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ADIABATIC TOPOLOGICAL QUANTUM COMPUTING

Chris Cesare, April 6th, 2011

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CQuIC
AQUARIUS™
Adiabatic quantum architectures in ultracold systems
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LABORATORY DIRECTED RESEARCH & DEVELOPMENT

THE UNIVERSITY of NEW MEXICO

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Sandia National Laboratories

We would like to do a quantum computation with as much "natural" robustness as possible. Topological quantum computing and adiabatic quantum computing both have their own flavors of robustness. Can we combine them?

TQC is usually formulated in continuous spacetimes. Can we perform TQC on a lattice? How?

TQC operations are always assumed adiabatic. Can we make this assumption explicit? How?

Stabilizer Quantum Codes

↓

Topological Quantum Codes

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Adiabatic Topological Quantum Computing

Stabilizer Quantum Codes

↓

Topological Quantum Codes

↓

Adiabatic Topological Quantum Computing

Preliminaries

- **Qu**antum **Bit** (qubit) – a two state quantum system

$$|\psi\rangle = a|0\rangle + b|1\rangle\,, \quad |a|^2 + |b|^2 = 1$$

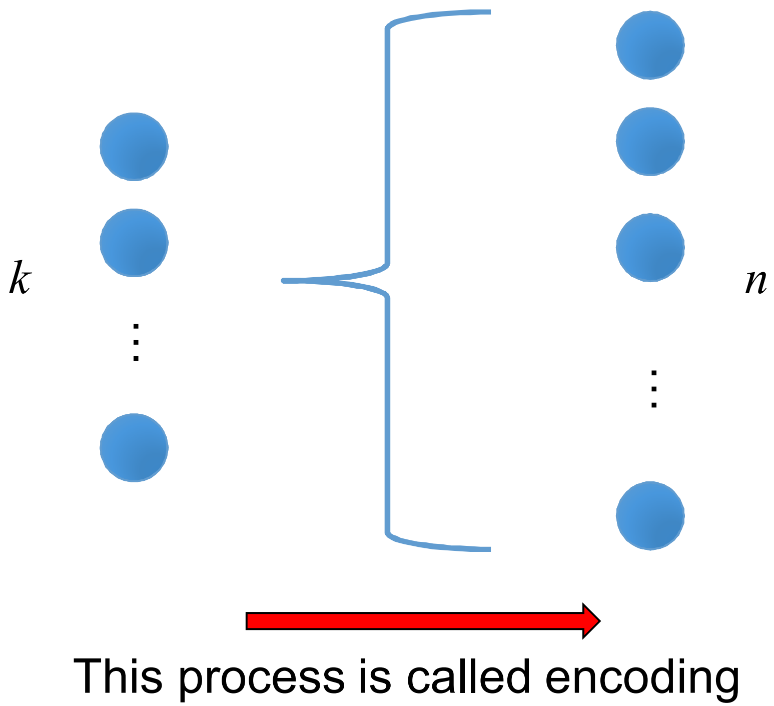
- The basis of the qubit Hilbert space is the span of σ_Z eigenstates. I will use Z instead of σ_Z, X instead of σ_X, etc.

$$Z|0\rangle = |0\rangle$$
$$Z|1\rangle = -|1\rangle$$

$$X|0\rangle = |1\rangle$$
$$X|1\rangle = |0\rangle$$

Quantum Codes

- Qubits are often fragile things. How can we combat this?
- The game: store k qubits in n qubits, where $n > k$



- How?

Stabilizer Quantum Codes

- One way: stabilizer codes
  - Pick a 2^k-dimensional subspace of the 2^n-dimensional total space; call this the **codespace** $\mathcal{C}$
  - Define the **stabilizer group** $\mathcal{S}$ as those Pauli group elements that act trivially on the codespace

$$S_i \in \mathcal{S}, \;\; |\psi\rangle \in \mathcal{C} \iff S_i |\psi\rangle = |\psi\rangle$$

  - These form a group since

$$S_i S_j |\psi\rangle = S_i |\psi\rangle = |\psi\rangle \implies S_i S_j \in \mathcal{S}$$

  - Additionally, since $[S_i, S_j] = 0$, the stabilizer group is Abelian

Gottesman Ph.D. Thesis
arXiv:quant-ph/9705052

Stabilizer Quantum Codes

The codespace

$$|\psi\rangle \in \mathcal{C}$$

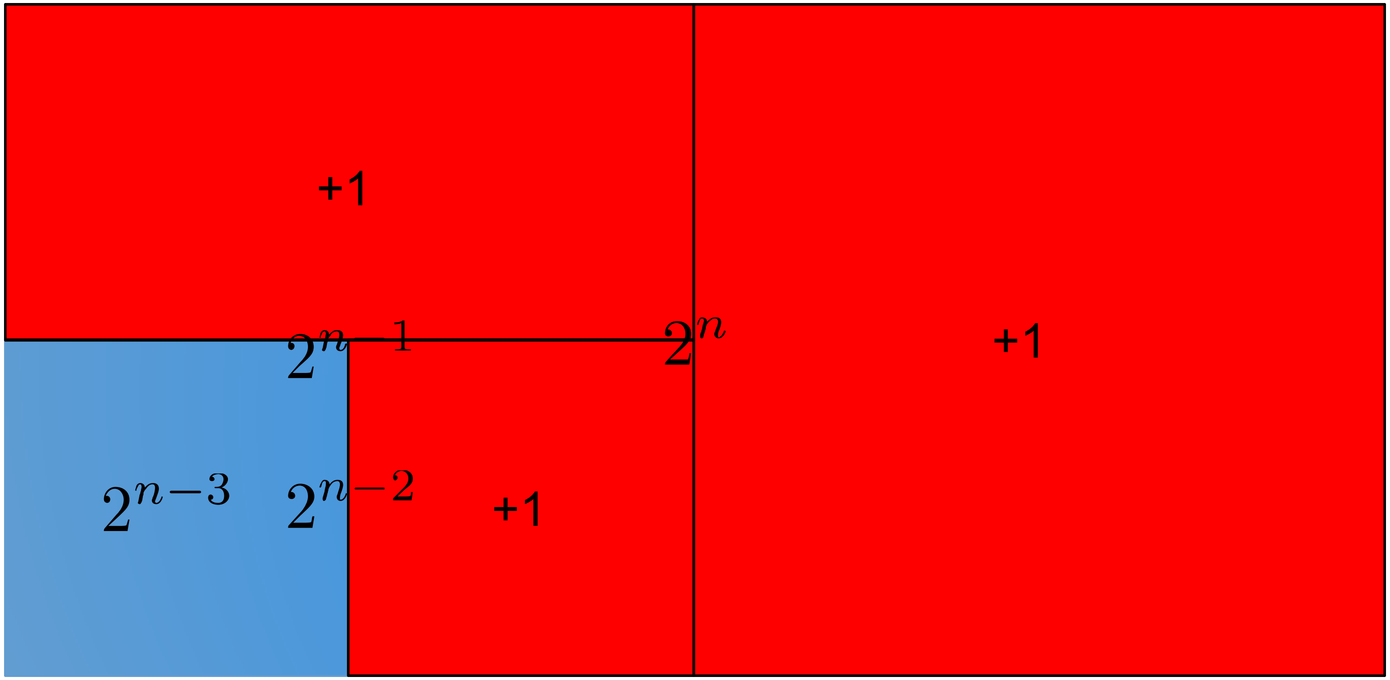
$$\Longleftrightarrow \quad S_i |\psi\rangle = |\psi\rangle$$

Stabilizer group element

- States in the codespace are +1 eigenstates of stabilizer group elements
- The stabilizer group has $n-k$ generators (“checks”)
- Eigenvalues of stabilizer generators are ±1, so each specified stabilizer generator halves the Hilbert space

Stabilizer Quantum Codes

- Eigenvalues are ±1, so each stabilizer generator halves the Hilbert space



Stabilizer Quantum Codes

- How do the stabilizer generators detect errors?
  - Note that elements of the Pauli group **either commute or anti-commute**
  - So, pick an operator O, $|\psi\rangle \in \mathcal{C}$

$$O|\psi\rangle = OS_i|\psi\rangle$$

commute

$$S_i\left(O|\psi\rangle\right) = O|\psi\rangle$$

Defining property of the codespace $\mathcal{C}$!

anti-commute

$$-S_i\left(O|\psi\rangle\right) = O|\psi\rangle$$

$$\implies S_i\left(O|\psi\rangle\right) = -\left(O|\psi\rangle\right)$$

$$\implies O|\psi\rangle \notin \mathcal{C}$$

This means O takes $|\psi\rangle$ out of the codespace. We call this a **detectable error**.

Stabilizer Quantum Codes

- So if we *measure* each stabilizer generator, there are two cases
  1. Measurement result is +1; two possibilities
     - O is in the stabilizer group and thus **a product of stabilizer generators**
     - O is not a product of stabilizer generators and we call it a logical operator; this is **an undetectable error**

$$S_i \left(O \left| \psi \right\rangle \right) = \left(O \left| \psi \right\rangle \right)$$

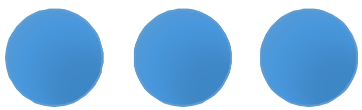
  2. Measurement result is −1
     - O is a **detectable error**, and we may be able to recover

$$S_i \left(O \left| \psi \right\rangle \right) = - \left(O \left| \psi \right\rangle \right)$$

Stabilizer Quantum Codes: An Example

Encoding:
$$|0\rangle \rightarrow |000\rangle$$
$$|1\rangle \rightarrow |111\rangle$$

3 qubits



Stabilizer generators

$Z \otimes Z \otimes I$	S_1
$I \otimes Z \otimes Z$	S_2

Consider $O = X \otimes I \otimes I$

Is O a detectable error?

$$\{O, S_1\} = 0$$
$$[O, S_2] = 0$$

$\Longrightarrow$ O is a detectable error, and it is detected by S_1. Additionally, O is correctable in this case.

$$O\,|000\rangle = |100\rangle$$

$$S_1\,|100\rangle = -\,|100\rangle$$
$$S_2\,|100\rangle = |100\rangle$$

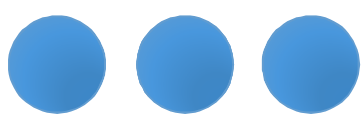
Stabilizer Quantum Codes

- **How do we operate on the encoded information?**
- From the previous example

Encoding:

$$|0\rangle \to |000\rangle \qquad S_1 = Z \otimes Z \otimes I$$
$$|1\rangle \to |111\rangle \qquad S_2 = I \otimes Z \otimes Z$$

- Logical operators commute with all stabilizer generators but are not in the stabilizer group



$$X_L \qquad X \otimes X \otimes X$$
$$Z_L \qquad Z \otimes I \otimes I$$

There is actually some freedom in these choices, since any product $X_L S_i$ or $Z_L S_i$ will also satisfy the conditions for logical operators

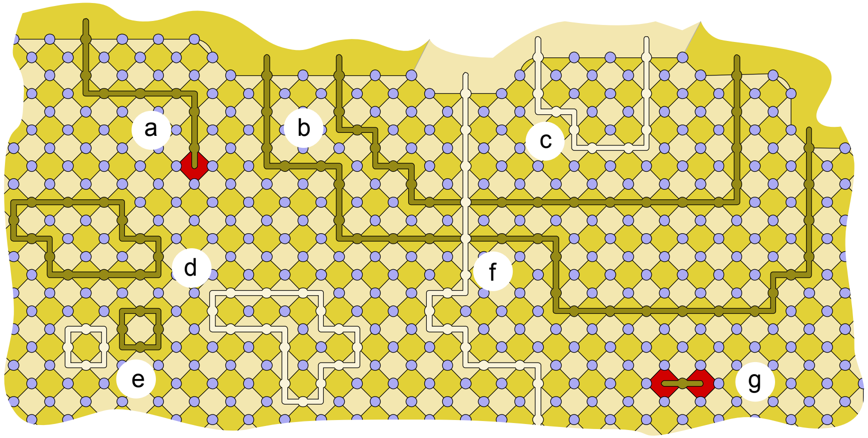
Stabilizer Quantum Codes

Topological Quantum Codes

Adiabatic Topological Quantum Computing

Topological Quantum Codes

- Topological codes are stabilizer codes with some nice properties
  - The stabilizer generators are local
  - The logical operators are topological in nature (nonlocal). This makes it hard for the environment to access the information.
- The codespace dimension depends solely on the topology



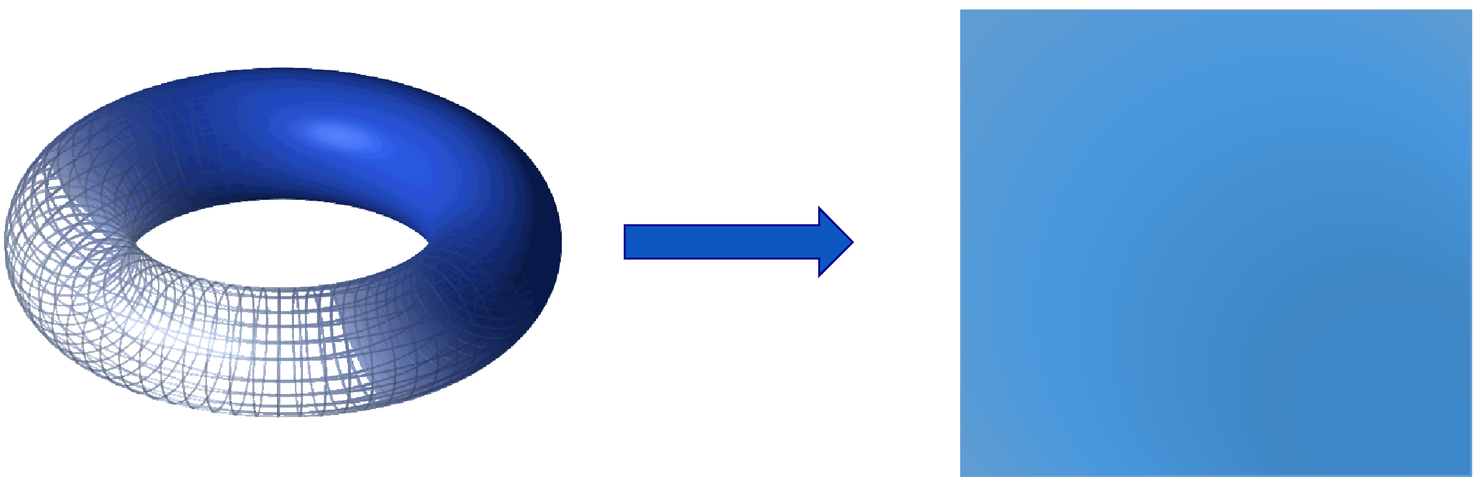
Bombin and Martin-Delgado, Topological Quantum Distillation, Phys.Rev.Lett. 97 (2006) 180501

Kitaev, Fault-Tolerant Quantum Computing with Anyons, Annals Phys. 303 (2003) 2-30

Bombin and Martin-Delgado, Quantum Measurements and Gates by Code Deformation, J. Phys. A: Math. Theor. 42 (2009) 095302

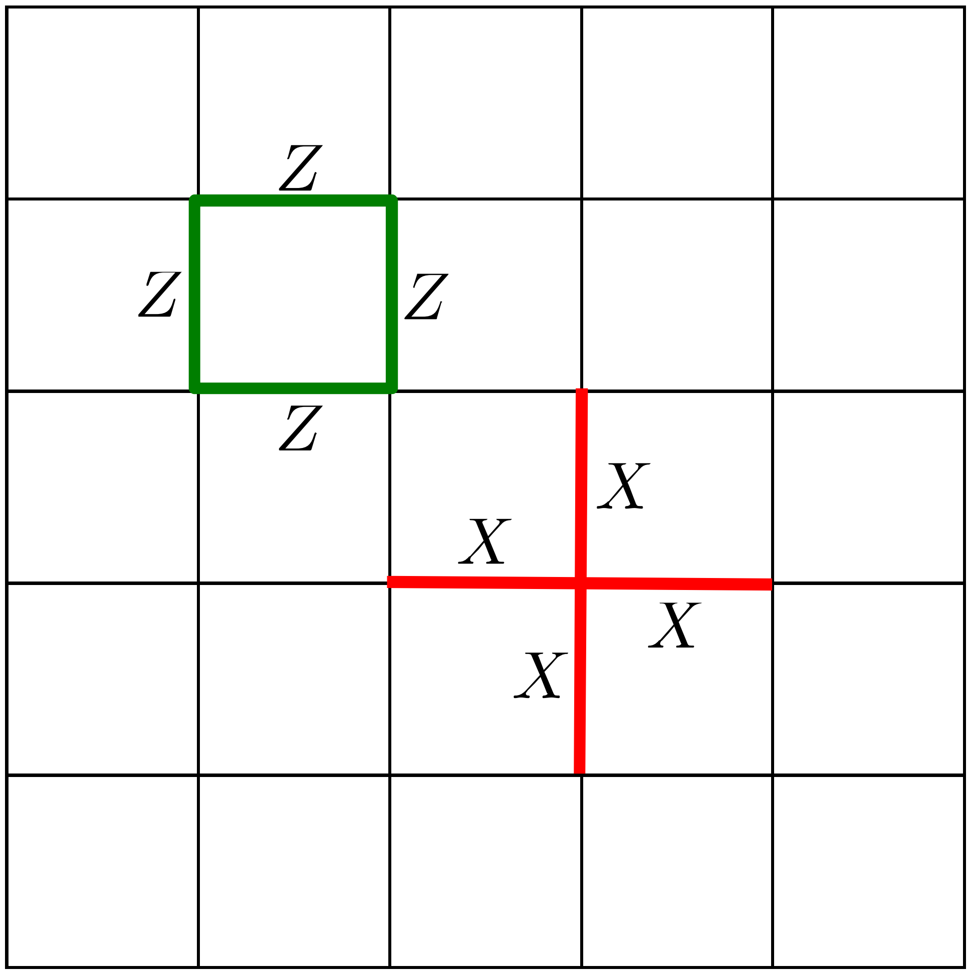
The Toric Code

- The Toric Code was originally defined on a lattice on the (surprise!) torus
- However, we will employ the planar version, which is defined on a lattice in the plane



Kitaev, Fault-Tolerant Quantum Computing with Anyons, Annals Phys. 303 (2003) 2-30

The Toric Code Stabilizer Generators



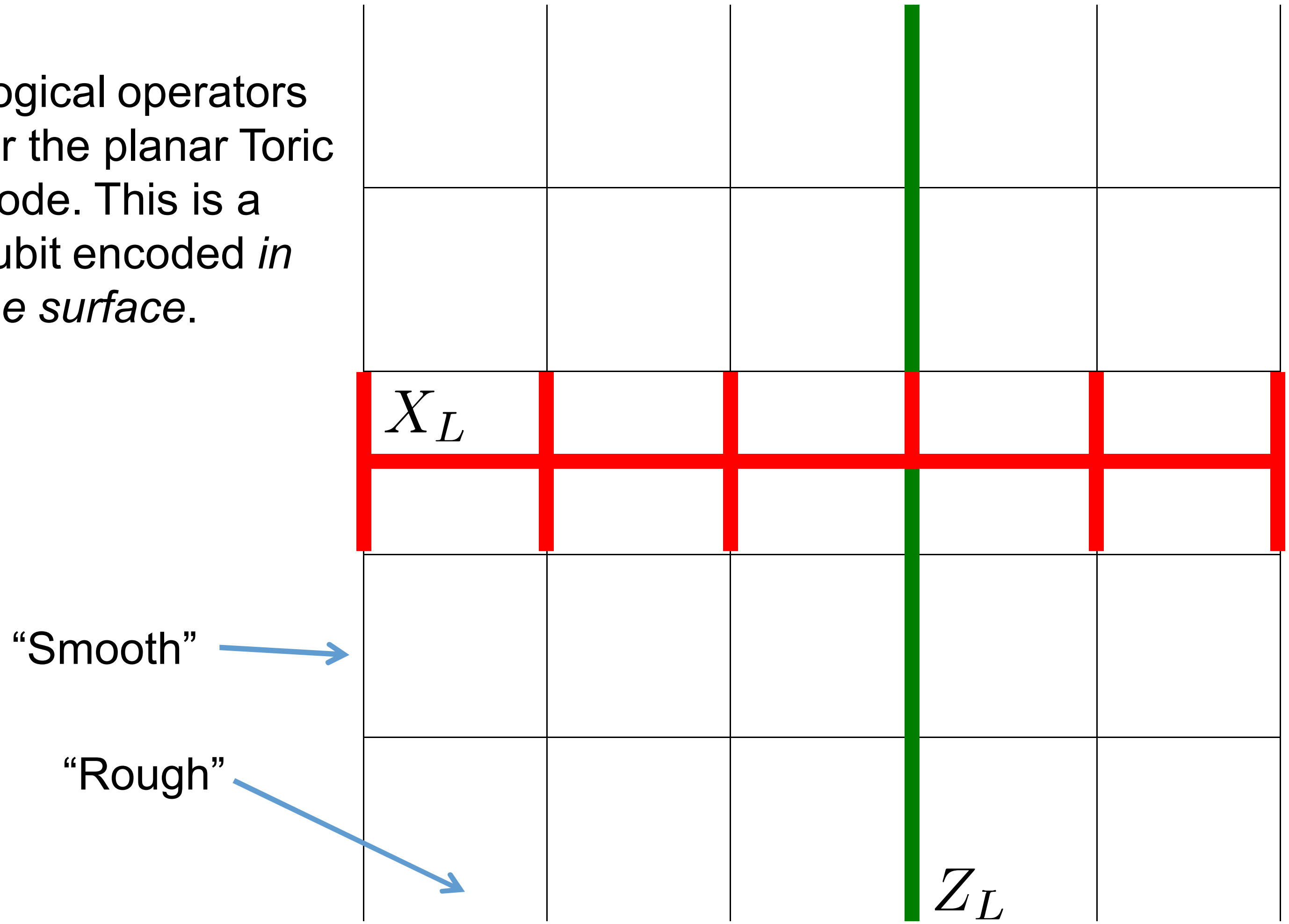
Edges: qubits

Faces: Z-type stabilizer generator

Vertices: X-type stabilizer generator

The Toric Code Logical Operators

Logical operators for the planar Toric Code. This is a qubit encoded *in the surface*.



Face checks: 25

Vertex checks: 24

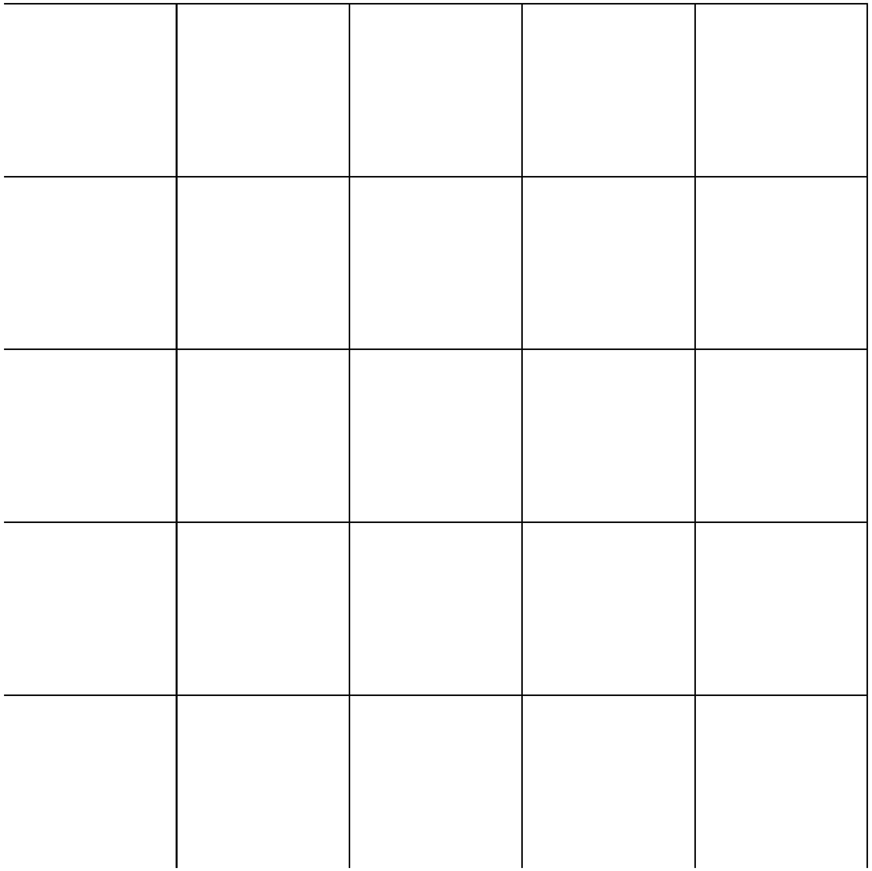
Qubits: 50

$$\frac{2^{50}}{2^{49}} = 2$$

This code has one logical qubit.

Topological Codes II: The Toric Code

- What happens if we change the boundary conditions?



Face checks: 25

Vertex checks: 35

Qubits: 50

$$\frac{2^{50}}{2^{50}} = 1$$

So with these boundaries, no qubits are stored in the surface. We will employ surfaces like these. Where will the qubits come from?

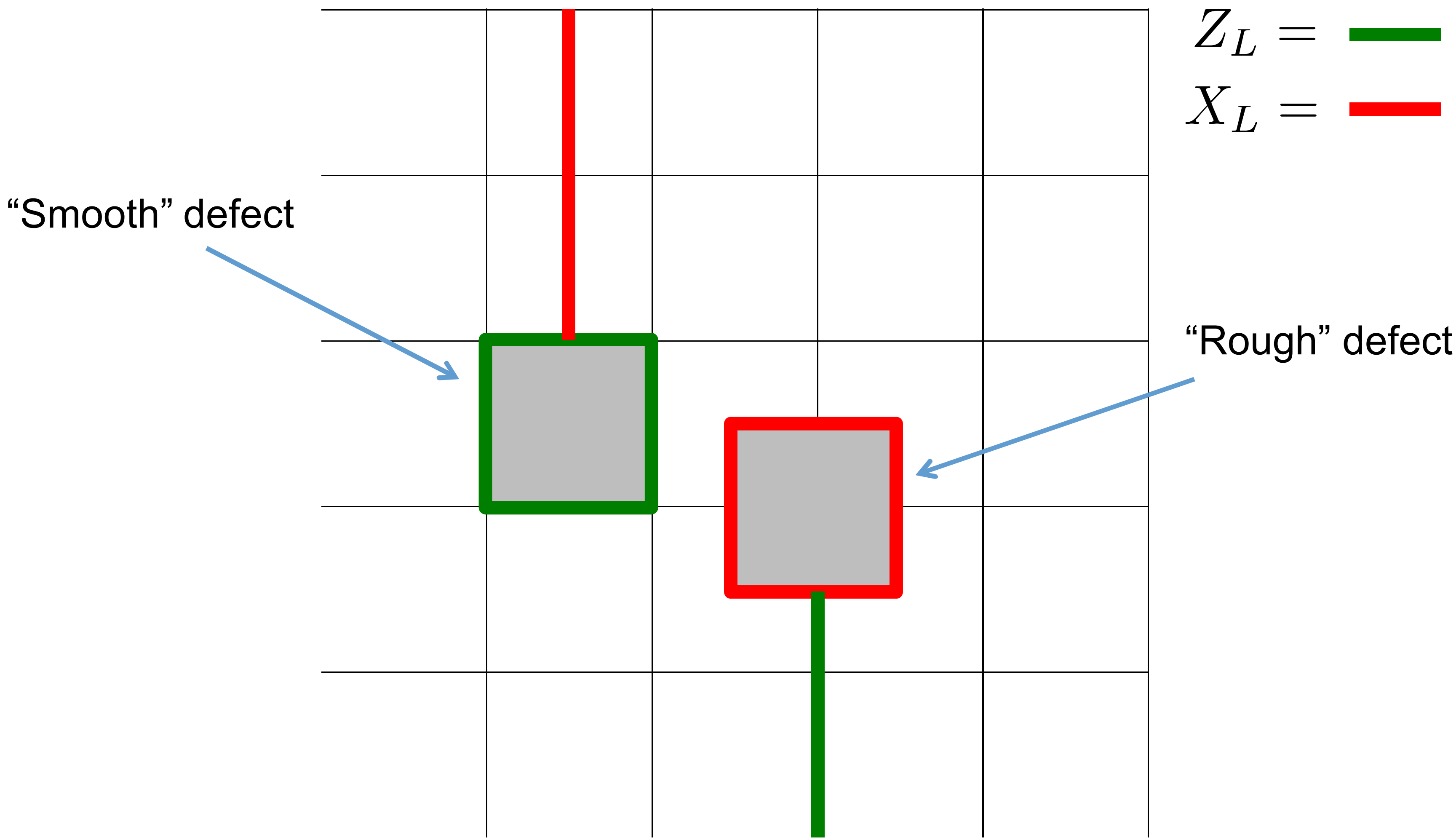
Topological Codes II: The Toric Code

- We will encode qubits in *defects* in the code.
- These defects will be introduced by removing some of the stabilizer generators.
- Removing generators has the opposite effect of introducing them: each generator removed doubles the dimension of the codespace.

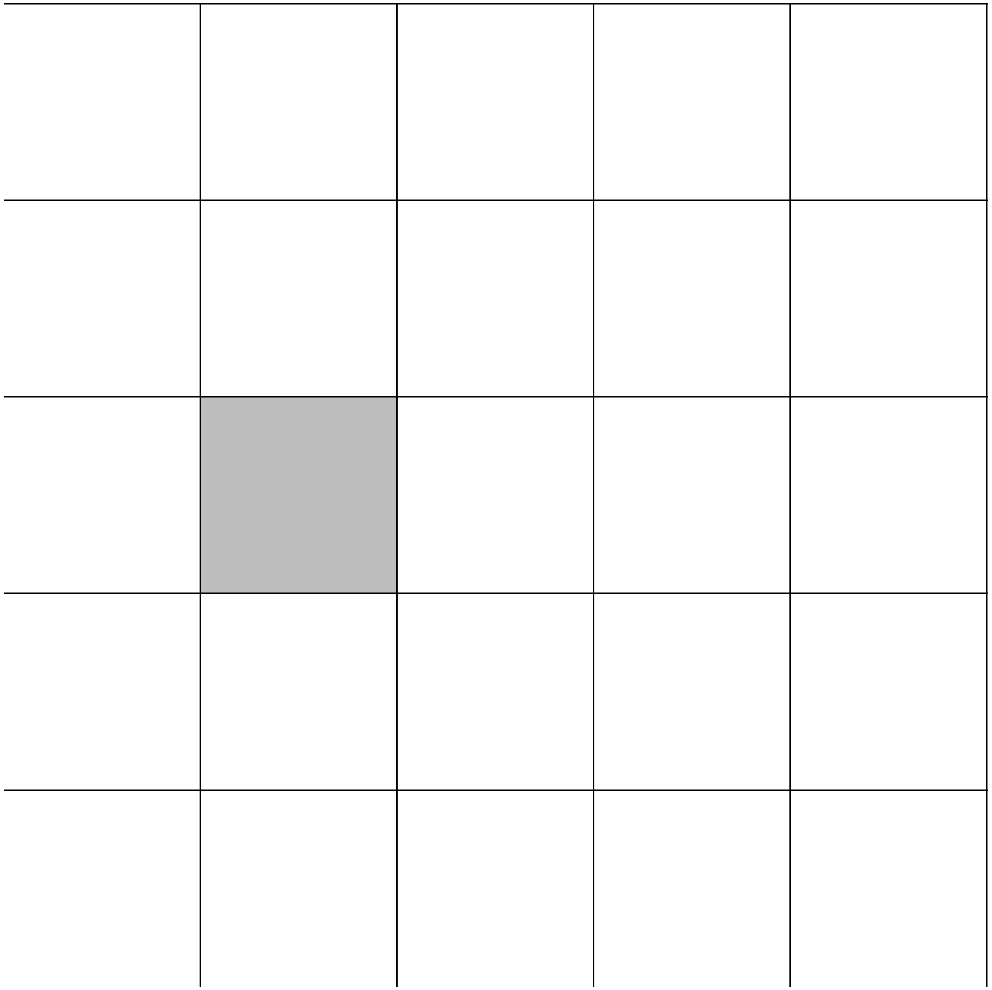
Raussendorf et. al., A fault-tolerant one-way quantum computer, Annals of Physics 321, 2242 (2006)

Fowler et. al., High threshold universal quantum computation on the surface code Phys. Rev. A 80, 052312 (2009)

Toric Code Defects



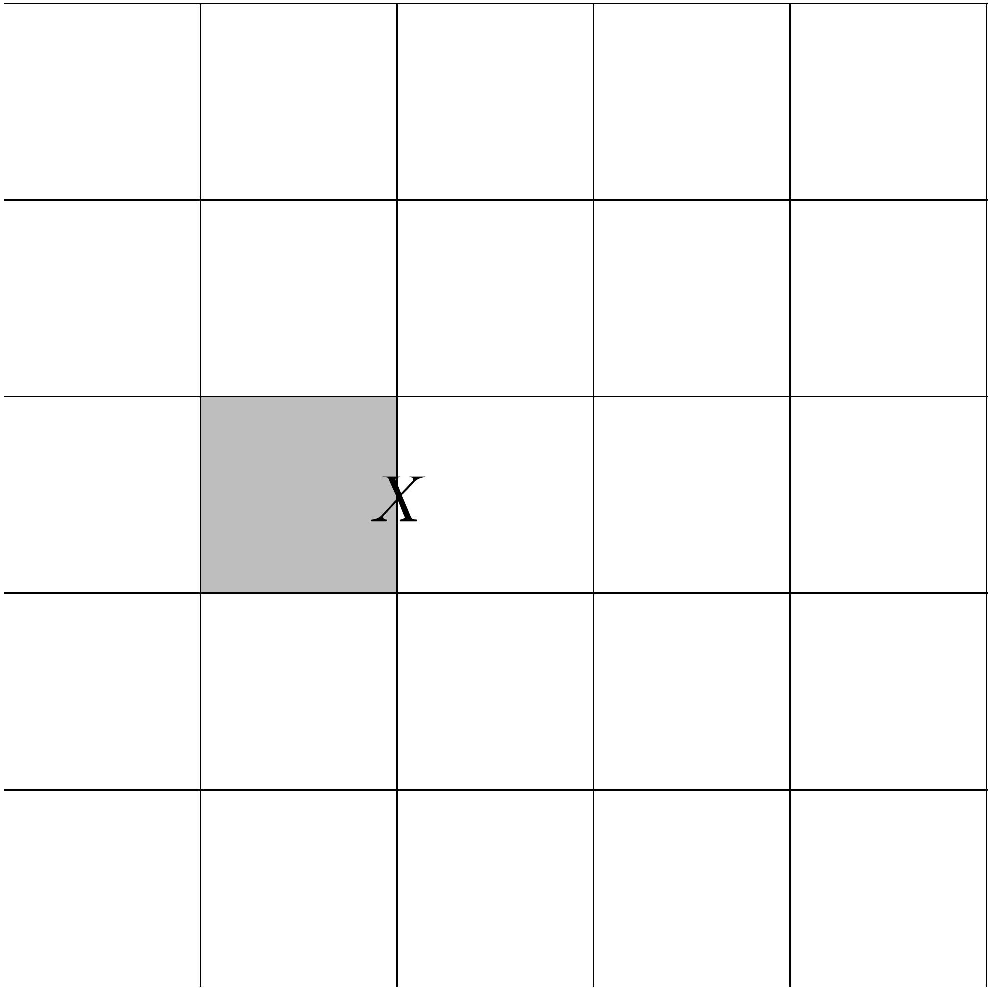
Toric Code Defect Deformation



It is possible to move a defect by measurements.

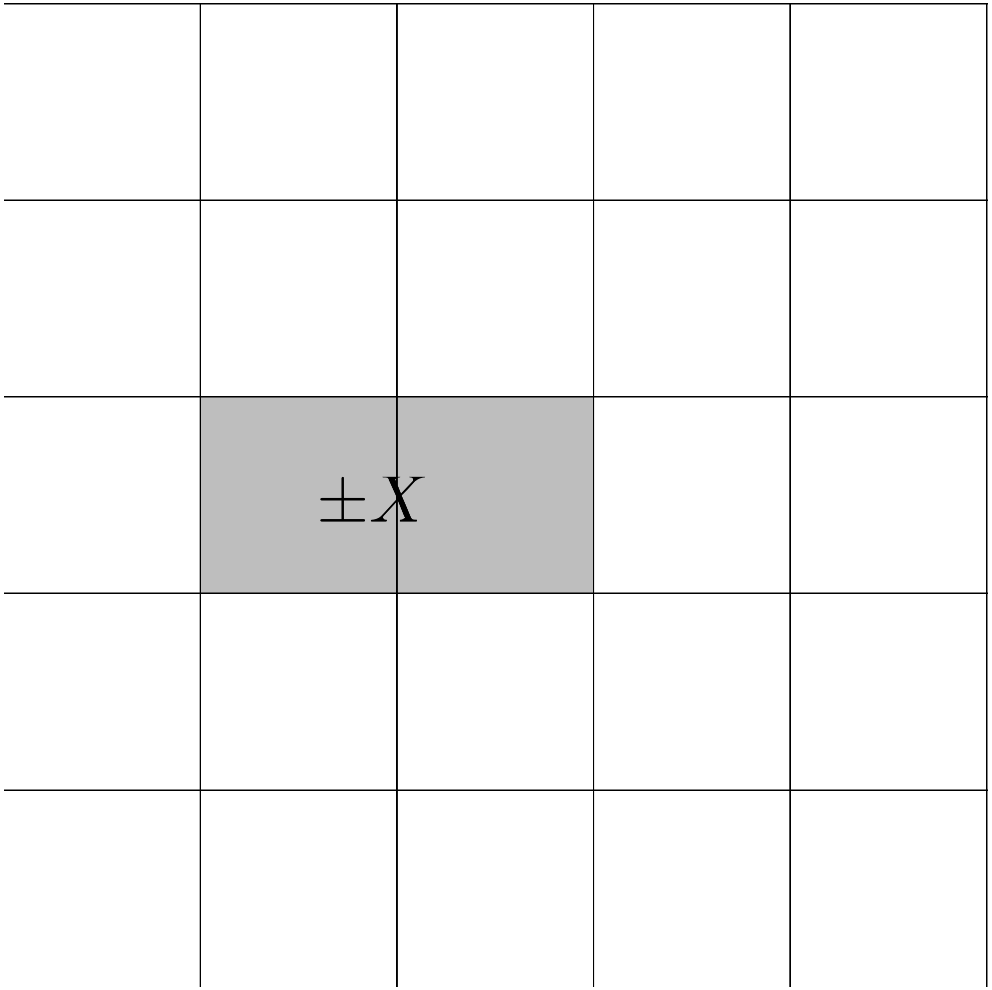
The surface begins with a smooth defect at a particular location.

Toric Code Defect Deformation



We begin by measuring X on the edge in the direction we want to move the defect.

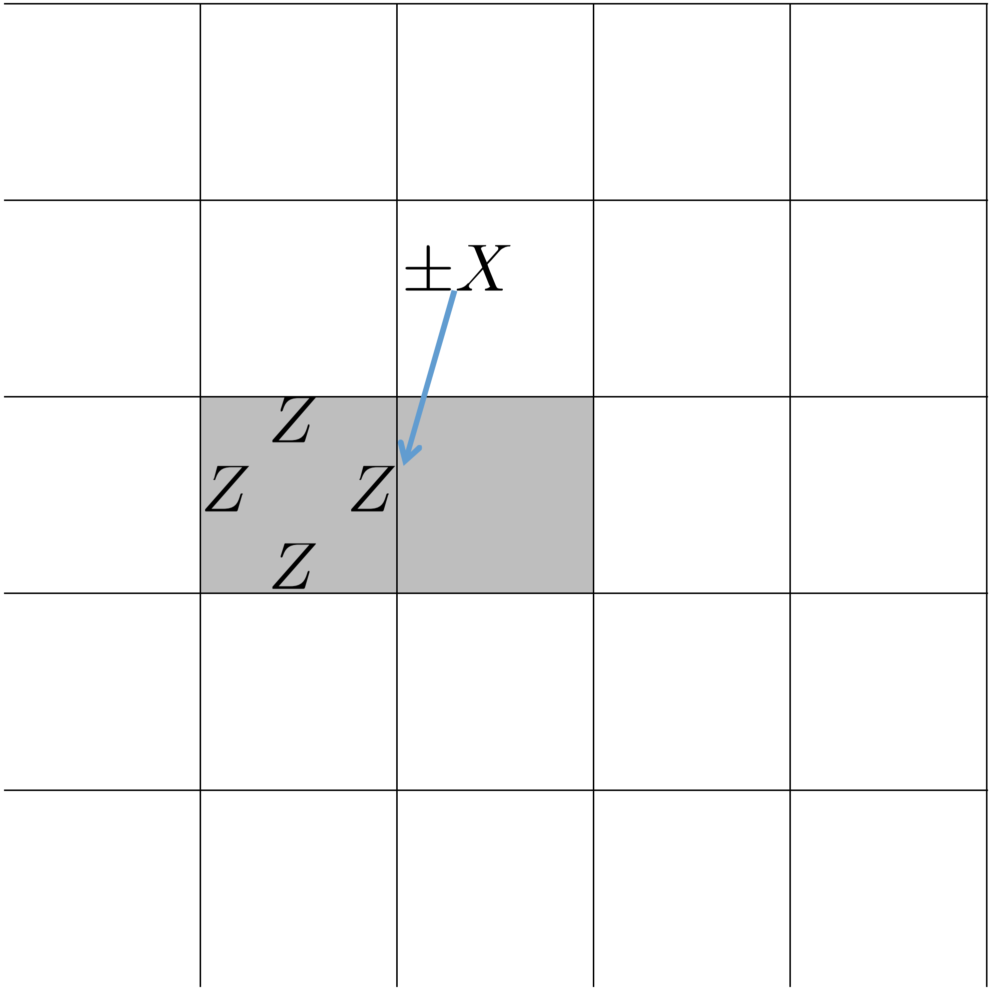
Toric Code Defect Deformation



The measurement result is ±1.

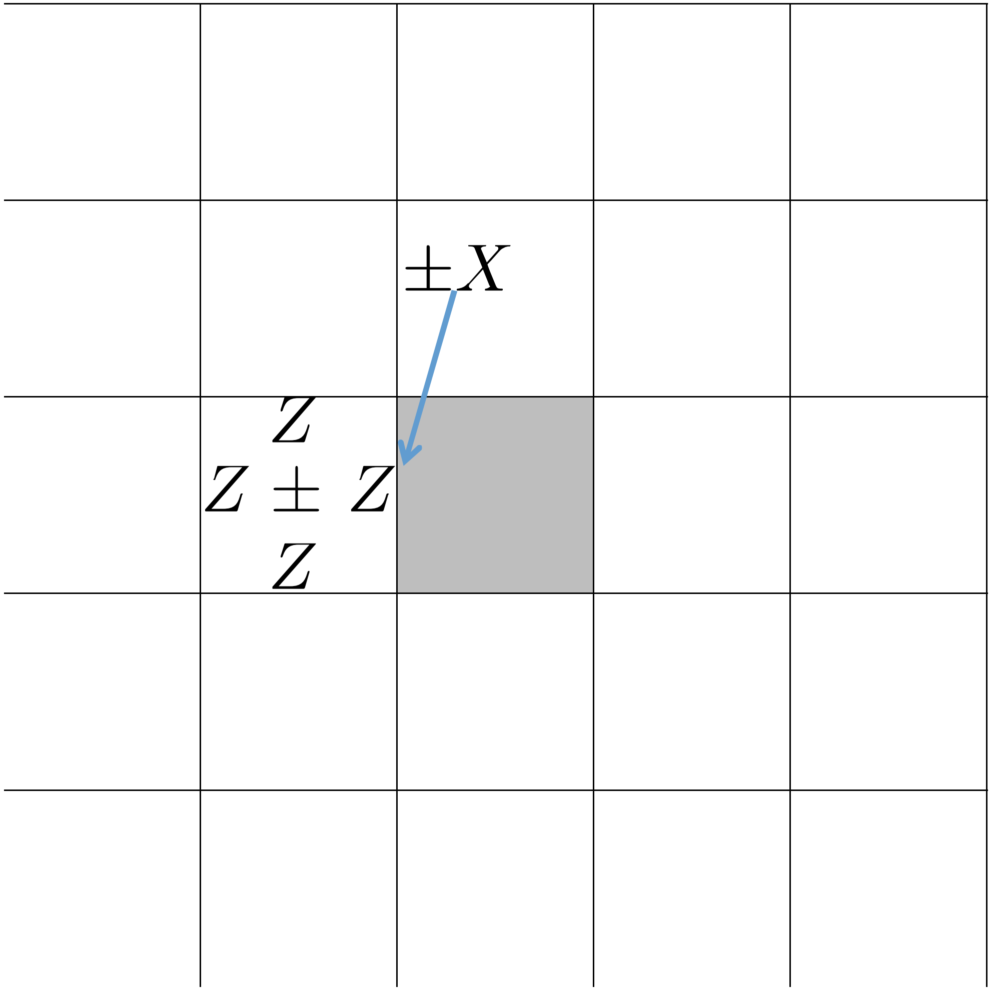
However, this single X measurement anti-commutes with the neighboring face check. Thus the face check is removed from the set of stabilizer generators and replaced with $\pm X$

Toric Code Defect Deformation



Next, we want to reintroduce the face check that was missing originally. This requires measuring $Z \otimes Z \otimes Z \otimes Z$ around the face.

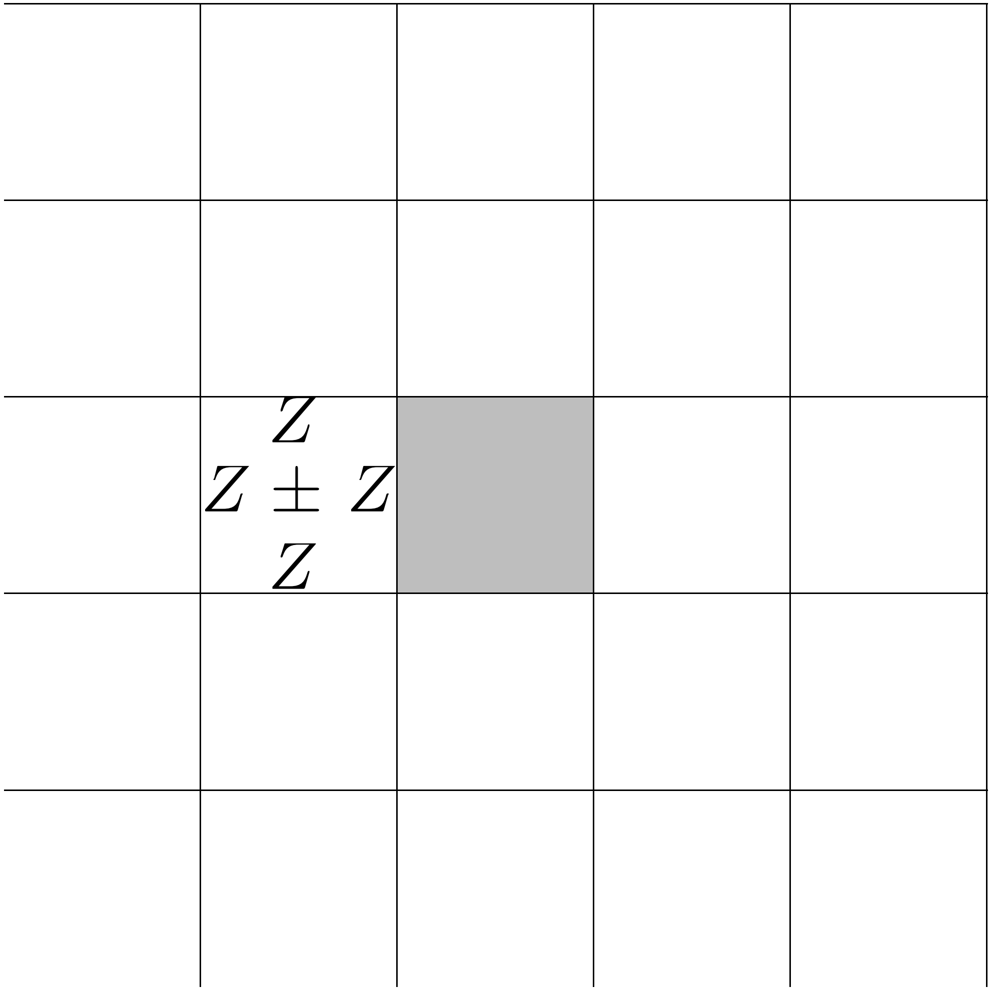
Toric Code Defect Deformation



The measurement result is ±1.

However, this measurement anti-commutes with the previously introduced stabilizer generator. Thus, that generator is removed and replaced with $\pm Z \otimes Z \otimes Z \otimes Z$

Toric Code Defect Deformation



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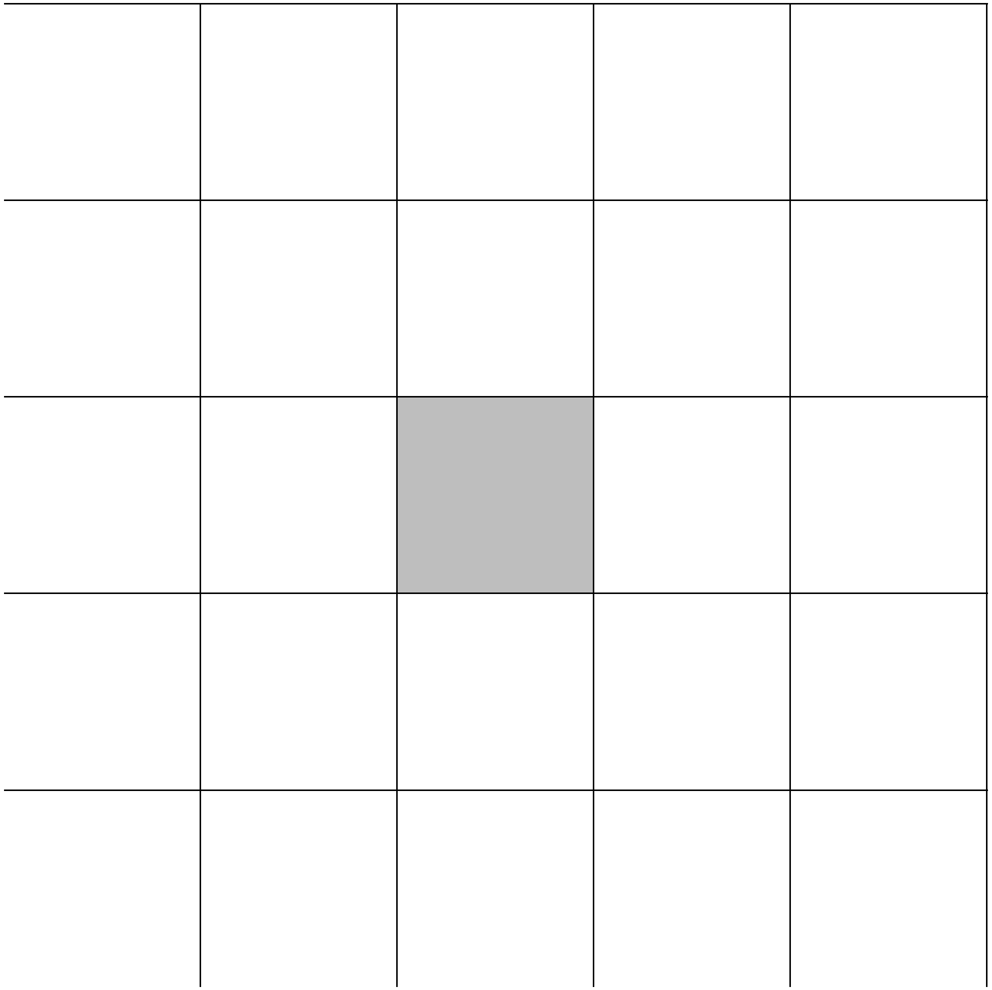
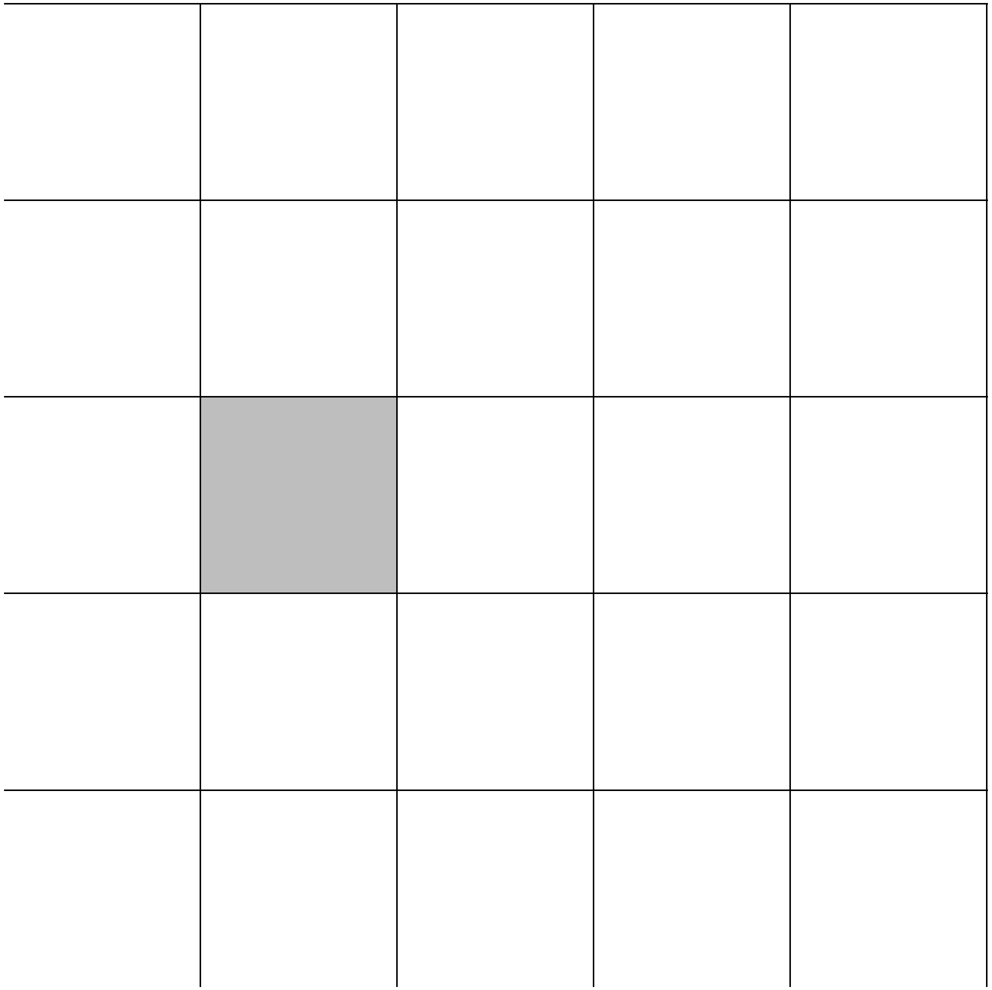
Toric Code Defect Deformation



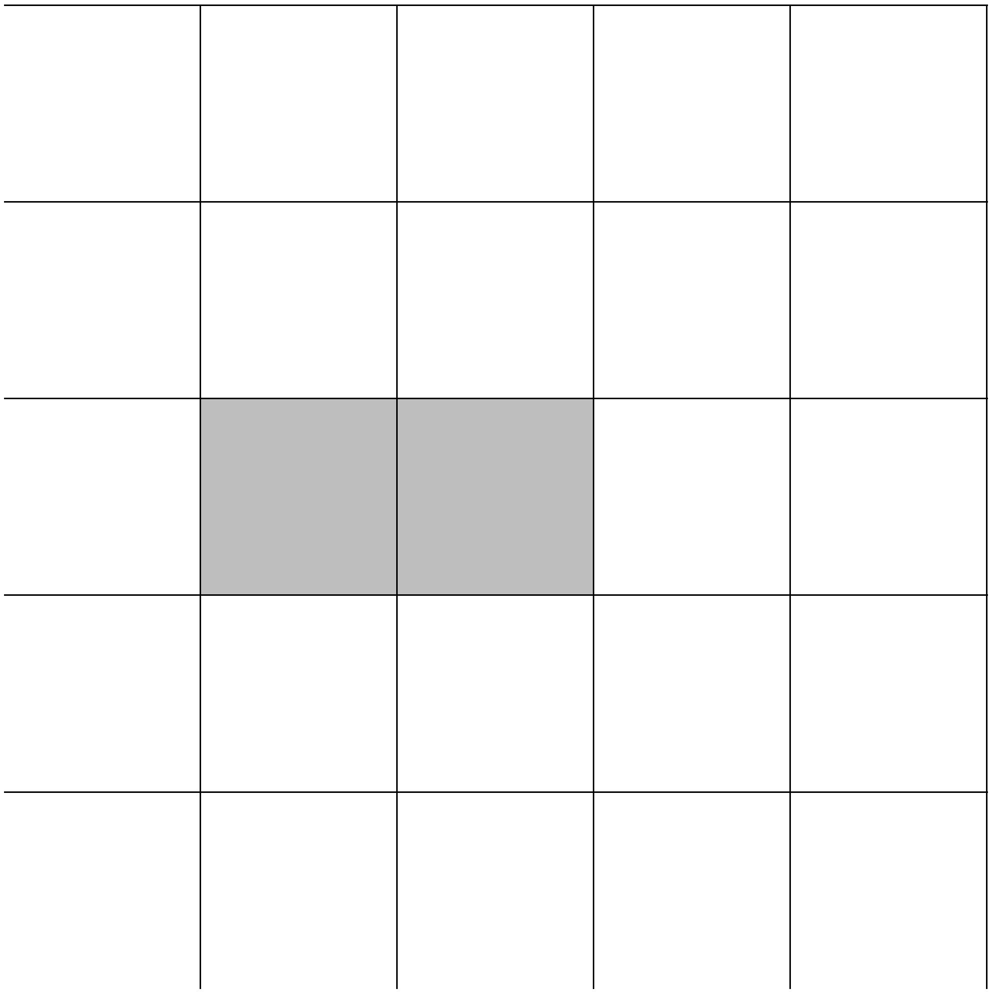
The defect has been moved!

Toric Code Defect Dynamics



We can use part of the preceding sequence to create larger defects which will be better protected from logical errors.

Toric Code Defect Dynamics



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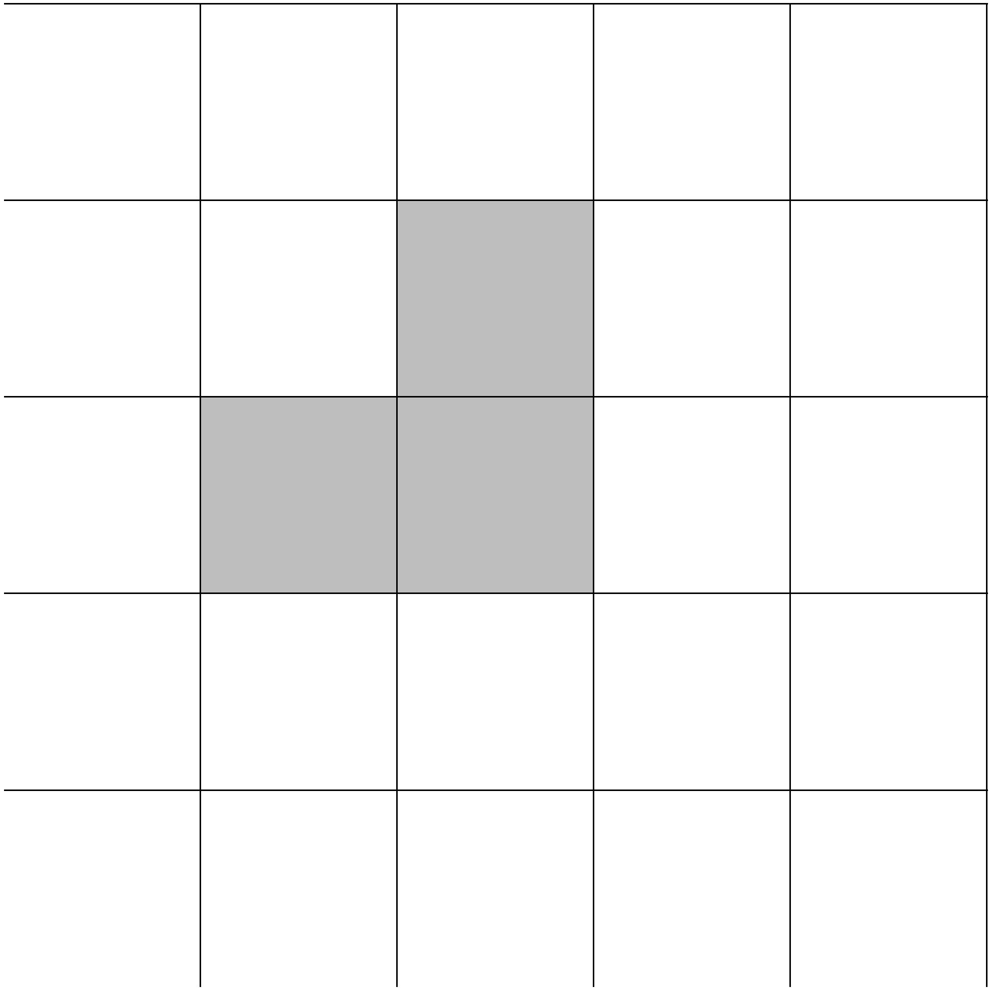

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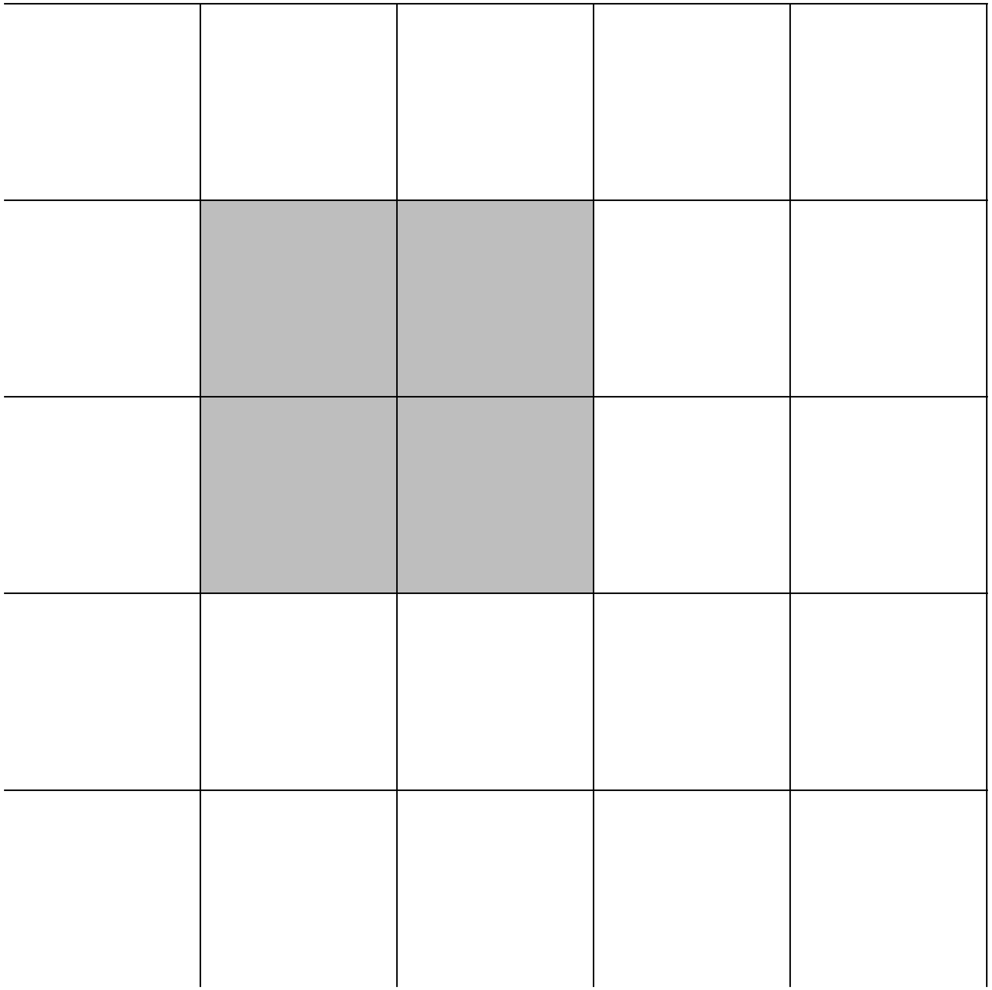

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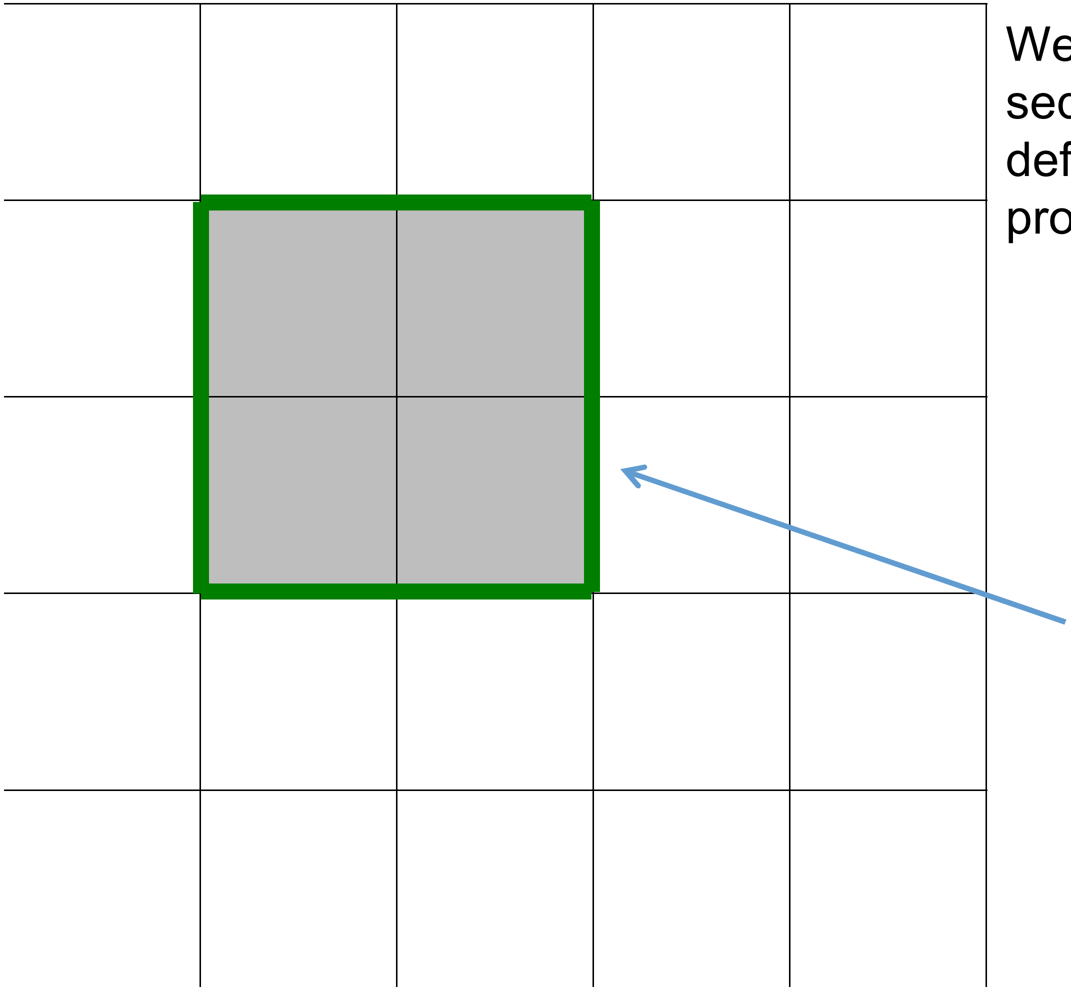
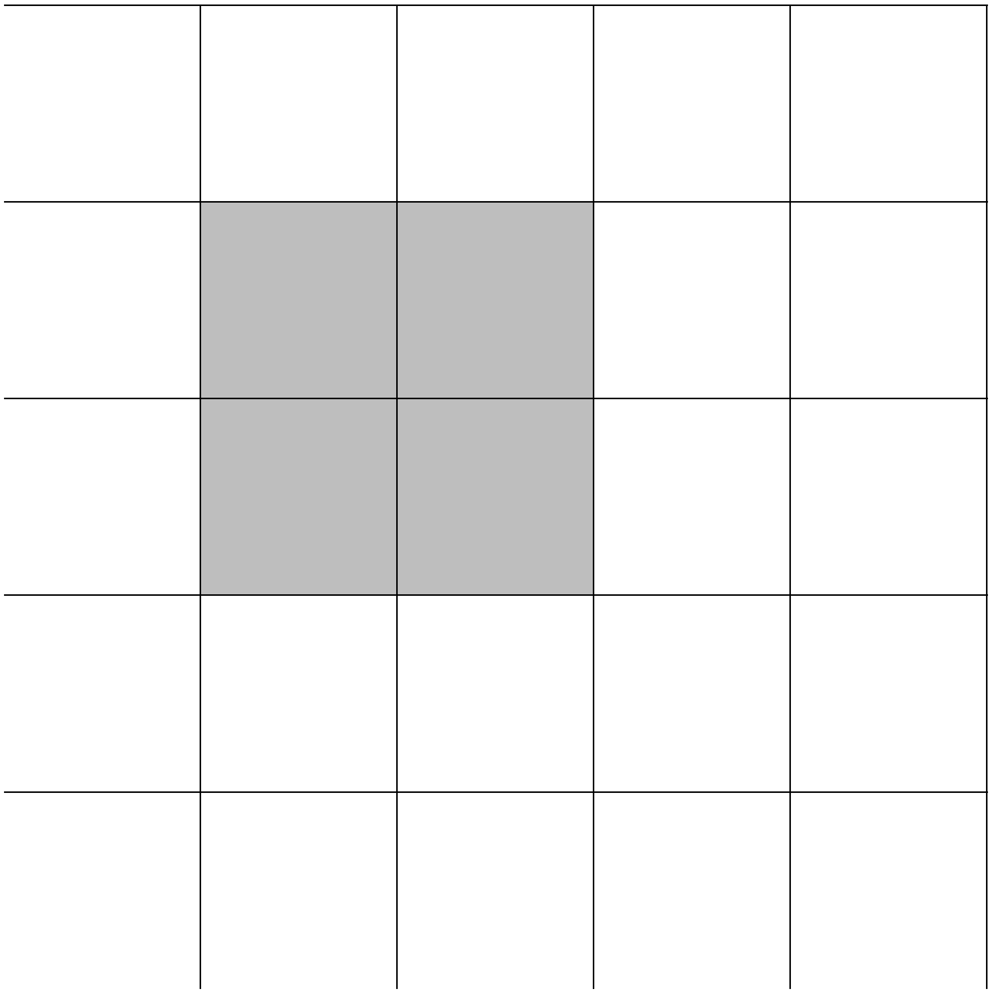


We can use part of the preceding sequence to create larger defects which will be better protected from logical errors.

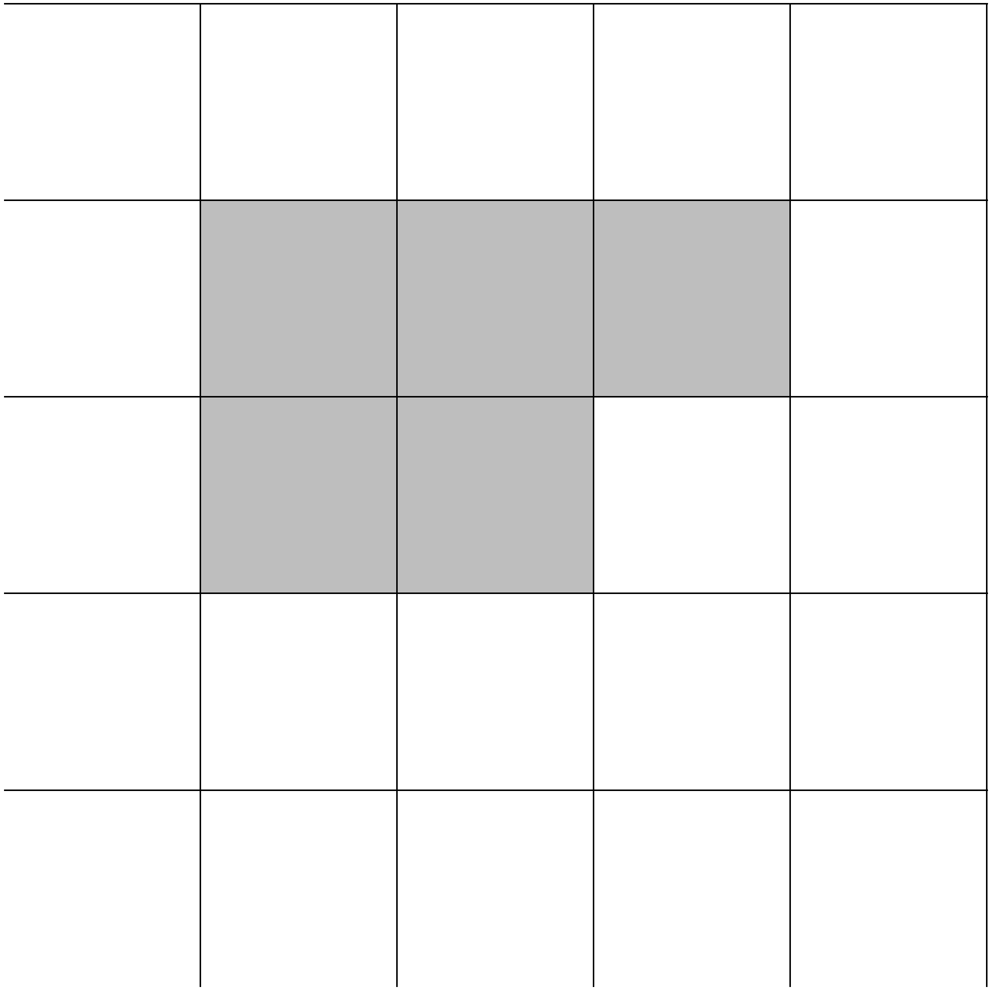
Higher weight Z_L

Toric Code Defect Dynamics



We can also move defects wholesale to another location.

Toric Code Defect Dynamics



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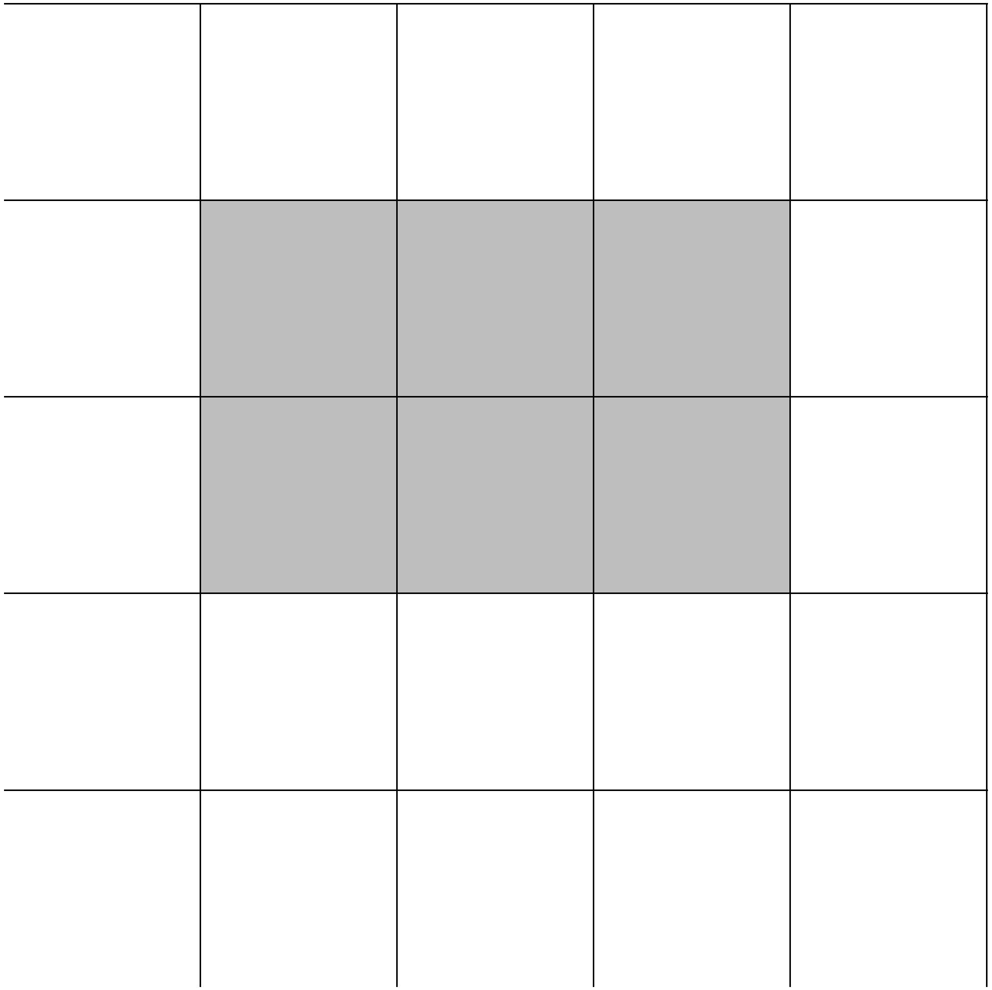

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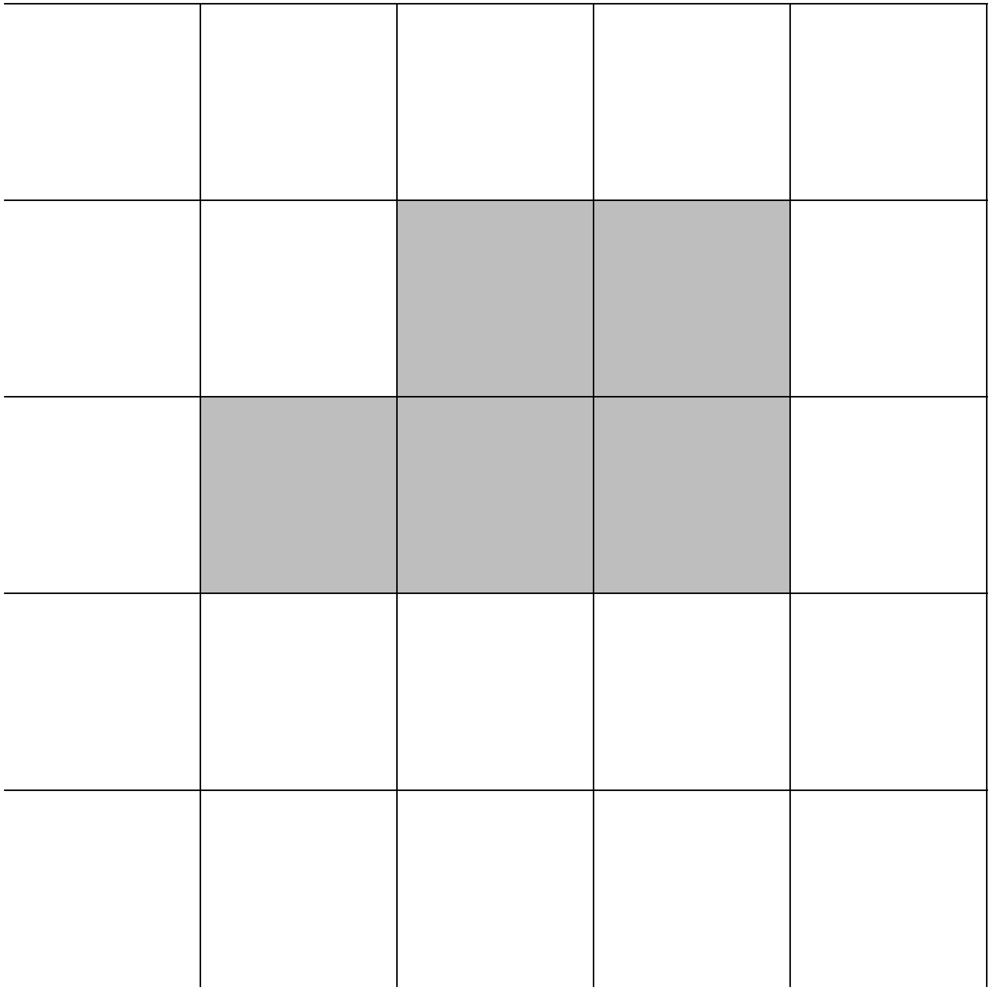

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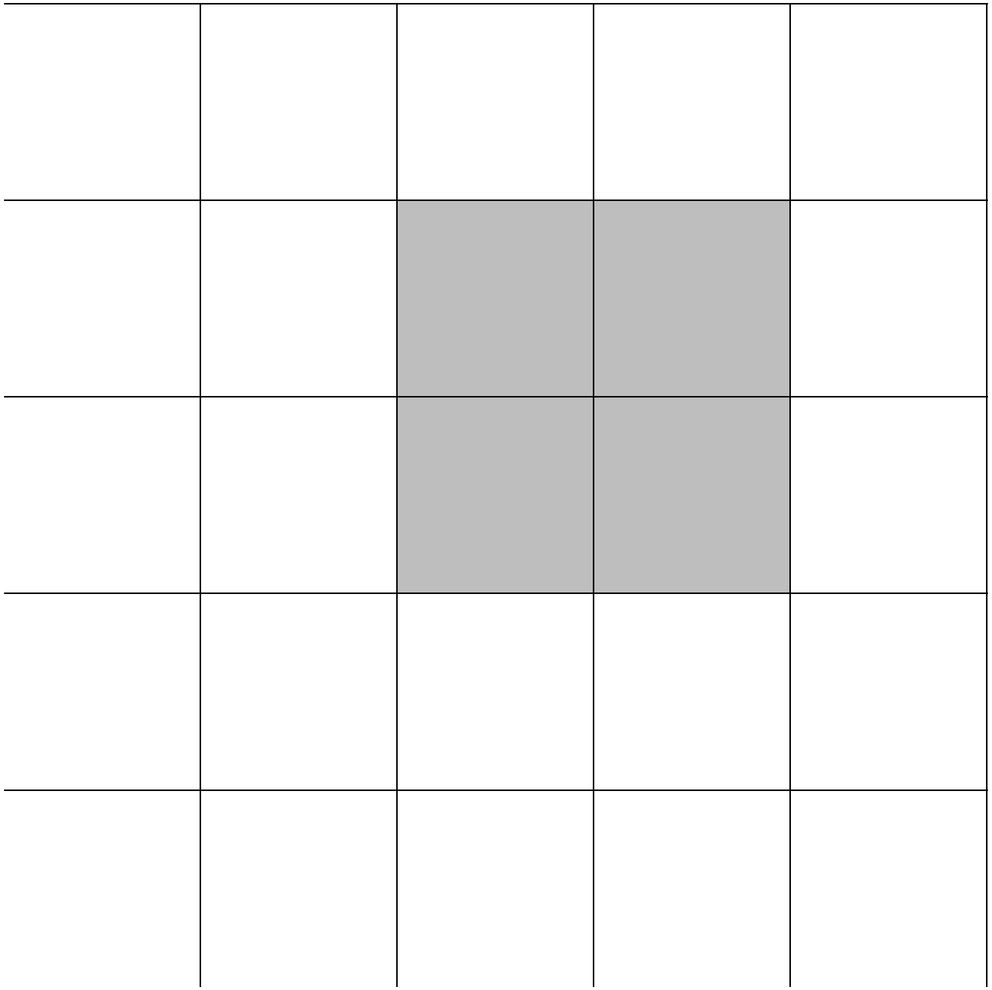
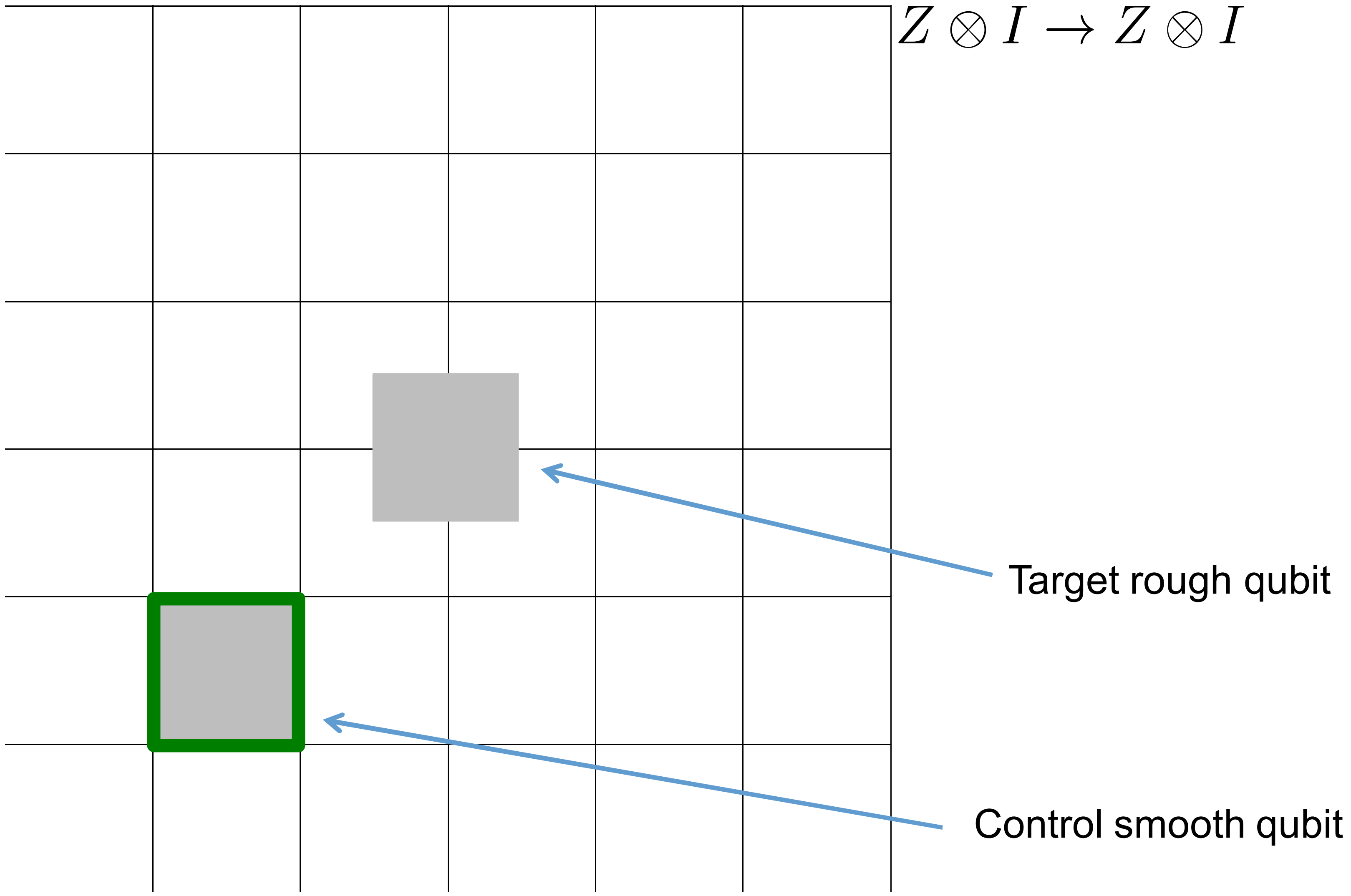


Moved!

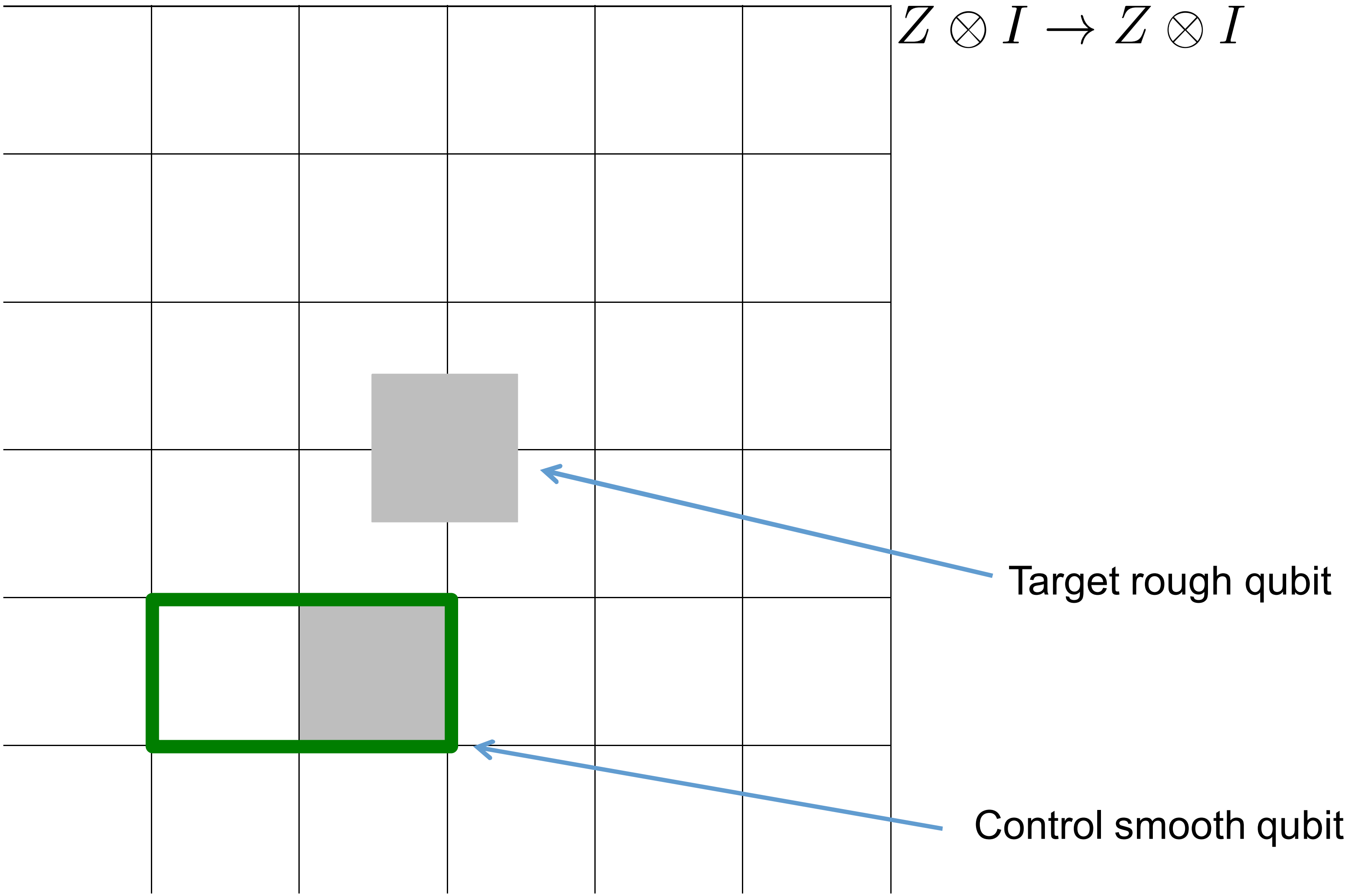
All of these procedures work equally well for rough defects with some slight modifications.

CNOT with Defects

$$Z \otimes I \to Z \otimes I$$



CNOT with Defects



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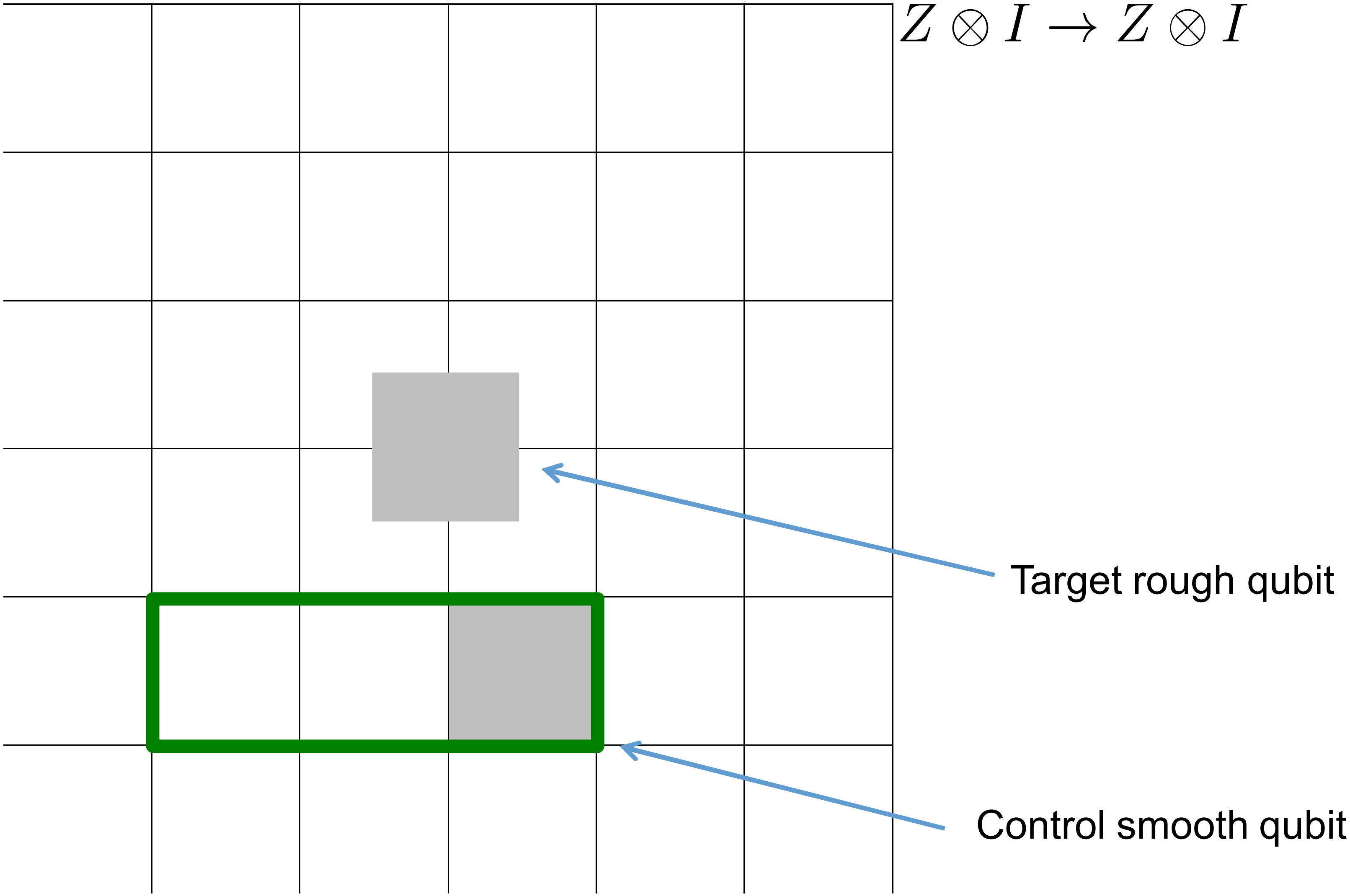
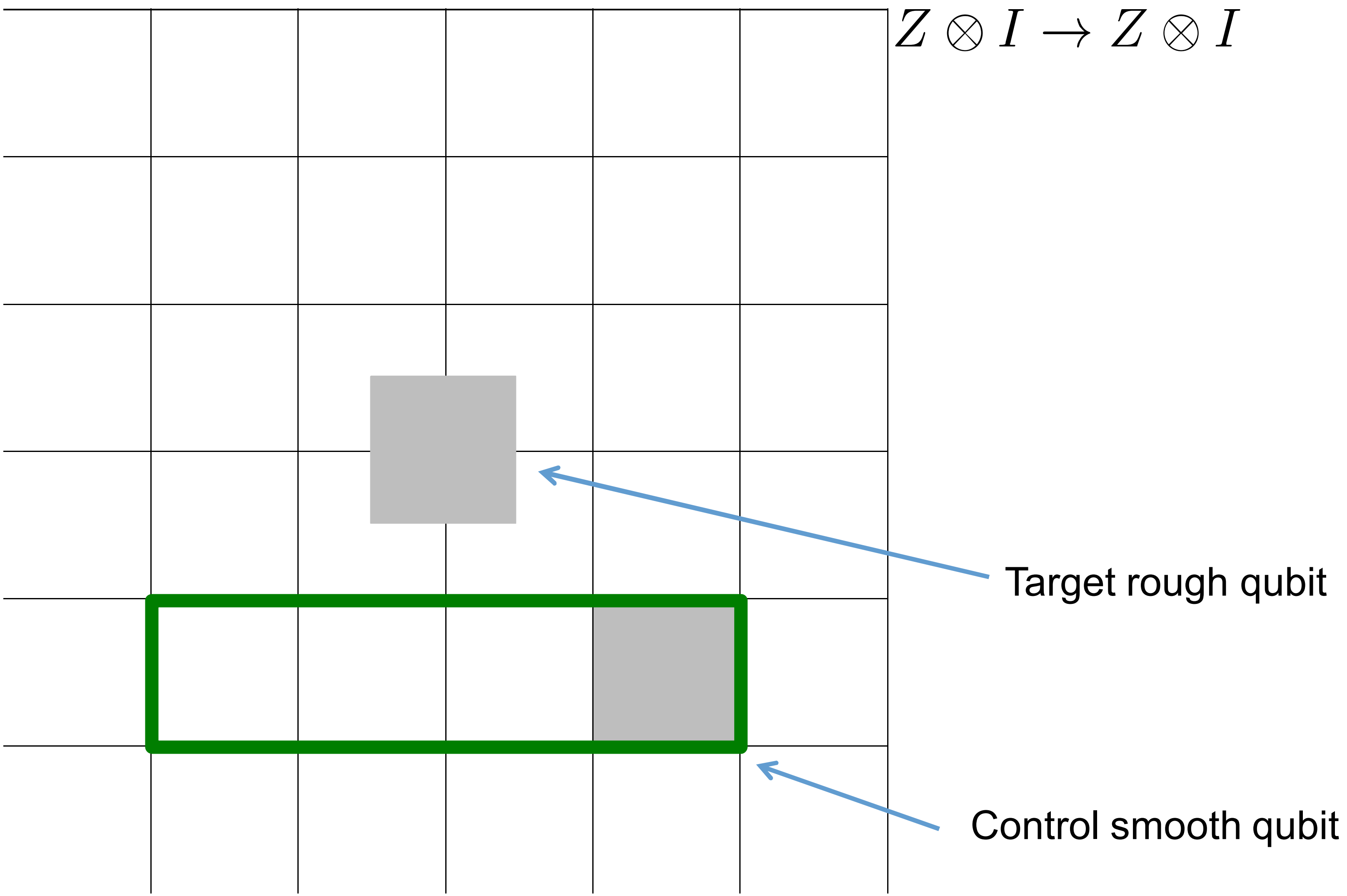
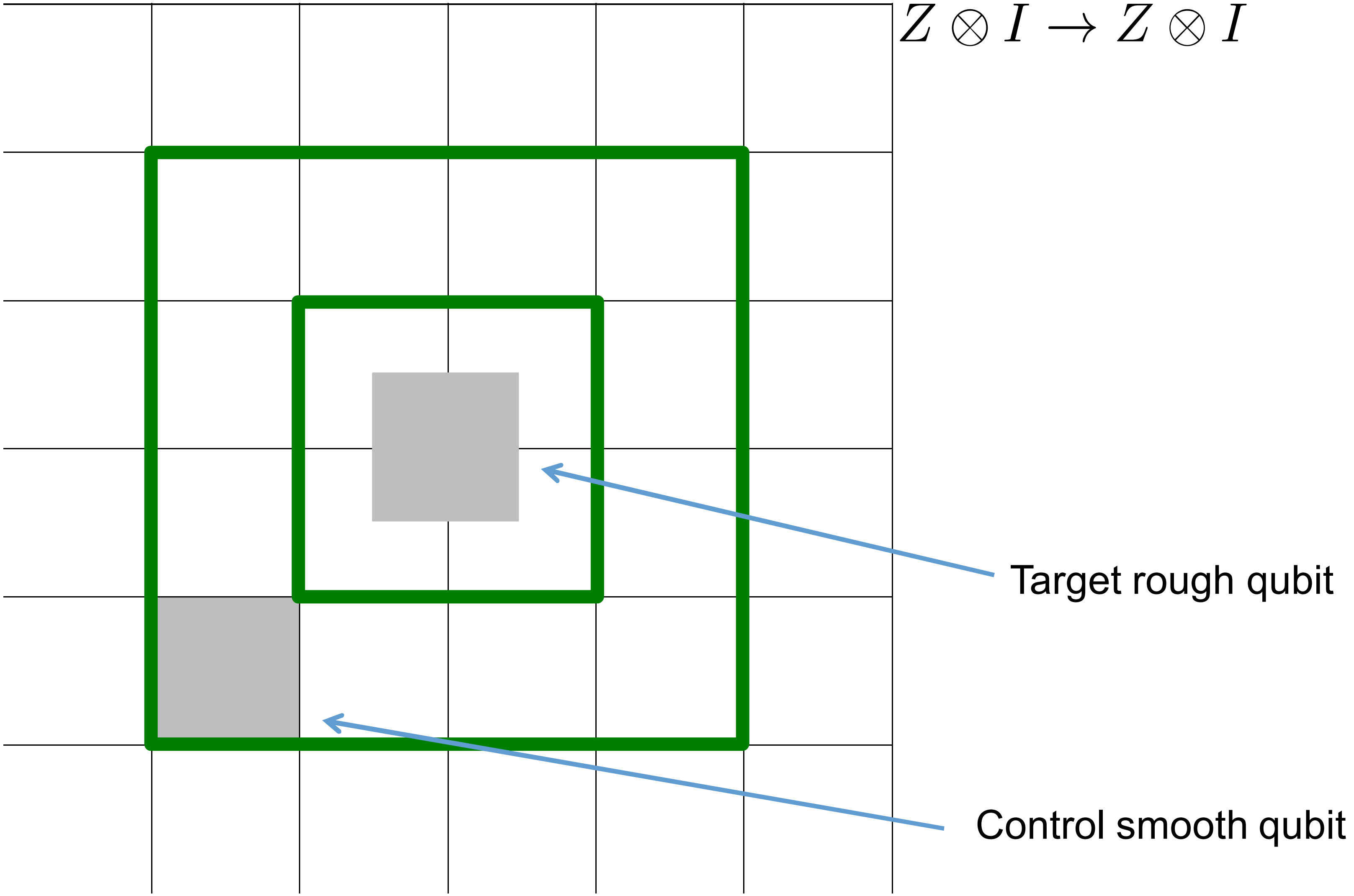


CNOT with Defects



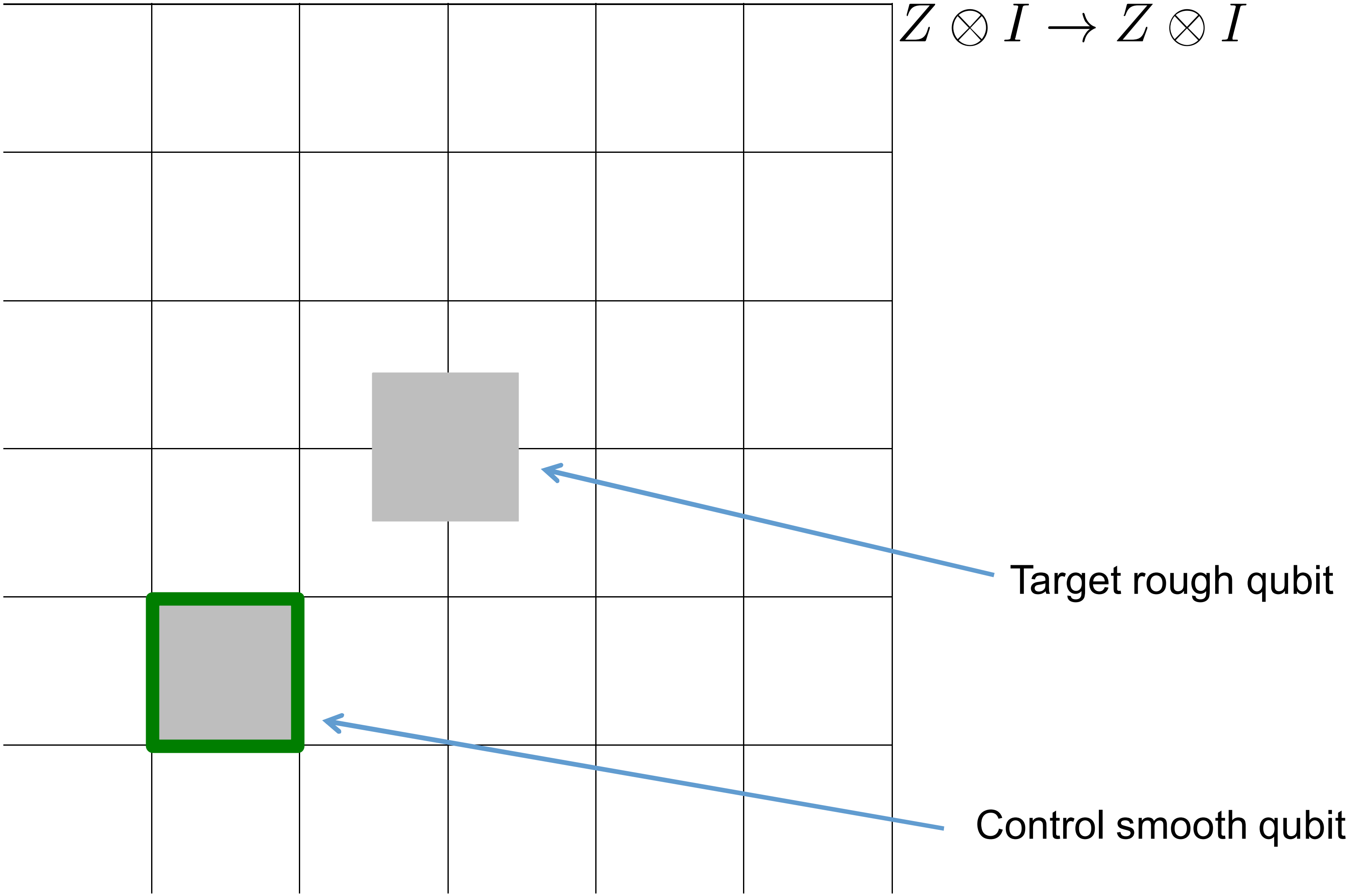
CNOT with Defects

$$Z \otimes I \rightarrow Z \otimes I$$



CNOT with Defects

$$Z \otimes I \to Z \otimes I$$



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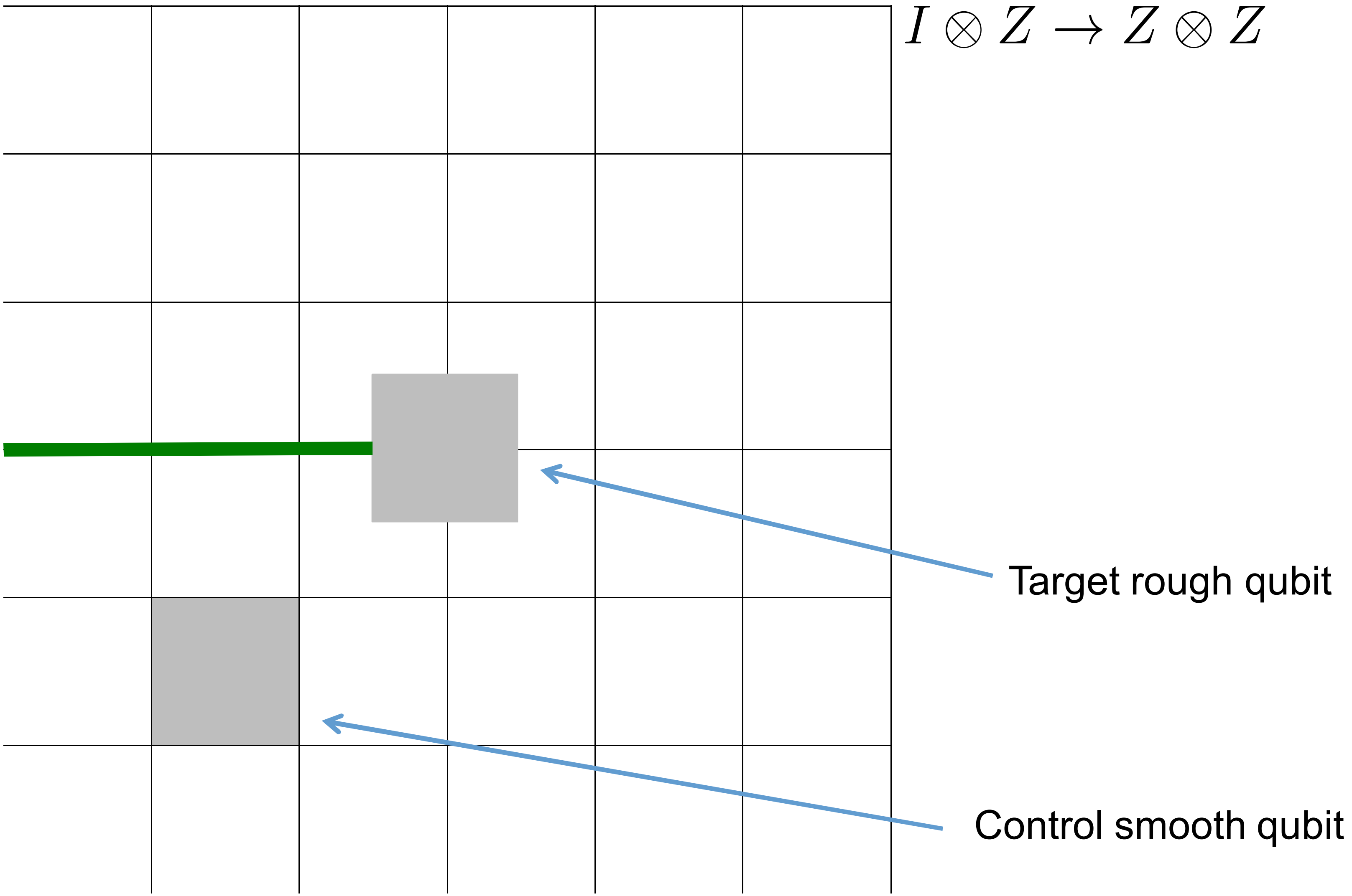
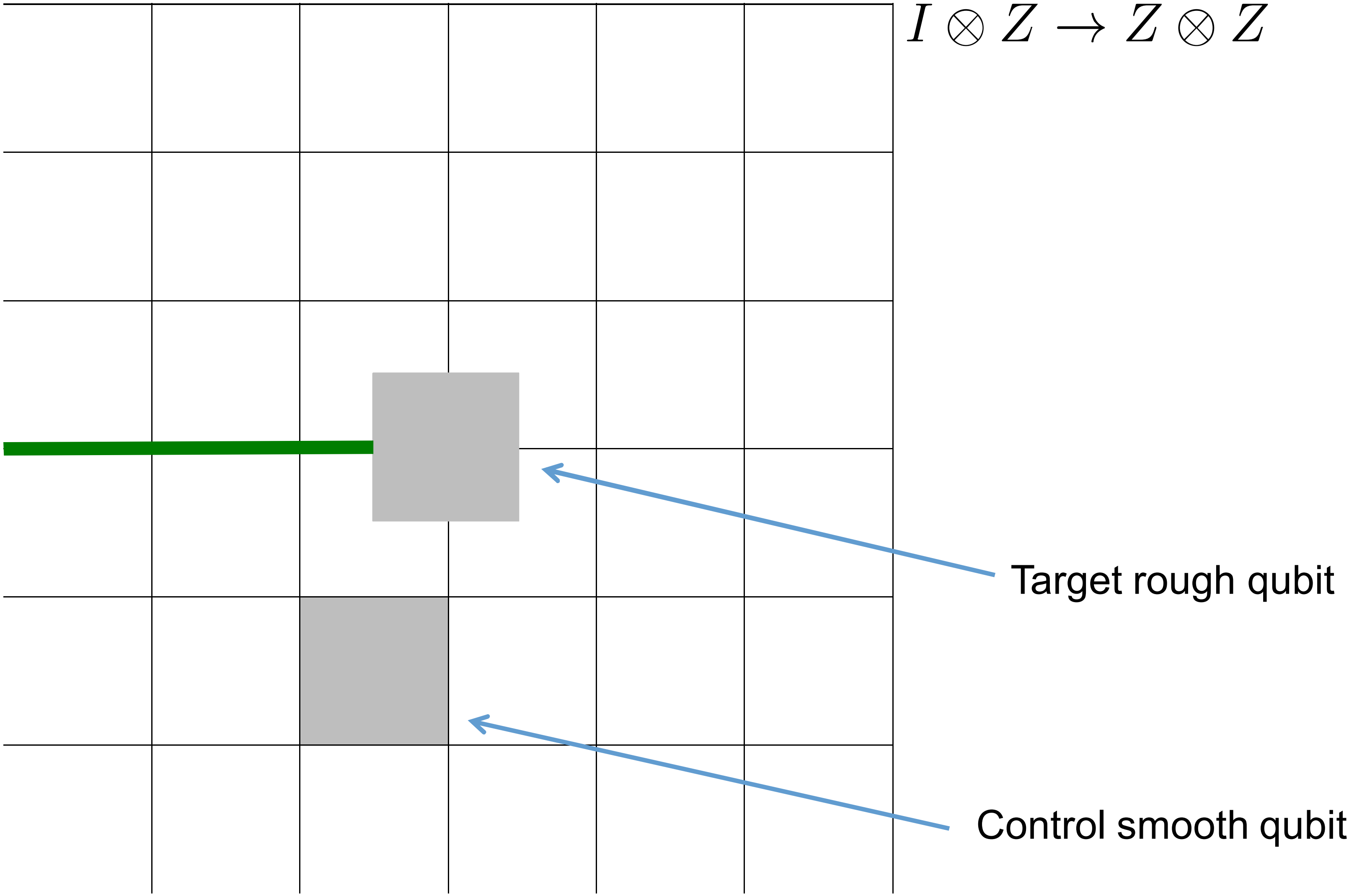


CNOT with Defects

$$I \otimes Z \to Z \otimes Z$$



CNOT with Defects

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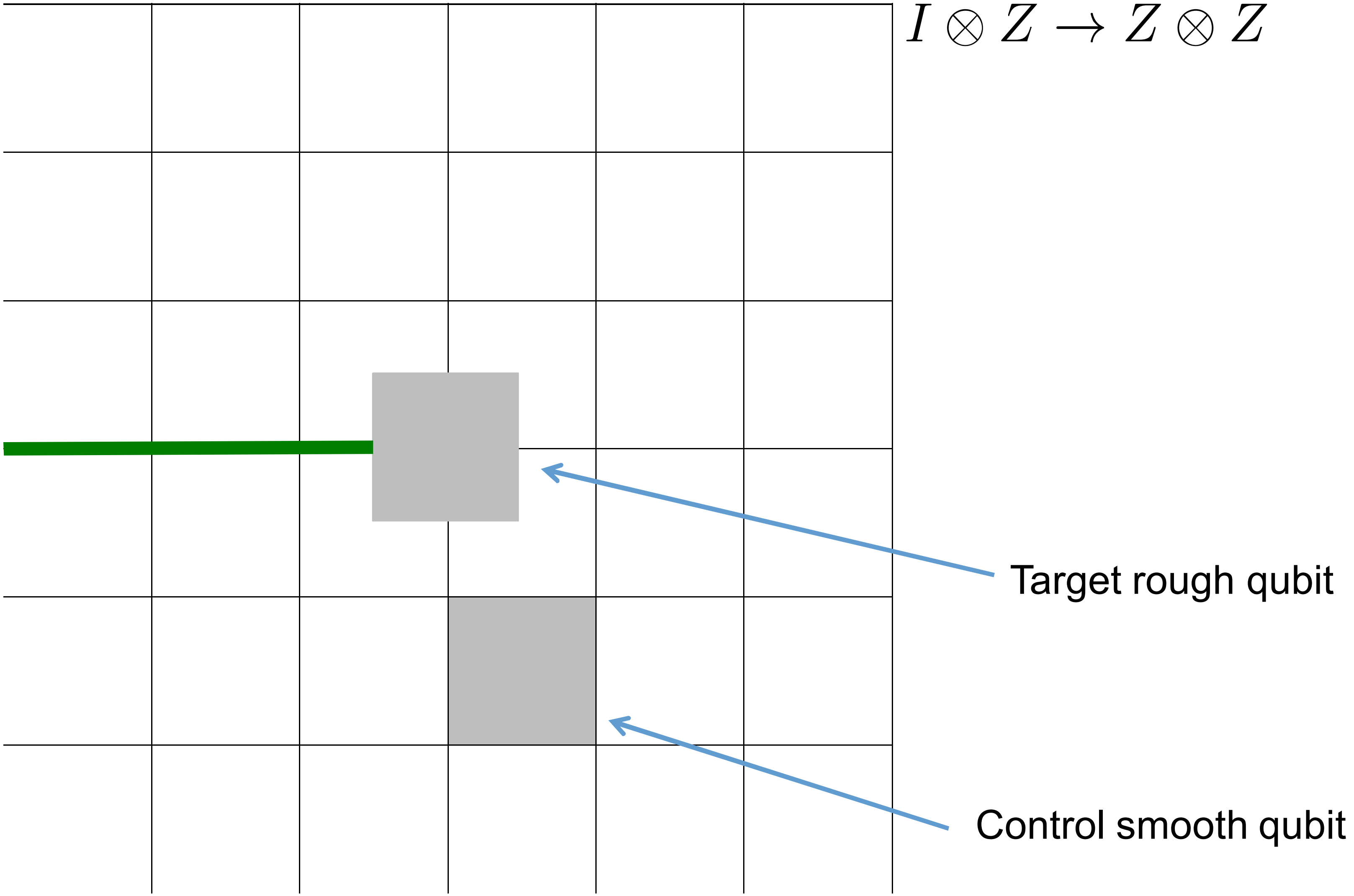

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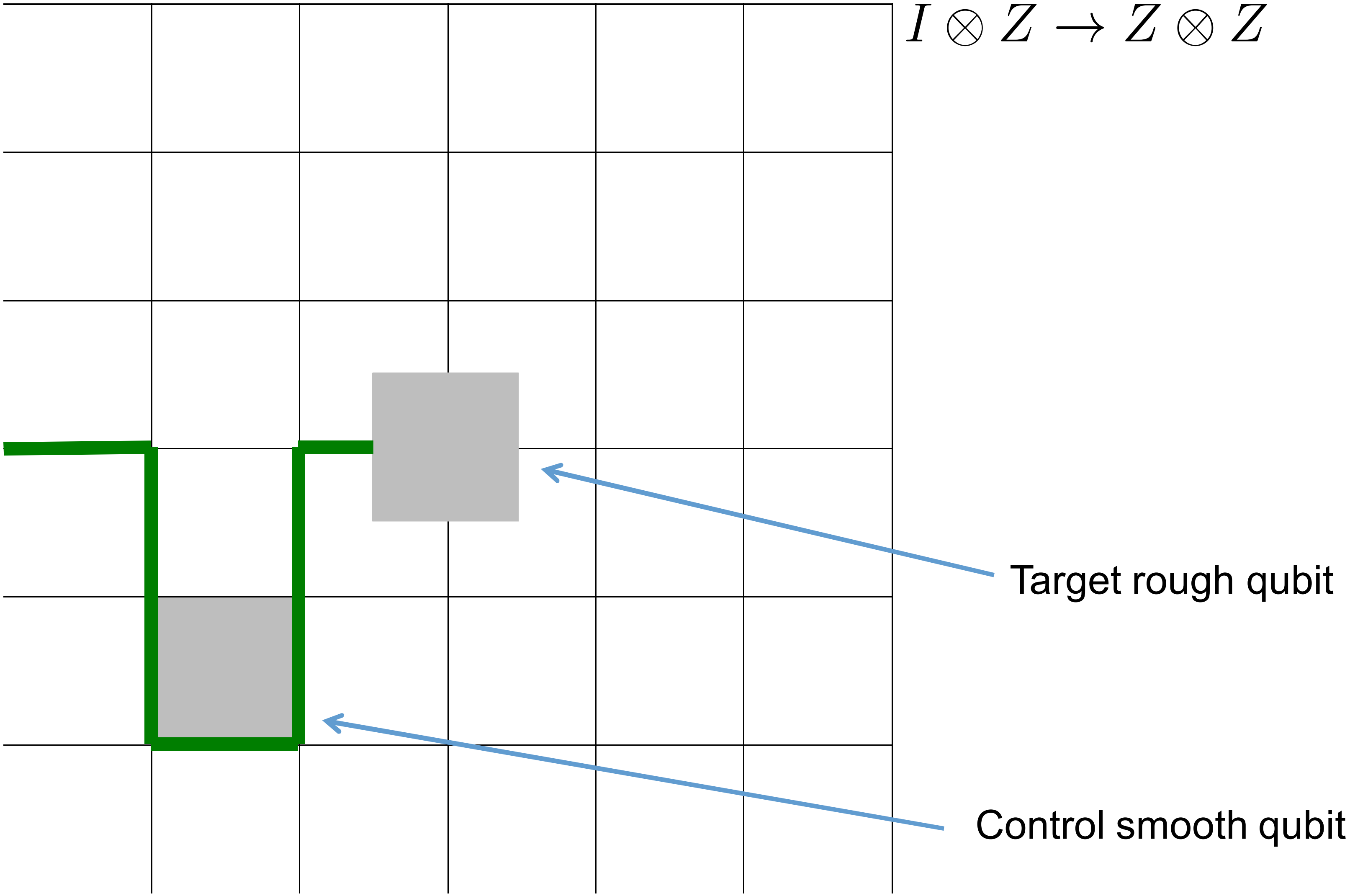

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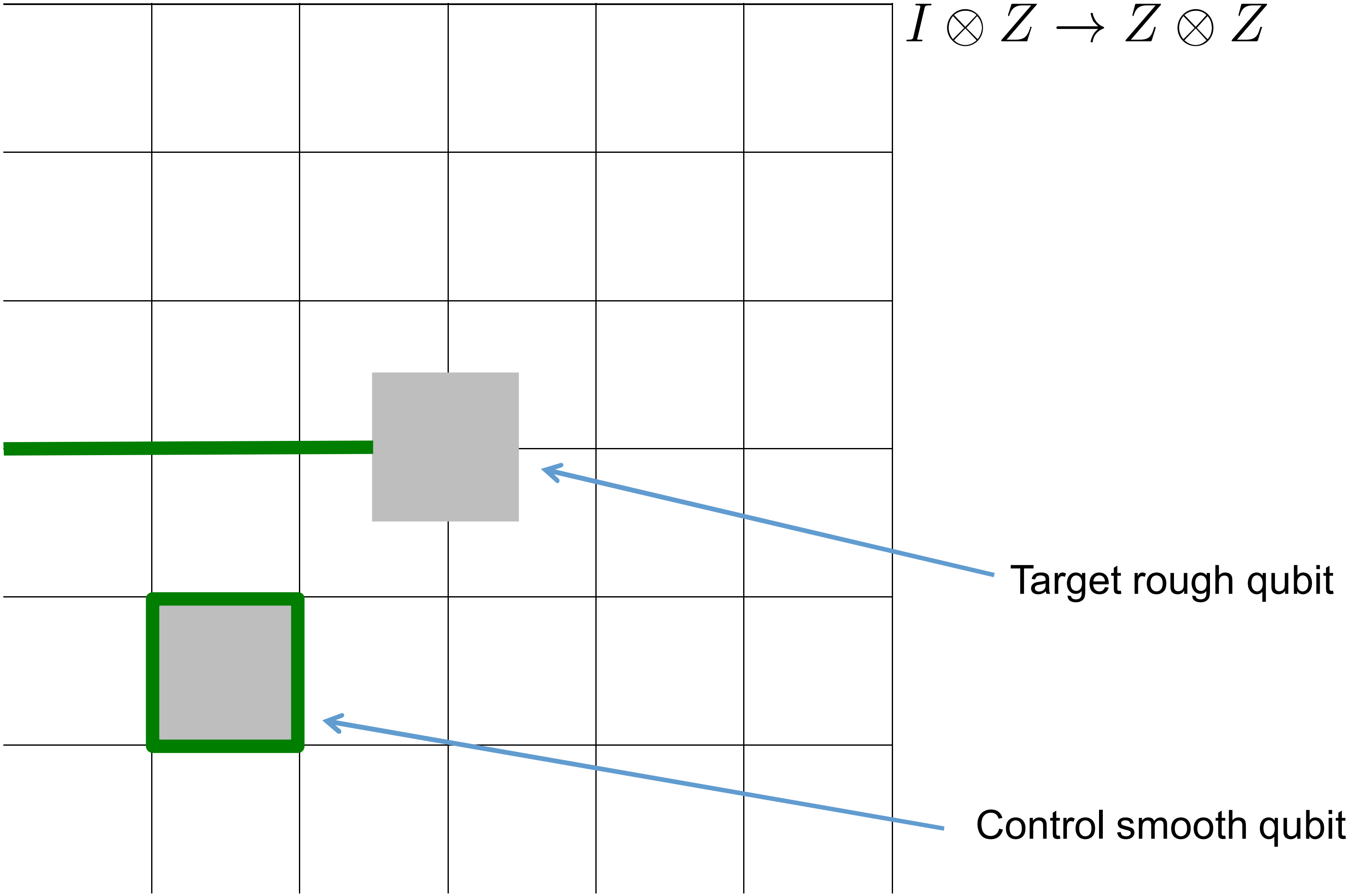
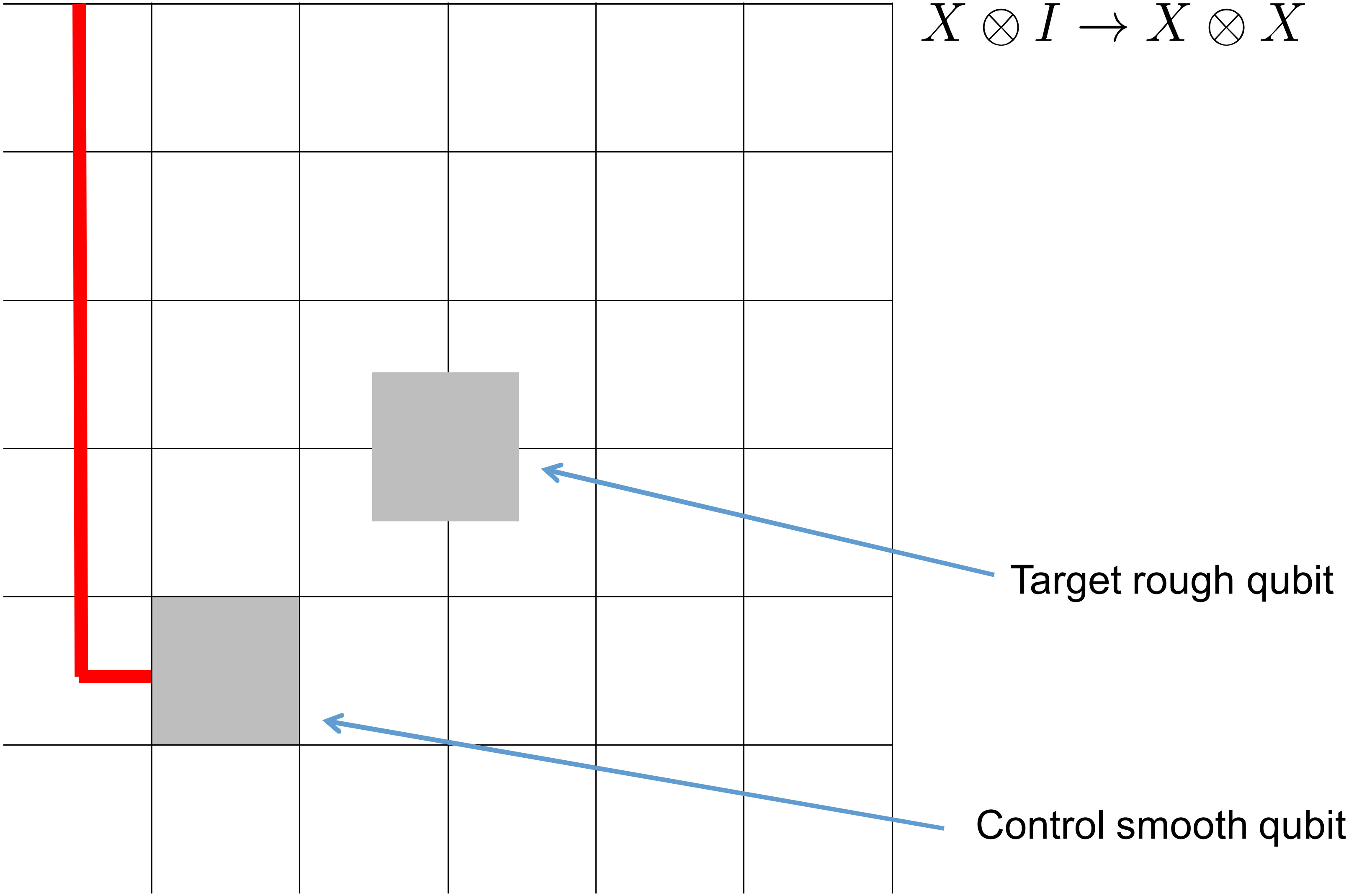


CNOT with Defects

$$X \otimes I \rightarrow X \otimes X$$



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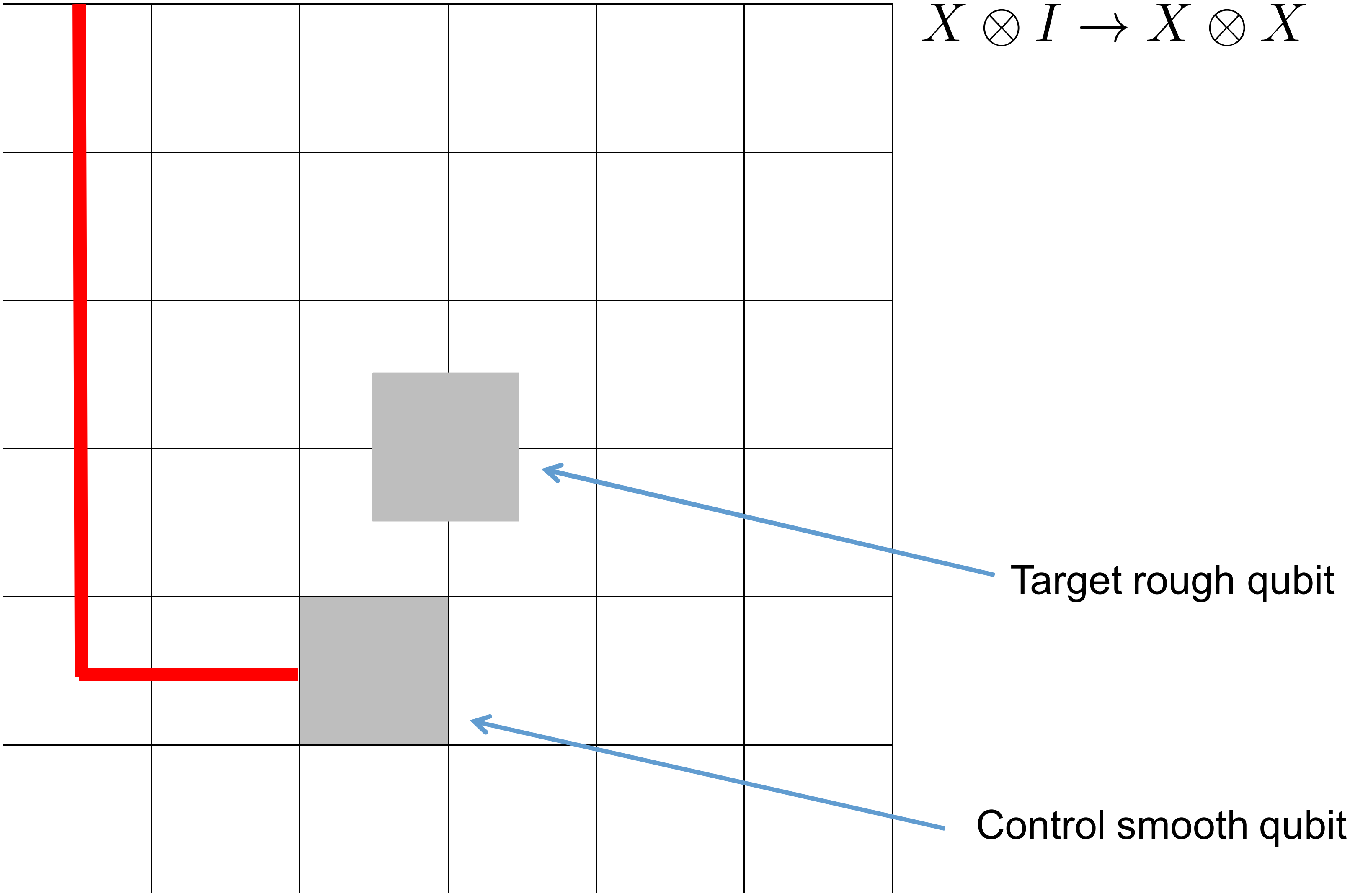

$$X \otimes I \to X \otimes X$$

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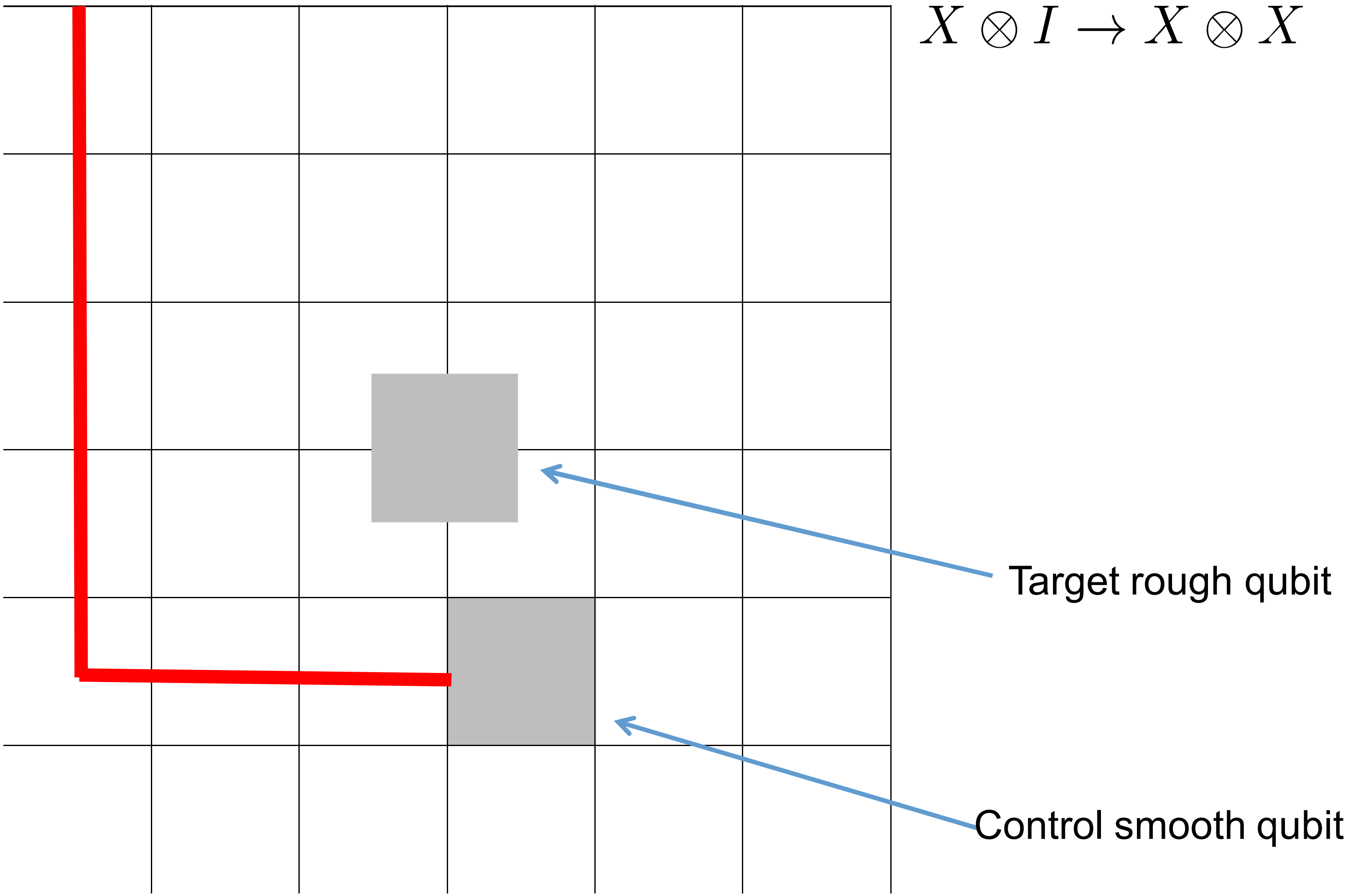

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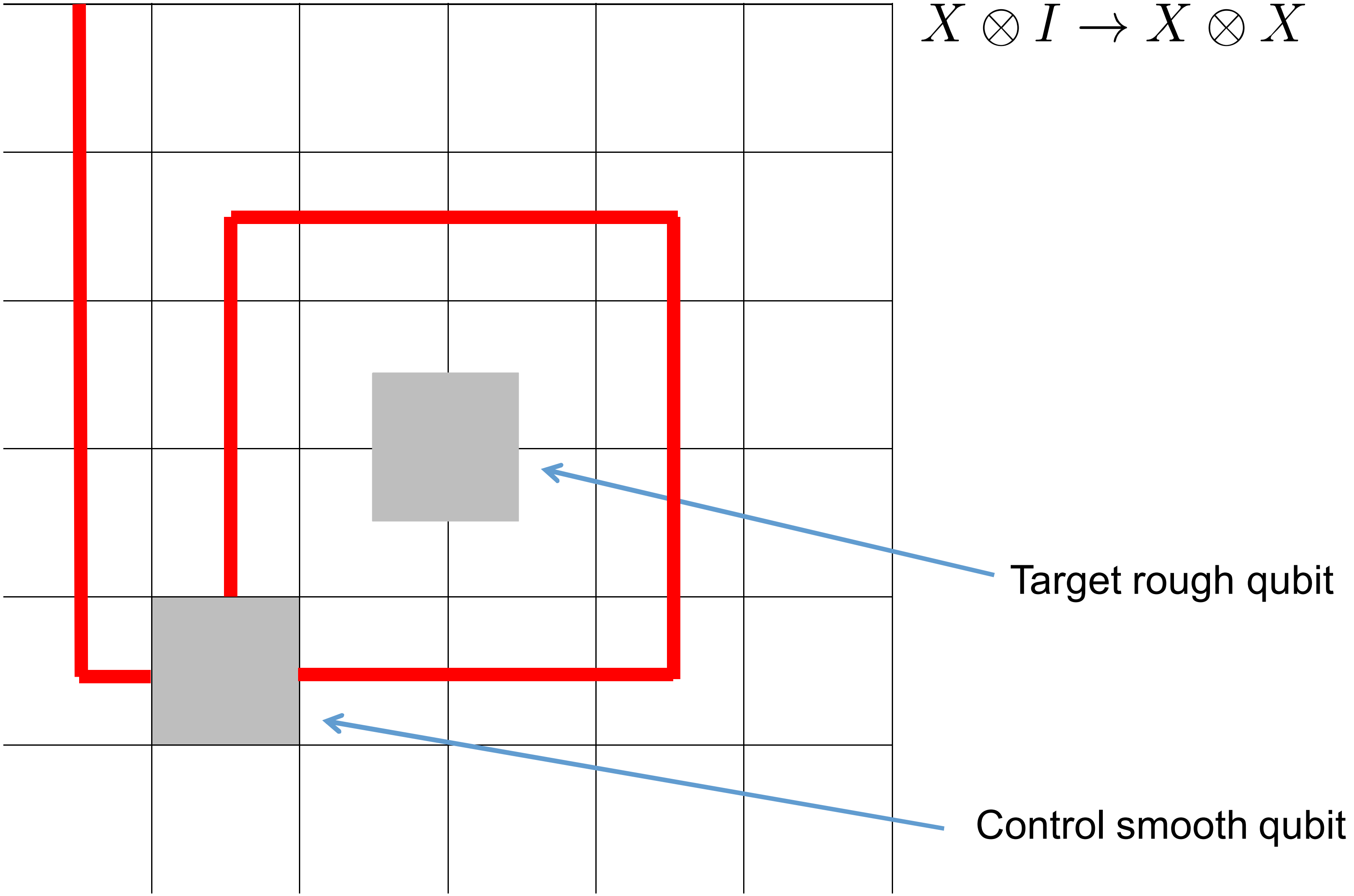

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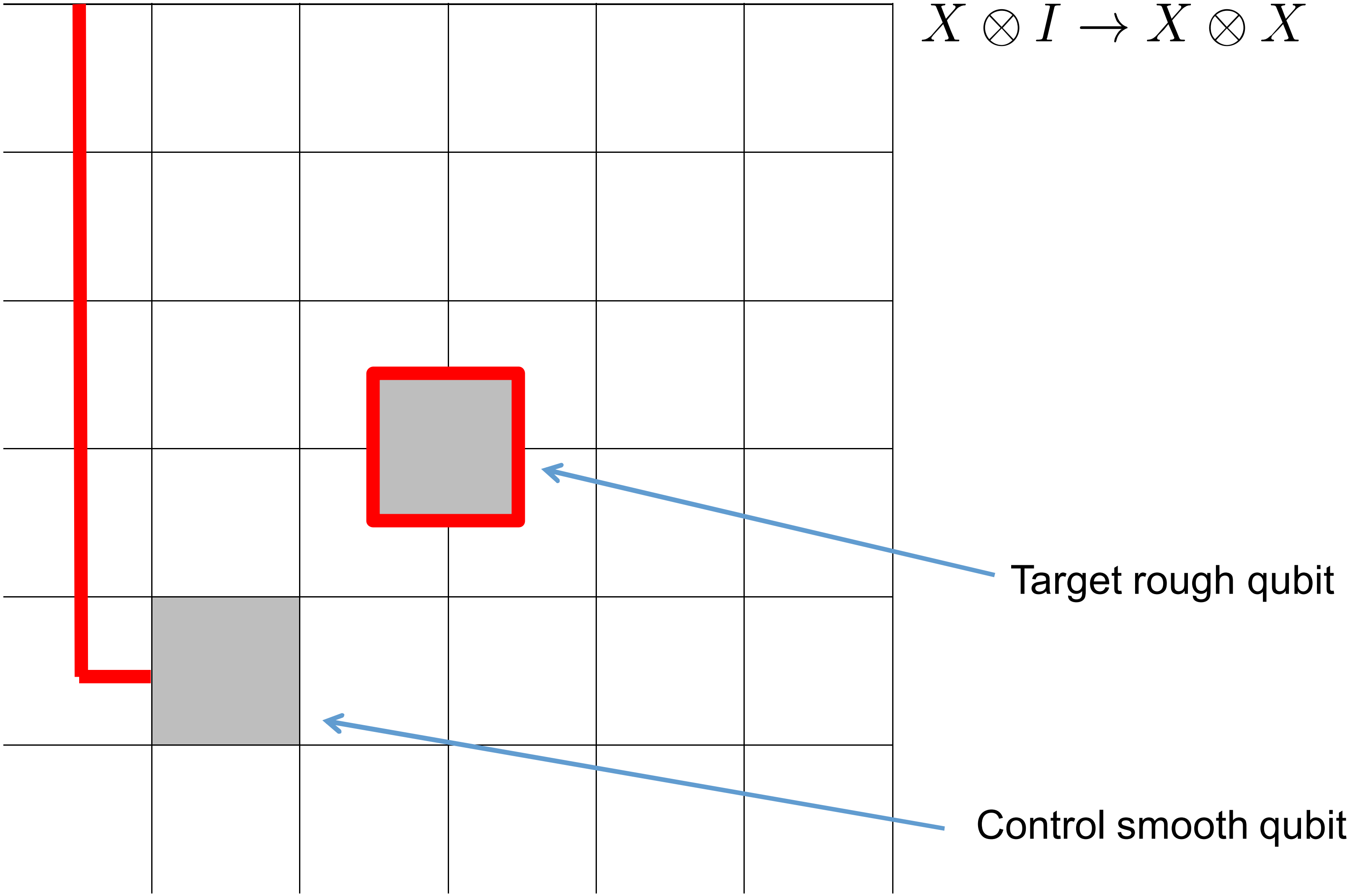

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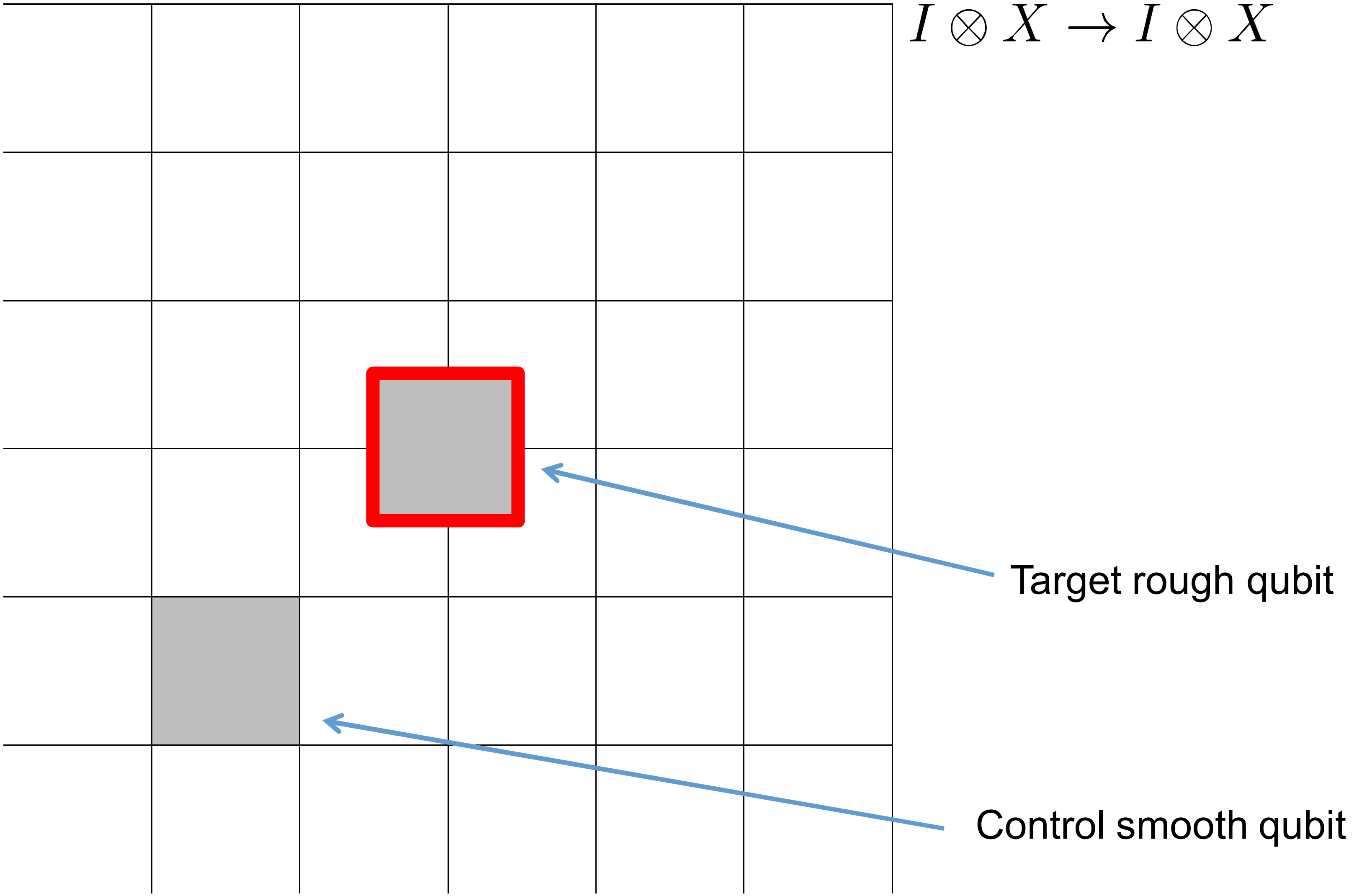

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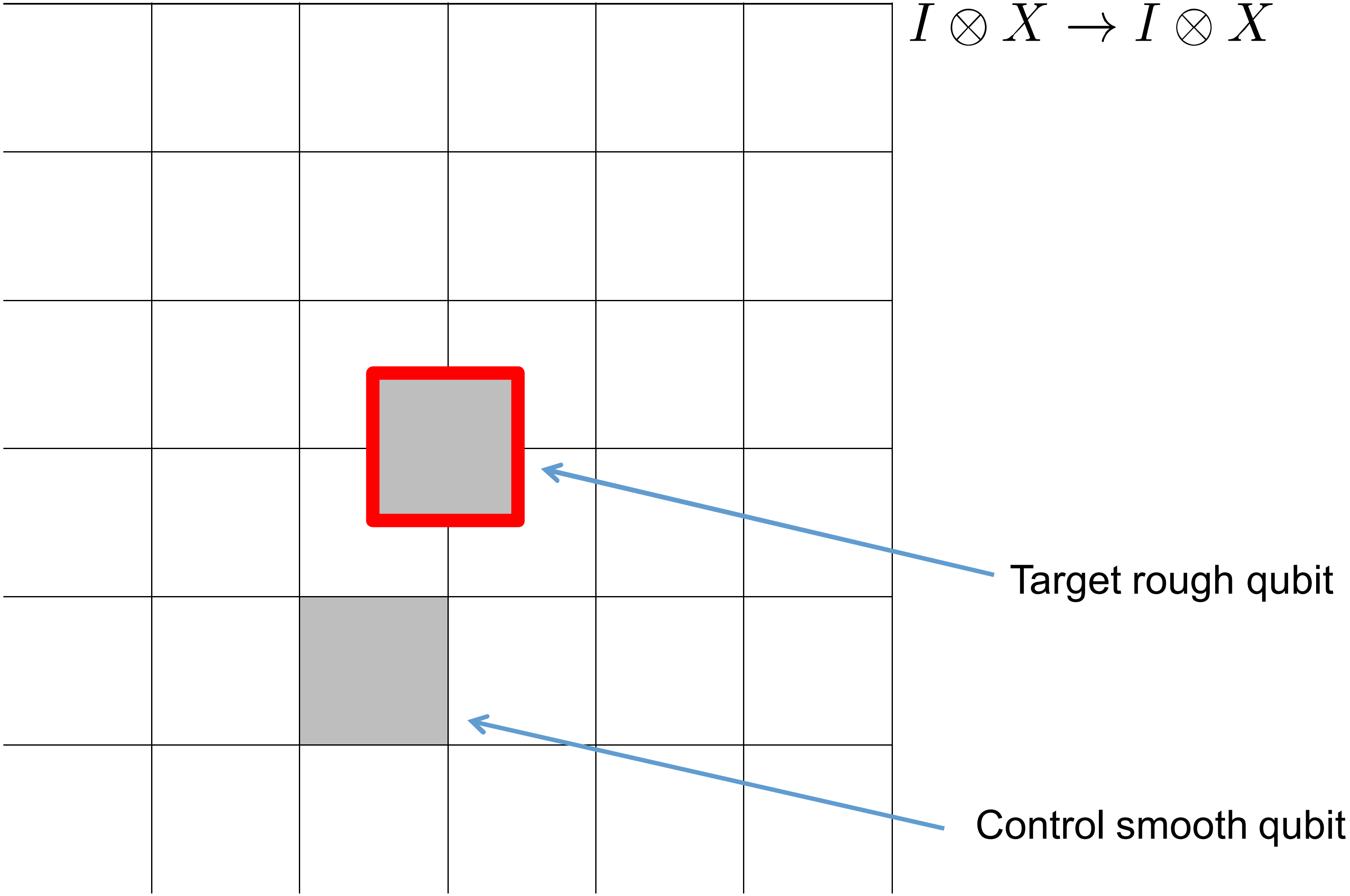

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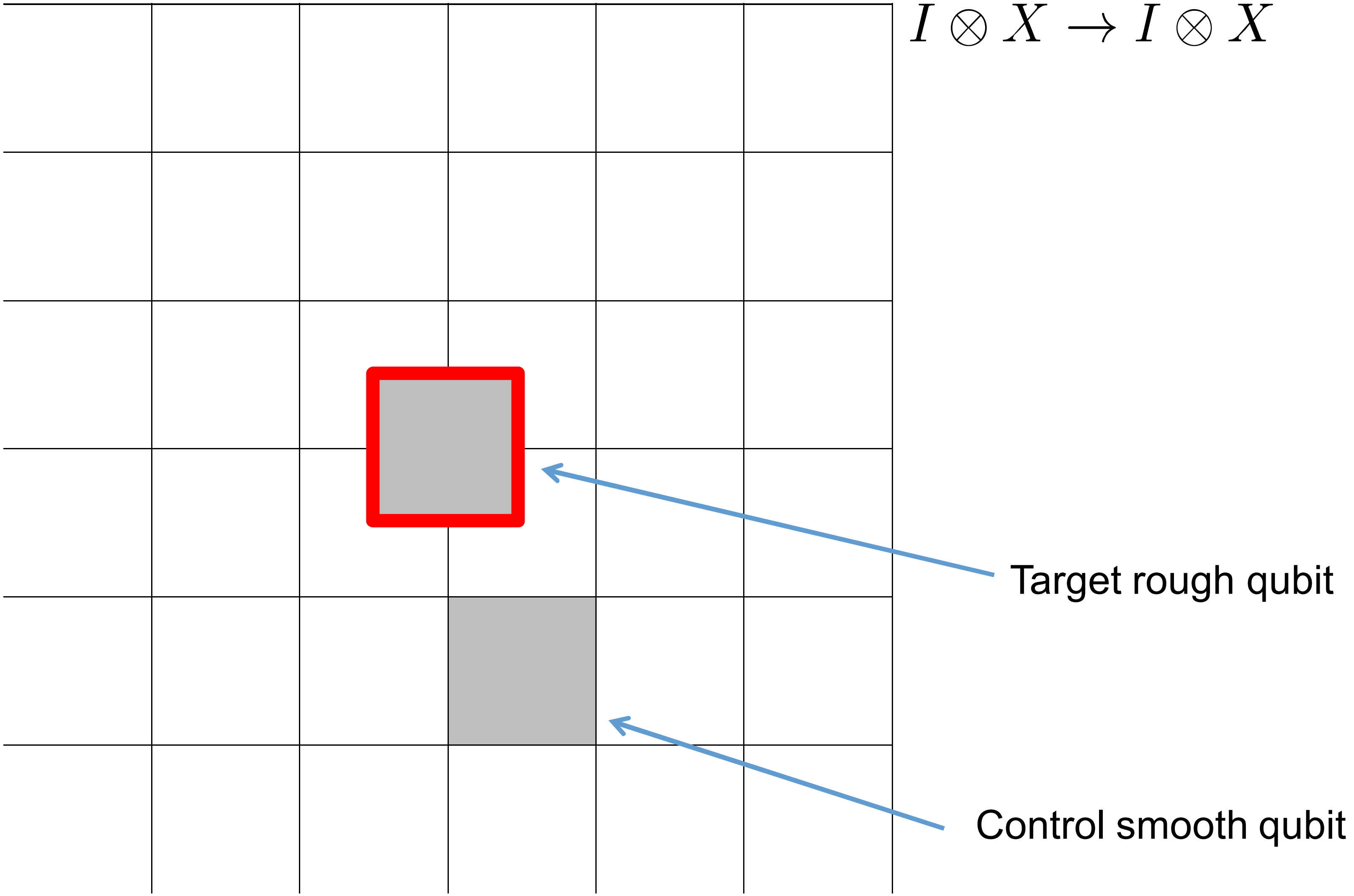
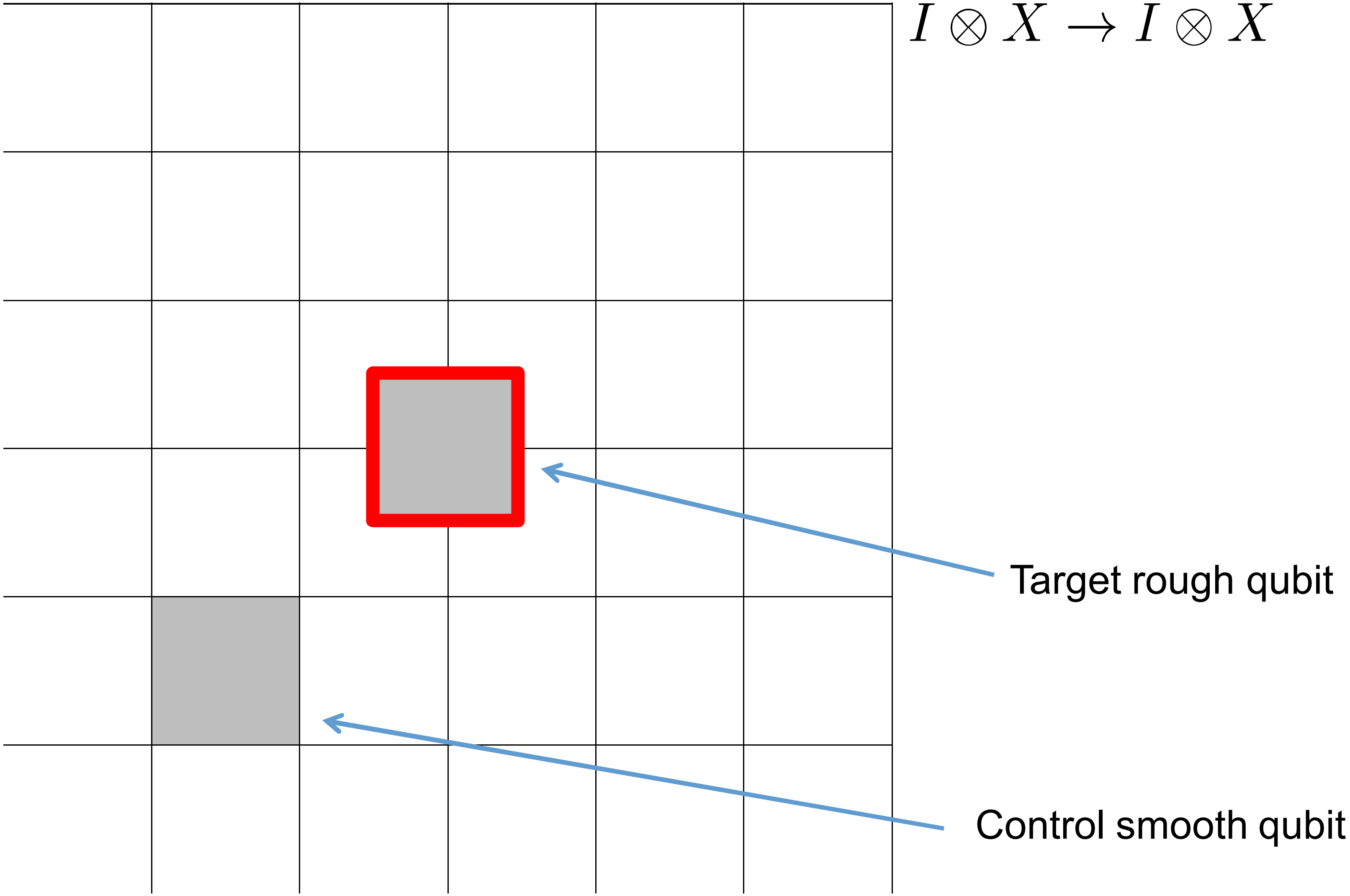


CNOT with Defects

$$I \otimes X \rightarrow I \otimes X$$



CNOT with Defects

$$I \otimes Z \to Z \otimes Z$$

$$X \otimes I \to X \otimes X$$

$$I \otimes X \to I \otimes X$$

$$Z \otimes I \to Z \otimes I$$

These relations are precisely how the CNOT gate acts. Is it possible to do this defect braiding adiabatically?

Stabilizer Quantum Codes

↓

Topological Quantum Codes

↓

Adiabatic Topological Quantum Computing

Codespace as Ground Space

- Consider the following Hamiltonian

$$H = -\sum S_i, \quad S_i \in \langle \mathcal{S} \rangle$$

- The ground space of this Hamiltonian is exactly the codespace of the stabilizer code with generators S_i
- Errors in the code appear as excitations out of the ground space → errors are suppressed energetically

Adiabatic TQC

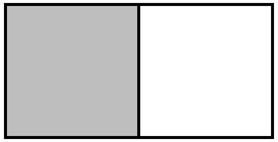
- In TQC, braiding is always assumed to be adiabatic
- This work introduces explicit adiabatic braiding and uses an explicit lattice model
- The tools can be extended more generally to “Adiabatic Code Deformation”

Bacon and Flammia, Adiabatic Gate Teleportation,
Phys. Rev. Lett. 103, 120504 (2009)

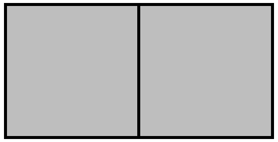
Bacon and Flammia, Adiabatic Cluster State Quantum Computing,
Phys. Rev. A 82, 030303(R) (2010)

Adiabatic TQC Defect Movement

- The adiabatic movement of defects simulates the preceding measurement-based approach



$$H = \ldots - Z \otimes Z \otimes Z \otimes Z - \ldots$$



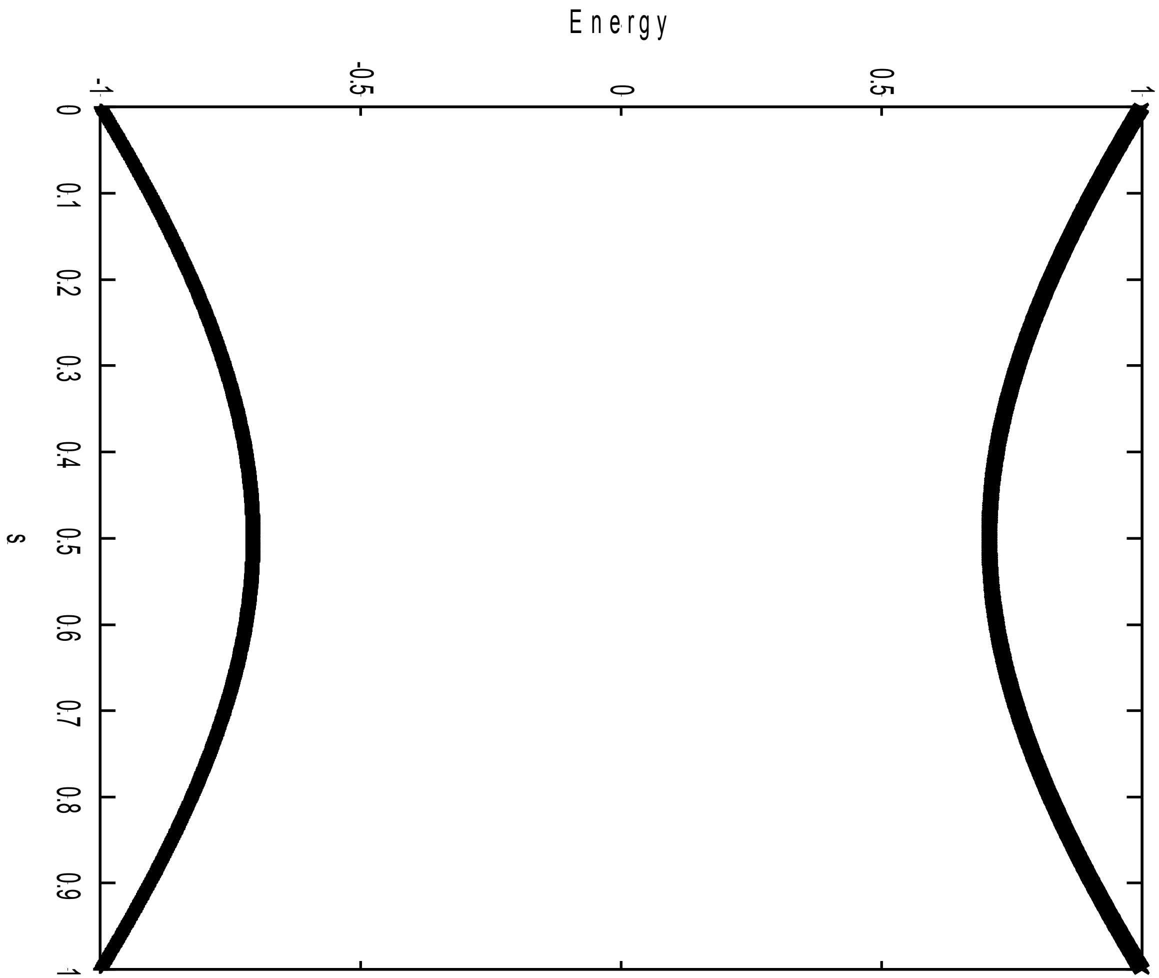
$$H = \ldots - X_{\mathrm{middle}} - \ldots$$

- The following evolution succeeds in growing the defect

$$H(s) = (1 - s)\left(-Z \otimes Z \otimes Z \otimes Z\right) + s\left(-X_{\mathrm{middle}}\right)$$

Adiabatic TQC Defect Movement

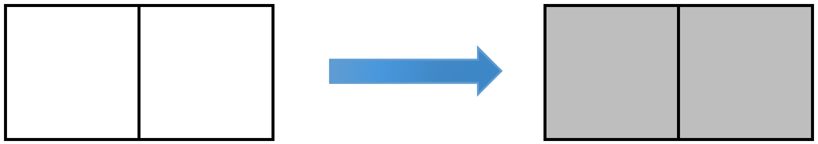
$$H(s) = (1-s)\left(-Z \otimes Z \otimes Z \otimes Z\right) + s\left(-X_{\text{middle}}\right)$$



Adiabatic TQC Defect Creation

- A defect can be created adiabatically

- Proposed evolution



$$H(s) = (1-s)\,(-Z \otimes Z \otimes Z \otimes Z - Z \otimes Z \otimes Z \otimes Z) + s\,(-X)$$

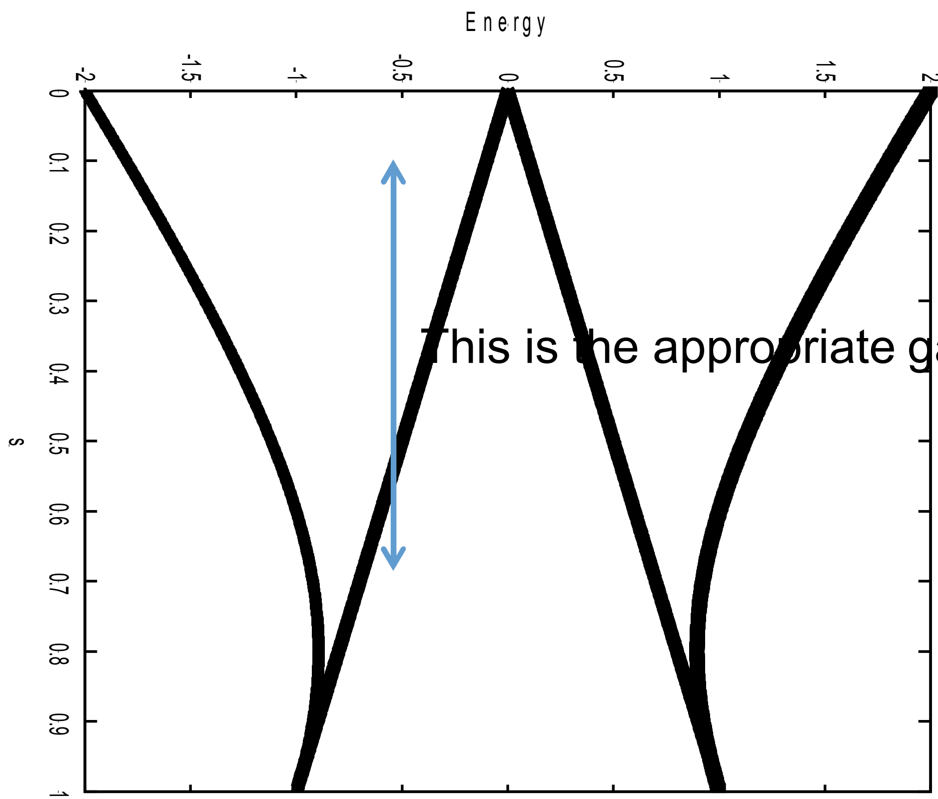
Adjacent face checks

Single new check in the middle

- How do we get to a degenerate ground space from a non-degenerate one in an adiabatic fashion?

Adiabatic TQC Defect Creation

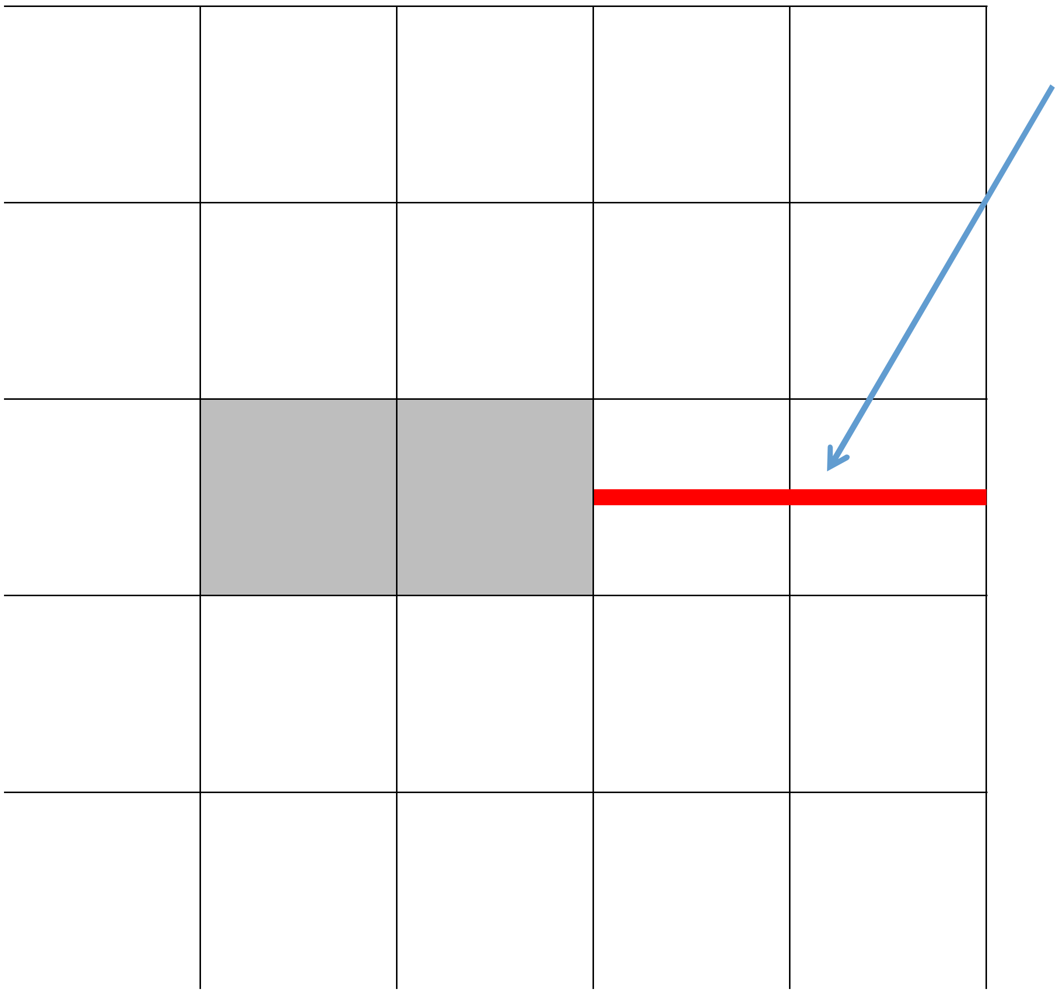
- Only a topologically nontrivial operator couples the lowest eigenstates that cross



This is the appropriate gap to examine

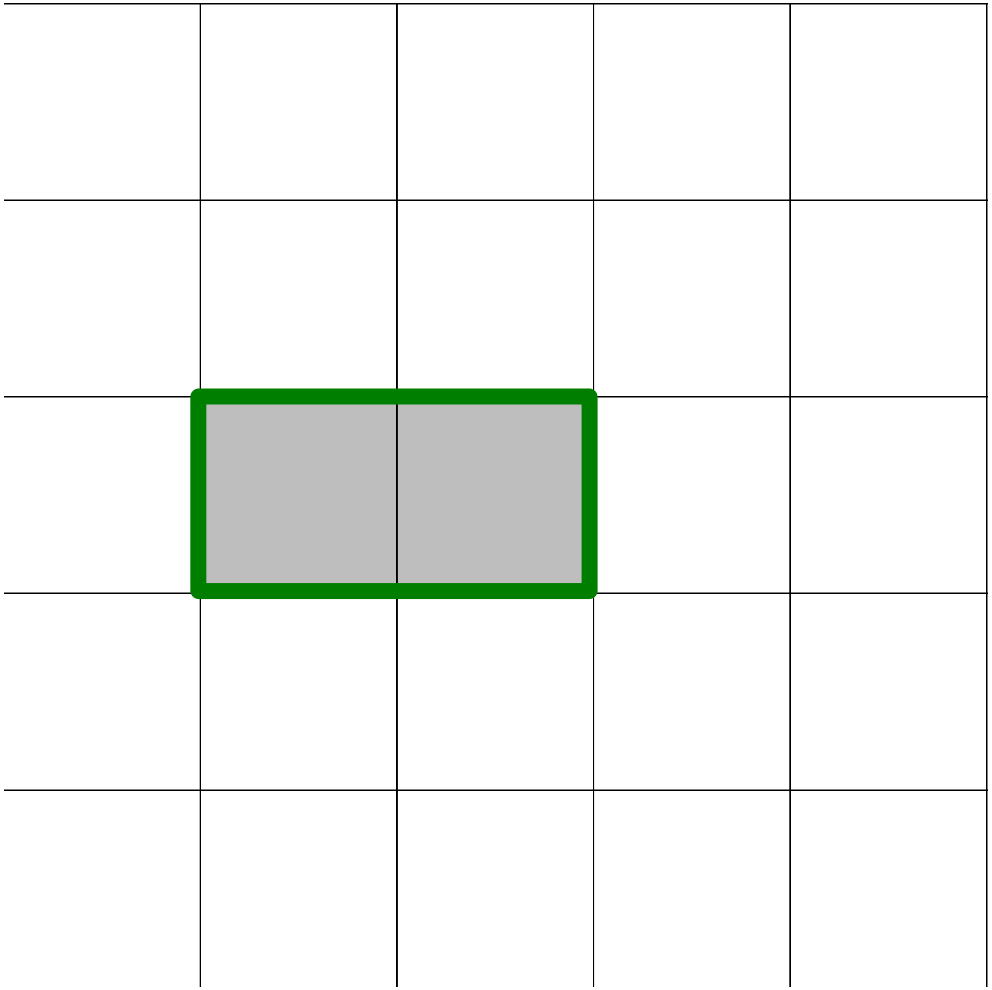
Aren’t crossings bad for adiabaticity?

Adiabatic TQC Defect Creation



This has to happen to couple those two states, and during a real computation this would be a high weight operator. The environment is unlikely to have access to this degree of freedom. This is why we use topological codes in the first place!

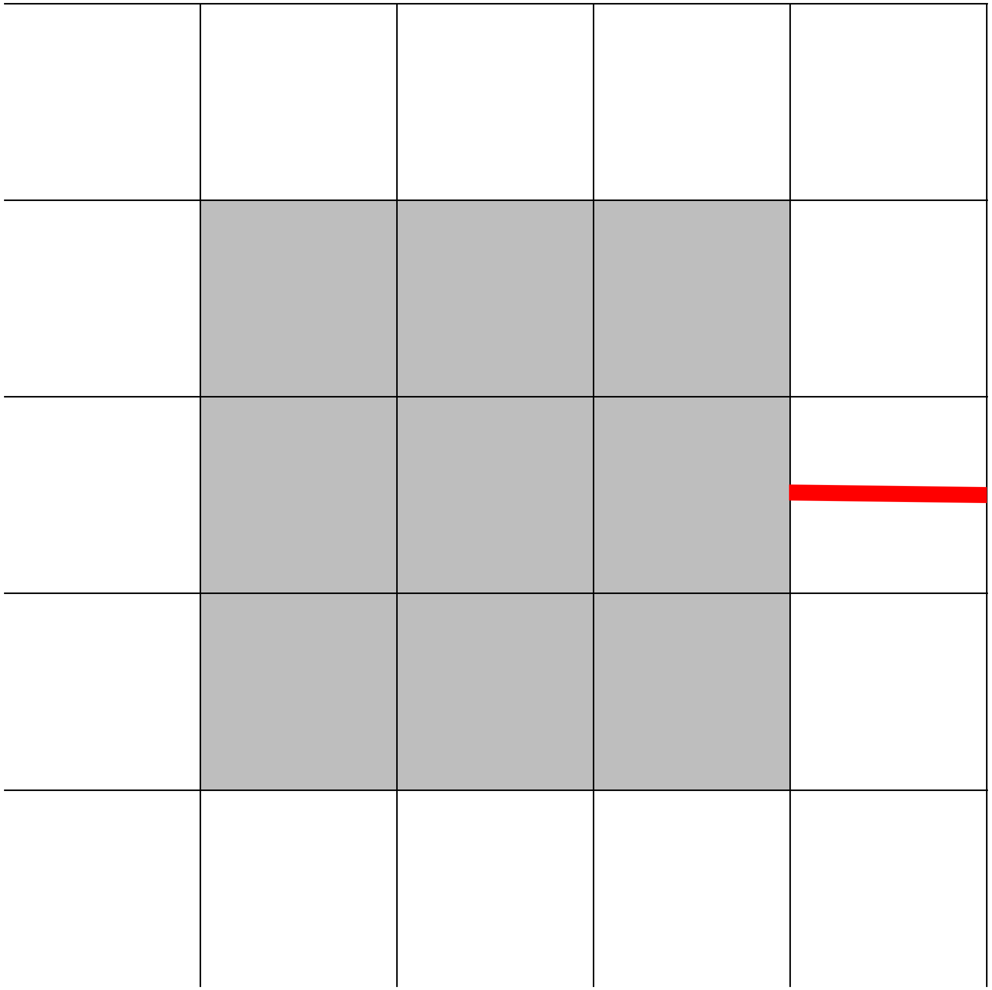
Adiabatic TQC Measurement



To measure Z_L shrink the defect to its original size and measure the weight six operator around the perimeter.

The qubit becomes exposed to this lower weight operator, but since we are measuring in this basis, the measurement outcome is unaffected.

Adiabatic TQC Measurement



To measure X_L keep the perimeter large and move the defect near the appropriate boundary to measure the weight two operator.

The qubit becomes exposed to this lower weight operator, but since we are measuring in this basis, the measurement outcome is unaffected.

Adiabatic TQC Universality

- Smooth defects can be created in the +1 eigenstate of Z_L
- Rough defects can be created in the +1 eigenstate of X_L

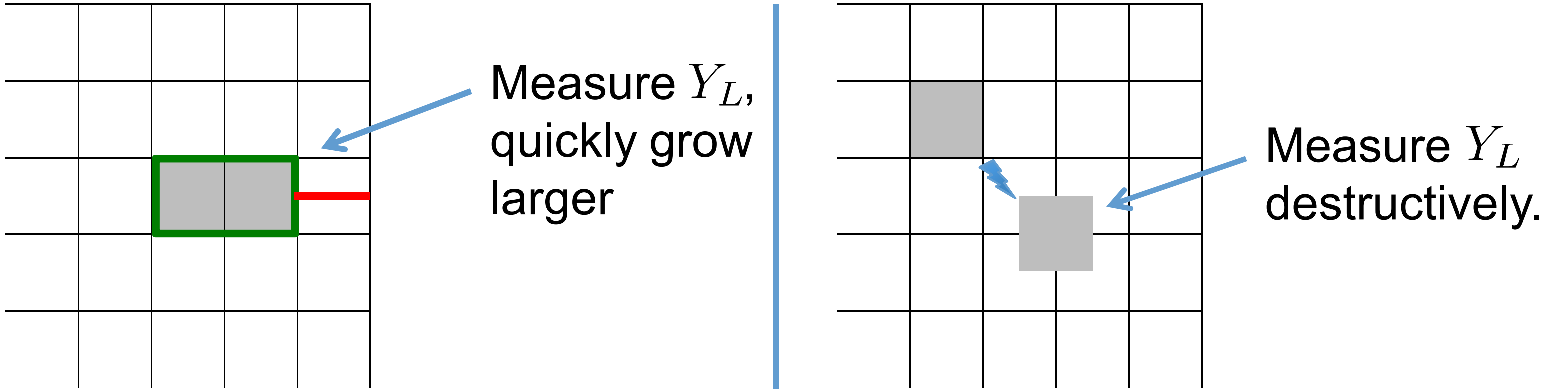
- Measurements allow us to prepare the other eigenstates of logical operators

- Braiding allows us to perform the CNOT gate

- We utilize magic states to achieve the other necessary encoded gates

Adiabatic TQC Magic States

- Two ideas
  - First, make the defect small and bring it near the appropriate boundary, and simply measure the operator whose eigenstate is the desired magic state, e.g. $|+i\rangle$



  - Or, entangle the defect you want the magic state in with an ancilla defect and measure the ancilla destructively
- Still a work in progress!

Other Projects

- GSQC
- Adiabatic code for gaps and spectra
- Hadamard via ACD
- Explicitly adiabatic quantum double computations
- Holonomic vs. Topological QC
- Lattice gauge theories

Quantum Doubles

- If time permits