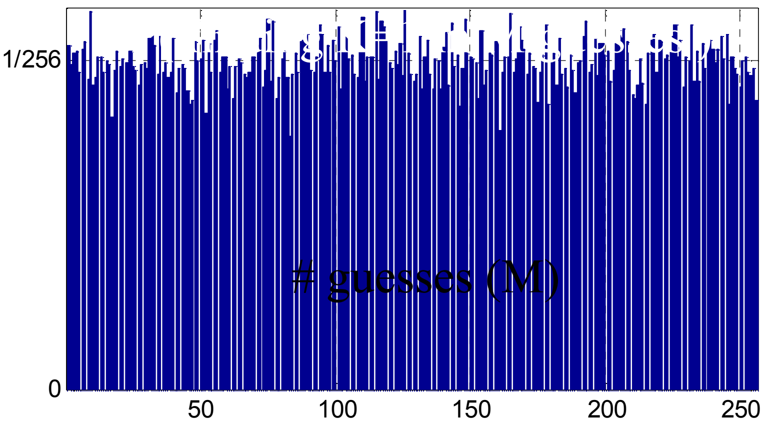
# A new twist to an old game

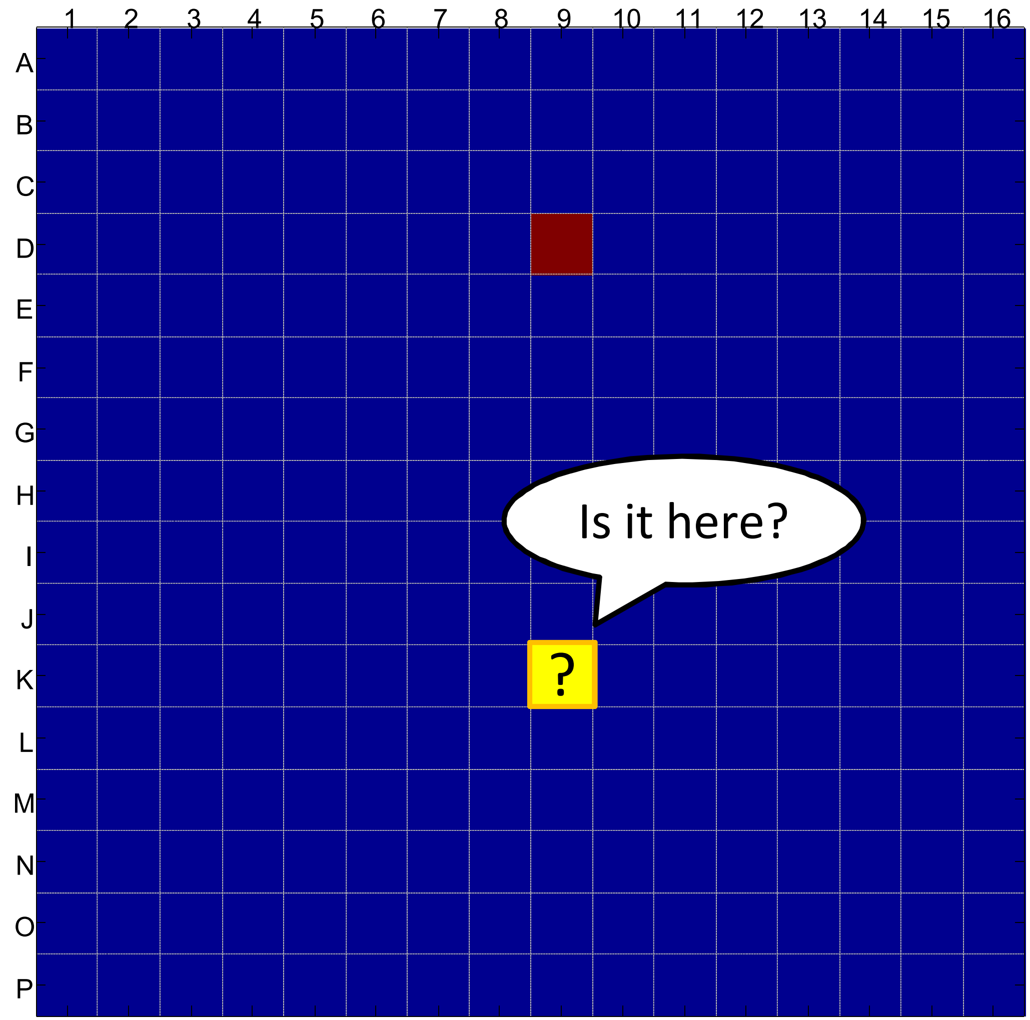
- Ships occupy 1 space
- Ask any question of the form: “Are there any ships (here)?”
- You win if you locate all ships in fewer guesses than competition



# Old strategy: lazy, but inefficient

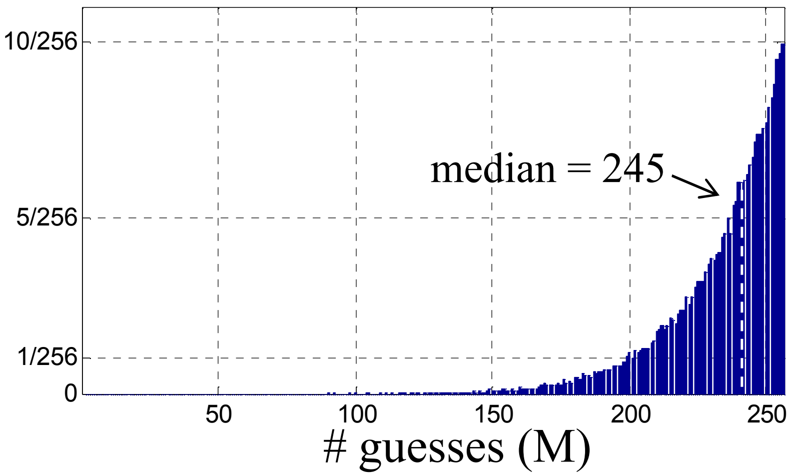
- Query one location at a time, chosen at random
- How many guesses until I locate K=1 ship?
- K=1: uniform



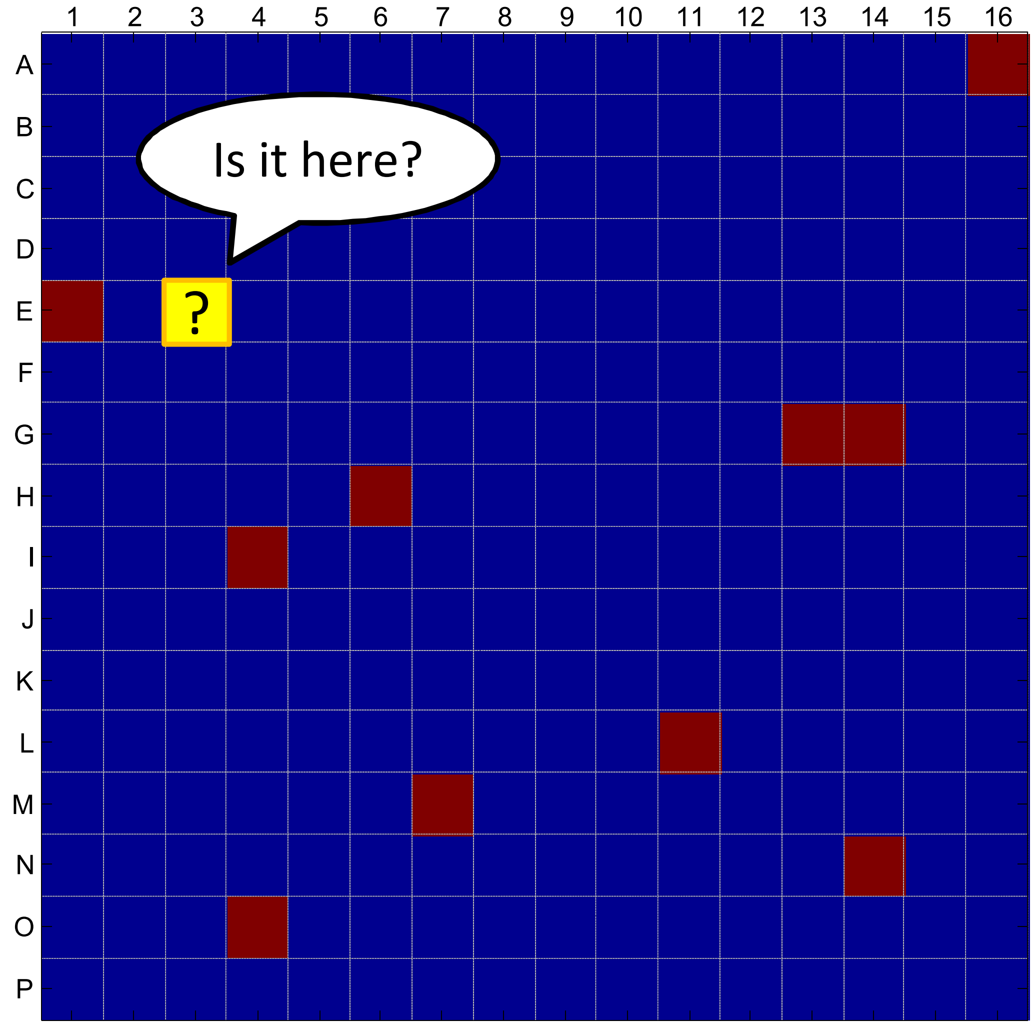



# Old strategy: lazy, but inefficient

- Query one location at a time, chosen at random
- How many guesses until I locate K=10 ships?
- K=10 ships: peaky

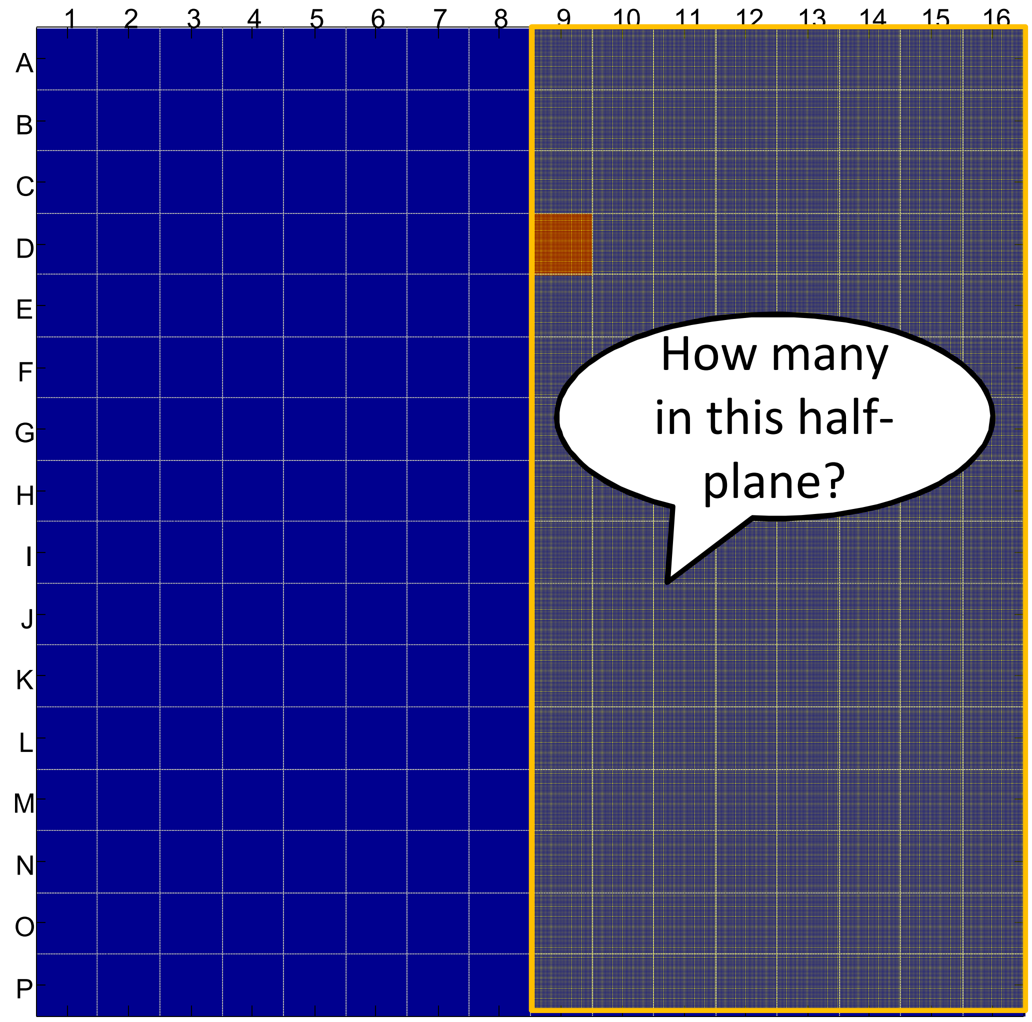



- **M → N as K gets larger**



# Smart strategy: optimal, adaptive

- Bisection: count number of hits in recursively subdividing half-planes



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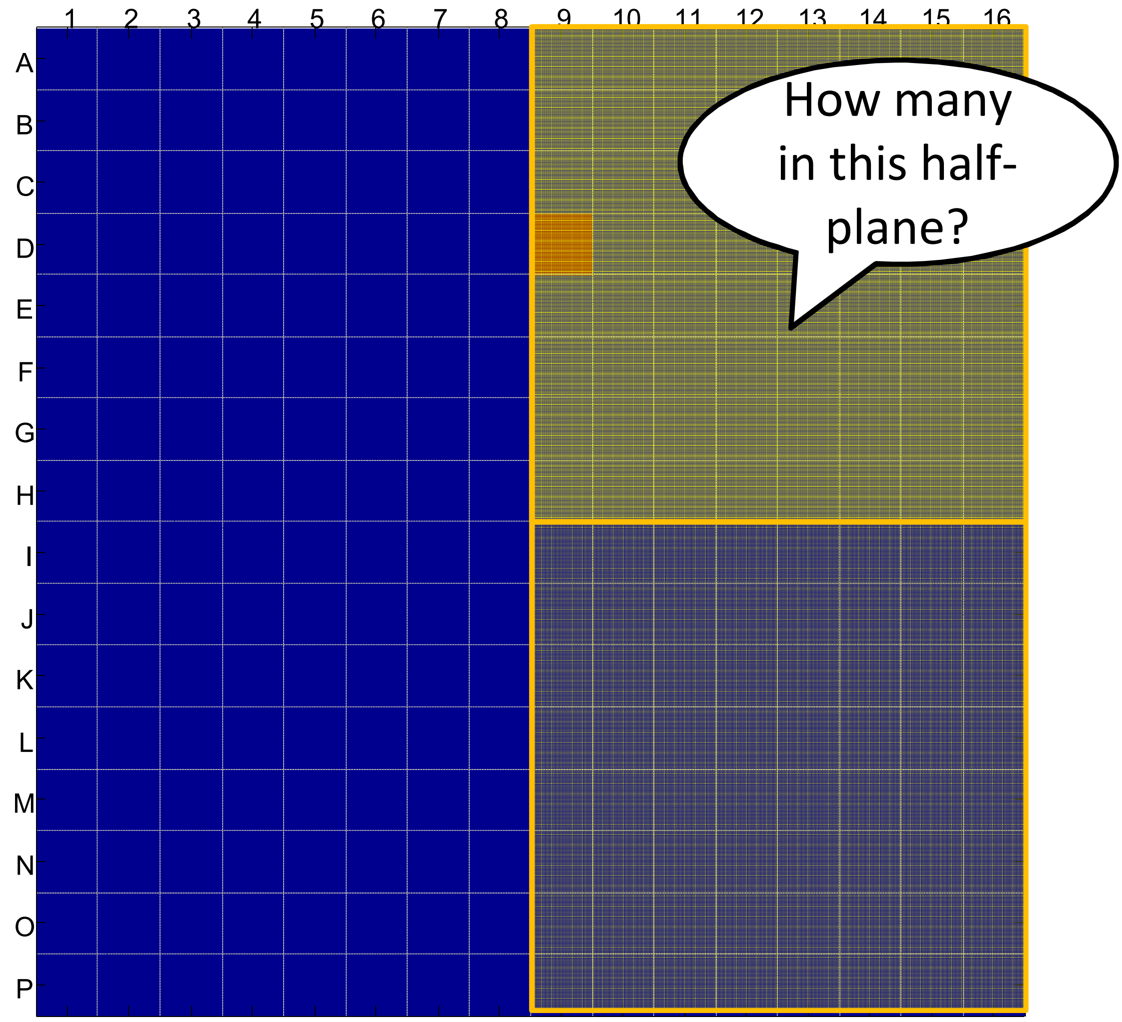

# Smart strategy: optimal, adaptive

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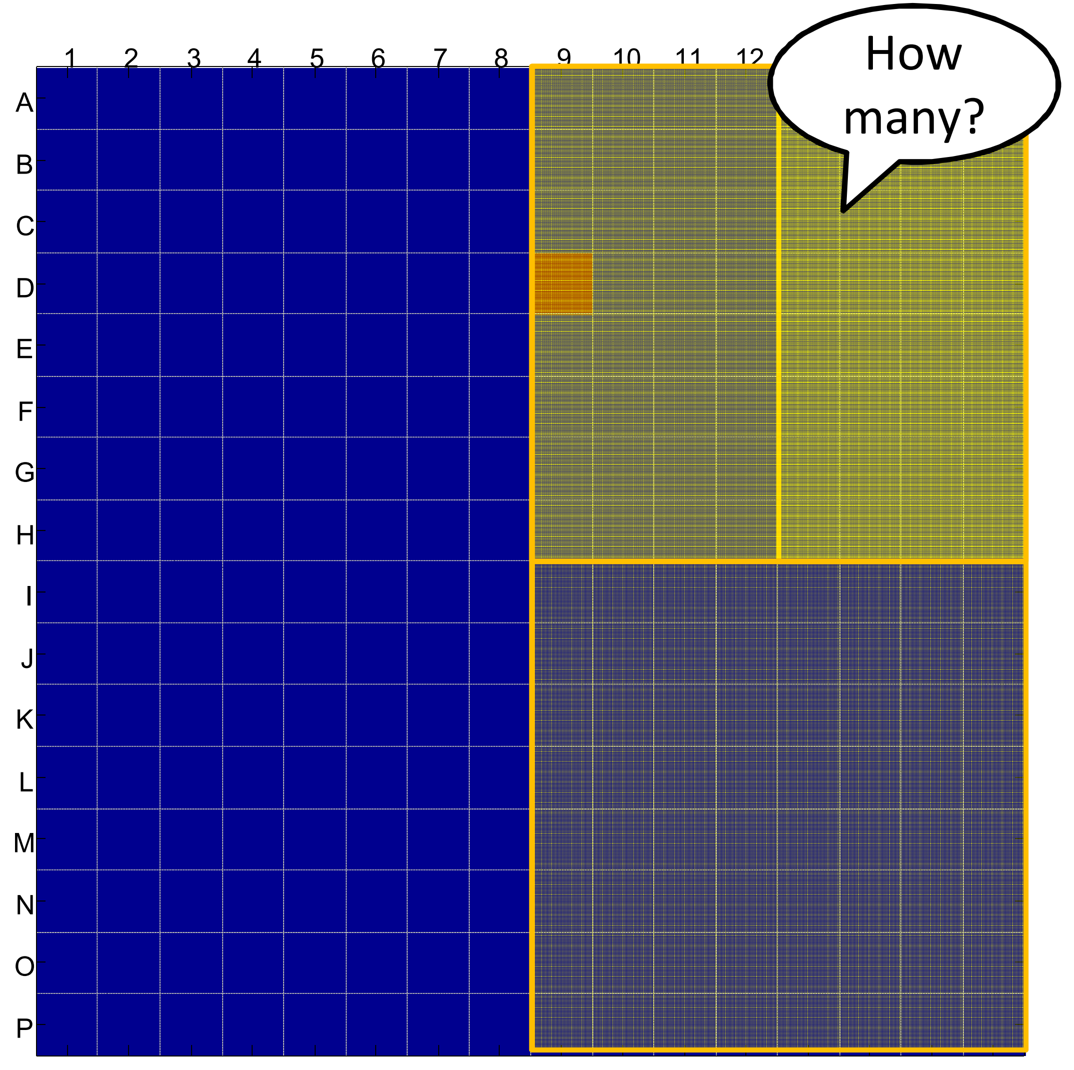
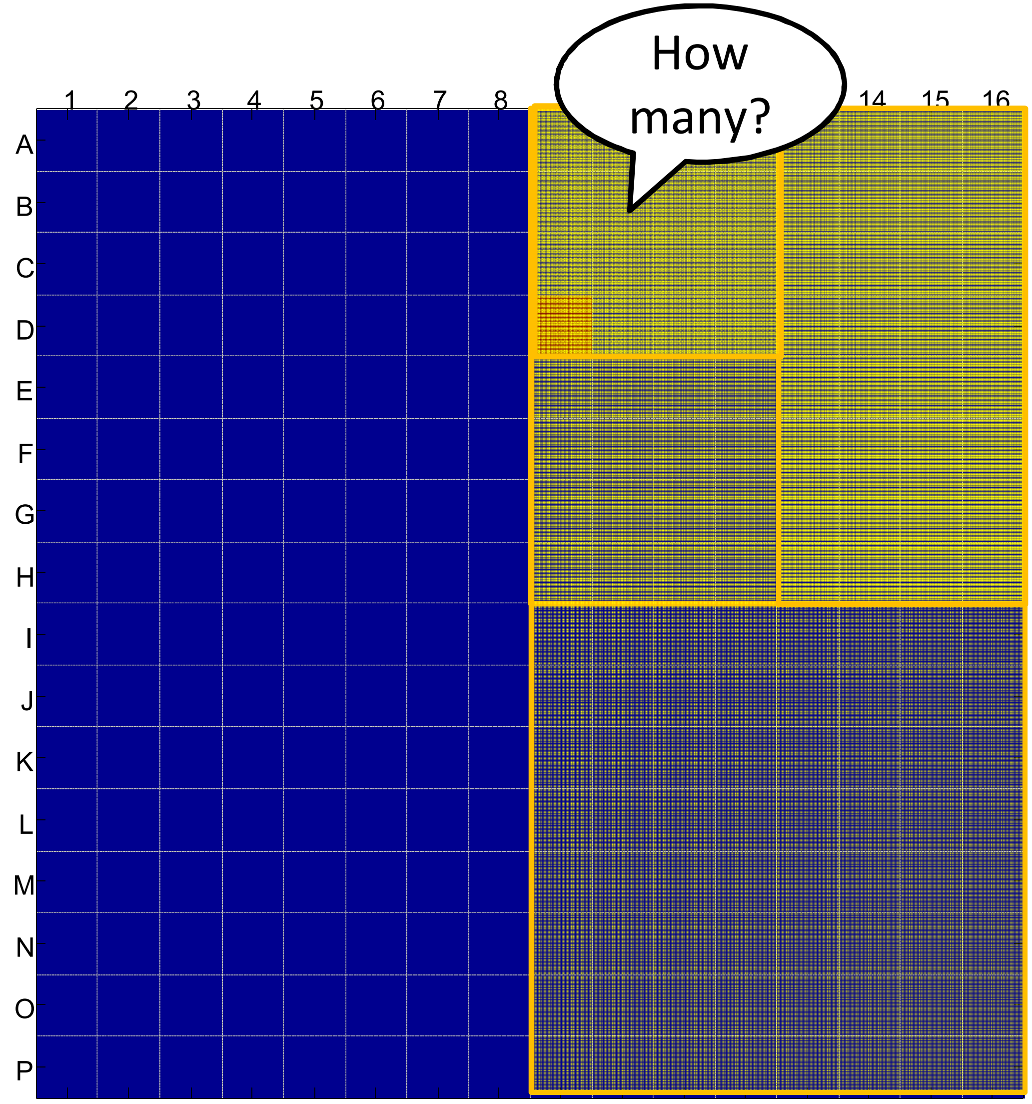


# Smart strategy: optimal, adaptive

- Adaptive
  - Requires book-keeping
  - Next query depends on answer from previous guess
- Optimal
  - Queries designed to maximize information gain from each guess
- # of queries needed

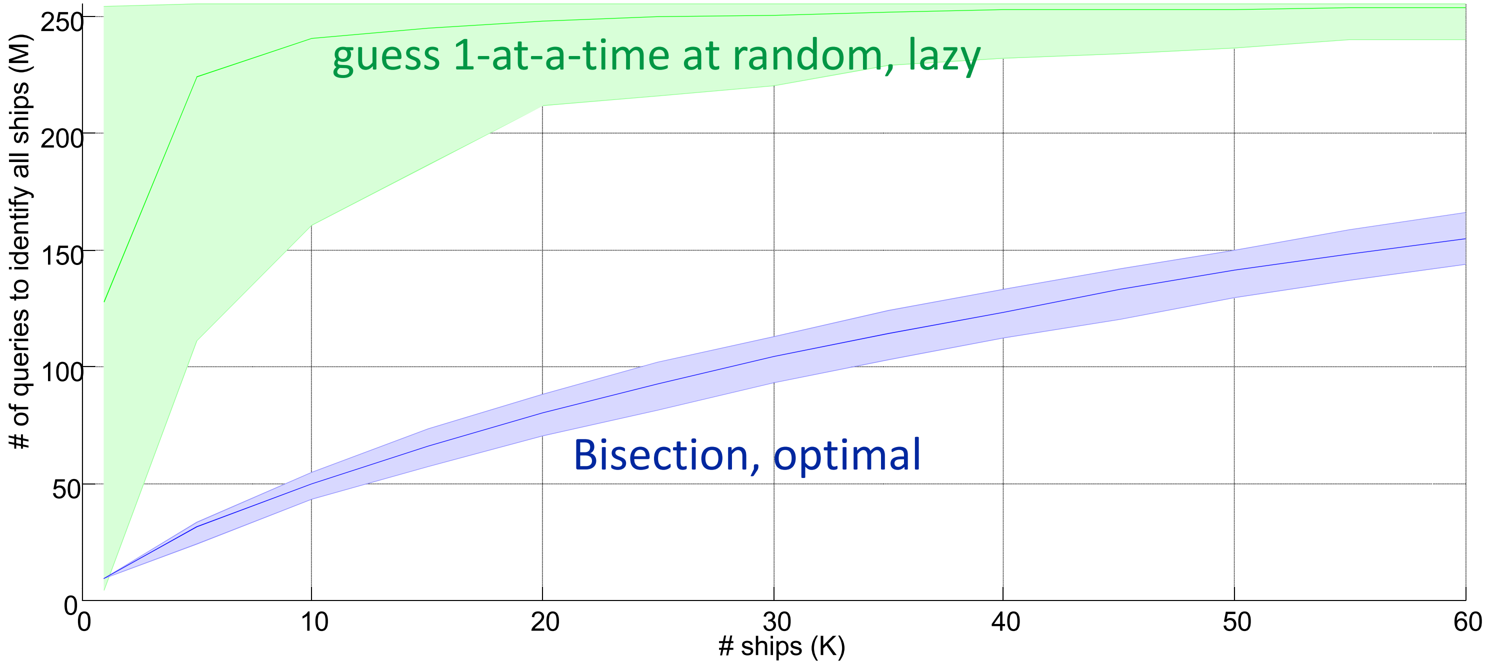
$$M \approx K \log_2 \frac{N}{K}$$

(tight around mean)



# Comparison

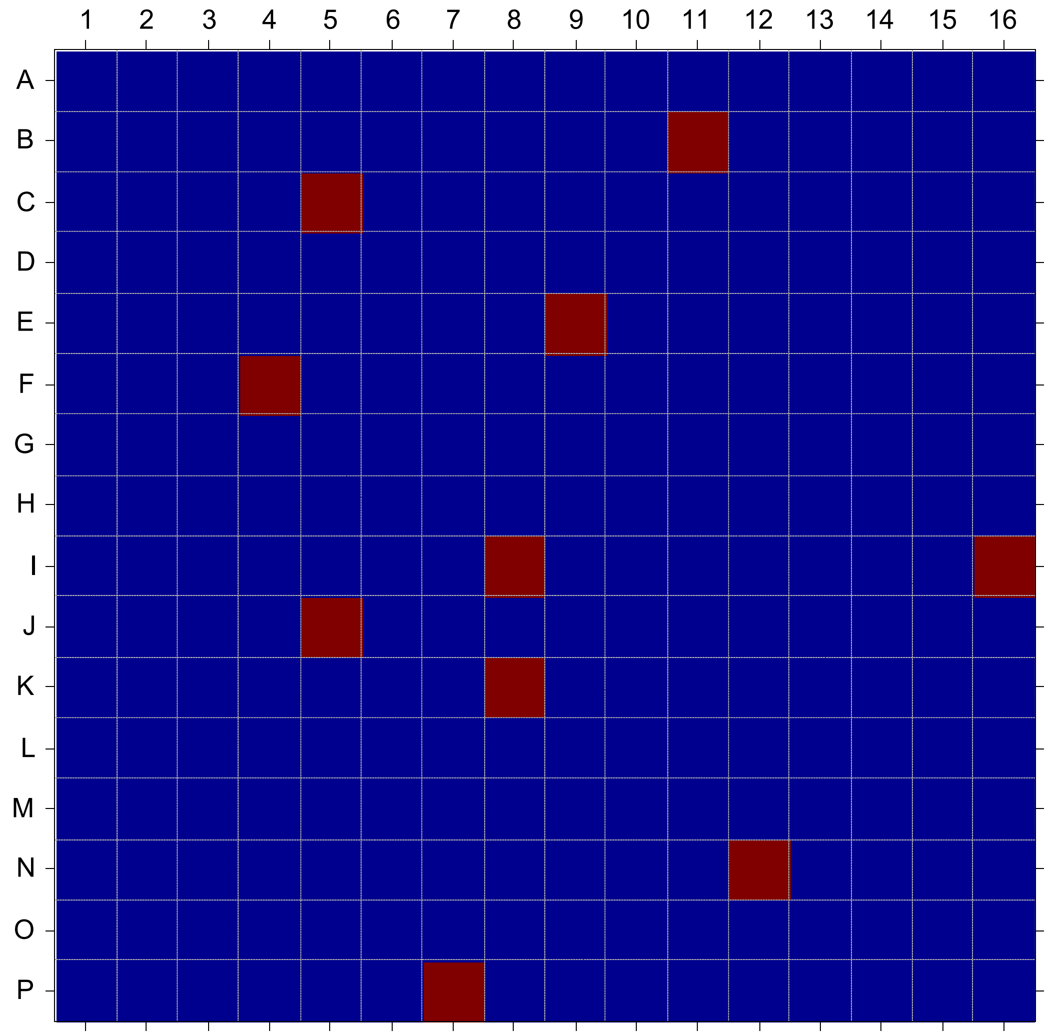
- Guess 1-at-a-time
  - Lazy: pick a location at random, independently from last guess
  - Inefficient: with even a few ships K, M → N (guess every location)
- Bisection method
  - Adaptive: next guess depends on previous answers
  - Optimal: tight around M = O( K log N/K )




Error bars at 1% and 99% quantiles

# Proposed strategy

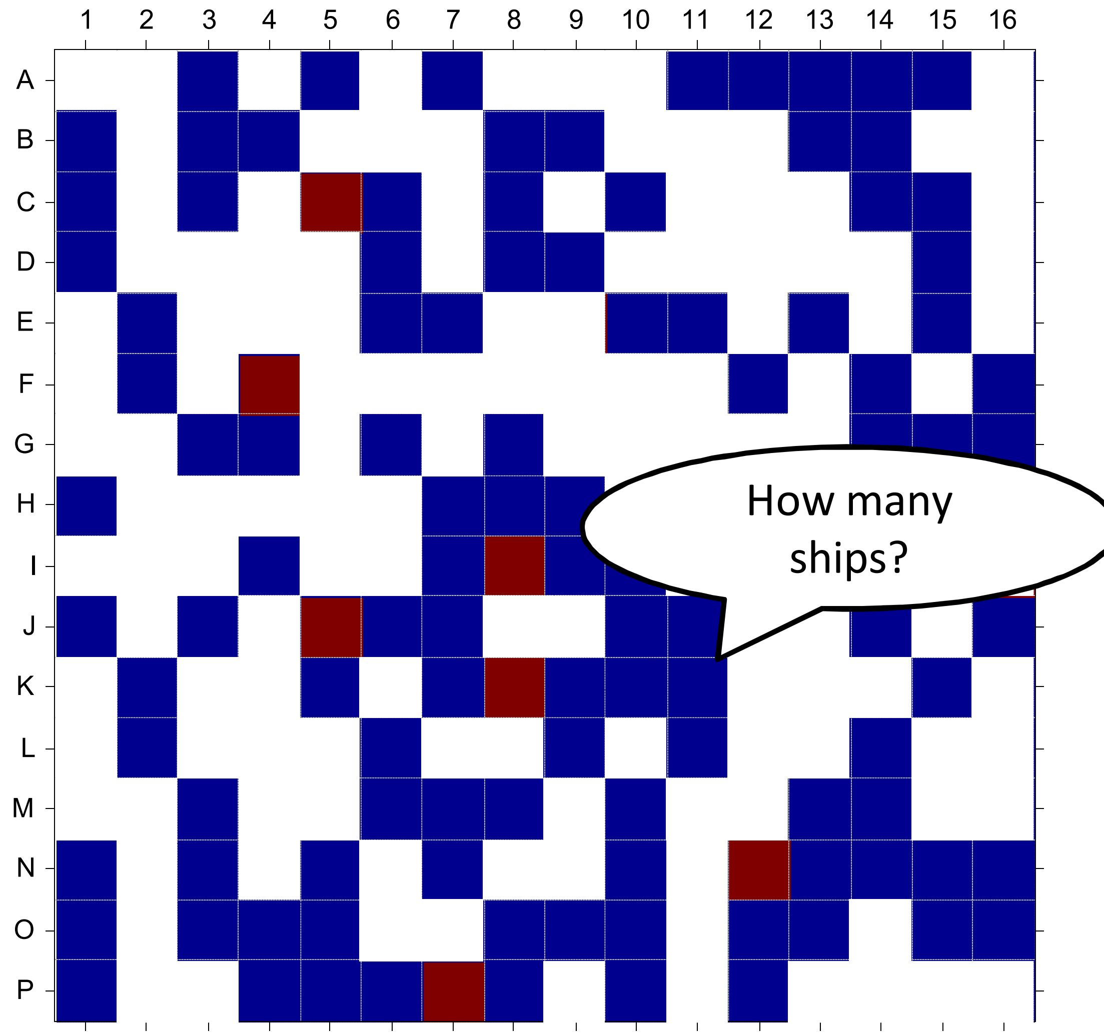
- Create pattern by randomly choosing N/2 of the N locations
- Measurement: Count # ships touching random pattern
- Keep a journal of patterns and counts



# Proposed strategy

- Create pattern/mask by randomly choosing N/2 of the N locations
- Measurement: Count # ships showing through the mask
- Keep a journal:

  $\mathbf{A_1}$ = [ 1  0 1 … 1 0 ]  =>  7



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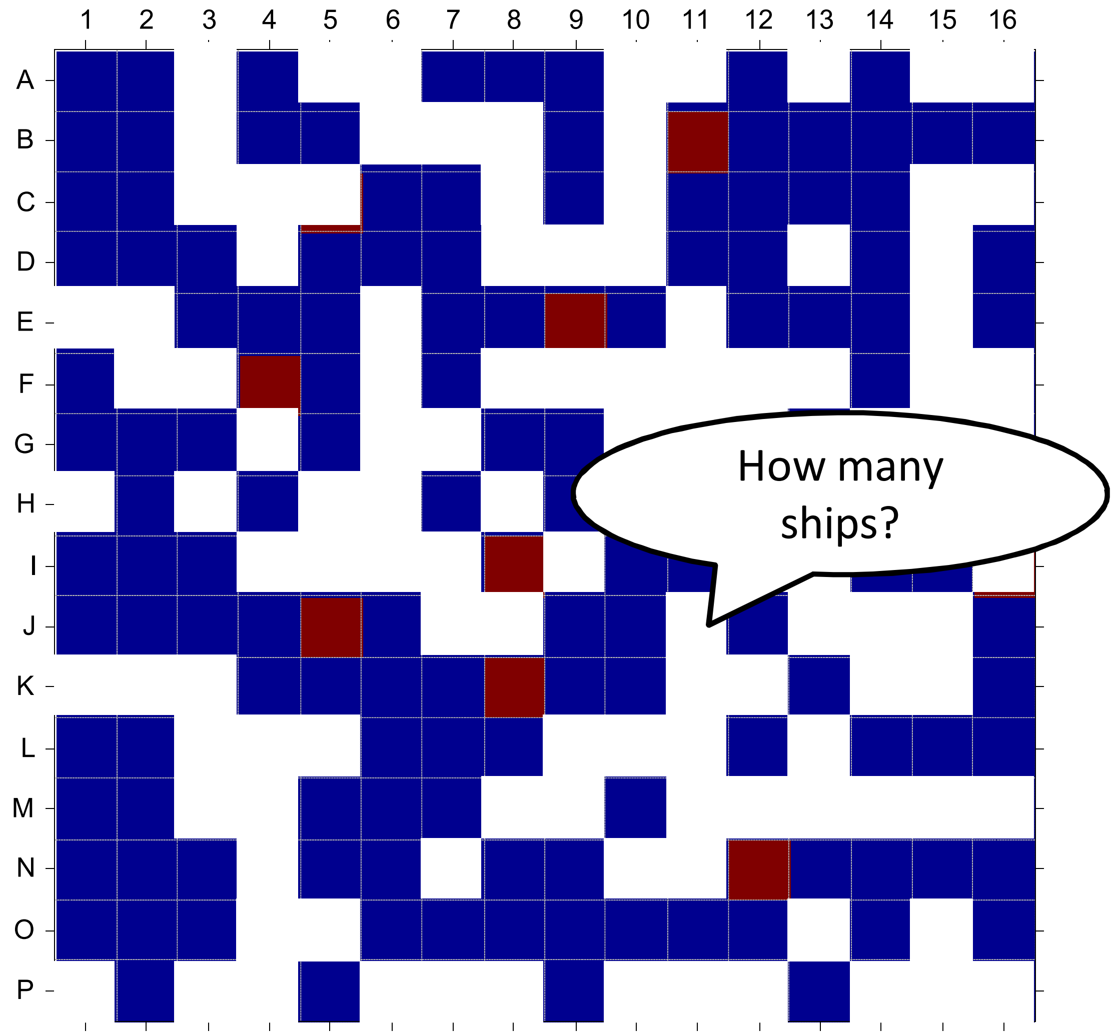
- Keep a journal:

  $\mathbf{A_1}$ = [ 1 0 1 … 1 0 ] => 7

  $\mathbf{A_2}$ = [ 0 1 0 … 1 0 ] => 7



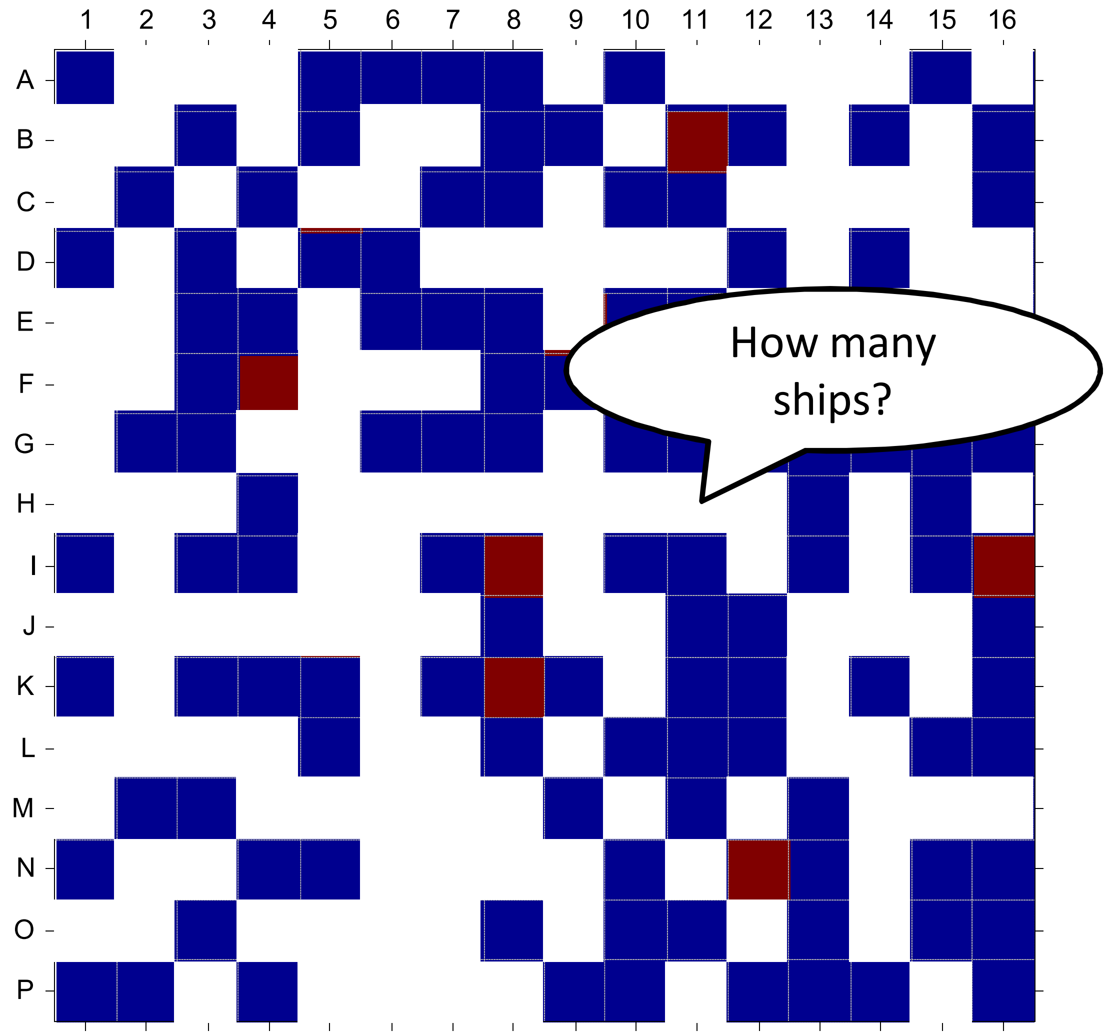
# Proposed strategy

- Create pattern/mask by randomly choosing N/2 of the N locations
- Measurement: Count # ships showing through the mask
- Keep a journal:

  $\mathbf{A_1}$ = [ 1 0 1 … 1 0 ] => 7

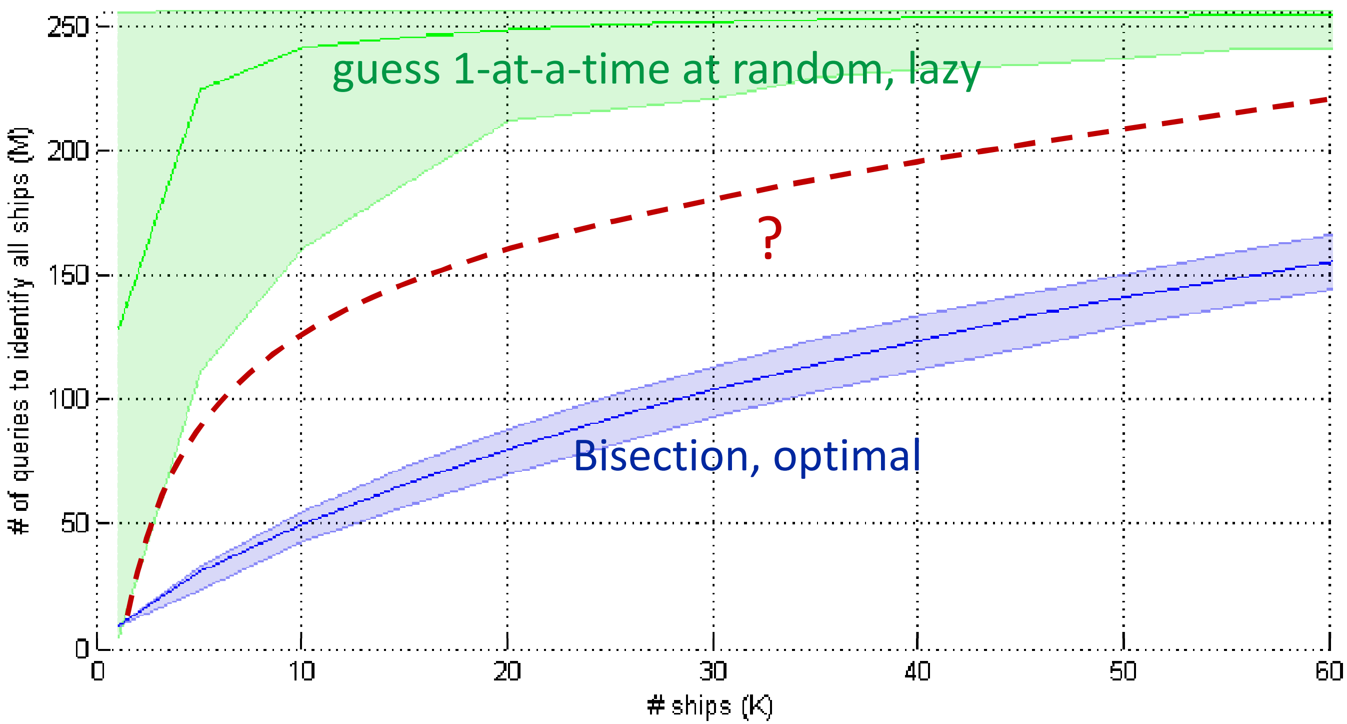
  $\mathbf{A_2}$ = [ 0 1 0 … 1 0 ] => 7

  $\mathbf{A_3}$ = [ 0 0 1 .. 1 1 ] => 6
- After *M* measurements, solve a logic problem



# Comparison

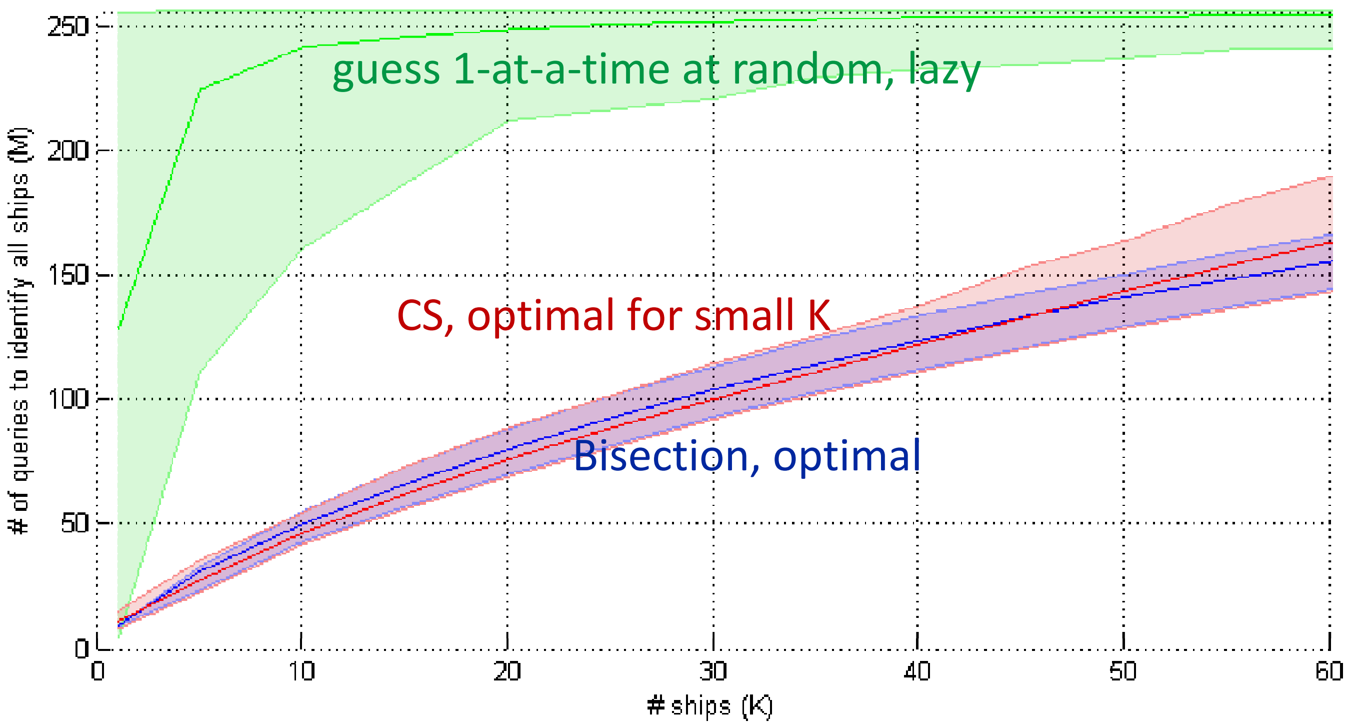
- **Guess 1-at-a-time**: non-adaptive, worst-case ($M \rightarrow N$)
- **Bisection method**: adaptive, optimal ($M \sim K \log N/K$)
- **Compressed sensing**: non-adaptive, ??????




**Error bars at 1% and 99% quantiles**

# Comparison

- **Guess 1-at-a-time**: non-adaptive, worst-case, $M \rightarrow N$
- **Bisection method**: adaptive, optimal, $M=O(K \log N/K)$
- **Compressed sensing**: non-adaptive, optimal, $M=O(K \log N/K)$
  Lazy, agnostic measurements and optimal solution
  – Price to pay: harder recovery problem after measurements are taken

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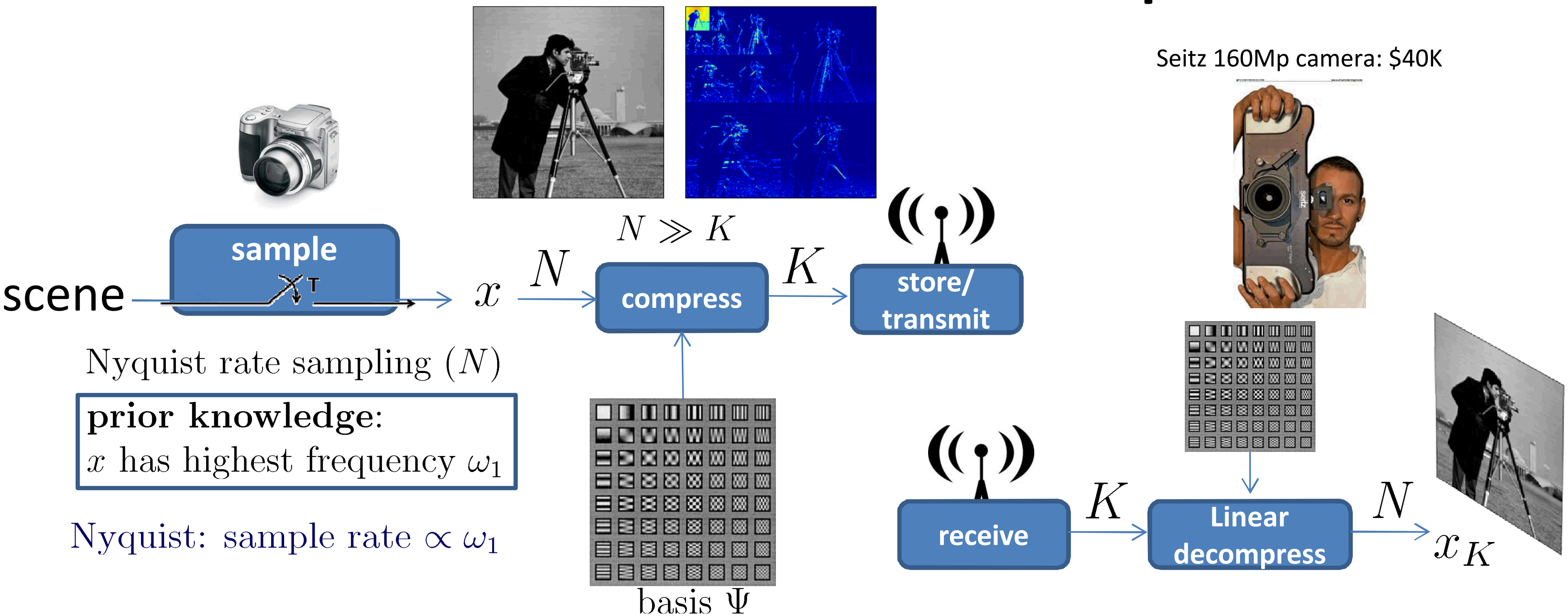
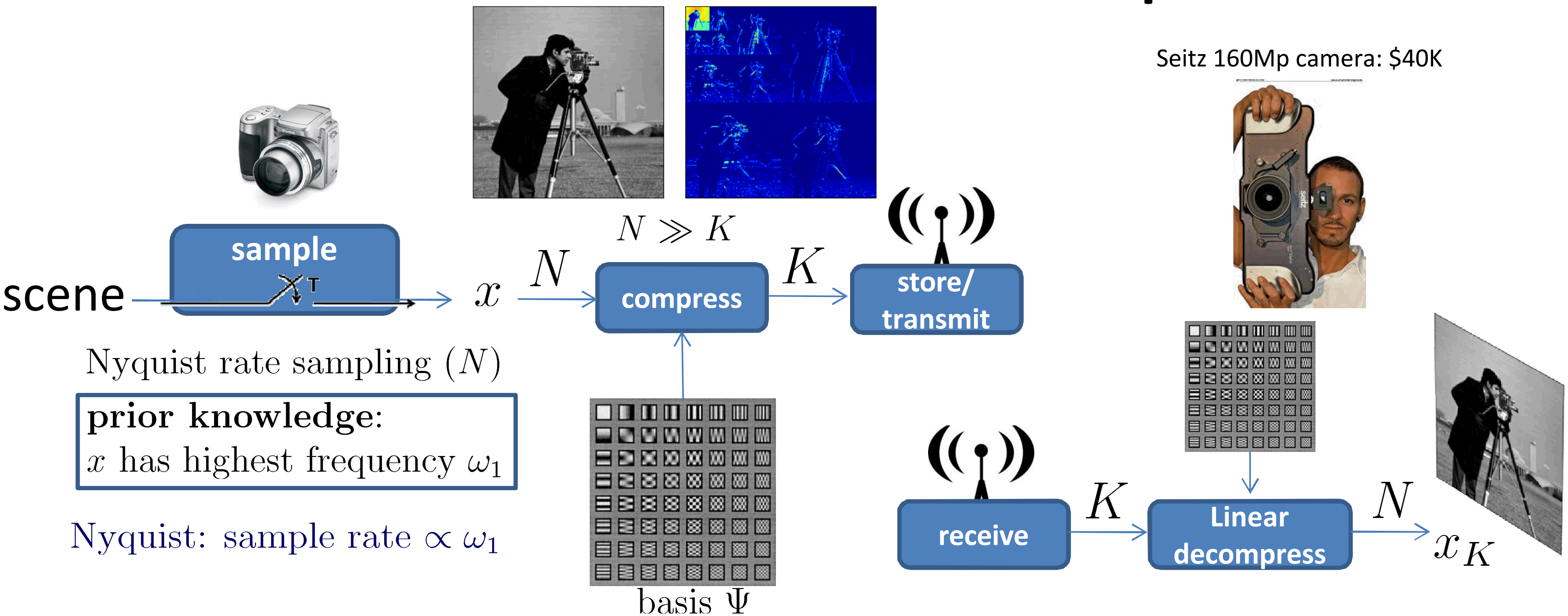
# Traditional vs. CS data acquisition



# Traditional vs. CS data acquisition

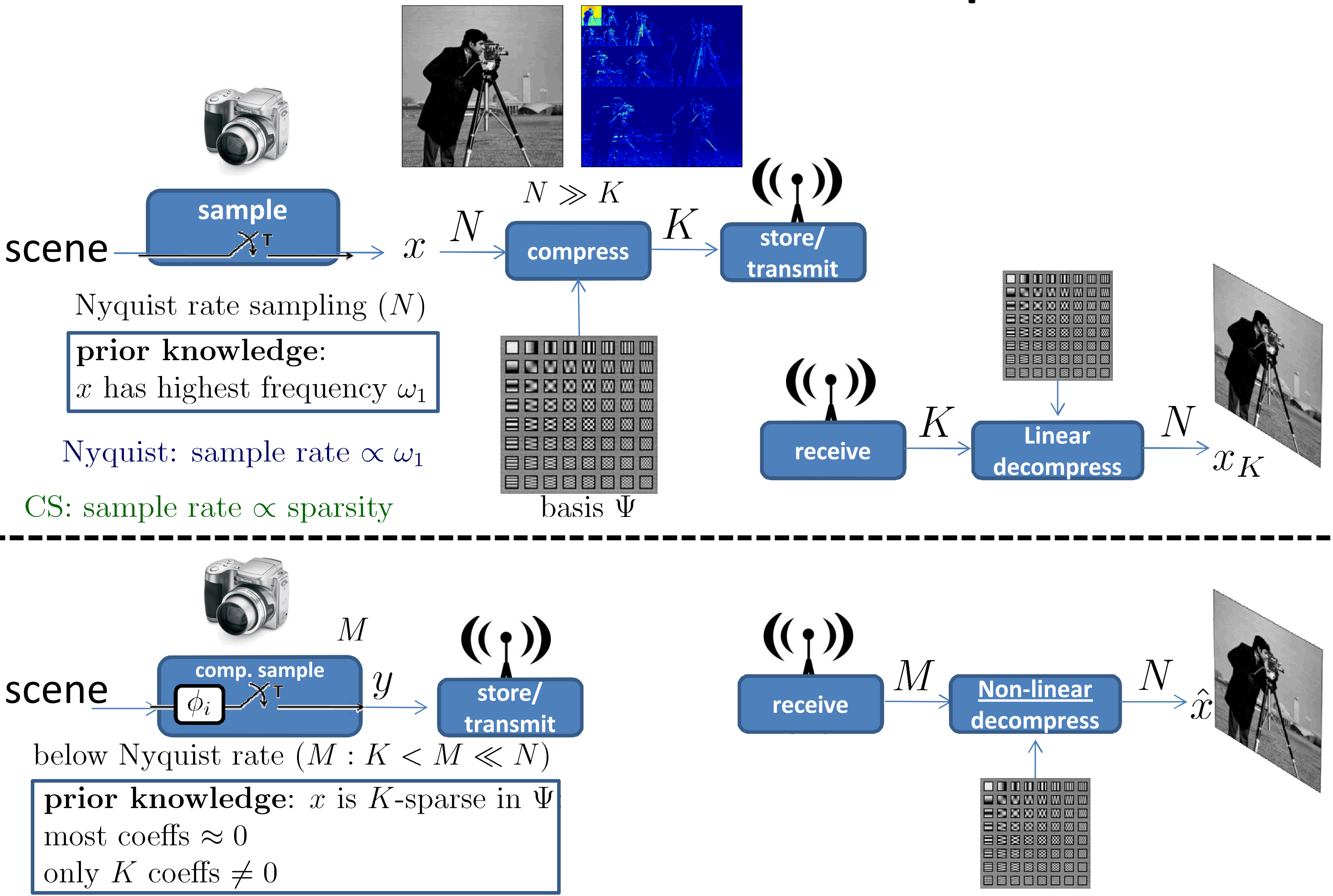



*One can regard the possibility of digital compression as a failure of sensor design. If it is possible to compress measured data, one might argue that too many measurements were taken.*

David Brady

*Can we acquire only the useful parts of the image?*

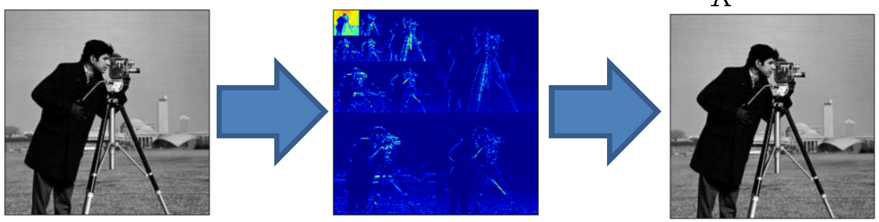
# Traditional vs. CS data acquisition



# Compressed Sensing Theory

## Digital compression:

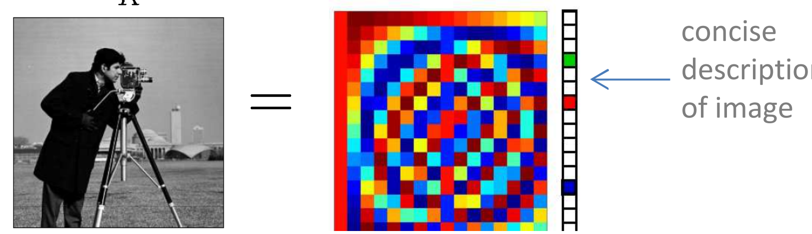
true image $x \in \mathbb{R}^N$ | transform domain $\Psi^* x$ | compressed image $x_K = \Psi \alpha$



(only $K$ significant values)

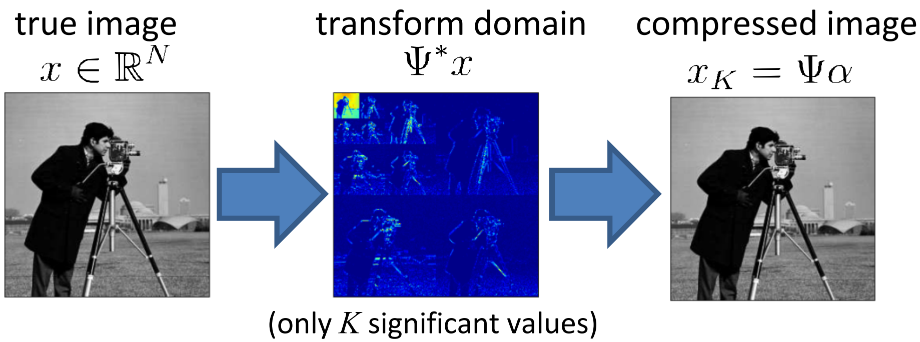
sparse image as a matrix-vector product using basis

$x_K$ = $\Psi$ $\alpha$

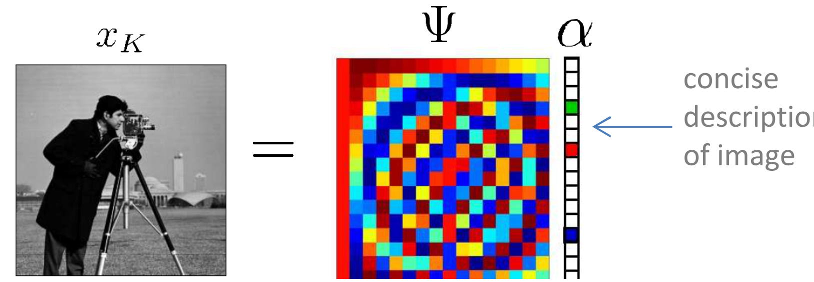


# Compressed Sensing Theory

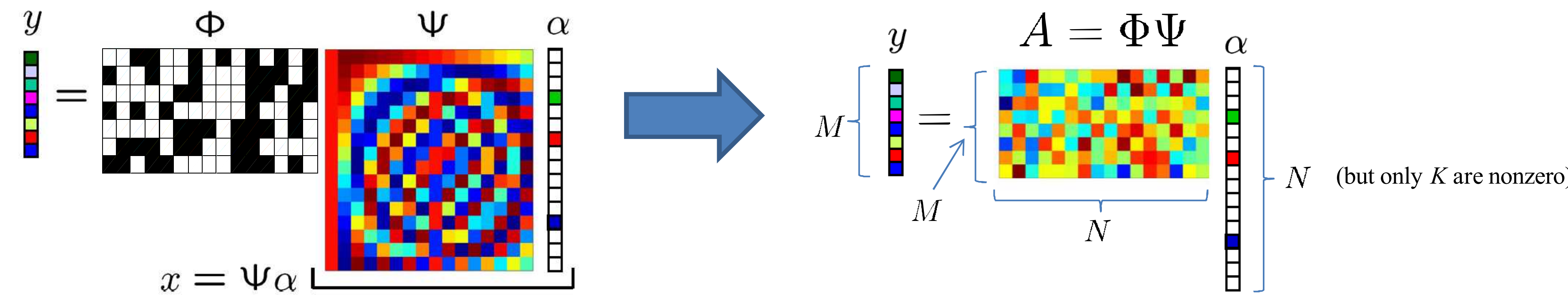
## Digital compression:




sparse image as a matrix-vector product using basis

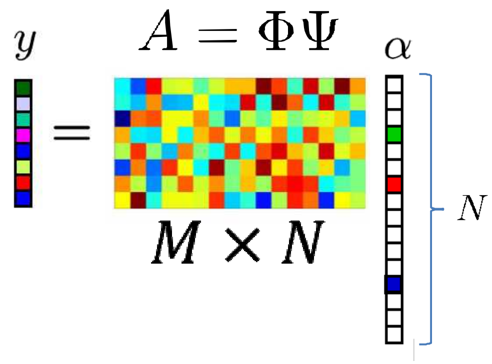



## Take M "mixture" measurements of $\alpha$:



# Compressed Sensing Theory

***Compressed sensing:***
*A underdetermined*
*($M < N$) but $M \geq K$.*
***Support of $\alpha$ is unknown.***




Theorem: If $A$ is properly designed, then solve

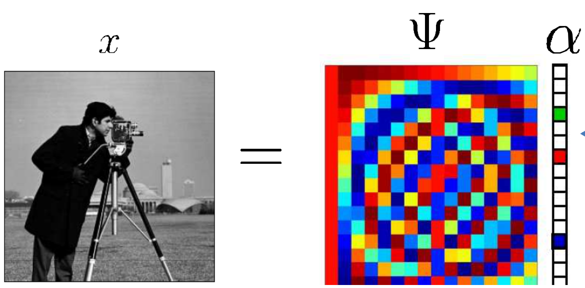
$$\min_{\alpha} \quad \|\alpha\|_1 \qquad s.t. \quad A\alpha = y$$

(convex relaxation
[polynomial time]
NP-hard problem)

If $\quad M \geq C_1 K \log N$

then $\|\alpha - \alpha^*\|_1 \leq C_2 \|\alpha^*_K - \alpha^*\|_1$

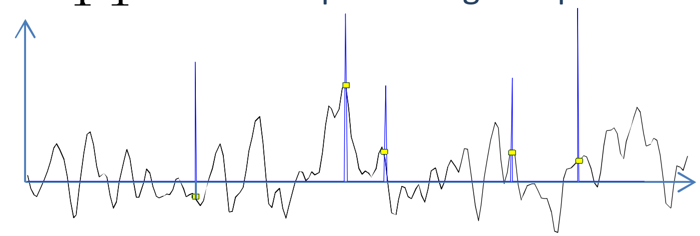
- Exact reconstruct of sparse images in polynomial time from few measurements
- Bounded reconstruction of compressible (almost sparse) images




concise
description
of image

How to properly design sensing matrix $\Phi$?

- **Incoherence:** sampling functions "global" in basis
- **RIP:** $\Phi\Psi$ is distance-preserving for sparse vectors



Most common choices:

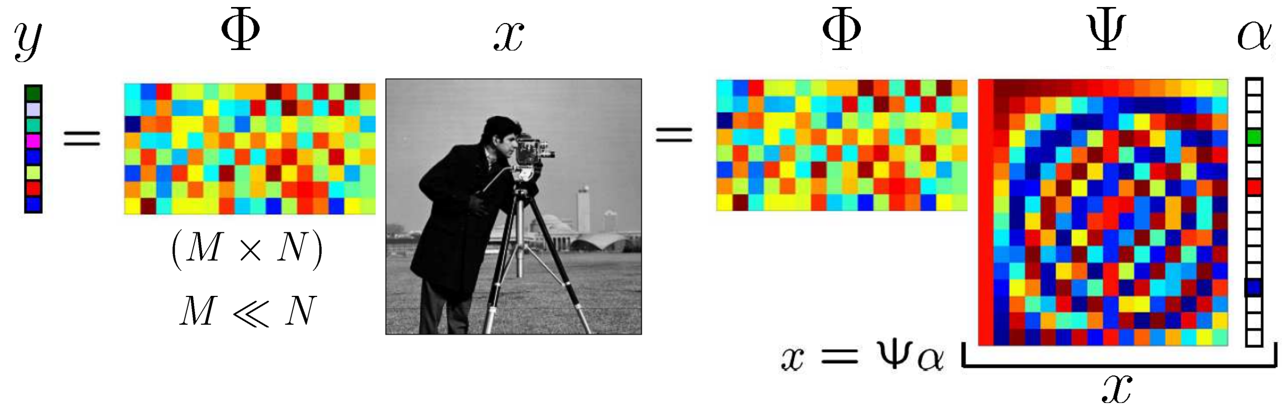
**low mutual coherence**

- if $\Psi$ is DFT, then $\Phi$ is random subset of identity
- if $\Psi$ is identity, then $\Psi$ is random subset of DFT

**satisfies RIP**

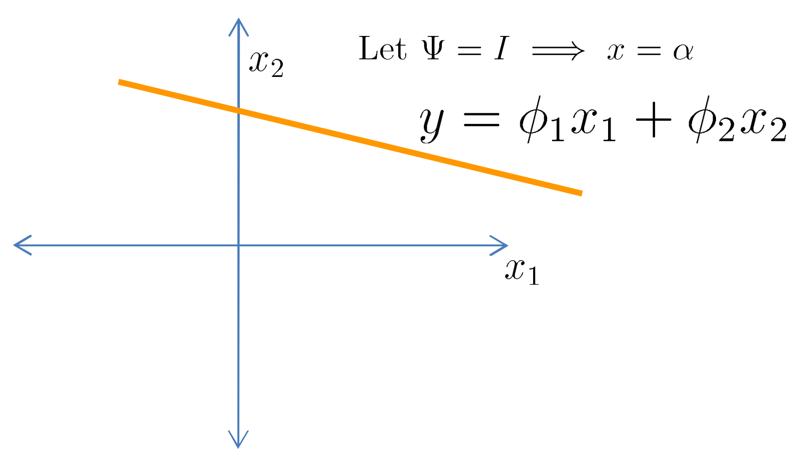
- If I choose elements of $\Phi$ at random,
  - $\Psi$ can be anything!
  - I can choose after data has been collected!

# Exact Recovery of Sparse Signals

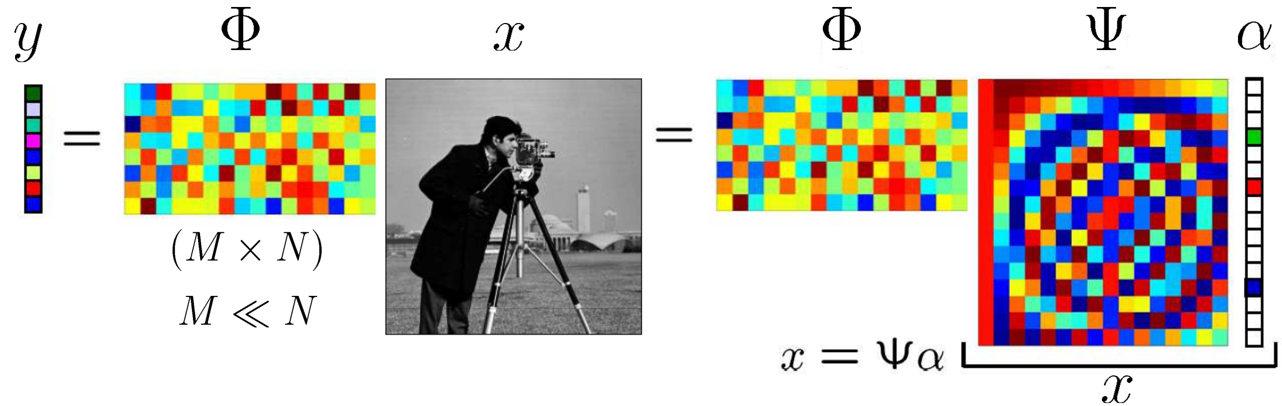



- Measurement equation is an underdetermined system

  (Fundamental theorem of linear algebra ==> cannot uniquely determine $x$)



# Exact Recovery of Sparse Signals



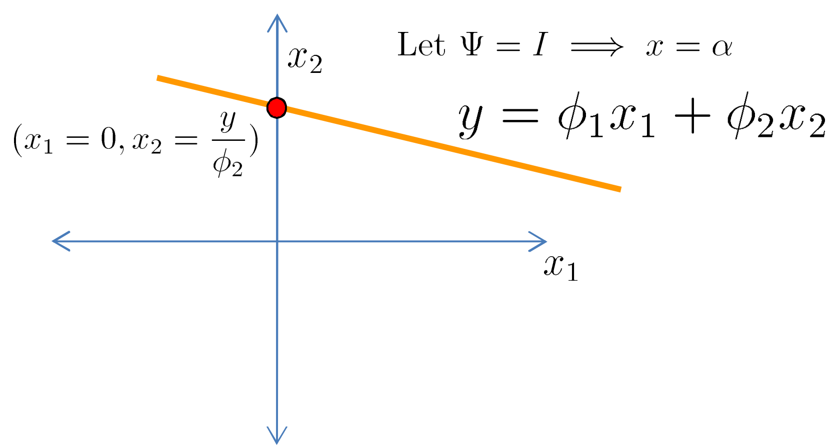

- Measurement equation is an underdetermined system
- Occam's razor: find ***sparsest solution*** that explains measurements

$$\min_{\alpha} \|\alpha\|_0 \ \text{ s.t. } \ \Phi\Psi\alpha = y$$

$$\|\alpha\|_0 = \#\{i : \alpha_i \neq 0\}$$

$\ell_0$ quasi-norm counts
\# of nonzero elements

$$\hat{x} = \Psi\hat{\alpha}$$



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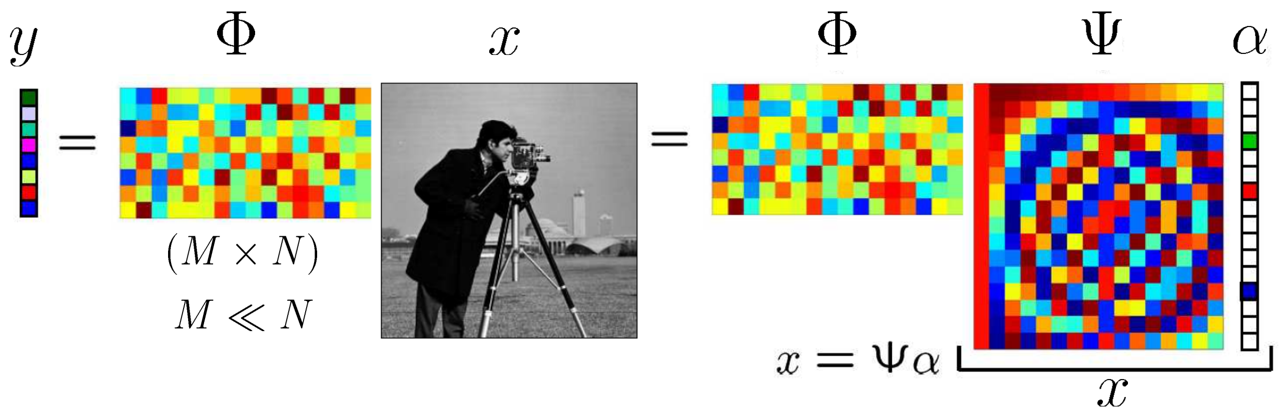


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$\ell_0$ quasi-norm counts # of nonzero elements

- Some algorithmic approaches (See [Tropp & Wright, 2010] for overview)
  - **Greedy pursuit** (approximate, fast). Iterative: refine sparse solution. Greedy, myopic.
  - **Convex relaxation** (exact, managable). Use $\ell_1$ norm as convex surrogate for sparsity.

$$\min_{\alpha} \|\alpha\|_1 \;\; \text{s.t.} \;\; \Phi\Psi\alpha = y \qquad\qquad \|\alpha\|_1 = \sum_i |\alpha_i|$$

(linear program)

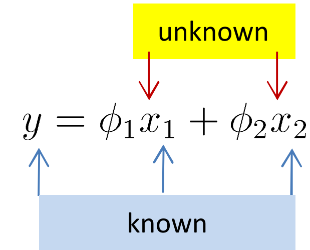
# Minimizing $\ell_1$ norm yields sparse solution

An example:

- Let $\Psi = I$
- Vector $\mathbf{x} = [x_1, x_2]^T$ known to be sparse
- Single measurement
- Family of solutions
- Find unique solution to

$$y = \Phi\,\mathbf{x}$$

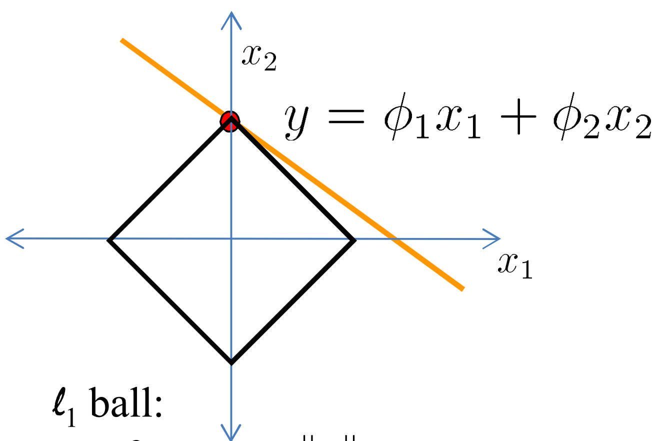
$$= [\phi_1 \quad \phi_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$




$$\min \|\mathbf{x}\|_{(1 \text{ or } 2)} \ \text{s.t.} \ \phi_1 x_1 + \phi_2 x_2 = y$$

$$\|\mathbf{x}\|_1 = |x_1| + |x_2|$$

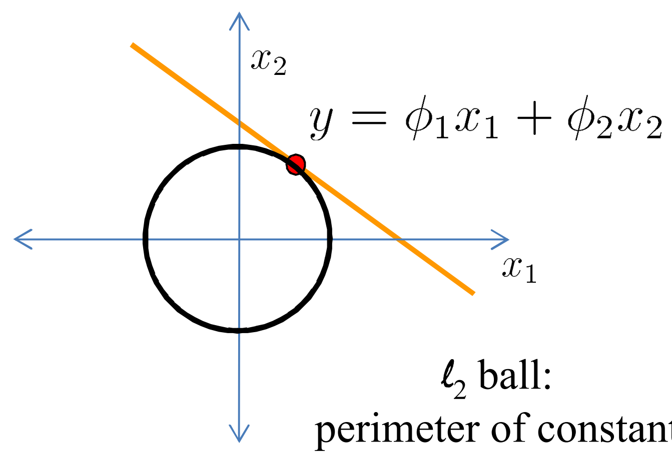
Sparse Solution




$\ell_1$ ball:
perimeter of constant $\|\mathbf{x}\|_1$

$$\|\mathbf{x}\|_2 = \sqrt{x_1^2 + x_2^2}$$

Non-sparse Solution




$\ell_2$ ball:
perimeter of constant $\|\mathbf{x}\|_2$

# Single-pixel camera [Rice University]

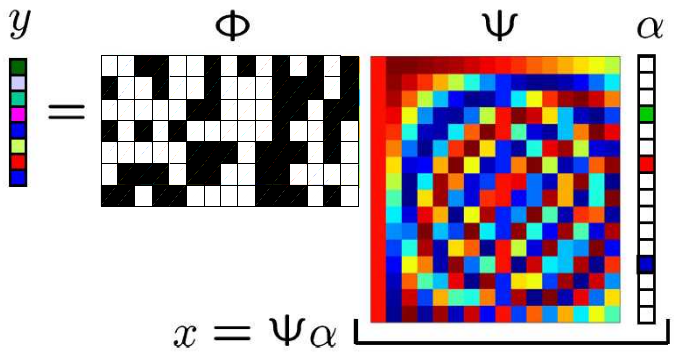
[Duarte, et al, 2008]

## What is sparse? Image coefficients in wavelet basis

## What sampling kernel? Randomly-generated binary

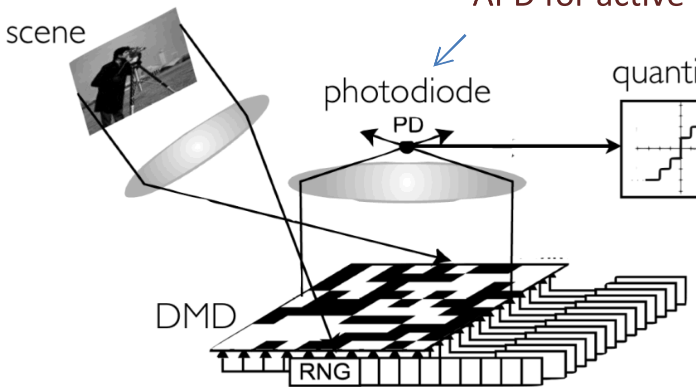
## How to implement CS?

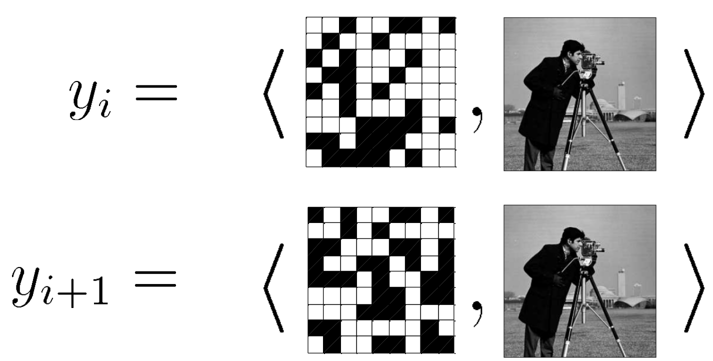
- Single photo-detector
- Micro-mirror array encodes sensing function



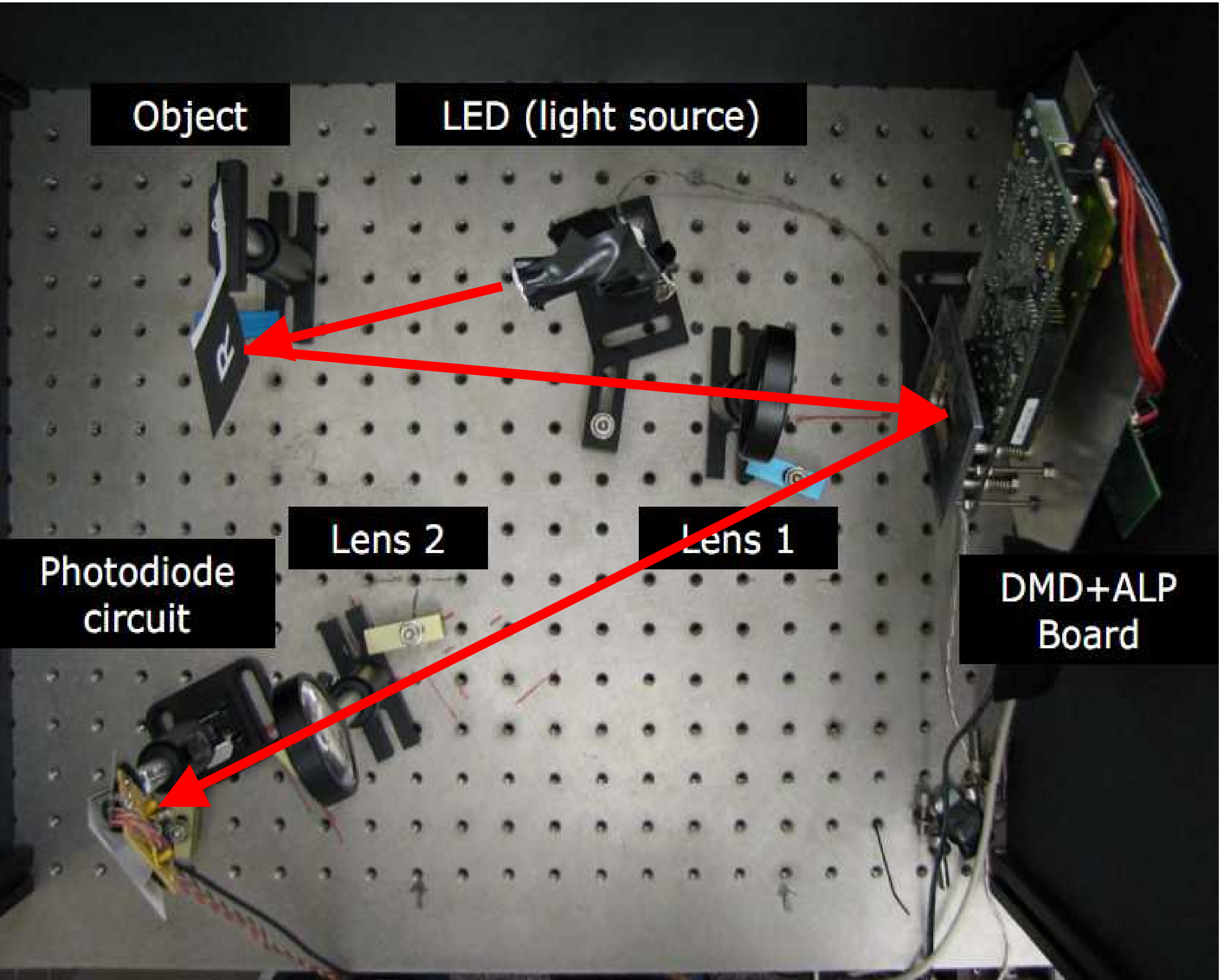

could be exotic: for example,

- line array of hyperspectral elements (InView Corp)
- PMT for low-light imaging
- APD for active-illuminated lidar (U of Rochester)



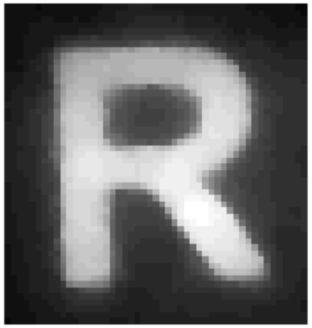



# Single-pixel camera [Rice University]



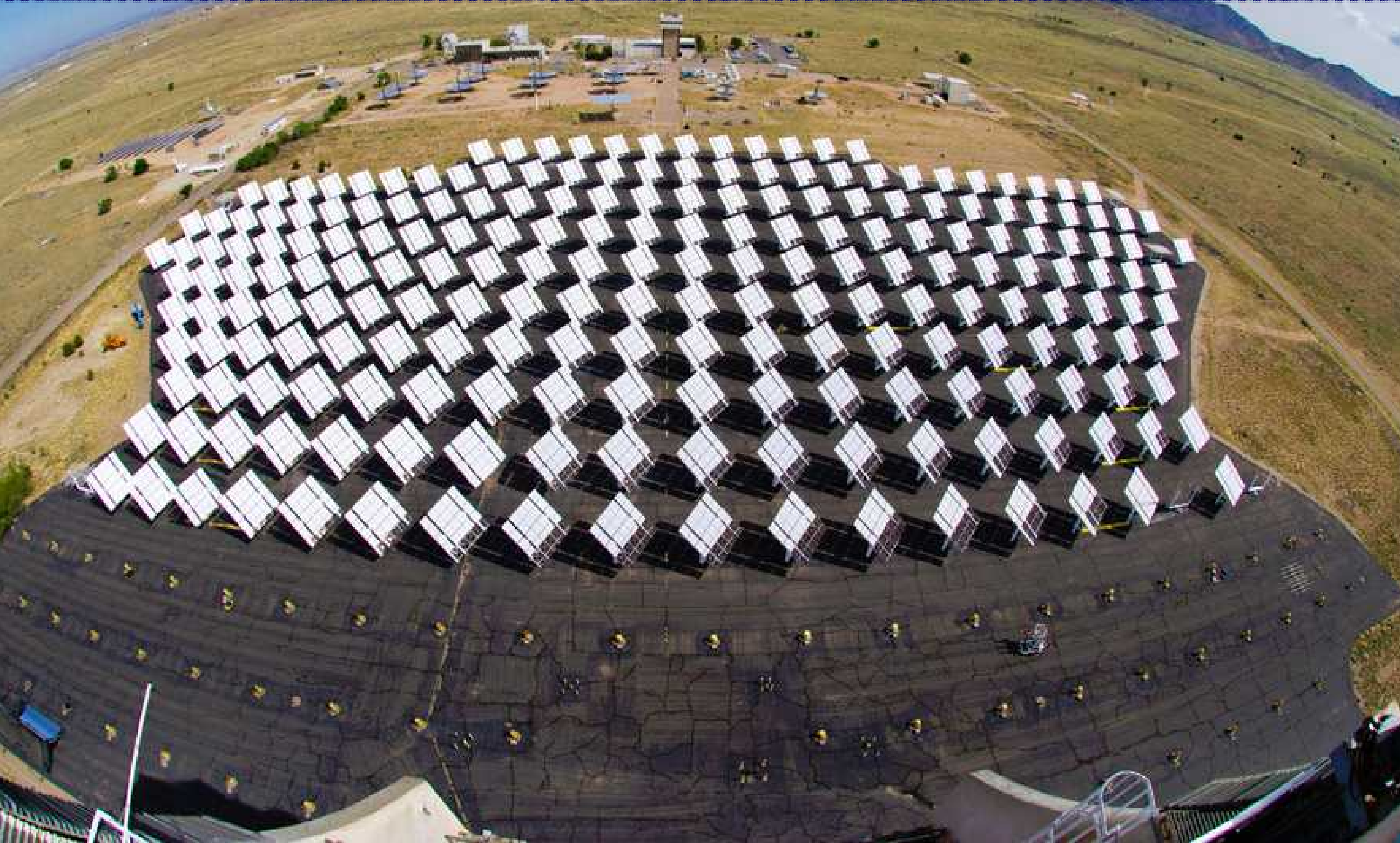

target
65536 pixels

11000 measurements
(16%)



1300 measurements
(2%)

# Sandia's Heliostat Array: world's largest single pixel camera?



# Sparse Magnetic Resonance Imaging

(Lustig, Donoho, Pauly, 2007)

## What is sparse? edges in proton density image

## Sampling kernel? Randomly-selected Fourier (k-space) samples

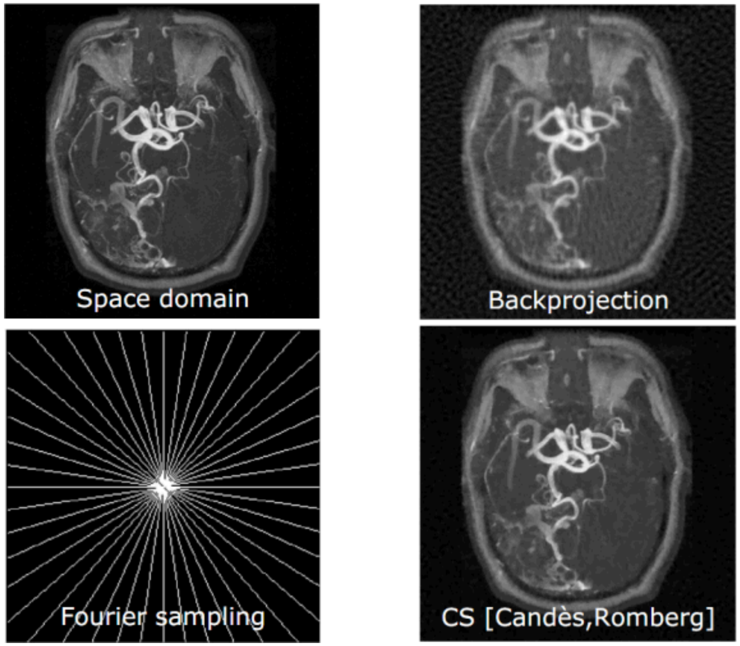
## How to implement CS?

- **MRI already samples in k-space! [Incoherent with canonical (pixel) domain]** (Fourier coefficients)

$$X(\omega_1, \omega_2) = \sum_{(t_1,t_2)} x(t_1, t_2) e^{-j2\pi(\omega_1 t_1 + \omega_2 t_2)}$$

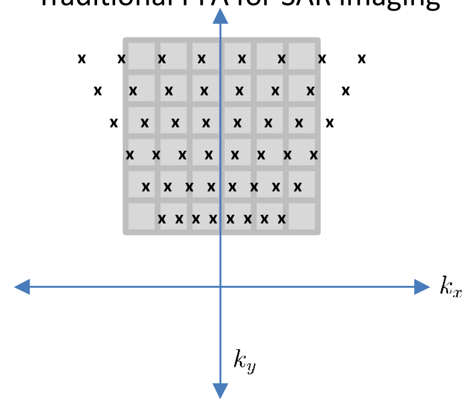
- Choose random subset of Fourier samples

**Life-saving application: pediatric MRI**



# Sandia SAR imagery (www.sandia.gov/RADAR)

# Compressive Sensing for SAR

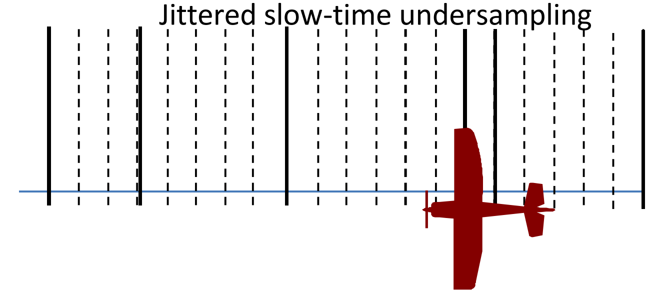



What is sparse? Resolved SAR image

What sensing matrix? Random slow-time sampling

How to encode sensing matrix?

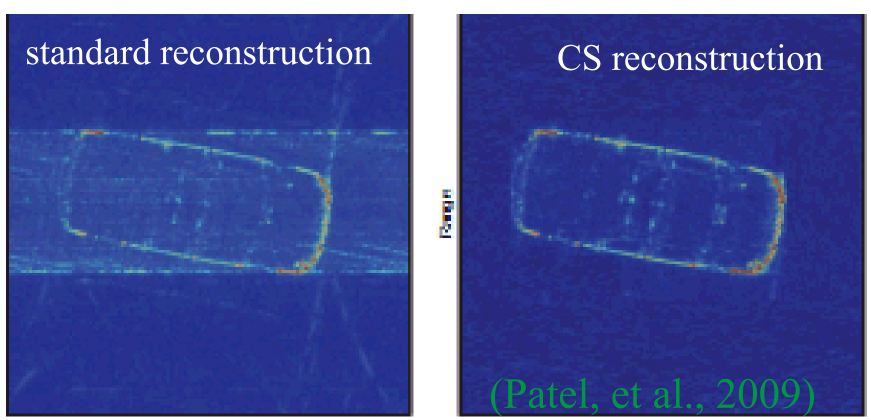
- SAR already gives Fourier measurements
- Random slow-time undersampling

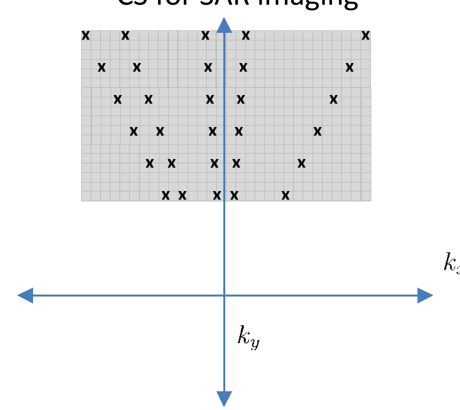



**Benefits:** **reduced transmit power**

**longer flight times**

**electronic counter-countermeasures**



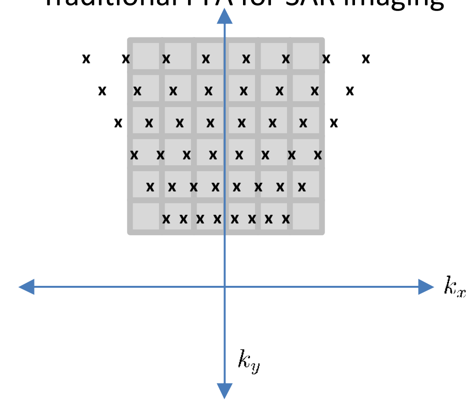



# SAR Imaging from undersampled data

## Standard Synthetic Aperture Radar (SAR):

- Regularly-spaced chirps to measure **k-space samples** in trapezoidal (Sandia) or annular region
- SAR image processing (PFA simplified):
  - interpolate onto regular k-space grid
  - FFT

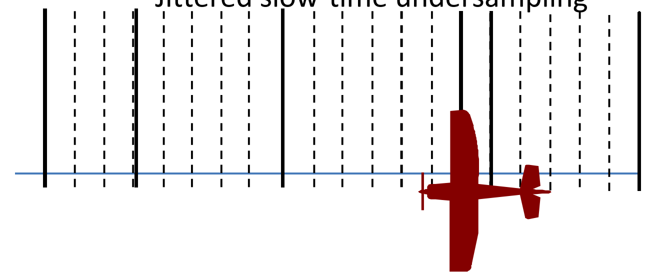
Traditional PFA for SAR imaging



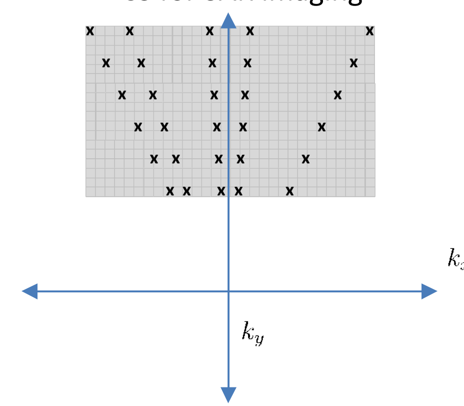

## Compressive SAR (akin to CS for MRI):

- Random-stagger pulse repetition interval (PRI)
  - Faster “logical” chirp timing rate (e.g., 4x)
  - Chirp only at a fraction of intervals (e.g., 10%)
  - Overall pulse reduction (e.g., 40% of typical)
- CS image reconstruction:
  - Embed samples on fine k-space grid
  - Reconstruct via Occam’s razor: what’s the simplest (sparsest) image that can describe my observations

Jittered slow-time undersampling



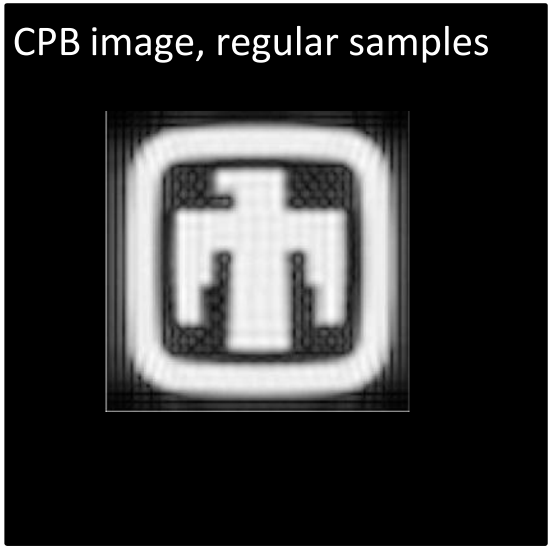
CS for SAR imaging

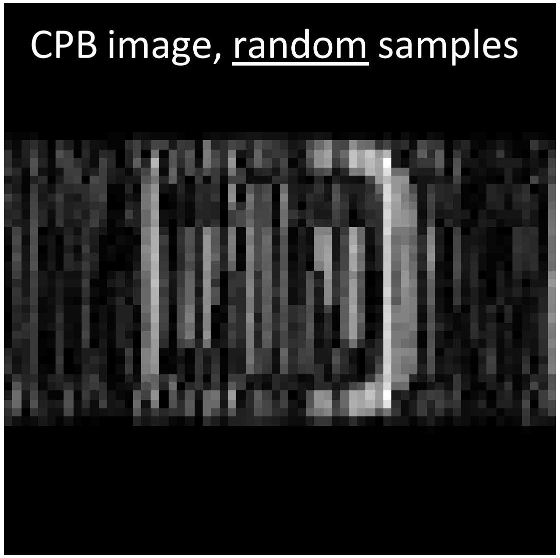


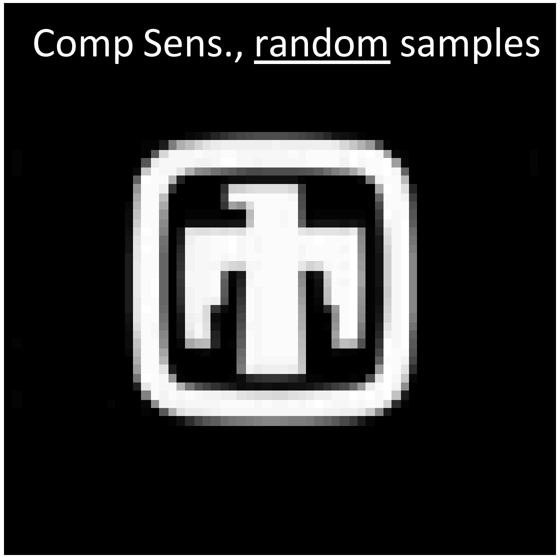

phase histories → $\mathbf{y} = A\mathbf{x}$ ← scene

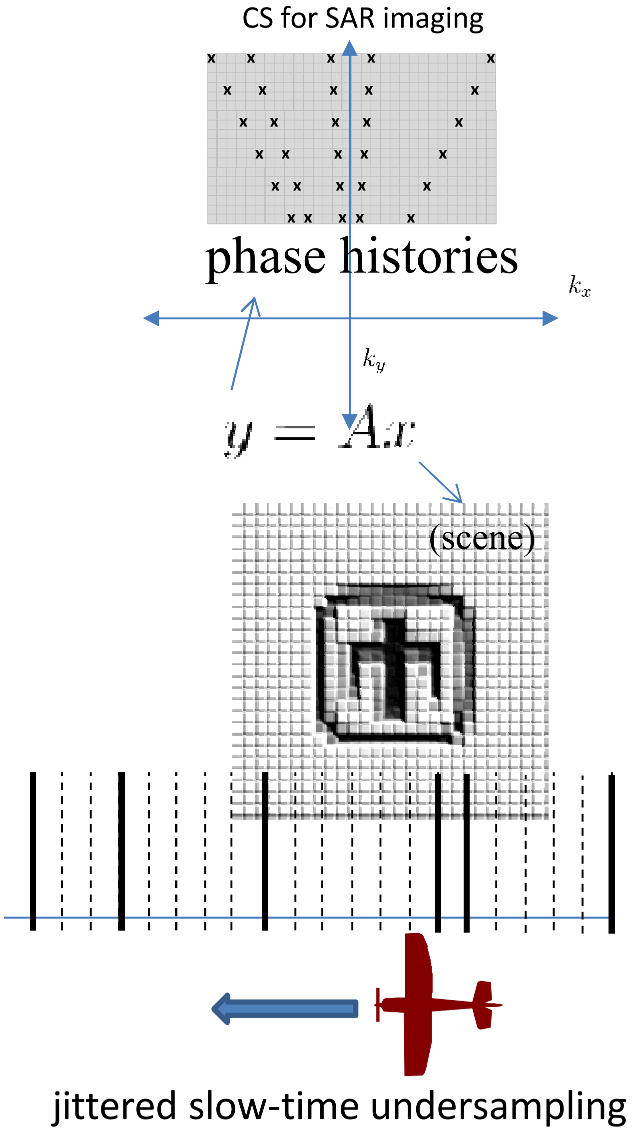
$$\min_x \|x\|_1 \quad \text{s.t.} \quad Ax = y$$

# Results (synthetic scene)

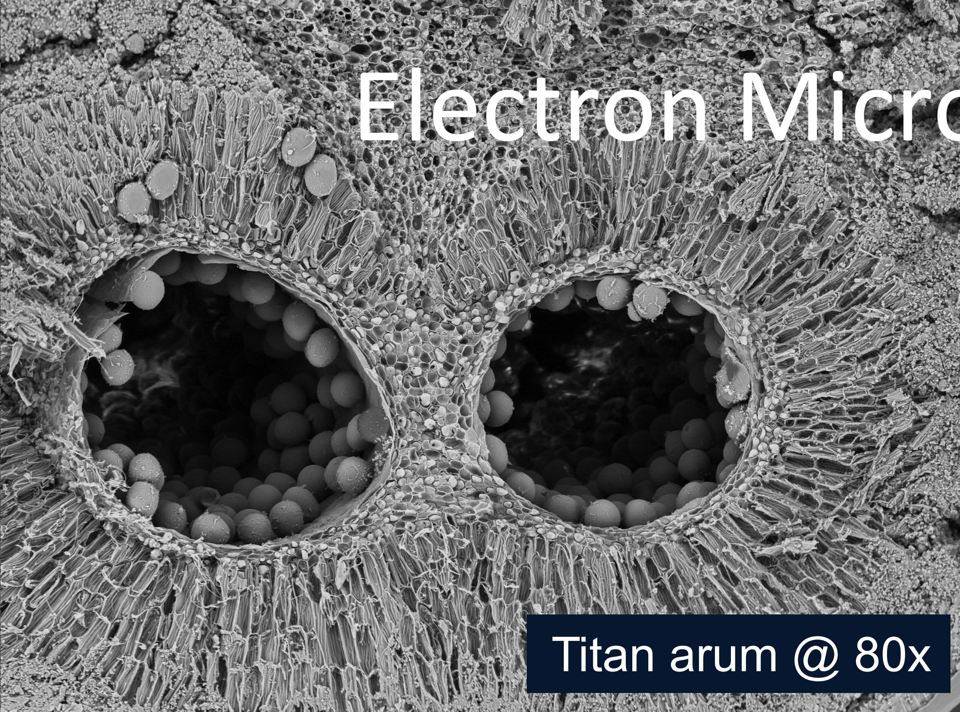



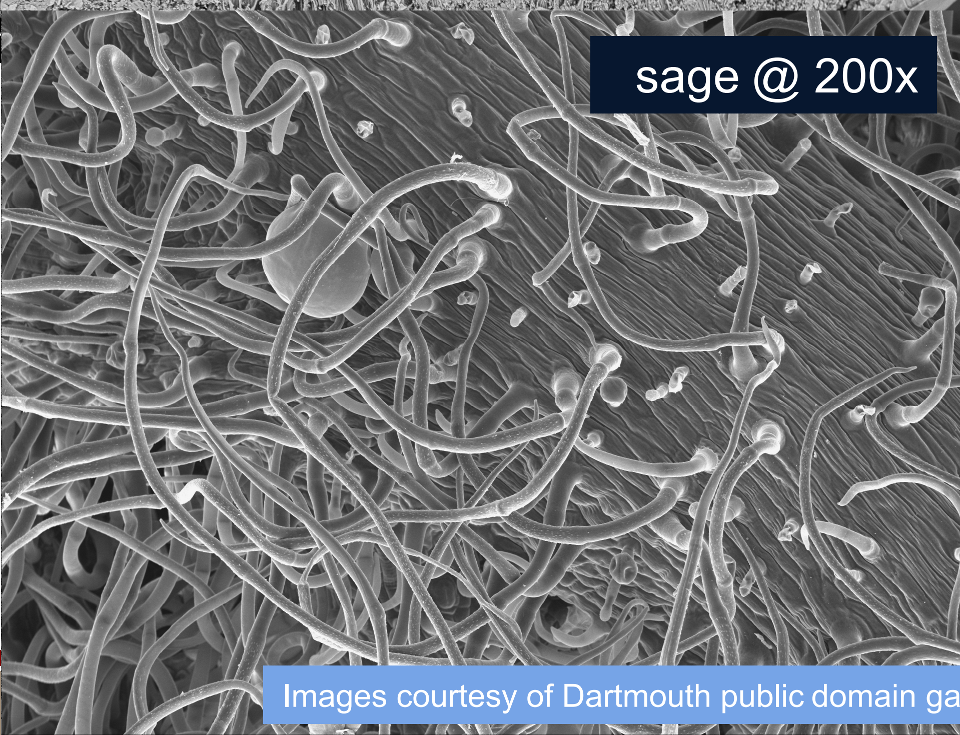



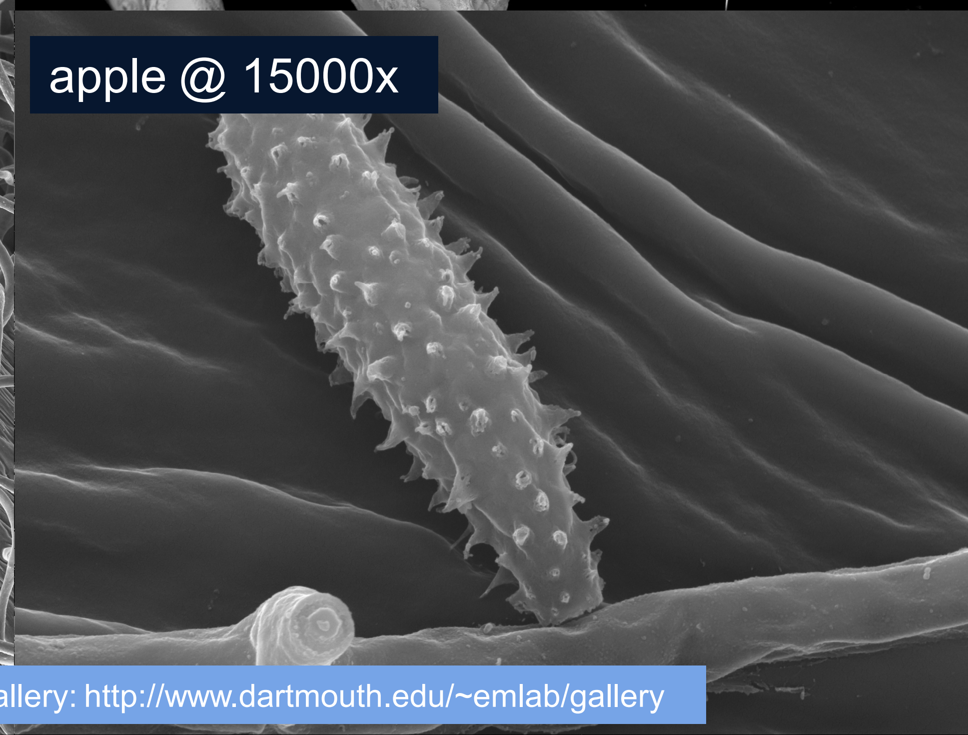



# Electron Microscopy Images





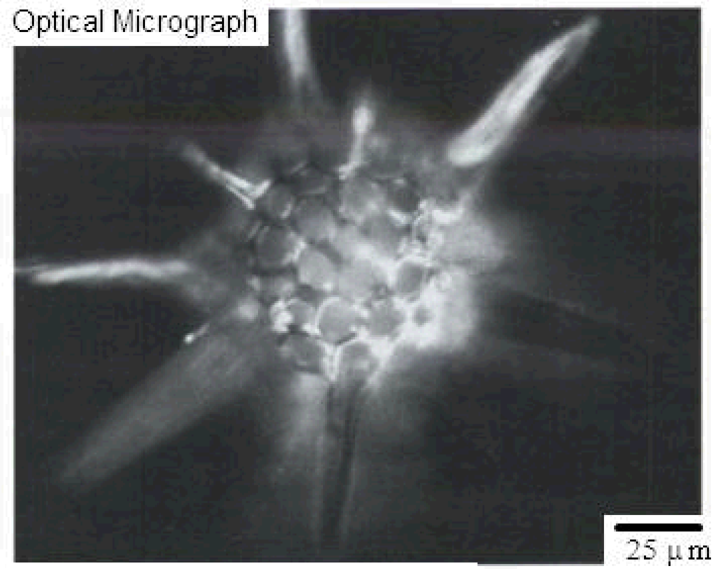



Images courtesy of Dartmouth public domain gallery: http://www.dartmouth.edu/~emlab/gallery

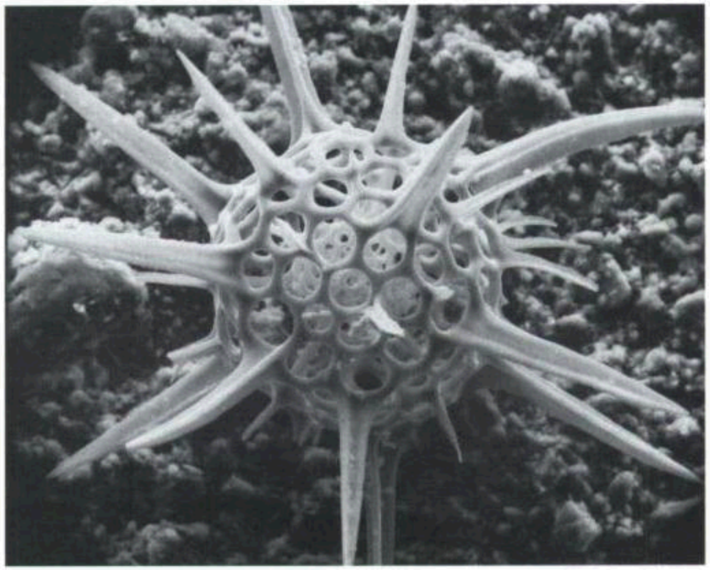
# Why SEM?

**Optical Microscope Image**




Radiolarian (marine organism)

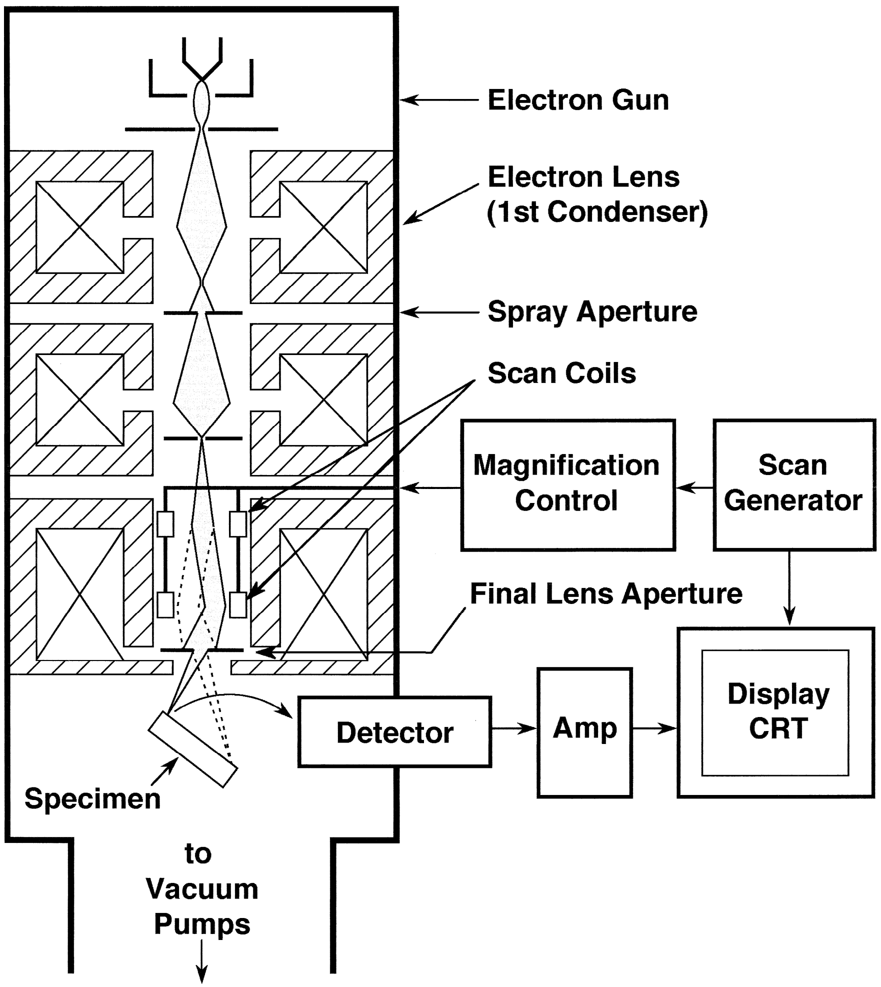
**SEM Image**



- Typical SEMs can resolve ~1 nm features ($10^3$x smaller diff. limit than optical)
- Large depth of focus
- Flexible viewing conditions, e.g., 10x to 500,000x mag

Image source: J. Goldstein, et al., *Scanning Electron Microscopy and X-Ray Microanalysis* (Springer, New York, 2003).

# SEM Electron Column

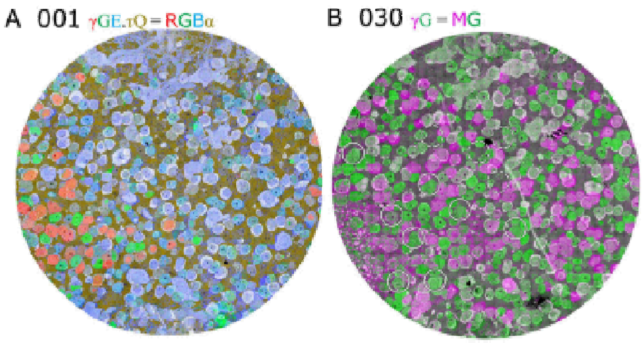



- Electron gun generates electrons
- Electromagnetic condenser lenses and apertures focus electrons into a beam w/ small spot size (~1 nm)
- Scan coils raster beam across sample area to be images
- Detector collects electrons at each point of raster pattern and plots on computer display (typically a single SE/BSE, but there may be other, specialized detectors).

Source: J. Goldstein, et. al, *Scanning Electron Microscopy and X-Ray Microanalysis* (Springer, New York, 2003).

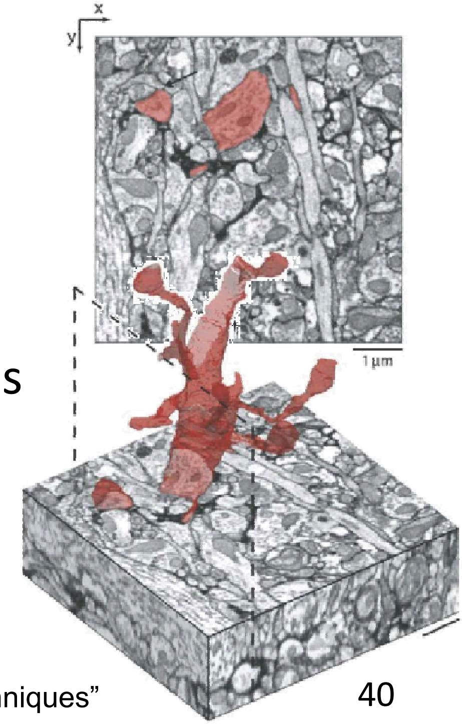
# SNR-limited image collection time

- Rabbit retina connectome (Anderson, et al., 2011)
  - Tissue ~ 0.25 mm in diameter
  - ~2 nm resolution
  - 350,000 image tiles (16.5 TB) in 5 months
  - Automated trans. electron microscope




Anderson, et al., 2011, "Exploring the retinal connectome"

- Mouse brain (Briggman and Denk, 2006)
  - Single cortical column from mouse ~0.1 $mm^3$
  - ~10nm / pixel per 30nm slice
  - Thousands of images ($10^8$ pixels each) over several months
  - Serial block-face SEM




- Many engineering efforts to reduce collection time

(Lichtman et al. @ Harvard)

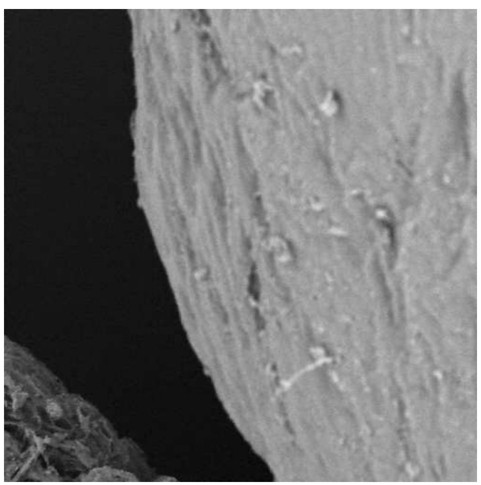
Briggman and Denk, 2006, "Towards neural circuit reconstruction with volume electron microscopy techniques"

# Approach

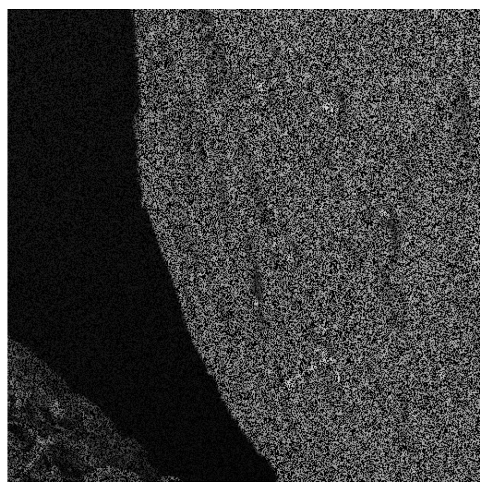
- Visit a random subset of pixel locations
- From $M < N$ measurements, reconstruct
- Compression basis chosen to be block-DCT
  - Good compressibility of SEM images
  - Low mutual coherence

$$\min_{\mathbf{x}} \quad \|\mathbf{\Psi}^T\mathbf{x}\|_1 + \|\nabla\mathbf{x}\|_1$$
$$\text{s.t.} \quad \|\mathbf{y} - \mathbf{\Phi}\mathbf{x}\| \leq \sigma^2$$

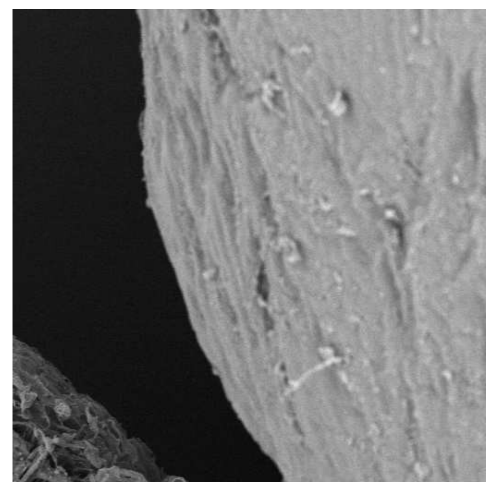
- Total variation regularizer $\|\nabla\mathbf{x}\|_1$ to denoise and promote smoothness between block boundaries



original



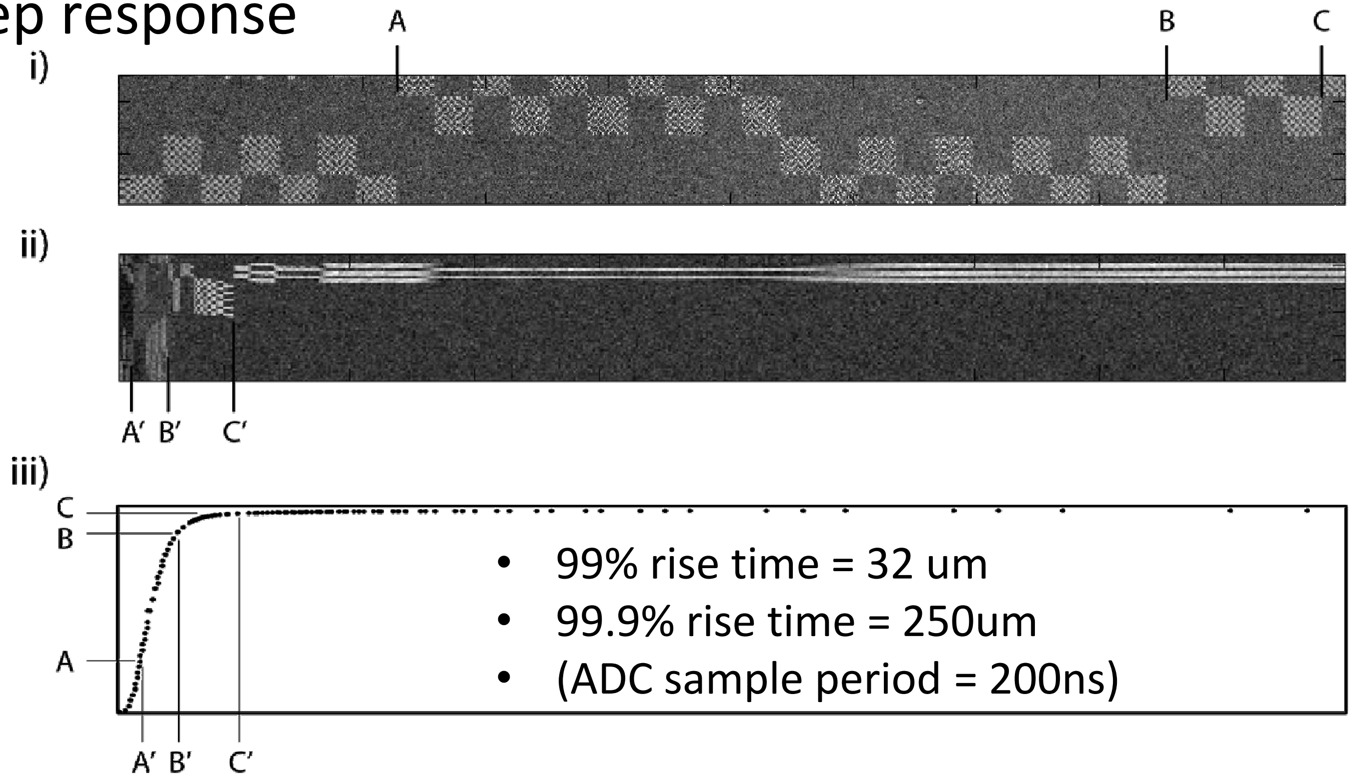
50% samples



reconstruction (36 dB PSNR)

# Scan coil dynamics

- Measure step response




- 99% rise time = 32 um
- 99.9% rise time = 250um
- (ADC sample period = 200ns)

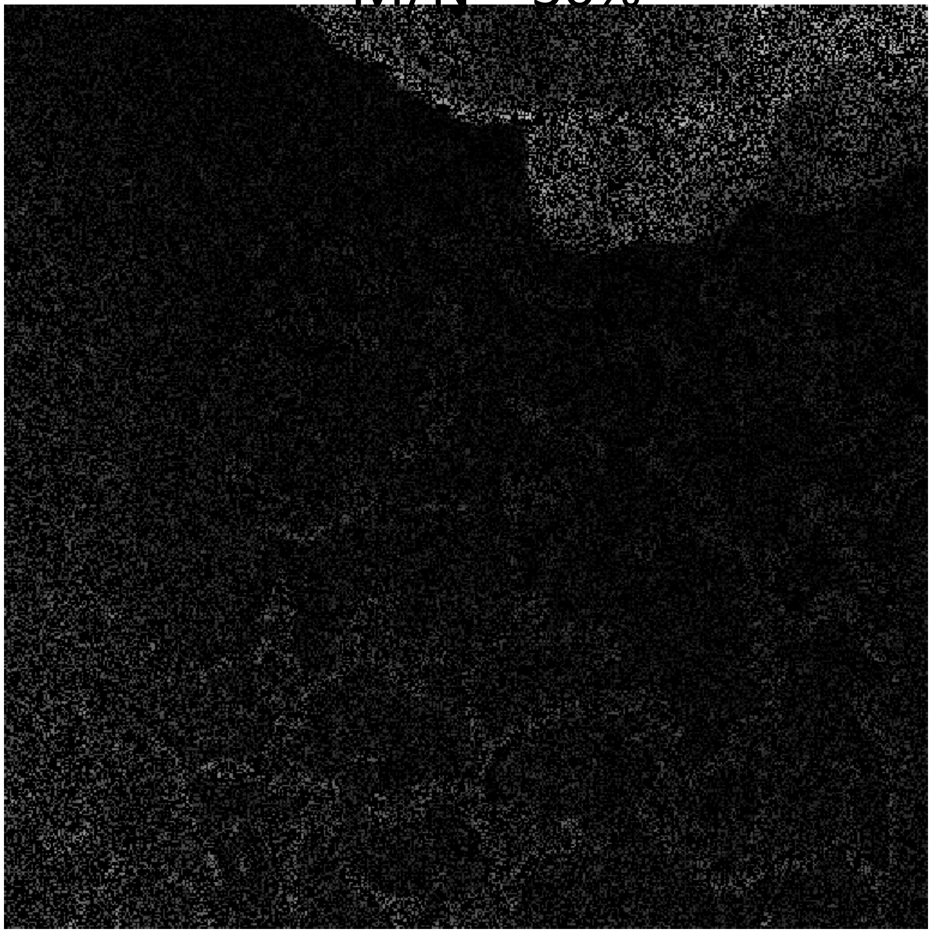
- Linear dynamical model to predict “actual” location

$$\frac{d^5x(t)}{dt^5} = a_0(\hat{x}(t) - x(t)) - a_1\frac{dx(t)}{dt} - a_2\frac{d^2x(t)}{dt^2} - a_3\frac{d^3x(t)}{dt^3} - a_4\frac{d^4x(t)}{dt^4}$$

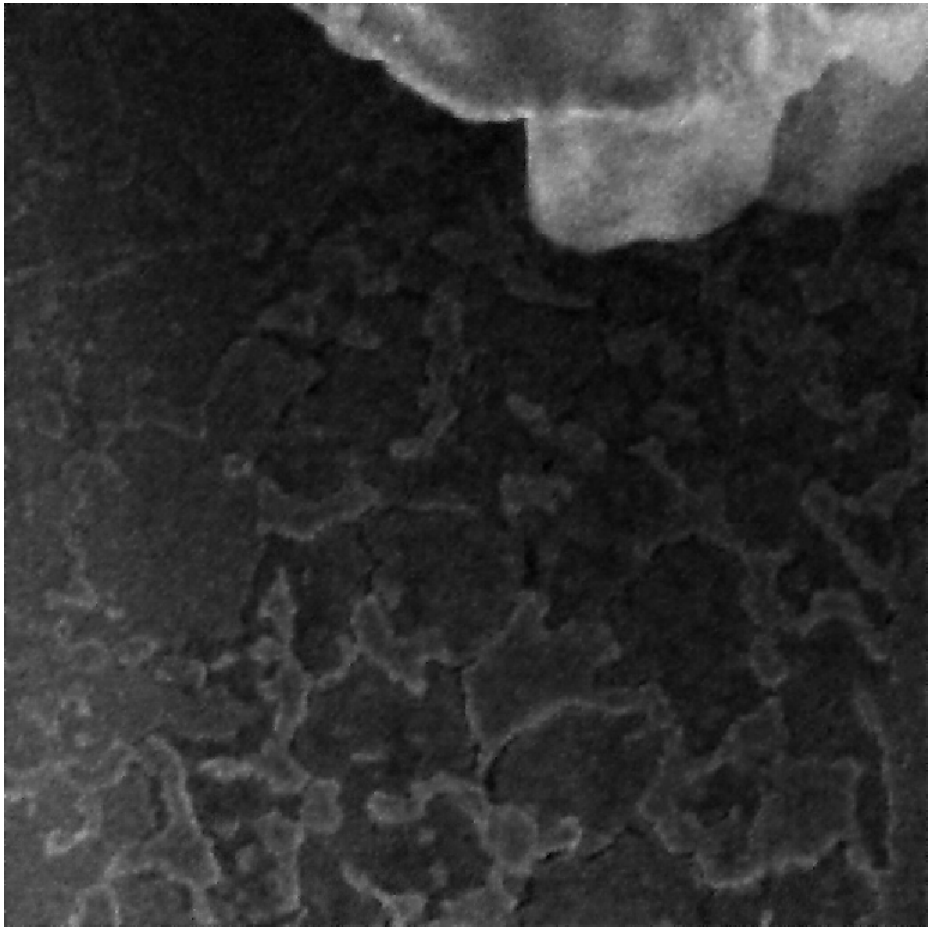
# Simulated: Gibeon meteorite surface

(noiseless simulated recover)

M/N = 30%



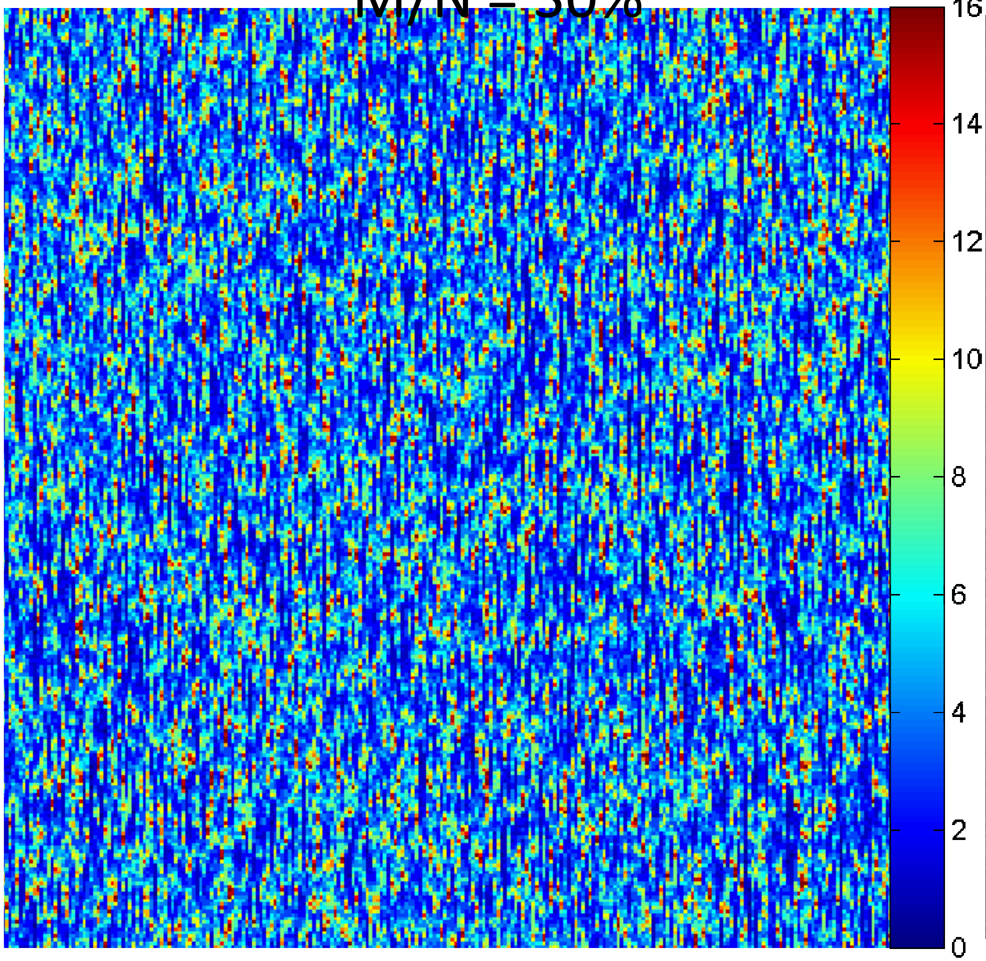
recovered



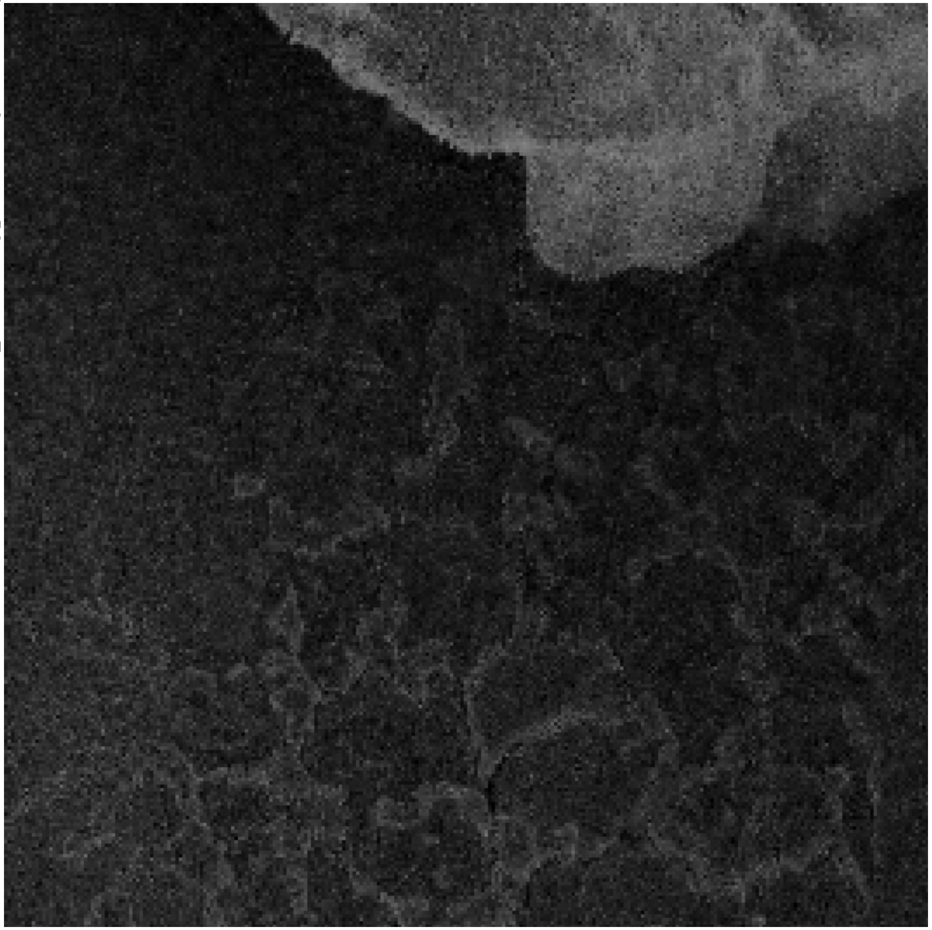
# Actual: Gibeon meteorite surface

(actual measurement location + recovery)

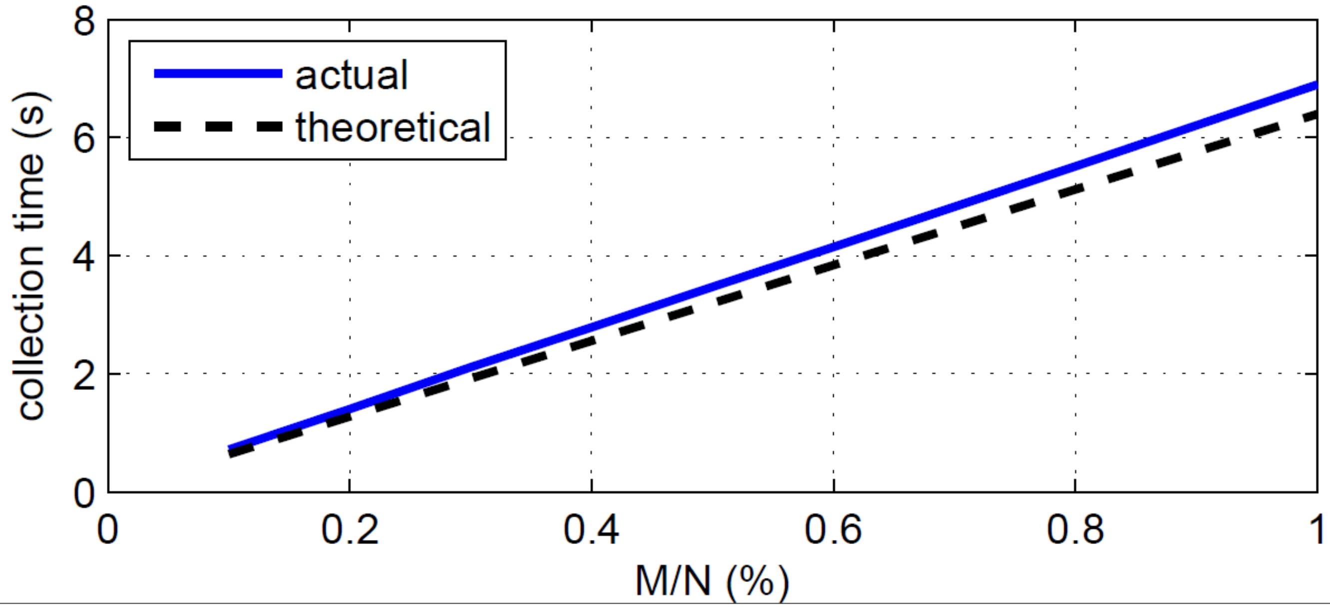
M/N = 30%



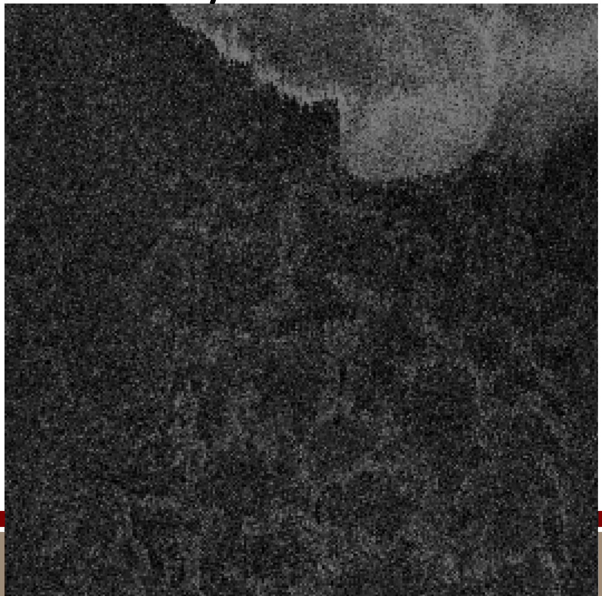

recovered



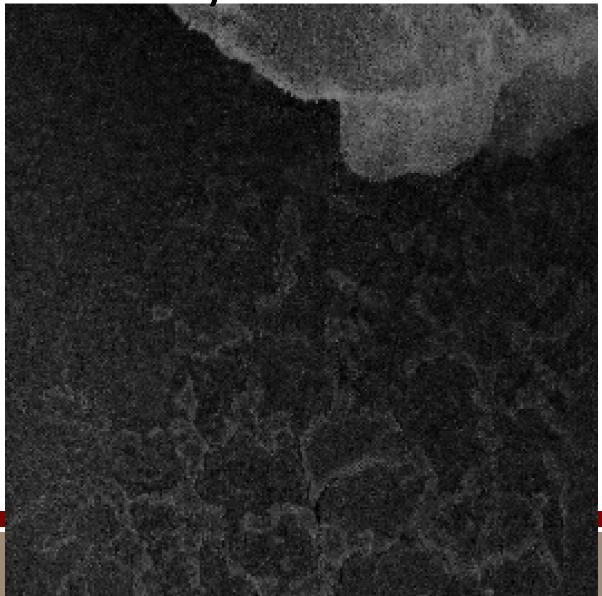
# Undersampling timing results



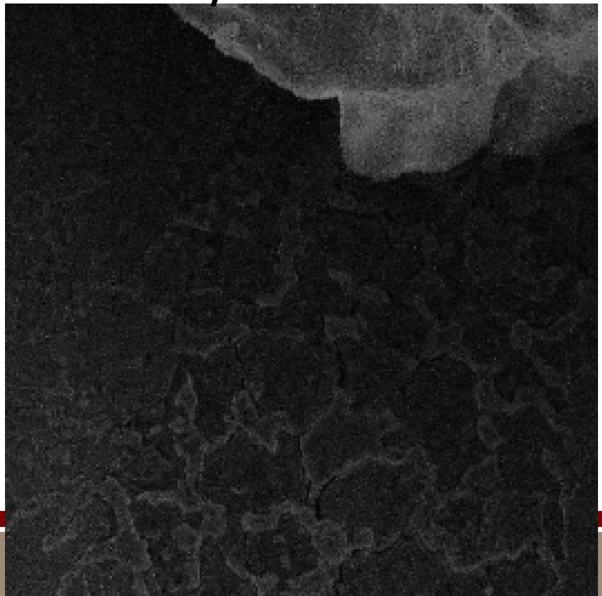

M/N = 10%



M/N = 30%



M/N = 50%



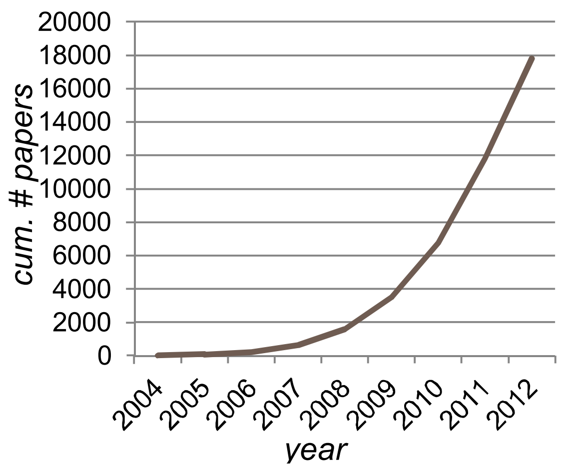
# Compressed Sensing: Hope or Hype?

## Explosion of interest

- Elements of theory known since 1950s (group testing, seismic, wavelets)
- A modern theory "assembled" in 2004
- ~200 papers by 2006; ~18,000 in early 2013

## A general theory that *in some cases* may provide

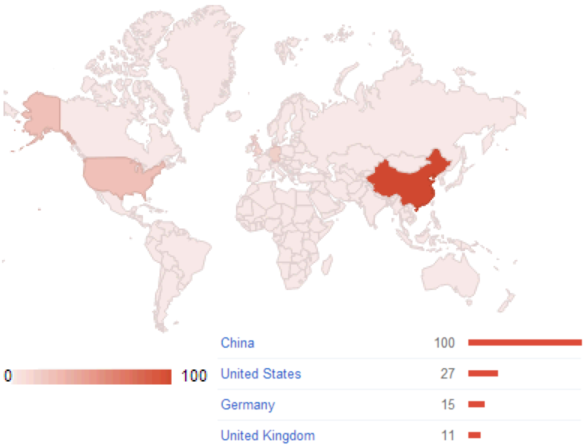
- **Better resolution** (radar images without a matched filter)
- **Faster** measurements (MRI from hours to minutes)
- **Cheaper** sensors (infrared cameras from InView Corp)
- **Lower SWaP** for small platforms (CS SAR on a UAVs)
- **Higher sampling rates** (spectral sensing at currently infeasible BWs)




## Trade for recovery burden, reduced SNR

## Compelling applications are nascent

- Single-pixel **infrared**, **multispectral**, **lidar** cameras
- Compressed **MRI**, **CT** & **ultrasound**
- "Analog-to-Information" converter
- Compressive **UWB communication**
- Compressive sensing **radar** / **SAR** **[Sandia!]**
- Microscopy **[Sandia!]**




*normalized Google search volume by country, 2004-present*