Natural Frequencies and Mode Shapes of a Coupled Cantilever Array

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Introduction

This is a study material to discuss during the 2013 Summer School at the Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas, Universidad Nacional Autónoma de México (UNAM), Mexico City.

Single Cantilever

From Euler-Bernoulli theory (EB), the natural frequency of a prismatic cantilever beam is

$$\omega_m = u_m{}^2 \sqrt{\frac{EI/L^3}{\rho AL}} \tag{1}$$

For a cantilever beam with no tip mass, EB gives $u_1{}^2 = 1.8751^2 = 3.5160$ [Blevins, 1981]. For realizing the natural frequency with a lumped spring-mass system, assume a spring constant k equal to the static stiffness of the cantilever at the tip, compute the effective lumped mass according to

$$\omega_m{}^2 = \frac{k}{m} = \frac{3EI/L^3}{m} = u_m{}^4 \frac{EI/L^3}{\rho AL} \tag{2}$$

Therefore, the effective mass is

— (3)

Cantilever Array Model

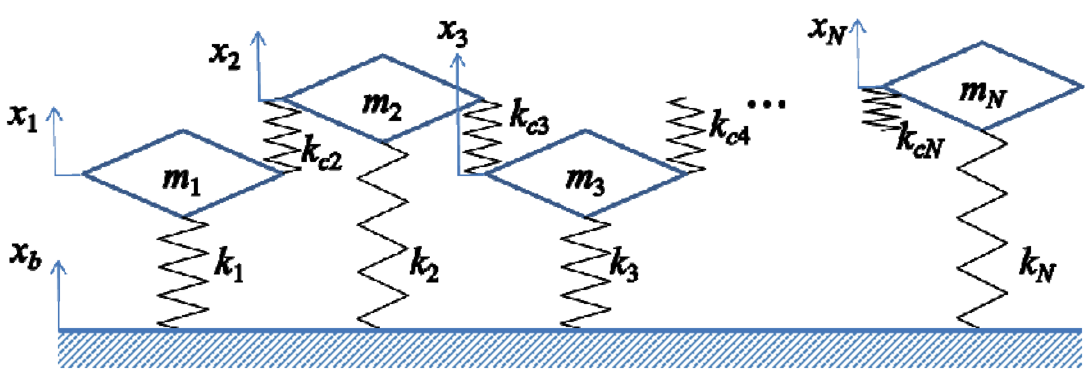


Figure 1: Lumped model of cantilever array

The bases of the cantilevers are excited by a common base displacement x_b. The stiffness of each cantilever i is k_i. The deflection of cantilever i pulls cantilevers i-1 and i+1 through a coupling stiffness k_{ci} and k_{ci+1}. Therefore, the model assumes equations of motion

$$\begin{gathered} m_1\ddot{x}_1 = k_1(x_b - x_1) + k_{c2}(x_2 - x_1) \\ m_2\ddot{x}_2 = k_{c2}(x_1 - x_2) + k_2(x_b - x_2) + k_{c3}(x_3 - x_2) \\ \cdots \\ m_i\ddot{x}_i = k_{ci}(x_{i-1} - x_i) + k_i(x_b - x_i) + k_{ci+1}(x_{i+1} - x_i) \\ \cdots \\ m_N\ddot{x}_N = k_{cN}(x_{N-1} - x_N) + k_N(x_b - x_N). \end{gathered} \tag{4}$$

If all the cantilevers are identical, $k_i = k$, and so are the coupling between all two adjacent cantilevers, $k_{ci} = k_c$, then the equations of motion can be written as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{K_b}x_b \tag{5}$$

where [Spletzer et al, 2008]

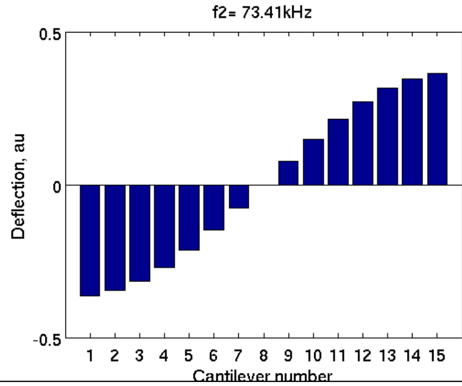
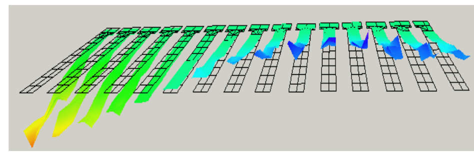
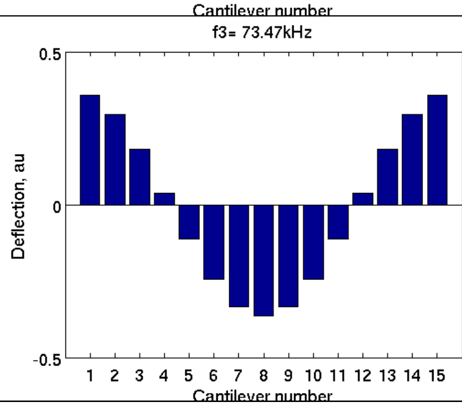
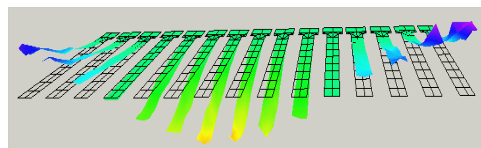
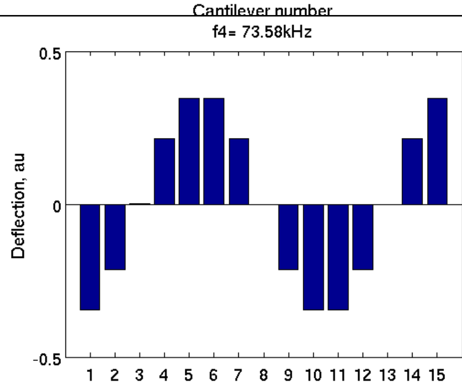
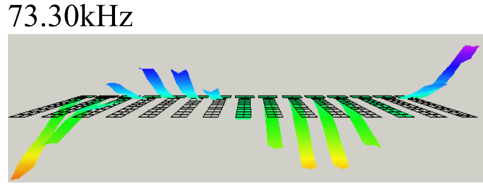
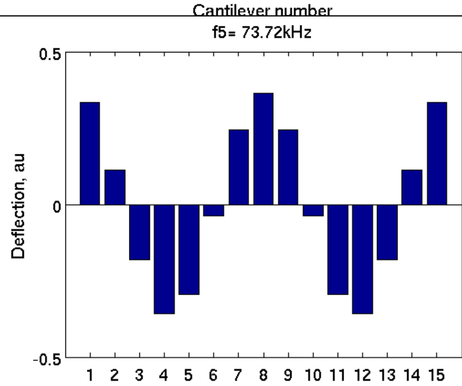
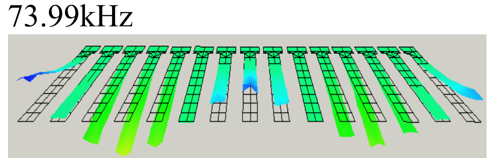
$$\begin{gathered} \mathbf{M} = \mathrm{diag}(m_1, m_2, \dots, m_N) \\ \mathbf{x} = (x_1, x_2, \dots, x_N)^{\mathrm{T}} \\ \mathbf{K_b} = \mathrm{diag}(k_1, k_2, \dots, k_N) \end{gathered} \tag{6}$$

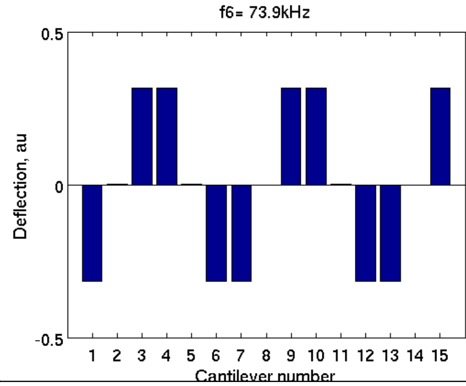
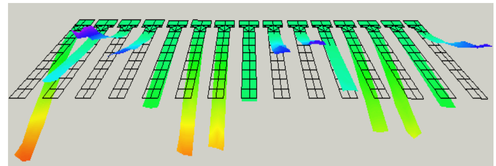
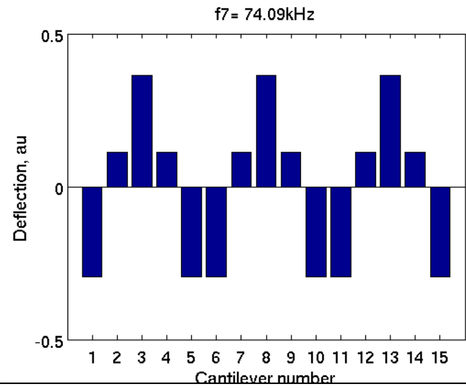
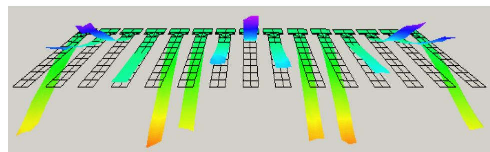
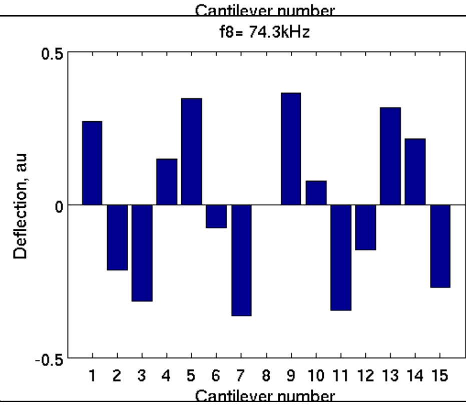
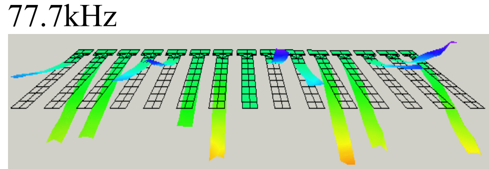
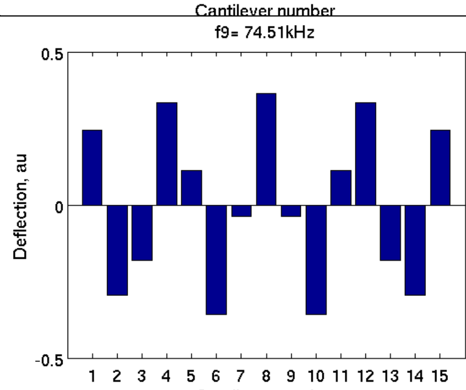
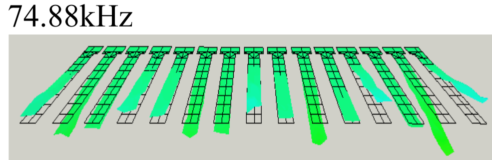
and

$$\mathbf{K} = \begin{bmatrix} k_1 + k_{c2} & -k_{c2} & \cdots & 0 \\ -k_{c2} & k_2 + k_{c2} + k_{c3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & -k_{cN-1} & k_N + k_{cN} \end{bmatrix} \tag{7}$$

The natural frequency and mode shapes can be obtained by solving the eigenvalue problem of Eq. (5).

Mode Shapes

Mode number	Predicted	Measured	
2		68.09kHz 	
3		71.80kHz 	
4		73.30kHz 	
5		73.99kHz 	

6		74.38kHz 	
7		74.60kHz 	
8		77.7kHz 	
9		74.88kHz 	

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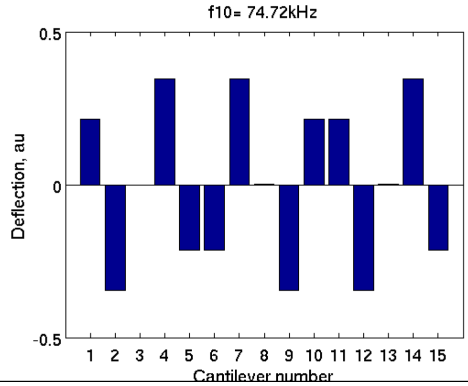
<tr><td>10</td><td>
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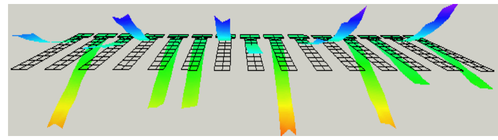
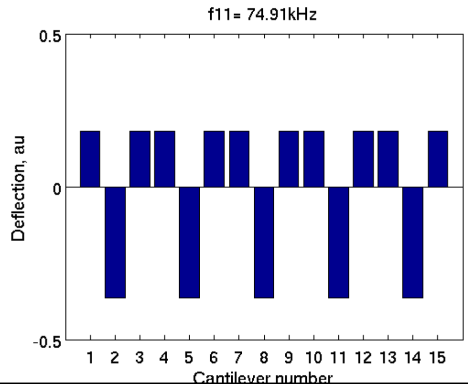
</td><td></td></tr>
<tr><td>11</td><td>
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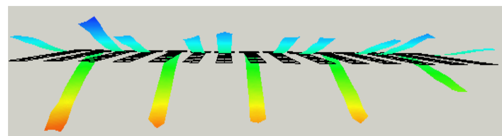</td><td></td></tr>
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