SAND2013-5483P

The Mathematical and Physical Theory of Lossless Beam Shaping

Louis A. Romero
Sandia National Laboratories, Albuquerque, New Mexico,

Fred M. Dickey
FMD Consulting,LLC, 850 West Highland St, Springfield Mo,

I. INTRODUCTION

In this chapter we will present the basic mathematics and physics that are required to understand the theory of lossless beam shaping. Figure 1 is a diagram of the physical situation that we are concerned with. We assume that a parallel beam of coherent light enters an aperture at the plane z = 0. At the aperture the light gets refracted by a combination of a Fourier transform lens with focal length f, and a beam shaping lens. We are interested in the irradiance of the beam at the focal plane z = f. The separation of the refractive elements at the aperture into a Fourier transform lens and a beam shaping lens is convenient for our analysis, and sometimes convenient in practice, but it should be emphasized that these two lenses could in fact be combined into a single lens.

The beam shaping problem is concerned with how to choose the beam shaping lens so that we can transform a beam with an initial irradiance distribution at the plane z = 0 into a beam with a desired irradiance distribution at the focal plane $z = f$. We assume that the beam shaping lens is lossless. This means that it does not absorb or block out any of the energy of the incoming beam. If we assume that the laws of geometrical optics apply, it is possible to transform any initial distribution into any desired output distribution, provided only that the total energy of the incoming and outgoing beams are the same. When we include the effects of diffraction it is in general not possible to accomplish our goal exactly.

One of the major themes of this chapter is to determine the scaling properties of beam shaping systems (what happens when we make our system bigger or smaller, or change the wavelength?). In particular we want to know when the laws of geometrical optics can successfully be applied to designing our system. Due to our emphasis on scaling, we choose to write many of our functions in terms of dimensionless coordinates. For example, if the incoming beam has a radially symmetric Gaussian irradiance distribution many authors would write the irradiance distribution as

$$I(r) = g(r), \tag{1a}$$

where

$$g(r) = e^{-r^2/R^2}. \tag{1b}$$

Here the parameter R determines the basic scale of the irradiance distribution. In this chapter, we would prefer to write this irradiance distribution as

$$I(r) = g(r/R), \tag{2a}$$

where

$$g(\xi) = e^{-\xi^2}. \tag{2b}$$

It might appear simpler to say that the initial irradiance is given by $g(r)$, rather than saying it is given by $g(r/R)$. However, when we consider the scaling properties, the second form is much more powerful. In particular, if we say that the initial distribution is given by $g(r/R)$, then it will be much clearer how to apply the analysis of a system with distribution $g(r/R_1)$ to a system with distribution $g(r/R_2)$.

This approach is motivated by the practice commonly used in fluid mechanics of writing equations in dimensionless form (1, 2). This approach in fluid mechanics allows one to show that different physical systems will have the same behavior provided only that certain “dimensionless parameters” are the same. For example, when fluid flows past a sphere, the behavior of the flow depends on the Reynolds number

$$\mathrm{Re} = \frac{RU_0}{\nu}, \tag{3}$$

where R is the radius of the sphere, U_0 is the velocity far from the sphere, and v is the kinematic viscosity. If two flows have the same Reynolds number, then the patterns of fluid flow will be identical, after rescaling our coordinates. However, if the Reynolds numbers are different, then the flow patterns can look dramatically different. For example, in one case the flow may be turbulent, and in the other case not.

Ideas similar to these can be applied to the theory of beam shaping. Suppose our initial irradiance distribution is given by $g(x/R, y/R)$, and that our desired output irradiance distribution is given by $Q(x/D, y/D)$. The parameter R gives the characteristic length of the incoming beam, and D is the characteristic length of the output beam. If the wavelength of the light is λ, and we are imaging our output at a distance f from the aperture, then the dimensionless parameter

$$\beta = \frac{2\pi RD}{\lambda f} \tag{4}$$

is very important to understanding beam shaping. In particular, suppose that we design a lens that solves the beam shaping problem in the geometrical optics limit, and now we analyze how this lens works when the wavelength is finite. We will see that the irradiance distributions of two beam shaping systems will be geometrically similar, provided only that they have the same shape functions $g(s, t)$, and $Q(s, t)$, and provided the parameters β for the two systems are the same. This means that we can transform the irradiance distribution of one system into the irradiance distribution of the other system by merely rescaling our axes. In particular, one system will suffer from diffraction effects if and only if the other system (with identical β) also does.

Geometrical optics is a short wavelength approximation, so it is clear that we would like β to be large in order for geometrical optics to hold. We will see that if β is large it is relatively simple to do beam shaping, but if it is small, the uncertainty principle of signal analysis shows that it is essentially impossible.

Another important feature in determining the difficulty of a beam shaping problem is the continuity of the beam shaping lens. If the surface of the element designed using geometrical optics is infinitely differentiate, then we will not need a very high value of β in order to achieve good results. To be more precise, the effects of diffraction will die down like $1/\beta^2$ as β gets to be large. However, if the lens has a discontinuity in its third derivative, the effects of diffraction will die down like $1/\sqrt{\beta}$ in parts of the image plane, and hence we will need a much larger value of β in order to approach the geometrical optics limit. If the lens has discontinuities in the first or second derivatives, we will need to use even larger values of β before we can ignore the effects of diffraction.

If the input beam is smooth (such as Gaussian), then the continuity properties of the lens designed using geometrical optics are controlled by the continuity of the desired output beam. If one has a good understanding of geometrical beam shaping, it is not too difficult to see how the continuity of the desired output beam will affect the continuity of the lens. However, if one is not familiar with this theory, the results can be somewhat surprising. For example, Fig. 2 shows examples of three desired output beams. One might naively think that all of these beams have abrupt discontinuities in them, so they may all lead to equally difficult beam shaping problems. It turns out, however, that the output in Fig. 2a will lead to an infinitely differentiable lens, the beam in Fig. 2b leads to a lens with a discontinuity in the second derivative, and the lens required to produce Fig. 2c will have a discontinuity in the first derivative. The outputs 2a, 2b, and 2c get progressively harder to achieve.

This whole chapter is devoted to understanding the points we have just discussed. We feel that it is worth writing them down as succinctly as possible.

- In the geometrical optics approximation it is possible to turn a beam with a given initial distribution into a beam with any desired output distribution, provided only that the total energy of the input and output beams are the same.
- Diffraction effects make it impossible to do beam shaping exactly when we take into account the finite wavelength of light. For given shapes of the input and output beams, the parameter $\beta = 2\pi RD/\lambda f$ determines the difficulty of the beam shaping problem. If β is large, then the laws of geometrical optics will be a good approximation.

- If the surface of the element designed using geometrical optics has discontinuities in its first, second, or third derivatives, then we will need higher values of β in order for geometrical optics to be a good approximation.

In Sec. II we discuss some mathematical prerequisites for understanding the theory of beam shaping. After a brief summary of the basics of Fourier transforms, we prove the uncertainty theorem from signal analysis. In Sec. VII) this theorem will be used to show why it is impossible to do a good job of beam shaping when β is small. Section II also includes a discussion of how to use the Hankel transform in order to obtain radially symmetric Fourier transforms. This is important when analyzing the effects of diffraction on radially symmetric problems.

In Sec. III we outline the theory of stationary phase, with an emphasis on how discontinuities in the higher derivatives of the phase function can slow down the convergence. In Sec. VIII we use the method of stationary phase in order to obtain the large β approximation to the diffractive theory of beam shaping. We will see that the first term in the stationary phase approximation is equivalent to the geometrical optics approximation. We also use the method of stationary phase in Sec. V in order to analyze the errors introduced by making the Fresnel approximation.

Sections IV–VI discusses the electromagnetic theory necessary to understand beam shaping. Section IV presents a review of Maxwell's equations, Sec. V discusses the geometrical optics limit with an emphasis on Fermat's principle, and Sec. VI discusses the theory of Fresnel diffraction. Fresnel diffraction theory allows us to turn the physical problem of beam shaping into a mathematical problem involving Fourier transforms.

In Sees. VII and VIII we bring all of our tools together and discuss the theory of beam shaping. Section VII gives the theory of beam shaping in the geometrical optics limit, and Sec. VIII discusses the theory of beam shaping with diffraction effects taken into account. When the diffractive equations for beam shaping are written in dimensionless form, the importance of the parameter β will become evident. We will use the method of stationary phase to analyze the large β limit of the equations. The fact that our geometrical optics solution is based on a stationarity condition (Fermat's principle), and our large β approximation is also based on a stationarity condition (stationary phase), causes these two analyses to look almost identical. We end Sec. VIII by giving some examples that illustrate the principles concerning the importance of β, and the smoothness of the shape of the lens.

II. MATHEMATICAL PRELIMINARIES

A. Basic Fourier Analysis

The theory of Fresnel diffraction will allow us to write our beam shaping problem as a problem in Fourier analysis. For this reason it is impossible to understand our theoretical treatment of beam shaping if one is not familiar with some of the basic concepts from Fourier analysis. In later sections we will use both one- and two-dimensional Fourier analysis.

There are several definitions of the Fourier transform used in the literature. The differences are very minor, concerning only the sign of the complex exponential, and the constant in front of the integral. However, these differences can be annoying when one is using a table of Fourier transforms, or applying theorems such as Parseval's equalilty. The definition we use here is probably the most commonly used (3, 4).

Definition 1 The Fourier transform of a function $f(x)$ is defined as

$$F(\omega) = Tf(x) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx. \tag{5}$$

An almost identical definition holds for two-dimensional functions.

Definition 2 The Fourier transform of a function $f(x, y)$ is defined as

$$F(\omega_x, \omega_y) = Tf(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y) e^{-i(\omega_x x + \omega_y y)} dx\, dy. \tag{6}$$

The following are some well-known theorems in Fourier analysis that we will use throughout this chapter.

Theorem 1 *One-dimensional Fourier Inversion Theorem—If* $F(\omega)$ *is the Fourier transform of* $f(x)$, *then*

$$f(x) = T^{-1}(F(\omega)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}\,d\omega. \tag{7}$$

Theorem 2 *Two-dimensional Fourier Inversion Theorem—If* $F(\omega_x, \omega_y)$ *is the Fourier transform of* $f(x, y)$*, then*

$$f(x,y) = T^{-1}(F(\omega_x,\omega_y)) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\omega_x,\omega_y)e^{i(\omega_x x+\omega_y y)}\,d\omega_x d\omega_y. \tag{8}$$

Theorem 3 *One-dimensional Parseval's Equality—A function* $f(x)$ *and its Fourier transform* $F(\omega)$ *satisfy*

$$\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega = 2\pi\int_{-\infty}^{\infty} |f(x)|^2\, dx. \tag{9}$$

Theorem 4 *Two-dimensional Parseval's Equality—A function* $f(x, y)$ *and its Fourier transform* $F(\omega_x, \omega_y)$ *satisfy*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F(\omega_x,\omega_y)|^2\, d\omega_x d\omega_y = 4\pi^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,y)|^2\, dx\,dy. \tag{10}$$

Theorem 5 *The One-dimensional Fourier Convolution Theorem—Suppose* $F(\omega)$ *and* $G(\omega)$ *are the Fourier transforms of the functions* $f(x)$ *and* $g(x)$*. The inverse Fourier transform of* $F(\omega)G(\omega)$ *is given by*

$$T^{-1}(F(\omega)G(\omega)) = \int_{-\infty}^{\infty} f(\xi)g(x-\xi)\,d\xi. \tag{11}$$

Theorem 6 *The Two-dimensional Fourier Convolution Theorem—Suppose* $F(\omega_x, \omega_y)$ *and* $G(\omega_x, \omega_y)$ *are the Fourier transforms of the functions* $f(x, y)$ *and* $g(x, y)$*. The inverse Fourier transform of* $F(\omega_x, \omega_y)G(\omega_x, \omega_y)$ *is given by*

$$T^{-1}(F(\omega_x,\omega_y)G(\omega_x,\omega_y)) = \int_{-\infty}^{\infty} f(\xi,\eta)g(x-\xi,y-\eta)\,d\xi\,d\eta. \tag{12}$$

Theorem 7 *Transforms of derivatives—The Fourier transform of the derivative is given by*

$$T\left(\frac{df}{dx}\right) = i\omega F(\omega). \tag{13}$$

Theorem 8 *Transforms of partial derivatives—The Fourier transform of the partial derivatives are given by*

$$T\left(\frac{\partial f}{\partial x}\right) = i\omega_x F(\omega_x,\omega_y) \tag{14}$$

and

$$T\left(\frac{\partial f}{\partial y}\right) = i\omega_y F(\omega_x,\omega_y). \tag{15}$$

Although the Cauchy–Schwartz inequality is not really a theorem in Fourier analysis, we will need it in our proof of the uncertainty principle, and hence now state it.

Theorem 9 *The Cauchy–Schwartz Inequality (for infinite integrals)—For any function* $f(x)$ *and* $g(x)$ *we must have*

$$\left|\int_{-\infty}^{\infty} f(x)\bar{g}(x)dx\right|^2 \le \int_{-\infty}^{\infty} |f(x)|^2\,dx\int_{-\infty}^{\infty} |g(x)|^2\,dx. \tag{16}$$

The two sides are equal if and only if there is a constant λ *such that* $f(x) = \lambda g(x)$*.*

B. The Uncertainty Principle and the Space Bandwidth Product

In this section we discuss the space bandwidth product, and the uncertainty principle of signal analysis (5). This discussion is crucial to understanding the theory of beam shaping. As we shall see in later sections, in a beam shaping system, the space bandwidth product is related to the parameter β discussed in the introduction. In Sec. VIII we will use the the uncertainty principle to show that it is impossible to do a good job of beam shaping if β is small.

The Heisenberg uncertainty principle of quantum mechanics (6) states that the product of the uncertainty in position times the uncertainty in momentum must be greater than $h/2\pi$:

$$\Delta p\,\Delta x > \frac{h}{2\pi}. \tag{17}$$

In order to make this precise we must define precisely what we mean by Δx and Δp. This principle was one of Heisenberg's basic assumptions in his development of matrix mechanics. However, it can also be derived by assuming the wave mechanics of Schroedinger. The derivation of the result depends on the fact that the wave function for momentum is the Fourier transform of the wave function for position, and on the subject of this section, the uncertainty principle from Fourier analysis.

All of our derivations will be limited to one-dimensional functions and their transforms, but almost identical derivations apply for two-dimensional transforms. Once we have derived the one-dimensional results, we will state the two-dimensional results without proof. We now define the uncertainty in $f(x)$ and $F(\omega)$.

Definition 3 The uncertainty in $f(x)$ and its transform $F(\omega)$ are given by

$$\Delta_f = \sqrt{\frac{\int_{-\infty}^{\infty} x^2 \,|\, f(x)\,|^2 dx}{\int_{-\infty}^{\infty} |\, f(x)\,|^2 dx}}, \tag{18}$$

and

$$\Delta_F = \sqrt{\frac{\int_{-\infty}^{\infty} \omega^2 \,|\, F(\omega)\,|^2 d\omega}{\int_{-\infty}^{\infty} |\, F(\omega)\,|^2 d\omega}}. \tag{19}$$

The uncertainty principle concerns the product of these two quantities, and is simply related to the space bandwidth product.

Definition 4 The space bandwith product of a function $f(x)$ is defined as

$$\text{space bandwith product} = \Delta_f \Delta_F. \tag{20}$$

It should be noted that the space bandwidth product of a function does not depend on the scaling of the function.

Lemma 1 *For any nonzero constant a, and nonzero real number b, the space bandwidth product of af* (*bx*) *is the same as the space bandwidth product of f*(*x*).

We are now ready to state the uncertainty principle of signal analysis.

Theorem 10 *The One-dimensional Uncertainty Principle—For any square integrable function f*(*x*) *the space bandwidth product must be greater than* 1/2. *In other words,*

$$\Delta_f \Delta_F \geq 1/2. \tag{21}$$

Proof: The Cauchy–Schwartz inequality implies that

$$\left| \int_{-\infty}^{\infty} (x\bar{f}) \frac{df}{dx} dx \right|^2 \leq \int_{-\infty}^{\infty} x^2 \,|\, f(x)\,|^2 \, dx \int_{-\infty}^{\infty} \left| \frac{df}{dx} \right|^2 dx. \tag{22}$$

Clearly

$$\left| \int_{-\infty}^{\infty} (x\bar{f}) \frac{df}{dx} dx \right|^2 \geq \left| \operatorname{Re} \int_{-\infty}^{\infty} (x\bar{f}) \frac{df}{dx} dx \right|^2. \tag{23}$$

We can write

$$\operatorname{Re} \int_{-\infty}^{\infty} x\bar{f} \frac{df}{dx} dx = \frac{1}{2} \int_{-\infty}^{\infty} x \left(f \frac{d\bar{f}}{dx} + \bar{f} \frac{df}{dx} \right) dx = -\frac{1}{2} \int_{-\infty}^{\infty} |\, f(x)\,|^2 \, dx. \tag{24}$$

The inequalities (22) and (23) now imply

$$\frac{1}{4} \left| \int_{-\infty}^{\infty} |\, f(x)\,|^2 \, dx \right|^2 \leq \int_{-\infty}^{\infty} x^2 \,|\, f(x)\,|^2 \, dx \int_{-\infty}^{\infty} \left| \frac{df}{dx} \right|^2 dx. \tag{25}$$

Since the Fourier transform of df/dx is $i\omega F(\omega)$, Parseval's equality implies that

$$\int_{-\infty}^{\infty}\left|\frac{df}{dx}\right|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}\omega^2 \,|\, F(\omega)\,|^2 \, d\omega. \tag{26}$$

The inequality (25) can now be written as

$$\frac{1}{4}\left|\int_{-\infty}^{\infty}|\, f(x)\,|^2 \, dx\right|^2 \le \frac{1}{2\pi}\int_{-\infty}^{\infty} x^2 \,|\, f(x)\,|^2 \, dx \int_{-\infty}^{\infty}\omega^2 \,|\, F(\omega)\,|^2 \, d\omega. \tag{27}$$

Using Parseval's equality we can write this as

$$\begin{aligned}&\frac{1}{4}\frac{1}{2\pi}\int_{-\infty}^{\infty}|\, F(\omega)\,|^2 \, d\omega \int_{-\infty}^{\infty}|\, f(x)\,|^2 \, dx \\ &\quad\le \frac{1}{2\pi}\int_{-\infty}^{\infty} x^2 \,|\, f(x)\,|^2 \, dx \int_{-\infty}^{\infty}\omega^2 \,|\, F(\omega)\,|^2 \, d\omega.\end{aligned} \tag{28}$$

If we now divide both sides of this inequality by the left-hand side we arrive at the desired result. QED

Lemma 2 *We have* $\Delta_f\,\Delta_F = 1/2$ *iff the function* $f(x)$ *is a real Gaussian,* $f(x) = Ae^{-\alpha x^2}$ *where a is α real number*.

Proof: In order to get an equality in the uncertainty relation we must have an equality in the Cauchy Schwartz inequality in (22). This implies that $df/dx = -2x\lambda f$, and hence $f(x) = Ae^{-\lambda x^2}$. It is also necessary that we get an equality in (23). This will be the case iff $\bar{f}\,(df\,/\,dx)$ is real, which will be true iff λ is real. QED

Although the space bandwidth product can never be less than 1/2, there is no limitation to how big it can be. For example, the function $f(x) = e^{ix^2}$ has an infinite space bandwidth product.

Suppose we change the phase of the function $f(x)$ by multiplying it by the phase function $e^{iq(x)}$. How should we choose the phase q so that the function $f(x)e^{iq(x)}$ has a minimum space bandwidth product? Note that the phase function does not change the uncertainty in x, but it does change the uncertainty in ω. This question has implications for the depth of field of a laser beam shaping system. The following theorem gives a very simple answer to this question.

Theorem 11 *The function* $q(x)$ *that minimizes the space bandwidth product of* $f(x)e^{iq(x)}$ *is the one that makes the phase of* $f(x)e^{iq(x)}$ *constant*.

Proof: The only integral in the space bandwidth product that changes with the function $q(x)$ is the integral

$$\int_{-\infty}^{\infty}\omega^2 \,|\, G(\omega)\,|^2 \, d\omega, \tag{29}$$

where $G(\omega)$ is the Fourier transform of $f(x)e^{iq(x)}$. Let $f(x)e^{iq(x)} = A(x)e^{i\psi(x)}$, where $A(x)$ is a positive real function. Parseval's equality, and the formula for the Fourier transform of a derivative, show that

$$\int_{-\infty}^{\infty}\omega^2 \,|\, G(\omega)\,|^2 \, d\omega = 2\pi\int_{-\infty}^{\infty}\left|\frac{d}{dx}(A(x)e^{i\psi(x)})\right|^2 dx. \tag{30}$$

This last integral can be written as

$$2\pi\int_{-\infty}^{\infty}\left(\left(\frac{dA}{dx}\right)^2 + A^2(x)\left(\frac{d\psi}{dx}\right)^2\right)dx \ge 2\pi\int_{-\infty}^{\infty}\left(\frac{dA}{dx}\right)^2 dx. \tag{31}$$

This clearly implies that this integral, and hence the space bandwidth product is minimized by choosing the function ψ so that it is constant. QED

This theorem will be used in Chapter 3 (Sec. III) of this book when discussing the collimation of beams.

We now summarize how these results apply for two-dimensional functions. In two dimensions, the uncertainty will be defined as

$$(\Delta_f)^2 = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x^2+y^2)\,|\,f(x,y)|^2\,dx\,dy}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\,f(x,y)|^2\,dx\,dy}, \tag{32}$$

$$(\Delta_F)^2 = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\omega_x^2+\omega_y^2)\,|\,F(\omega_x,\omega_y)|^2\,d\omega_x\,d\omega_y}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\,F(\omega_x,\omega_y)|^2\,d\omega_x\,d\omega_y}. \tag{33}$$

The space bandwidth product is once again defined as $\Delta_f\Delta_F$. The two-dimensional uncertainty principle gives

Theorem 12 *The Two-dimensional Uncertainty Principle—For any square integrable function* $f(x, y)$ *the space bandwidth product must be greater than* 1. *In other words,*

$$\Delta_f\Delta_F \geq 1. \tag{34}$$

C. Separation of Variables in Cylindrical Coordinates

When you take the Fourier transform of a function $f(x, y)$ that has radial symmetry, you end up with a Fourier transform $F(\omega_x, \omega_y)$ that has radial symmetry in the Fourier domain. That is, if we can write

$$f(x,y) = g(r), \tag{35}$$

where $r = \sqrt{x^2+y^2}$, then we can write

$$F(\omega_x,\omega_y) = G(\alpha), \tag{36}$$

where $\alpha = \sqrt{\omega_x^2+\omega_y^2}$. The transformation that takes the function $g(r)$ into the function $G(\alpha)$ is known as a Hankel transform (4). This transform allows us to find the two dimensional Fourier transform of a radially symmetric function by performing a one-dimensional integral. The Hankel transform can be very useful when analyzing diffraction effects in beam shaping problems with radial symmetry.

In order to understand Hankel transforms it is necessary to be familiar with an identity in the theory of Bessel functions. In order to understand this identity we begin by considering the reduced wave equation in polar coordinates:

$$\nabla^2 p + k^2 p = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p}{\partial\phi^2} + k^2 p = 0. \tag{37}$$

If we assume solutions of the form

$$p(r,\phi) = f(r)e^{im\phi}, \tag{38}$$

we find that the function $f(r)$ must satisfy

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \frac{m^2 f}{r^2} + k^2 f = 0. \tag{39}$$

If $g(r)$ is a solution to

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dg}{dr}\right) - \frac{m^2 g}{r^2} + g = 0, \tag{40}$$

then $f(r) = g(kr)$ is a solution to Eq. (39).

The equation (40) is known as Bessel's equation. The solutions that are regular at $r = 0$ are called Bessel functions. They are written as $J_m(r)$. If we were interested in the waves emitted from a circular cylinder, we would not require that the solution was finite at $r = 0$, but that as $r\to\infty$ the solution represented only outgoing waves. In this case we would use the solution to Bessel's equation $H_m^1(kr)$. This is known as the Hankel function of the first kind. Our goal is to understand the Hankel transform as a circularly symmetric Fourier transform. For this purpose we only need the regular solutions to Bessel's equation, which means we only need to consider the function $J_n(kr)$ where n is an integer.

One of the most elegant ways of approaching the theory of Bessel functions (7) is through the use of an integral identity, which we will now derive. This identity allows us to derive almost all of the most commonly known properties of Bessel functions such as their asymptotic behavior for large indices, asymptotic behavior for large argument, recursion formulas, and the behavior near the origin. This identity is almost the only property of Bessel functions that we will need in order to understand the Hankel transform.

The integral identity can be derived by considering the function

$$F(x,y) = e^{ix}. \tag{41}$$

This clearly satisfies the two-dimensional reduced wave equation

$$\nabla^2 F + F = 0. \tag{42}$$

We can express this in terms of polar coordinates, and then expand the function in a Fourier series. If we do this we find that

$$e^{ir\cos(\theta)} = \sum_{k=-\infty}^{\infty} a_k(r) e^{ik\theta}. \tag{43}$$

From our discussion at the begining of the section we know that this last infinite sum will satisfy the reduced wave equation if the functions $a_k(r)$ satisfy Bessel's equation. Due to the rotational symmetry of the reduced wave equation, it can be shown that in order for this infinite sum to satisfy the reduced wave equation it is necessary that each individual term satisfy the reduced wave equation. This means that it is necessary (not just sufficient) that the functions $a_k(r)$ satisfy Bessel's equation. It is also clear that they must be bounded at $r = 0$. It follows that they are multiples of the Bessel functions $J_k(r)$. We will in fact define the Bessel functions so that the multiplicative factor is unity. This gives us the result

$$e^{ir\cos(\theta)} = \sum_{k=-\infty}^{\infty} J_k(r) e^{ik\theta}. \tag{44}$$

Using the fact that the right-hand side is the Fourier expansion of the function $e^{ir\cos(\theta)}$ we arrive at the identity

$$J_k(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(r\cos(\theta)-k\theta)} d\theta. \tag{45}$$

D. Hankel Transforms

The Fourier transform of $f(x, y)$ can be written as

$$F(\omega_x,\omega_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\omega_x x+\omega_y y)} f(x,y)\,dx\,dy. \tag{46}$$

Suppose we write both the original function $f(x, y)$ and the Fourier transform in terms of polar coordinates:

$$(x,y) = r(\cos(\theta),\sin(\theta)), \tag{47}$$

$$(\omega_x,\omega_y) = \alpha(\cos(\phi),\sin\phi)). \tag{48}$$

The Fourier transform can be written as

$$F(\alpha,\phi) = \int_0^{\infty}\int_{-\pi}^{\pi} e^{-ir\alpha\cos(\theta-\phi)} f(r,\theta) r\,dr\,d\theta. \tag{49}$$

If the function $f(x, y)$ is independent of θ, then the transform $F(\alpha, \phi)$ will be independent of ϕ. It follows that we can write

$$F(\alpha) = \int_0^{\infty}\int_{-\pi}^{\pi} e^{-ir\alpha\cos(\theta)} f(r) r\,dr\,d\theta. \tag{50}$$

If we perform the integral with respect to θ first, and use the integral representation of J_0, we get

$$F(\alpha) = 2\pi\int_0^{\infty} J_0(\alpha r) f(r) r\,dr. \tag{51}$$

The function $F(\alpha)$ is known as the Hankel transform of the function $f(r)$. We can apply the same steps to show that the inverse Hankel transform is given by

$$f(r) = \int_0^{\infty} J_0(kr) F(k) k\,dk. \tag{52}$$

III. THE METHOD OF STATIONARY PHASE

A. The Basic Idea of Stationary Phase

The method of stationary phase (8) is an asymptotic method, first used by Stokes and Kelvin, for evaluating integrals whose integrands have a very rapidly varying phase. The method is very important in the theory of dispersive wave propagation where it motivates the concept of group velocity (9, 10). In the theory of beam shaping it can be used to derive the geometrical optics limit from the theory of Fresnel diffraction, and more importantly, it gives us bounds on when the geometrical theory is applicable.

We will now give a brief heuristic derivation of the lowest-order term in the approximation. Suppose we have an integral of the form

$$H(\gamma) = \int_{-\infty}^{\infty} e^{i\gamma q(\xi)} f(\xi)\, d\xi, \tag{53}$$

and we are interested in evaluating this integral for large values of γ. Intuitively we expect that intervals where the function $\gamma q(\xi)$ is changing rapidly will give neglibible contributions to this integral. If the derivative of q vanishes at $\xi = \xi_0$, we expect the main contribution to come from the region very near ξ_0. To a first approximation we can write

$$H(\gamma) \approx f(\xi_0) e^{i\gamma q(\xi_0)} \int_{-\infty}^{\infty} e^{i\gamma q''(\xi_0)(\xi-\xi_0)^2/2} d\xi. \tag{54}$$

We have arrived at this expression by assuming that the major contribution comes from a small region around ξ_0, and hence we have approximated the function $f(\xi)$ as being constant, and equal to $f(\xi_0)$. We have also expanded the function $q(\xi)$ in a Taylor series about ξ_0, keeping only the terms up to the quadratic. The integral can now be evaluated analytically to give

$$H(\gamma) \approx f(\xi_0) e^{i\mu\pi/4} e^{i\gamma q(\xi_0)} \sqrt{\frac{2\pi}{\gamma\, |q''(\xi_0)|}}, \tag{55}$$

where

$$\mu = \operatorname{sgn}\left(\left. \frac{d^2 q(\xi)}{d\xi^2} \right|_{\xi_0} \right). \tag{56}$$

Here we have assumed that there is exactly one point where the phase is stationary. If there is more than one point, then we must sum over all points that are stationary in order to get our asymptotic expansion. If there are no stationary points, then the integral will die down exponentially fast with γ provided the functions $q(\xi)$ and $f(\xi)$ are infinitely differentiate, and the function $f(\xi)$ and all of its derivatives decay as $|\xi|\to\infty$. If there is no stationary point, but the function $f(\xi)$ has a discontinuity in it, we are at least guaranteed that the integral dies down like $1/\gamma$ as $\gamma\to\infty$. It is not too difficult to make this heuristic derivation more rigorous.

B. The Rate of Convergence of the Method of Stationary Phase

In our discussion of beam shaping we will see that the lowest-order term in the stationary phase approximation to the diffraction integral gives us the geometrical optics approximation. In this case the parameter β discussed in the introduction will serve as our large parameter in the phase of the integrand. In order to understand what sorts of errors are produced when we use the geometrical optics approximation, we need to understand the higher-order terms in the method of stationary phase. It is not important for us to have exact expressions for the higher-order terms, but we need to know how fast they die down with γ.

The subject of how to correct the lowest-order term in the method of stationary phase gets somewhat technical, so we feel that it is best if we begin by summarizing the main results. In our analysis of beam shaping we will have another parameter in our phase function, so our integrals will be of the form

$$H(x,\gamma) = \int_{-\infty}^{\infty} e^{i\gamma q(\xi,x)} f(\xi) d\xi. \tag{57}$$

Here the parameter ξ represents a point on the aperture, and x represents a point at the focal plane. The function $q(\xi, x)$ will be proportional to the travel time required to get from a point ξ on the aperture to a point x in physical space. In practice ξ and x will be two-dimensional vectors, but we assume they are scalars here in order to simplify the presentation. This one-dimensional case will be directly relevent for the case where our input and output beams can be written as a direct product of two one-dimensional distributions.

Let $\xi_0(x)$ be the point at which the phase is stationary. We will show that if the functions $q(\xi, x)$ and $f(\xi)$ are infinitely differentiate at the stationary point $\xi_0(x)$, and

$$\left.\frac{\partial^2 q(\xi,x)}{\partial \xi^2}\right|_{\xi_0} \neq 0, \tag{58}$$

then the next order correction dies down like $1/\gamma^{3/2}$. This gives us an expression of the form

$$H(x,\gamma) = \frac{A(x)}{\gamma^{1/2}} + \frac{iB(x)}{\gamma^{3/2}} + \cdots. \tag{59}$$

In this case the relative error between the first-order term and the exact solution will die down like $1/\gamma$. If the function $f(\xi)$ is real, then the functions $A(x)$ and $B(x)$ will have the same phase. This implies that the relative error between $|H(x, \gamma)|^2$ and the value predicted by the first term in the method of stationary phase will be $O(1/\gamma^2)$.

This expression will remain valid provided the functions $f(\xi)$ and $\partial^2 q/\partial\xi^2$ are differentiable at $\xi_0(x)$. If these functions are continuous, but not differentiable at some point x^*, then at the point x^*, the next order term in the method will be of the form

$$H(x^*,\gamma) = \frac{A(x^*)}{\gamma^{1/2}} + \frac{B(x^*)}{\gamma} + \cdots. \tag{60}$$

We see that in this case the relative error between the first order-term and the exact solution will die down like $1/\gamma^{1/2}$. Furthermore, in this case the functions $A(x)$ and $B(x)$ are not phase shifted by 90° when $f(\xi)$ is real. This means that the relative error between $|H(x, \gamma)|^2$ and the term predicted by the first term in the method of stationary phase will be $O(1/\sqrt{\gamma})$. This means that we will need a much larger value of γ before the first-order term is a good approximation. In terms of our beam shaping problem this will imply that if the surface of the beam shaping lens designed using geometrical optics has a discontinuity in the third derivative, then we will require much larger values of β in order for the results of geometrical optics to be a good approximation.

Suppose that at some point x_0, the function $f(\xi)$ or $\partial^2 q/\partial\xi^2$ is discontinuous at $\xi_0(x_0)$. Since the first-order term in the method of stationary phase requires us to know $f(\xi)$ and $\partial^2 q(\xi, x)/\partial\xi^2$ at $\xi_0(x)$, it is clear that we need to modify the results of the lowest-order term in our stationary phase approximation when $x = x_0$. More importantly, the method of stationary phase will hold for values of x near x_0, but the convergence near these points will be dramatically affected. The analysis of this situation is based on the Fresnel integral (11), and we see that this situation is related to the diffraction by an semi-infinite half plane. As with that case, we end up getting oscillations near the point x_0. For the beam shaping problem, this implies that if the surface of the lens designed using geometrical optics has a discontinuity in the second derivative, then we will get even worse convergence, and this will be accompanied by oscillations in the amplitude. When the surface of the lens has a discontinuity in the first derivative, the convergence towards the geometrical optics limit is affected even more dramatically.

Clearly, if the discontinuities are small enough, they will have little effect on the convergence towards the geometrical optics limit. For example, the elements are often manufactured by approximating the element by a piecewise constant element. This should not be any problem if the steps are small enough.

So far we have assumed that the second derivative of $q(\xi, x)$ does not vanish at the stationary point $\xi_0(x)$. In optics, points where this condition is violated are said to lie on a caustic surface. Suppose we have a point source of light whose rays get refracted by an inhomogeneous medium. It is possible that at certain points in the medium we might have more than one ray arriving from this point source, or possibly none at all. The surfaces separating regions where there are different numbers of rays are known as the caustic surfaces. When we analyze the diffraction integral using the method of stationary phase we find that on the caustic surface the second derivative of $q(\xi, x)$ vanishes.

We are not so much interested in computing the integral for $H(x, \gamma)$ right at a point where the stationary point ξ_0 has a vanishing second derivative. Instead, we are interested in analysing the integral $H(x, \gamma)$, for x near x_0 where

$$\left.\frac{\partial^2 q(\xi, x_0)}{\partial \xi^2}\right|_{\xi=\xi_0(x_0)} = 0. \tag{61}$$

In optics, the point x_0 would be a point on the caustic surface. We would find that for points on one side of x_0, there are no stationary points, and on the other side there are two stationary points. In order to understand the behavior of $H(x, \gamma)$ near such points we need to include cubic terms in the Taylor series expansion of the phase near the stationary point, and this analysis is based on the Airy integral.

We will not give any further discussion of the Airy integral or caustics since when discussing beam shaping we do not present any examples where caustics occur in the classical sense of the word. All of the problems we analyze lead to lenses whose phase functions do not have inflection points. However, in some of the lenses, the phase function grows linearly as we move far away from the center of aperture. This results in a situation where the caustic occurs at a value of $\xi_0 = \infty$.

C. A Preliminary Transformation

In our analysis of the higher-order terms in the method of stationary phase we will begin by analyzing the situation where $q(\xi) = \xi^2$. This leads us to consider integrals of the form

$$P(\gamma) = \int_{-\infty}^{\infty} f(\xi) e^{i\gamma\xi^2} d\xi. \tag{62}$$

By making a preliminary transformation, we can transform the analysis of the integral in Eq. (53):

$$H(\gamma) = \int_{-\infty}^{\infty} e^{i\gamma q(\xi)} f(\xi) d\xi, \tag{63}$$

into the analysis of this simpler problem. In order to do this we assume that $q(\xi)$ has a single stationary point at ξ_0. In this case we can introduce a new variable s such that

$$s^2 = \mu(q(\xi) - q(\xi_0)), \tag{64}$$

$$\mu = \operatorname{sgn}\left(\frac{d^2 q(\xi_0)}{d\xi}\right). \tag{65}$$

in the neighborhood of ξ_0. Making the change of variables $s(\xi) = \sqrt{\mu(q(\xi) - q(\xi_0))}$, we end up with an integral of the form

$$H(\gamma) = e^{i\gamma q(\xi_0)} \int_{-\infty}^{\infty} e^{i\mu\gamma s^2} f(\xi(s)) \frac{d\xi}{ds} ds. \tag{66}$$

This gives us an integral of the same form as (62), but with the function

$$g(s) = f(\xi(s)) \frac{d\xi}{ds} \tag{67}$$

replacing the function $f(\xi)$.

When we apply the method of stationary phase to the integral (62) we see that there is a stationary point at $\xi = 0$. The continuity properties of $f(\xi)$ are important in determining how quickly the first-order term in the method of stationary phase converges towards the exact answer. For the general case it is important to know the continuity properties of the function $g(s)$. A discontinuity in the *kth* derivative of $g(s)$ can arise by the *kth* derivative of $f(\xi)$ being discontinuous at $\xi = \xi_0$, or by the *kth* derivative of $d\xi/ds$ being discontinuous at $s = 0$. The derivatives of $\xi(s)$ depend on the derivatives with respect to ξ of $q(\xi)$ at ξ_0. These derivatives can be calculated using implicit differentiation. In particular, note that

$$2s \frac{ds}{d\xi} = \mu \frac{dq}{d\xi}. \tag{68}$$

If we evaluate this at $\xi = \xi_0$ we find that both sides of this equation vanish, and we have not determined any derivatives. However, if we differentiate once more with respect to ξ we get

$$2\left(\frac{ds}{d\xi}\right)^2 + 2s\frac{d^2s}{d\xi^2} = \mu\frac{d^2q}{d\xi^2}, \tag{69}$$

and when we evaluate this at $\xi = \xi_0$ we get

$$2\left(\frac{ds(\xi_0)}{d\xi}\right)^2 = \mu\frac{d^2q(\xi_0)}{d\xi^2}. \tag{70}$$

This gives us two possible values of $ds(\xi_0)/d\xi$. We can choose either sign we want to. When we take further derivatives we find that the $(d^k/d\xi^k)s(\xi_0)$ is determined by the derivatives of $q(\xi)$ up to $k + 1$. This means that the $(d^k/ds^k)\xi(0)$ is also determined by these same derivatives. Finally we see that the derivatives of $g(s)$ up to k will be continuous only if the derivatives of q up to $k + 2$ are continuous. In particular, we see that the function $g(s)$ will have a continuous first derivative if and only if the derivative of $f(\xi)$ and the third derivative of $q(\xi)$ are both continuous.

D. Generalized Functions

Before discussing the higher-order terms in the method of stationary phase we consider some relevant concepts from the theory of generalized functions (12). The Dirac delta function and its derivatives are examples of generalized functions. The definition of these functions often arise as infinite integrals whose integrands do not decay at infinity. For example the delta function is the inverse Fourier transform of a constant, and hence is defined by a divergent integral.

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}d\omega. \tag{71}$$

One way of thinking of this function is to imagine that it is defined by taking the inverse Fourier transform of $e^{-\alpha\omega^2}$, and then letting $\alpha\to 0$. The function that we get by doing this is the Dirac delta function. Even though it is a rather unusual function, it is extremely useful in practice.

In finding the higher-order terms for the method of stationary phase it will be useful to consider the integrals

$$R_k(\gamma) = \int_{-\infty}^{\infty} e^{i\gamma\xi^2}\xi^k d\xi. \tag{72}$$

These integrals can be confusing since if γ is real then the integrands of these integrals do not approach zero as $\xi\to\infty$. However, if we evaluate these integrals over a finite interval, and let the region of integration go to infinity, we find that these are in fact convergent integrals. Furthermore, if we give γ a very small positive imaginary part, then the integrands approach zero. After evaluating these integrals we could then let the imaginary part go to zero. When we let the imaginary part go to zero we find that all of the integrals R_k are well defined. We could also get the integrals by taking the derivatives of the integral $R_0(\gamma)$ with respect to γ. If we do this we find that

$$R_{k+2} = i\frac{dR_k}{dg}. \tag{73}$$

Due to the asymmetry of the integrand we get

$$R_k = 0 \text{ for } k \text{ odd}. \tag{74}$$

Carrying out this process we find that the first few of these integrals are given by

$$R_0(\gamma) = \frac{C_0}{\gamma^{1/2}}, \tag{75}$$

$$R_1(\gamma) = 0, \tag{76}$$

$$R_2(\gamma) = \frac{iC_0}{2\gamma^{3/2}}, \tag{77}$$

where

$$C_0 = e^{i\pi/4}\sqrt{\pi}. \tag{78}$$

We will also be concerned with the integrals

$$S_k(\gamma) = \int_0^\infty \xi^k e^{i\gamma\xi^2} d\xi. \tag{79}$$

We can use the same sorts of reasoning on these integrals. If k is even then

$$S_k(\gamma) = \frac{1}{2} R_k(\gamma) \text{ for } k \text{ even.} \tag{80}$$

However, unlike R_k these integrals do not vanish when k is odd. In particular, when $k = 1$

$$S_1(\gamma) = \frac{i}{2\gamma}. \tag{81}$$

The rest of the integrals can be evaluated using

$$S_{k+2}(\gamma) = -i\frac{d}{d\gamma} S_k(\gamma). \tag{82}$$

E. Higher Order Terms in the Method of Stationary Phase

We begin our analysis of the higher-order terms in the method of stationary phase by considering the special case

$$H(\gamma) = \int_{-\infty}^\infty e^{i\gamma\xi^2} f(\xi) d\xi. \tag{83}$$

This has the stationary point at $\xi = 0$. To obtain the first term in the method of stationary phase we argued that the major contribution to this integral came from the region around $\xi = 0$. For this reason we expanded $f(\xi)$ in a Taylor series about $\xi = 0$, and then kept only the first term in the series. It makes sense that we should get more accurate answers if we keep more terms in the Taylor series. For example, if we kept three terms in the Taylor series this would lead to an approximation of the form

$$H(\gamma) \approx f(0)R_0(\gamma) + \frac{df(0)}{d\xi} R_1(\gamma) + \frac{1}{2}\frac{d^2 f(0)}{d\xi^2} R_2(\gamma) + \cdots \tag{84}$$

where $R_k(\gamma)$ are the integrals that we discussed in the previous section. We conclude that the higher-order approximations for $H(\gamma)$ can be written as

$$H(\gamma) \approx e^{i\pi/4}\sqrt{\frac{\pi}{\gamma}}\left(f(0) + \frac{i}{4\gamma}\frac{d^2 f(0)}{d\xi}\right) + \cdots \tag{85}$$

In this special case, this shows that the next order term in the method of stationary phase dies down like $1/\gamma^{3/2}$ provided $f(\xi)$ is sufficiently differentiable.

If the derivative of $f(\xi)$ has a discontinuity at $\xi = 0$, then if we keep two terms in our Taylor series about $\xi = 0$, we end up with an expression

$$H(\gamma) \approx f(0)R_0(\gamma) + \frac{df(0_+)}{d\xi}\int_0^\infty e^{i\gamma\xi^2}\xi d\xi + \frac{df(0_-)}{d\xi}\int_{-\infty}^0 e^{i\gamma\xi^2}\xi d\xi + \cdots \tag{86}$$

We can write this as

$$H(x,\gamma) \approx f(0)R_0(\gamma) + S_1(\gamma)\left(\frac{df(0_+)}{d\xi} - \frac{df(0_-)}{d\xi}\right) + \cdots \tag{87}$$

Using our values of R_0 and S_1 we get

$$H(\gamma) \approx f(0)e^{i\pi/4}\sqrt{\pi/\gamma} + \frac{i}{2\gamma}\left(\frac{df(0_+)}{d\xi} - \frac{df(0_-)}{d\xi}\right) + \cdots \tag{88}$$

We see that if $f(\xi)$ has a discontinuous derivative at $\xi = 0$, then the relative error between the first-order term and the exact answer will die down like $1/\gamma^{1/2}$. This is much slower than when the derivative of $f(\xi)$ is continuous.

As we noted earlier, the general case where $q(\xi)$ is not quadratic can be transformed into the quadratic case, but replacing the function $f(\xi)$ by the function $g(s)$ in Eq. (67). We saw that the function $g(s)$ will have a discontinuous derivative if the function $f(\xi)$ has a discontinous derivative, or if $q(\xi)$ has a discontinuous third derivative. It follows that as long as the first derivative of $f(\xi)$ and the third derivative of $q(\xi)$ are continuous, then the next order term in the method of stationary phase will die down like $1/\gamma^{3/2}$. If either of these derivatives are discontinuous, then the next order term will die down like $1/\gamma$. This can be used to justify our earlier statement concerning the effect of a discontinuity in the third derivative of the lens surface on the rate of convergence towards the geometrical optics limit.

F. Lower-Order Discontinuities in the Phase Functions

When the functions $f(\xi)$ or $(d^2/d\xi^2)q(\xi, x)$ are discontinuous, we can get very slow convergence from the first term in the stationary phase approximation. We will begin with a simple example illustrating this point. By suitably changing coordinates, more general problems can in fact be related to this simple example.

Consider the integral

$$V(x,\gamma) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{i\gamma(\xi - x)^2} d\xi. \tag{89}$$

This is a special case of our general problem where $q(\xi, x)$ has a quadratic dependence on ξ and where $f(\xi) = 1$ for $\xi>0$ and 0 for $\xi<0$. This is an example where the function $f(\xi)$ is discontinuous.

If we apply the method of stationary phase to this integral we see that there is no stationary point for $x<0$, and that the method predicts that the integral is independent of x for $x > 0$. More specifically the method predicts

$$V(x,\gamma) \approx 0 \text{ for } x < 0, \tag{90}$$

$$V(x,\gamma) \approx \sqrt{\frac{1}{\gamma}} e^{i\pi/4} \text{ for } x > 0. \tag{91}$$

In the stationary phase approximation, the magnitude of $V(x, \gamma)$ is a multiple of the Heavyside function.

$$|V(x,\gamma)|^2 \approx 0 \text{ for } x < 0, \tag{92}$$

$$|V(x,\gamma)|^2 \approx \frac{1}{\gamma} \text{ for } x > 0. \tag{93}$$

We now consider this integral in more detail. A simple change of variables allows us to write

$$V(x,\gamma) = \frac{1}{\gamma^{1/2}} \mathrm{Fr}(-x\gamma^{1/2}), \tag{94}$$

where

$$\mathrm{Fr}(s) = \frac{1}{\sqrt{\pi}} \int_s^\infty e^{it^2} dt. \tag{95}$$

The function $\mathrm{Fr}(s)$ is known as a complex Fresnel integral. Figure 3 shows a plot of the intensity $|\mathrm{Fr}(s)|^2$ of the function $\mathrm{Fr}(s)$. This graph shows that for $x<0$ the function $V(x, \gamma)$ has a monotonic decay towards zero while for $x > 0$ we get an oscillatory approach towards the constant value of 1. We see that if $\left|x\sqrt{\gamma}\right| \gg 1$, then $V(x, \gamma)$ will agree very well with the stationary phase solution. The difference between the exact solution and the stationary phase solution is that the stationary phase solution approximates the lower limit of the integrand as being equal to $-\infty$. Since $\gamma \gg 1$, this is usually a good approximation, but when x is close to zero, this is not so good. This is the root of the slow convergence of the method of stationary phase for all problems that have a discontinuity in $f(\xi)$ or $d^2q/d\xi^2$.

We can understand the behavior of $\mathrm{Fr}(s)$ by considering the behavior of $\mathrm{Fr}(s)$ for large values of $|s|$. For $s \gg 0$ we can integrate by parts to show that

$$\mathrm{Fr}(s) = \frac{1}{\sqrt{\pi}} e^{is^2} \frac{i}{2s} + \frac{1}{\sqrt{\pi}} \int_s^\infty \frac{e^{it^2}}{2it} dt. \tag{96}$$

This shows that

$$\mathrm{Fr}(s) = \frac{1}{\sqrt{\pi}} e^{is^2} \frac{i}{2s} + O(1/s^2). \tag{97}$$

Similarly for $s \ll 0$ we can write

$$\mathrm{Fr}(s) = e^{i\pi/4} - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{s} e^{it^2}\, dt. \tag{98}$$

An integration by parts now shows that

$$\mathrm{Fr}(s) = e^{i\pi/4} - \frac{1}{\sqrt{\pi}} e^{is^2} \frac{i}{2s} + O(1/z^2). \tag{99}$$

When we compute the amplitude $|\mathrm{Fr}(s)|^2$ we see that

$$|\,\mathrm{Fr}(s)\,|^2 = \frac{1}{4\pi s^2} + \cdots \text{ for } s \ll 0,$$

and

$$|\,\mathrm{Fr}(s)\,|^2 = \left(1 - \frac{\sin(s^2 - \pi/4)}{\sqrt{\pi} s}\right) \text{for } s \gg 0. \tag{101}$$

We see that the solution approaches its asymptotic value much slower for $s \gg 0$ than for $s \ll 0$, and that it approaches it in an oscillatory fashion. This is only the asymptotic behavior, but it gives quite an accurate picture of the function Fr(s).

This shows that the function $V(x, \gamma)$ will have oscillations for $x > 0$, and be smooth for $x<0$. Note that the convergence for large values of γ will be very slow if x is near 0, since $x\sqrt{\gamma}$ will be relatively small in this case as $\gamma \to \infty$.

A very slight generalization of this problem is to consider the function

$$V(x,\gamma) = \int_{0}^{\infty} e^{i\gamma(s-x)^2}\, ds + \alpha \int_{\infty}^{0} e^{i\gamma(s-x)^2}\, ds. \tag{102}$$

This reduces to the previous problem when $\alpha = 0$. When α is non-zero we can analyze this problem in a similar fashion. In this case we see that we will get oscillations on both sides of $x = 0$, but the oscillations will be bigger on the side where the stationary phase solutions predicts that V is bigger. Generalizing further we see that when we have an integral of the form

$$H(x,\gamma) = \int_{-\infty}^{\infty} e^{i\gamma q(x,\xi)} f(\xi)\, d\xi, \tag{103}$$

and $f(\xi)$ has a discontinuity at ξ^*, then we need to break the integral up into two parts.

$$H(x,\gamma) = \int_{-\infty}^{\xi^*} e^{i\gamma q(x,\xi)} f(\xi)\, d\xi + \int_{\xi^*}^{\infty} e^{i\gamma q(x,\xi)} f(\xi)\, d\xi. \tag{104}$$

Suppose that the function $q(x,\xi)$ has a stationary point at $\xi(x)$, and that $\xi(x^*) = \xi^*$. When we apply the reasoning behind the method of stationary phase to the integral from ξ^* to ∞ we get

$$\int_{\xi^*}^{\infty} e^{i\gamma q(x,\xi)} f(\xi)\, d\xi \approx e^{i\gamma q(x,\xi(x))} f(\xi(x)) \int_{\xi^*}^{\infty} e^{i\gamma 1/2 (d^2 q(x,\xi(x))/d\xi^2)(\xi - \xi(x))^2} d\xi. \tag{105}$$

We now make the substitution

$$s^2 = \gamma \frac{1}{2} \frac{d^2 q(x,\xi(x))}{d\xi^2} (\xi - \xi(x))^2. \tag{106}$$

This gives us the approximation

$$\int_{\xi^*}^{\infty} e^{i\gamma q(x,\xi)} f(\xi)\, d\xi \approx e^{i\gamma q(x,\xi(x))} f(\xi(x)) \frac{\sqrt{\pi}}{\sqrt{\gamma/2 \left| \dfrac{d^2 q(x,\xi(x))}{d\xi^2} \right|}} \mathrm{Fr}(s^*), \tag{107}$$

where

$$s^* = (\xi^* - \xi(x)) \sqrt{\frac{d^2 q(x,\xi(x))}{d\xi^2} \gamma / 2} \tag{108}$$

If $s^* \ll 0$, we can make the approximation

$$F(s^*) \approx \mathrm{Fr}(-\infty) = e^{i\pi/4}, \tag{109}$$

and we get back the first term in the method of stationary phase. However, if $\xi(x)$ is too close to ξ^*, this will not be a very good approximation unless γ is extremely large. However, our results should be good if we keep s^* in our expression rather replacing it by $-\infty$. This is exactly what we did in our analysis of the integral in Eq. (89).

As in our analysis of Eq. (89), we can patch up our approximation for points x such that $\xi(x)\approx\xi^*$ by using the Fresnel integral. If we were to do this in the general case, the formulas would get extremely cumbersome. However, it is clear that in this general case we will get the same qualitative behavior as in Eq. (89).

G. The Method of Stationary Phase in Higher Dimensions

The method of stationary phase carries over to higher-dimensional integrals. In particular, suppose we have an integral of the form

$$H(\gamma) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta) e^{i\gamma q(\xi,\eta)} d\xi\, d\eta. \tag{110}$$

Once again if the function $q(\xi, \eta)$ has a stationary point where

$$\nabla q(\xi_0,\eta_0) = 0, \tag{111}$$

then the major contribution to the integral will come from points right around this stationary point, and we can approximate the integral by

$$H(\gamma) \approx f(\xi_0,\eta_0) e^{i\gamma q(\xi_0,\eta_0)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i1/2\gamma Q(\xi,\eta)} d\xi\, d\eta, \tag{112}$$

where

$$\begin{aligned} Q(\xi,\eta) = {} & \frac{\partial^2 q(\xi_0,\eta_0)}{\partial \xi^2}(\xi-\xi_0)^2 \\ & +2\frac{\partial^2 q(\xi_0,\eta_0)}{\partial \xi\, \partial \eta}(\xi-\xi_0)(\eta-\eta_0) + \frac{\partial^2 q(\xi_0,\eta_0)}{\partial \eta^2}(\eta-\eta_0)^2. \end{aligned} \tag{113}$$

This integral can be evaluated to give

$$H(\gamma) \approx \frac{2\pi i f(\xi_0,\eta_0)}{\gamma\sqrt{J(\xi_0,\eta_0)}} e^{i\gamma q(\xi_0,\eta_0)}, \tag{114}$$

where

$$J(\xi_0,\eta_0) = \frac{\partial^2 q(\xi_0,\eta_0)}{\partial \xi^2}\frac{\partial^2 q(\xi_0,\eta_0)}{\partial \eta^2} - \frac{\partial^2 q(\xi_0,\eta_0)}{\partial \xi\, \partial \eta}\frac{\partial^2 q(\xi_0,\eta_0)}{\partial \xi\, \partial \eta}. \tag{115}$$

IV. MAXWELL'S EQUATIONS

A. Maxwell's Equations

The theory of beam shaping is based on diffraction theory, which is itself based on electromagnetic field theory. For this reason we will now review basic eletromagnetic theory (13). The governing equations of eletromagnetic field theory are

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0, \tag{116a}$$

$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{J} \tag{116b}$$

and

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \tag{117a}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{117b}$$

In these equations **E** and **B** are the electric and magnetic fields, and ρ and **J** are the charge and current densities. The densities ρ and **J** are related to each other through the law of conservation of charge.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0. \tag{118}$$

The first of Eqs. (116) is the differential form of Faraday's principle of electromagnetic induction. The second equation describes both Ampere's law and Maxwell's displacement current. The first of equations (117) is known as Gauss's law, and the second equation describes the fact that there is no such thing as a magnetic monopole. It should be noted that the second set of Maxwell's equations almost follows from the first set. If we take the divergence of each of the equations in (116), use the fact that the divergence of a curl is zero, and use the law of conservation of charge, we get

$$\frac{\partial}{\partial t} \nabla \cdot \mathbf{B} = 0, \tag{119}$$

and

$$\frac{\partial}{\partial t} (\nabla \cdot \mathbf{E} - 4\pi\rho) = 0. \tag{120}$$

We see that if the second set of Maxwell's equations is true initially, then the first set requires that they be true for all time.

We are frequently concerned with wave propagation through some medium such as air, water, or glass. In this case there is an interaction between the charge distributions and the electromagnetic field. This interaction is usually taken into account by assuming that the electric field induces a polarization charge **P** such that the charge density is given by

$$\rho = -\nabla \cdot \mathbf{P}. \tag{121}$$

Assuming that there are no other charges other than those induced by the electric field this gives us the equation

$$\nabla \cdot (\mathbf{E} + 4\pi \mathbf{P}) = 0. \tag{122}$$

The assumption is typically made that the polarization is given by

$$\mathbf{P} = \chi \mathbf{E}. \tag{123}$$

Gauss's law can now be written as

$$\nabla \cdot \mathbf{D} = 0, \tag{124a}$$

where

$$\mathrm{D} = \varepsilon \mathrm{E}, \tag{124b}$$

and

$$\varepsilon = 1 + 4\pi\chi. \tag{124c}$$

The linearity between the polarization and the electric field is usually valid unless the electric field gets to be very large. Here we have also assumed that the medium is isotropic, so there are no preferred directions. In a non-isotropic medium, the polarization is related to the applied electric field by a symmetric second rank tensor. In order to describe the phenomenon of birefringence in crystals it is necessary to use the general tensor form for the polarization (this tensor is diagonal for the special case of an isotropic medium). This relation between the electric field and the polarization also assumes that the polarization depends only on the local value of the electric field. Using such a theory it is not possible to describe the rotation of the polarization field by an optically active material.

A similar approximation is used to take into account the effect of currents that are produced by the magnetic field. In this case the currents in the material induce a polarization current **M** such that the current is given by

$$\mathbf{J}_M = c \nabla \times \mathbf{M}. \tag{125}$$

It follows that the magnetic field must satisfy

$$\nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = 4\pi \nabla \times \mathbf{M}. \tag{126}$$

If we introduce the quantity **H** defined by

$$\mathbf{H} = \mathbf{B} - 4\pi \mathbf{M}, \tag{127}$$

then assuming the only currents are those arising from the induced current $\mathbf{J}_M$, we can write

$$\nabla \times \mathrm{H} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = 0. \tag{128}$$

In the simplest case it is assumed that the fields **B** and **H** are linearly proportional to each other

$$\mathbf{B} = \mu \mathbf{H}. \tag{129}$$

For most materials that are used in optics the linear relation between the **B** and **H** fields is totally satisfactory. In fact, the constant μ is very nearly equal to unity for most materials of optical interest.

We now collect the macroscopic form of Maxwell's equations in a linear isotropic material:

$$\frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} - \nabla \times \mathbf{H} = 0, \tag{130a}$$

$$\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0. \tag{130b}$$

$$\nabla \cdot \mathbf{D} = 0, \tag{131a}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{131b}$$

where

$$\mathrm{D} = \varepsilon \mathrm{E}, \tag{132a}$$

$$\mathbf{B} = \mu \mathbf{H}. \tag{132b}$$

We have omitted any sources of charges and currents other than those produced by the interaction of the fields with the materials.

B. The Wave Equation

Our analysis of diffraction effects in Sec. VI is based on the fact that in a linear, homogeneous, and isotropic medium, each component of the electric and magnetic fields satisfies the wave equation. We now give a derivation of this fact. We begin by deriving an equation for **E** that does not assume that ε and μ are constants.

To begin with we write the second of Eqs. (130) as

$$\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} + \frac{1}{\mu}\nabla \times \mathbf{E} = 0. \tag{133}$$

We now take the curl of this equation, and use the first of Eqs. (130) to arrive at the result

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = -c^2 \frac{1}{\varepsilon}\nabla \times \left(\frac{1}{\mu}\nabla \times \mathbf{E}\right). \tag{134}$$

This is the form of the wave equation for **E** in a medium where ε and μ are not assumed to be constant. If we assume that μ is constant, we can write this equation as

$$\frac{\partial^2 \mathbf{E}}{\partial t} = -c^2 \frac{1}{\varepsilon\mu}\nabla \times \nabla \times \mathbf{E}. \tag{135}$$

We can simplify this equation by using the identity $\nabla\times\nabla\times\mathbf{A} = -\nabla^2\mathbf{A} + \nabla\nabla \cdot \mathbf{A}$, along with the fact that $\nabla \cdot \mathbf{E} = 0$ (assuming that ε is constant). For a homogeneous medium this gives us the equation

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{c^2}{\varepsilon\mu}\nabla^2 \mathbf{E}. \tag{136}$$

This shows that each component of the electric field satisfies the wave equation. If the fields are time harmonic, with frequency ω, the spatial dependence of the electric field must satisfy

$$k^2 \mathbf{E} + \nabla^2 \mathbf{E} = 0, \tag{137}$$

where

$$k = \frac{\omega\varepsilon\mu}{c} \tag{138}$$

We refer to this equation as the reduced wave equation, or the Helmholtz equation.

Similar arguements show that the field **H** satisfies

$$\frac{\partial^2 \mathbf{H}}{\partial t^3} = -c^2 \frac{1}{\mu} \nabla \times \left(\frac{1}{\varepsilon} \nabla \times \mathbf{H} \right). \tag{139}$$

Note that this is not quite the same as Eq. (134) for **E** since we have put ε inside the curl and μ outside the curl. However, if μ and ε are constant, we once again arrive at the conclusion that each component of **H** (and hence **B**) will satisfy the scalar wave equation.

C. The Energy Flux

We will now derive an expression for the flux of energy in an electromagnetic field. If we dot the first of Eqs. (130) with respect to **E**, and the second equation with respect to **H**, and add the results, we get the equation

$$\frac{1}{2c} \frac{\partial}{\partial t} = (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}) - \mathbf{E} \cdot \nabla \times \mathbf{H} + \mathbf{H} \cdot \nabla \times \mathbf{E} = 0. \tag{140}$$

If we use the identity

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}. \tag{141}$$

We see that

$$\frac{1}{2c} \frac{\partial}{\partial t} = (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}) + \nabla \cdot (\mathbf{E} \times \mathbf{H}) = 0. \tag{142}$$

When put in integral form this equation can be written as

$$\frac{1}{2c} \int_V (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}) \, dV = -\int_S (\mathbf{E} \times \mathbf{H}) \cdot \mathbf{n} \, ds, \tag{143}$$

where **n** is the outward facing normal to the surface. This equation can be interpreted as the fact that the quantity $1/2c(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$ is the energy density, and $\mathbf{E}\times\mathbf{H}$ is the flux of energy. The interpretation of $1/2c(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$ as the energy density of the field is actually clearer when we include charges in Maxwell's equations. In this case we would have to add a term to these equations that would represent the change in kinetic energy of the particles in the system. The vector

$$\mathbf{S} = c\mathbf{E} \times \mathbf{H}, \tag{144}$$

is referred to as the Poynting vector.

We will use the Poynting vector to justify the fact that the rays in geometrical optics are in fact the direction that energy is begin transported.

V. GEOMETRICAL OPTICS

A. Fermat's Principle

In order to understand our discussion of beam shaping it is essential to know how to use the laws of geometrical optics. Although understanding the derivation of the basic principles clearly leads to a deeper understanding, this is not essential for our presentation. For this reason we begin by stating the basic principles, and showing how we will use them. Once this is done, we will discuss the derivation of the principles.

Our treatment of geometrical optics is based on Fermat's principle (11). Fermat's principle is often stated as saying that the ray that gets from a point **a** to a point **b** will take the path that minimizes the travel time. This is a very concise statement of the principle, but it is not technically correct. Rather than saying that the true path minimizes the travel time, we need to say that the true path is stationary with respect to travel time. In many situations, the travel time is in fact minimized, but it is not always the case.

Before discussing what stationarity means in geometrical optics, we will clarify what we mean by stationarity in a simpler setting. The function $F(x, y) = (x–x_0)^2 + (y–y_0)^2$ has a minimum at $(x, y) = (x_0, y_0)$. A necessary condition that it has a minimum at (x_0, y_0) is that the partial derivatives of F vanish at (x_0, y_0). The vanishing of the partial derivatives is equivalent to saying that the function $F(x, y)$ is stationary at (x_0, y_0). Another way of putting this is to say that if we take any numbers $(\hat{x}, \hat{y})$, then

$$F(x_0 + \varepsilon\hat{x}, y_0 + \varepsilon\hat{y}) = F(x_0, y_0) + O(\varepsilon^2) \quad \text{as} \quad \varepsilon \to 0. \tag{145}$$

At a point that is not stationary we would have $F(x_0 + \varepsilon\hat{x}, y_0 + \varepsilon\hat{y}) = F(x_0, y_0) + O(\varepsilon)$ as $\varepsilon \to 0$. In order for a function to have a minimum at (x_0, y_0) it must be stationary, but stationarity does not imply that the function is a minimum. For example, the function

$$F(x,y) = -x^2 - y^2 \tag{146}$$

is stationary at (0, 0), but it has a maximum rather than a minimum. The function

$$F(x,y) = x^2 - y^2 \tag{147}$$

is stationary at (0, 0), but it has neither a minimum nor a maximum.

Returning to geometrical optics, we will parametrize curves going from point **a** to **b** by a parameter s such that $0 \le s \le 1$. Let $\mathbf{x}(s) = (x(s), y(s), z(s))$ be a curve such that $\mathbf{x}(0) = \mathbf{a}$ and $\mathbf{x}(1) = \mathbf{b}$, then we will denote the travel time along this curve as

$$T(\mathbf{x}(s)) = \text{travel time}. \tag{148}$$

Suppose that $\mathbf{x}_0(s)$ is the true path that a light ray takes to get from **a** to **b**. The stationarity condition implies that for any functions $\hat{\mathbf{x}}(s)$ such that $\hat{\mathbf{x}}(0) = \hat{\mathbf{x}}(1) = 0$,

$$T(\mathbf{x}_0(s) + \varepsilon\hat{\mathbf{x}}(s)) = T(\mathbf{x}_0(s)) + O(\varepsilon^2). \tag{149}$$

Fermat's principle applies in an enormous variety of situations. Many times we put constraints on the travel paths. For example we can use Fermat's principle to show that the angle of incidence equals the angles of reflection for a light ray bouncing off of a mirror. In this case we use the constraint that a ray goes from point **a** to **b** after first touching a surface.

If a light ray gets from point **a** to point **b** by first passing through an intermediate point **c**, it can be shown that the paths from **a** to **c** and from **c** to **b** must each be stationary.

We will now give some concrete examples illustrating Fermat's principle. The first few examples that we give are not directly relevant to the beam shaping problem, but the last example is absolutely essential to understanding our discussion of beam shaping.

Example 1 Suppose we have a medium that has a constant speed of light. For a ray to get from a point **a** to a point **b** in the least amount of time it is clear that it must travel in a straight line. Since the travel path to get from **a** to **b** is a minimum, it is clear that it is also stationary. It can be shown that in this case, straight lines are the only stationary paths.

Example 2 Suppose the plane $z = 0$ separates medium I with a velocity of c_I from medium II with velocity c_{II}. What path does a light ray take to get from a point **a** in medium I to a point **b** in medium II. From our last example we already know that the path must be a straight line in each medium. For simplicity we will assume that the light ray travels in the plane $y = 0$. Suppose that $\mathbf{a} = (x_1, 0, z_1)$, and $\mathbf{b} = (x_2, 0, z_2)$. Suppose that in going from **a** to **b** the ray goes through $\mathbf{c} = (\xi, 0, 0)$ on the interface between the two media. We do not know the value of ξ ahead of time, but it can be determined using Fermat's principle. The total travel time to get from **a** to **b** by going through **c** is

$$T(\xi) = \frac{1}{c_I}\sqrt{(x_1-\xi)^2 + z_1^2} + \frac{1}{c_{II}}\sqrt{(x_2-\xi)^2 + z_2^2}. \tag{150}$$

In order for the travel time to be stationary, we must have

$$\frac{dT}{d\xi} = 0. \tag{151}$$

This implies that

$$\frac{\sin(\theta_I)}{c_I} = \frac{\sin(\theta_{II})}{c_{II}}, \tag{152}$$

where

$$\sin(\theta_I) = \frac{\xi - x_1}{\sqrt{(x_1-\xi)^2 + z_1^2}}, \tag{153}$$

$$\sin(\theta_{II}) = \frac{x_2 - \xi}{\sqrt{(x_2-\xi)^2 + z_2^2}}. \tag{154}$$

This is equivalent to Snell's law of refraction.

Example 3 Suppose we would like to design a mirror that focuses all of the light rays coming from a point **a** to a point **b**. For simplicity we will consider this problem to take place in two dimensions. We also assume that the speed of light is constant throughout our medium. Suppose a ray comes from **a** at an angle of θ with the horizontal. Suppose that this ray bounces off the mirror at a point $\mathbf{p}(\theta)$, and then goes to the point **b**. Let $T(\theta)$ be the travel time to get from **a** to $\mathbf{p}(\theta)$ and then to **b**. In order for Fermat's principle to hold we must have

$$\frac{dT}{d\theta} = 0. \tag{155}$$

This means that if $\mathbf{q}_1$ is any point on the mirror, then the distance from **a** to $\mathbf{q}_1$ plus the distance from $\mathbf{q}_1$ to **b** must be the same as for any other point $\mathbf{q}_2$ on the mirror. This implies that the mirror must in fact have the shape of an ellipse, with foci at **a** and **b**.

It should be noted that as we move the point **a** off to ∞, this ellipse ends up turning into a parabola. This gives us the solution of how to focus rays coming in from ∞ to a single point **b.**

Example 4 When using Fermat's principle for parallel beams of light it is necessary to be familiar with the following argument. Suppose we have a parallel beam of rays coming in from ∞. We can think of such rays as coming from a very distant point source. Suppose the point source is at $\mathbf{p} = (-L, 0, 0)$. Assuming a homogeneous medium, the time to get from **p** to a point (x, y, z) is given by

$$T(x, y, z) = \frac{1}{c}\sqrt{(x+L)^2 + y^2 + z^2}. \tag{156}$$

As $L \to \infty$ we can make the approximation

$$T(x, y, z) = \frac{1}{c}(L + x + O(1/L)). \tag{157}$$

This shows that the travel time to get to any point in space (x, y, z) is independent of y and z. When applying Fermat's principle, the travel time L/c will not matter since it is the same for all paths. We will use this fact whenever we are applying Fermat's principle to rays that are coming in parallel.

Example 5 We now give an example that shows that the ray paths do not always minimize the travel time, but they are still stationary with respect to travel time. Suppose we have a cylindrical mirror (see Fig. 4) whose surface is given by

$$(x, y) = R(\cos(\theta), \sin(\theta)) \quad -\pi/2 \le \theta \le \pi/2. \tag{158}$$

We are interested in finding the paths of rays that are coming in parallel from $x = -\infty$. As we have already mentioned, this can be thought of as rays coming from a distant source at $\mathbf{p} = (-\mathrm{L}, 0)$ where L is very large. The travel time to get from **p** to a point on the surface of the mirror $R(\cos(\theta), \sin(\theta))$, and then to a point (x, y) is (assuming $L \gg 1$)

$$T(\theta) = \frac{1}{c}(L + R\cos(\theta) + \sqrt{(x - R\cos(\theta))^2 + (y - R\sin(\theta))^2}). \tag{159}$$

Given a point (x, y) the equation $dT/d\theta = 0$ will determine where the ray that reaches (x, y) reflects off the mirror. For simplicity we will limit ourselves to points such that $y = 0$. In this case we can write the stationarity condition as

$$\frac{dT}{d\theta} = \frac{R\cos(\theta)}{c}\left(-1 + \frac{x}{\sqrt{(x - R\cos(\theta))^2 + R^2\sin^2(\theta)}}\right). \tag{160}$$

This equation has the solutions

$$\sin(\theta) = 0, \tag{161}$$

or

$$x^2 = (x - R(\cos)(\theta))^2 + R^2\sin^2(\theta). \tag{162}$$

This last equation can be written as

$$x = \frac{R}{2\cos(\theta)}. \tag{163}$$

For any value of *x* the first of these equations gives us the solution $\theta = 0$. However, the second equation will have no solutions if $x<R/2$, and will have two solutions if $x>R/2$. When we look throughout the *xy* plane we find that there will be a region that has three reflected rays reaching each point, and another region with only one reflected ray reaching each point. The curve separating the two regions is an example of a caustic surface. These caustic surfaces are easily observed since the irradiance at the surface becomes much larger than at a typical point in the plane. This particular situation can be observed when looking into a cup of tea, or a bowl of sugar under the light from a concentrated source such as an incandescent light bulb.

Note that not all of the rays have minimum travel time. If we compute $d^2T/d\theta^2$ we get

$$\begin{aligned}\frac{d^2T}{d\theta^2} &= \frac{R\cos(\theta)}{c}\left(-1+\frac{x}{\sqrt{(x-R\cos(\theta))^2+R^2\sin^2(\theta)}}\right)\\ &\quad -\frac{R^2\sin^2(\theta)}{c}\frac{x^2}{((x-R(\cos(\theta))^2+R^2\sin^2(\theta))^{3/2}}.\end{aligned} \tag{164}$$

If we restrict our attention to the ray that hits the mirror at $\theta = 0$, we have

$$\frac{d^2T(0)}{d\theta^2} = \frac{R}{c}\left(-1+\frac{x}{|x-R|}\right). \tag{165}$$

This shows that if $x>R/2$, then the travel time is a minimum, but if $x<R/2$ the travel time is in fact a local maximum.

Example 6 This last example plays a large role in our theory of beam shaping since it shows that our Fourier transform lens has a quadratic time delay, and hence a quadratic phase function. Suppose we would like to place a lens at $x = 0$ that focuses all of the rays coming in from $x = -\infty$ to a single point at $(x, y) = (f, 0)$. We will make several approximations. To begin with, we assume that the lens is thin. This means that the rays that enter the lens at $(0, y)$ emerge at very nearly the same point. We can model the effect of the lens by saying that it introduces a time delay of $t_L(y)$. This means that a ray that enters the lens at $(0, y)$ takes a time $t_L(0, y)$ to emerge from the lens. In practice this time delay can be introduced by making the lens out of a material that has a different index of refraction than the medium that the rays are traveling in, and by varying the thickness of the lens.

We would like to determine the function $t_L(y)$ such that all of the rays from ∞ get focused to the point $(f, 0)$. The time for a ray to go from a distant point $(-L, 0)$ to a point $(0, y)$ and then to the point $(f, 0)$ is approximately

$$t(y) = \frac{1}{c}\left(L+\sqrt{f^2+y^2}\right)+t_L(y). \tag{166}$$

Here we have made the approximation that $L \gg 1$, and used the simplified expression for the distance from $(-L, 0)$ to $(0, y)$. We will now make the paraxial approximation. This assumes that all points on the lens satisfy $y^2/f^2 \ll 1$. In this case we can approximate the square root using

$$\sqrt{f^2+y^2} \approx f + \frac{y^2}{2f}. \tag{167}$$

Using this approximation, we can write

$$t(y) \approx \frac{1}{c}\left(L+f+\frac{y^2}{2f}\right)+t_L(y). \tag{168}$$

Fermat's principle requires that the path that gets from $(-L, 0)$ to $(f, 0)$ must be stationary with respect to all nearby paths. This implies that

$$\frac{dt}{dy} = 0 \text{ for all } y. \tag{169}$$

This means that we must have $t(y) = \text{constant}$, and hence

$$t_L(y) = -\frac{y^2}{2fc}. \tag{170}$$

This shows that in the paraxial approximation, we must use a quadratic lens in order to focus an initially parallel beam of rays to a point.

B. The Eikonol Equation

The laws of geometrical optics can be derived as a high-frequency approximation to the solutions of Maxwell's equations. The rays of light are related to the direction of energy propagation. There is a very strong connection between Fermat's principle and the method of stationary phase. Both the method of stationary phase and the laws of geometrical optics are high frequency limits, they both are centered about the phase of the wave field, and they both use a stationarity condition.

Before considering the high-frequency limit of Maxwell's equations, we will begin by considering the high-frequency limit of the scalar wave equation. Suppose we have a solution $p(\mathbf{x}, \omega)$ to the equation

$$\nabla^2 p + \frac{\omega^2}{c^2(\mathbf{x})} p = 0. \tag{171}$$

This is the time harmonic wave equation, also known as the reduced wave equation. We are interested in determining the behavior of these solutions for large values of ω. In particular we ask what solutions that are coming from a single point source look like. The theory of Green's functions shows that the general high-frequency limit (not necessarily from a point source) can be built up by integrating over many such point sources. If the velocity is constant we know that the point source solutions can be written as

$$p(\mathbf{x},\omega) = A\frac{e^{i\omega r/c}}{r}. \tag{172}$$

Here $r^2 = x^2 + y^2 + z^2$. This solution has a very rapidly varying phase (it varies more rapidly the bigger ω is), and a slowly varying amplitude (that is independent of ω). In the case of variable $c(\mathbf{x})$ we assume that even though the amplitude may not be completely independent of ω, it depends very weakly on ω. Generalizing to the case of non-homogeneous media, in the high-frequency limit we will assume that $p(\mathbf{x}, \omega)$ can be written as

$$p(\mathbf{x},\omega) = A(\mathbf{x})e^{i\omega\phi(\mathbf{x})}. \tag{173}$$

This is only the first term in an asymptotic expansion. The general solution needs to include corrections to the amplitude that depend on ω.

We can write

$$\nabla p = (\nabla A + Ai\omega\nabla\phi)e^{i\omega\phi}. \tag{174}$$

Using this we can now write

$$\nabla^2 p = \nabla\cdot\nabla p = (\nabla^2 A + 2i\omega\nabla A\cdot\nabla\phi + i\omega A\nabla^2\phi - \omega^2 A\nabla\phi\cdot\nabla\phi)e^{i\omega\phi}. \tag{175}$$

If we substitute this expression into Eq. (171) and keep only the highest-order term in ω, we find that

$$|\nabla\phi|^2 = \frac{1}{c(\mathbf{x})^2}. \tag{176}$$

This equation is usually referred to as the eikonol equation. The next higher-order term gives us

$$2\nabla A\cdot\nabla\phi + A\nabla^2\phi = 0. \tag{177}$$

This last equation can be written as

$$\nabla\cdot(A^2\nabla\phi) = 0. \tag{178}$$

The fact that this equation can be written in divergence form suggests that the quantity $A^2\nabla\phi$ is the flux of some quantity that is conserved. When we apply these arguments to optical systems we will see that this quantity is in fact proportional to the flux of energy.

We mentioned that the general high-frequency approximation can be built up by integrating or summing over a family of point sources. As a simple example, if our wave field comes from two point sources, the high-frequency limit of the wave field will look like

$$p(\mathbf{x},\omega) = A_1(\mathbf{x})e^{i\omega\phi_1(\mathbf{x})} + A_2(\mathbf{x})e^{i\omega\phi_2(\mathbf{x})}. \tag{179}$$

It should be mentioned that even for a single point source there may be points in space where the high-frequency limit consists of a sum of terms as in the previous equation. This will be the case if the rays are bent so that more than one ray from the same source reaches the same point. The surfaces separating the regions where there are different numbers of rays are the caustic surfaces.

C. The Eikonol Equation and Maxwell's Equations

In the previous section we derived the eikonol equation from the scalar wave equation. Each component of the eletromagnetic wave field satisfies this equation, so it is not surprising that the eikonol equation also arises when considering the high-frequency limit of Maxwell's equations. In this section we will derive the eikonol equation using Maxwell's equations, and we will see that Poynting's theorem shows that the energy of the electromagnetic field is in fact being propagated normal to the surfaces of constant phase. The derivation of the eikonol equation from Maxwell's equations is almost identical to the analysis of plane monochromatic plane waves given in most textbooks on electrodynamics (12).

The time-harmonic Maxwell's equations are

$$i\omega\frac{1}{c_0}\mathbf{D} - \nabla\times\mathbf{H} = 0, \tag{180a}$$

$$i\omega\frac{1}{c_0}\mathbf{B} + \nabla\times\mathbf{E} = 0, \tag{180b}$$

where

$$\mathbf{D} = \varepsilon(\mathbf{x})\mathbf{E}, \tag{181a}$$

$$\mathbf{B} = \mu(\mathbf{x})\mathbf{H}. \tag{181b}$$

Similar to our derivation of the eikonol equation for the scalar wave equation, we assume a solution of the form.

$$\mathbf{E}(\mathbf{x},\omega) = \mathbf{E}_0(\mathbf{x})e^{i\omega\phi(\mathbf{x})}, \tag{182}$$

$$\mathbf{H}(\mathbf{x},\omega) = \mathbf{H}_0(\mathbf{x})e^{i\omega\phi(\mathbf{x})}. \tag{183}$$

Substituting this expression into Maxwell's equations and using the vector identity

$$\nabla\times(f(\mathbf{x})\mathbf{A}(\mathbf{x})) = f(\mathbf{x})\nabla\times\mathbf{A}(\mathbf{x})\times\mathbf{A}(\mathbf{x})\nabla f(\mathbf{x}), \tag{184}$$

we get

$$i\omega\frac{\varepsilon}{c_0}\mathbf{E}_0 - (\nabla\times\mathbf{H}_0 + i\omega\mathbf{H}_0\times\nabla\phi) = 0, \tag{185a}$$

$$i\omega\frac{\mu}{c_0}\mathbf{H}_0 + (\nabla\times\mathbf{E}_0 + i\omega\mathbf{E}_0\times\nabla\phi) = 0, \tag{185b}$$

If we only keep the highest-order terms in ω in this equation we get

$$\frac{\varepsilon}{c_0}\mathbf{E}_0 - \mathbf{H}_0\times\nabla\phi = 0, \tag{186a}$$

$$\frac{\mu}{c_0}\mathbf{H}_0 + \mathbf{E}_0\times\nabla\phi = 0. \tag{186b}$$

If we dot each of these equations with $\nabla\phi$ we find that

$$\mathbf{E}_0\cdot\nabla\phi = 0, \tag{187}$$

$$\mathbf{H}_0\cdot\nabla\phi = 0. \tag{188}$$

If we dot the first of Eqs. (186) with respect to $\mathbf{H}_0$, or the second with respect to $\mathbf{E}_0$, we find that

$$\mathbf{E}_0\cdot\mathbf{H}_0 = 0. \tag{189}$$

If we eliminate $\mathbf{H}_0$ from Eqs. (186) we find that

$$\frac{\epsilon\mu}{c_0}\mathbf{E}_0 + c_0(\mathbf{E}_0 \times \nabla\phi) \times \nabla\phi = 0. \tag{190}$$

Using the identity

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{d}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{a}(\mathbf{b} \cdot \mathbf{c}), \tag{191}$$

we find that this can be written as

$$\frac{\varepsilon\mu}{c_0^2}\mathbf{E}_0 - (\mathbf{E}_0 \nabla\phi \cdot \nabla\phi - \nabla\phi \mathbf{E}_0 \cdot \nabla\phi) = 0. \tag{192}$$

Using the fact that $\mathbf{E}_0 \cdot \nabla\phi = 0$ this can be written as

$$\mathbf{E}_0 \left(\frac{1}{c^2(\mathbf{x})} - \nabla\phi \cdot \nabla\phi \right) = 0, \tag{193}$$

where

$$c^2(\mathbf{x}) = \frac{c_0^2}{\varepsilon\mu}. \tag{194}$$

We see that we have once again arrived at the eikonol equation.

The direction of energy flux is given by the Poynting vector

$$\mathbf{S} = c\mathbf{E} \times \mathbf{H}. \tag{195}$$

Using the fact that **E** and **H** are orthogonal to each other, and also to $\nabla\phi$, it follows that this vector is in the direction of $\nabla\phi$. We see that the direction of energy flux is in fact normal to the surfaces of constant phase.

D. First-Order Non-linear Partial Differential Equations

The theory of ray tracing from the eikonal equation is a special case of the solution of non-linear first-order partial differential equations (14, 15). In this section we will give a brief outline of this theory. Suppose we have an equation of the form

$$F(x, y, z, p, q, r) = 0, \tag{196a}$$

where

$$p = \frac{\partial\phi}{\partial x}, \tag{196b}$$

$$q = \frac{\partial\phi}{\partial y}, \tag{196c}$$

$$r = \frac{\partial\phi}{\partial z}. \tag{196d}$$

For optical applications we are especially concerned with the case where

$$F(x, y, z, p, q, r)\frac{1}{2}\left(p^2 + q^2 + r^2 - \frac{1}{c^2(\mathbf{x})} \right), \tag{197}$$

which is just the eikonol equation. In this section we will consider the case for a general function F rather than limiting ourselves to the eikonal equation. Our analysis could be extended to the case where the function F also depends on the function ϕ, but that case is just a bit more complicated, and it never arises in optical applications, so we will not consider it here.

Suppose we know the function ϕ and all of its first derivatives at some point (x_0, y_0, z_0). Is it possible to determine the solution ϕ in the neigborhood of the point (x_0, y_0, z_0)? In particular, is it possible to determine the second derivatives of the function ϕ at the point (x_0, y_0, z_0)? If we take the derivatives of our equation with respect to x, y, and z we end up with the equations

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial F}{\partial r}\frac{\partial r}{\partial x} = 0, \tag{198a}$$

$$\frac{\partial F}{\partial y}+\frac{\partial F}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial y}+\frac{\partial F}{\partial r}\frac{\partial r}{\partial y}=0, \tag{198b}$$

$$\frac{\partial F}{\partial z}+\frac{\partial F}{\partial p}\frac{\partial p}{\partial z}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial z}+\frac{\partial F}{\partial r}\frac{\partial r}{\partial z}=0. \tag{198c}$$

The partial derivatives of p, q and r can all be expressed in terms of the six second-order partial derivatives of ϕ:

$$\frac{\partial^2\phi}{\partial x^2},\frac{\partial^2\phi}{\partial y^2},\frac{\partial^2\phi}{\partial z^2},\frac{\partial^2\phi}{\partial x\partial y},\frac{\partial^2\phi}{\partial x\partial z},\frac{\partial^2\phi}{\partial y\partial z}. \tag{199}$$

By differentiating our equation $F(x, y, p, q, r) = 0$ we have arrived at three equations for the six second-order partial derivatives of ϕ. Clearly we do not have enough equations to determine the second-order partial derivatives. The question now arises, is it possible to determine the derivatives p, q and r in a particular direction? It turns out that this is in fact possible. To do this we use the fact that

$$\frac{\partial p}{\partial y}=\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial t\partial x}=\frac{\partial p}{\partial x}. \tag{200a}$$

Similarly

$$\frac{\partial r}{\partial x}=\frac{\partial p}{\partial z}. \tag{200b}$$

It follows that the first of Eqs. (198) can be written as

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial F}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial F}{\partial r}\frac{\partial p}{\partial z}=0. \tag{201a}$$

Similarly, by switching the order of the other mixed partial derivatives we can get

$$\frac{\partial F}{\partial y}+\frac{\partial F}{\partial p}\frac{\partial q}{\partial x}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial y}+\frac{\partial F}{\partial r}\frac{\partial q}{\partial z}=0, \tag{201b}$$

and

$$\frac{\partial F}{\partial z}+\frac{\partial F}{\partial p}\frac{\partial r}{\partial x}+\frac{\partial F}{\partial q}\frac{\partial r}{\partial y}+\frac{\partial F}{\partial r}\frac{\partial r}{\partial z}=0. \tag{201c}$$

These equations can be written as

$$\frac{\partial F}{\partial x}+\nabla p\cdot\mathbf{a}=0, \tag{202a}$$

$$\frac{\partial F}{\partial y}+\nabla q\cdot\mathbf{a}=0, \tag{202b}$$

and

$$\frac{\partial F}{\partial z}+\nabla\mathbf{r}\cdot\mathbf{a}=0, \tag{202c}$$

where

$$\mathbf{a}=\left(\frac{\partial F}{\partial p},\frac{\partial F}{\partial q},\frac{\partial F}{\partial r}\right). \tag{202d}$$

These equations show that although we do not know the derivatives of p, q and r in any arbitrary direction,, we do know the derivatives in the direction $\mathbf{a} = (\partial F/\partial p,\partial F/\partial q,\partial F/\partial r)$. This suggests that there may be special curves $(x(s),y(s),z(s))$ such that we can determine $(p(s),q(s),r(s))$. In particular, if

$$\dot{x}=\frac{\partial F}{\partial p}, \tag{203}$$

$$\dot{y} = \frac{\partial F}{\partial q}, \tag{204}$$

$$\dot{z} = \frac{\partial F}{\partial r}, \tag{205}$$

then we have

$$\dot{p} = -\frac{\partial F}{\partial x}, \tag{206}$$

$$\dot{q} = -\frac{\partial F}{\partial y}, \tag{207}$$

$$\dot{r} = -\frac{\partial F}{\partial z}. \tag{208}$$

The function ϕ changes according to the equation

$$\dot{\phi} = \frac{\partial \phi}{\partial x}\dot{x} + \frac{\partial \phi}{\partial y}\dot{y} + \frac{\partial \phi}{\partial z}\dot{z} = p\dot{x} + q\dot{y} + r\dot{z} = p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q} + r\frac{\partial F}{\partial r}. \tag{209}$$

This is a seventh-order ordinary differential equation for the unknowns (x, y, z, p, q, r, ϕ). We can solve this system of equations provided we specify initial values of (x, y, z, p, q, r,ϕ). It should be noted that we cannot specify these values arbitrarily, but must require that they satisfy the equation $F(x, y, z, p, q, r) = 0$.

In optics the function ϕ is the phase of our wave field. The curves ($x(s)$,$y(s)$,$z(s)$) along which we propagate our solution are know as the rays. In optics they are what we intuitively think of as being the rays of light. If we know the phase $\phi(x, y, z)$ on some plane $z = z_0$, then we can parametrically map out the phase in all of space by tracing all the rays from the plane $z = z_0$.

This process of tracing out the phase field assumes that one and only one ray passes from a given point (x, y, z) to the plane $z = z_0$. In practice it is possible that no rays pass through some points, and multiple rays pass through other points. The surfaces separating regions with different numbers of rays are once again the caustic surfaces.

Example 7 We will now apply this theory to the eikonol equation where

$$F(x,y,z,p,q,r) = \frac{1}{2}\left(p^2 + q^2 + r^2 - \frac{1}{c^2(\mathbf{x})}\right). \tag{210}$$

We will use the shorthand notation $\mathbf{p} = (p, q, r)$ and $\mathbf{x} = (x, y, z)$. The theory we have just derived shows that

$$\frac{d\mathbf{x}}{ds} = \mathbf{p}, \tag{211a}$$

$$\frac{d\mathbf{p}}{ds} = -\frac{\nabla c}{c^3}, \tag{211b}$$

and

$$\frac{d\phi}{ds} = \frac{1}{c^2(\mathbf{x})}, \tag{211c}$$

where $\mathbf{p}$ is required to satisfy the initial condition

$$\mathbf{p}.\mathbf{p} = 1/c^2. \tag{212}$$

we can eliminate $\mathbf{p}$ from this equation to get

$$\frac{d^2\mathbf{x}}{ds^2} = -\frac{\nabla c}{c^3}. \tag{213}$$

Note that the equation for ϕ can be written as

$$\frac{d\phi}{ds} = \frac{|\dot{\mathbf{x}}|}{c(\mathbf{x})}. \tag{214}$$

This equation can be interpreted as saying that the change is ϕ in going from $\mathbf{x}(s_0)$ to $\mathbf{x}(s_1)$ is the travel time to get from $\mathbf{x}(s_0)$ to $\mathbf{x}(s_1)$ along the curve $\mathbf{x}(s)$.

Example 8 The system of equations (213) is in some ways the simplest set of equations we could write down for the paths of the light rays. However, it suffers from one problem. The equations are not invariant under a change of parametrization. If we parameterize our curves by $\xi = \xi(s)$, we will end up getting a different differential equation for $\mathbf{x}(\xi)$ than we got for $\mathbf{x}(s)$. The solutions will result in the same curve in physical space, but the differential equations will be different. Unless one is extremely concerned with the aesthetic properties of their equations this is not a real problem. However, it turns out that the equations that we derive using Fermat's principle will be invariant under a change of parametrization, and hence it will be difficult to compare the two sets of equations unless we write equations (213) so that they are also invariant.

In order to do this we note that the first of Eqs. (213) requires that

$$|\dot{\mathbf{x}}| = |\mathbf{p}| = \frac{1}{c}. \tag{215}$$

It follows that along our solution curve we have

$$|\dot{\mathbf{x}}|\, c(\mathbf{x}) = 1. \tag{216}$$

It follows that we will not change the solutions to Eqs. (211) if we divide the left-hand sides by $|\dot{\mathbf{x}}|\, c(\mathbf{x})$. In this case we get the equations

$$\frac{1}{|\dot{\mathbf{x}}|\, c(\mathbf{x})}\dot{\mathbf{x}} = \mathbf{p}, \tag{217}$$

$$\frac{1}{|\dot{\mathbf{x}}|\, c(\mathbf{x})}\dot{\mathbf{p}} = -\nabla c \frac{1}{c^3}. \tag{218}$$

Now if we eliminate $\mathbf{p}$ from these equations we end up with the system of equations

$$\frac{1}{|\dot{\mathbf{x}}|\, c(\mathbf{x})}\frac{d}{ds}\left(\frac{\dot{\mathbf{x}}}{|\dot{\mathbf{x}}|\, c(\mathbf{x})}\right) = -\frac{\nabla c}{c^3}. \tag{219}$$

This is the final system of equations that we will use to compare to the curves obtained by using Fermat's principle. Note that if we make a change of variables $\xi = \xi(s)$, the differential equation in terms of ξ will be identical to the differential equation in terms of s.

E. Fermat's Principle without Reflections

In the last example we derived the equations for the path $\mathbf{x}(s)$ that a light ray follows in an inhomogeneous medium. We will now show that the path that gets from point $\mathbf{x}_0$ to point $\mathbf{x}_1$ is stationary with respect to the travel time between these two points. Suppose we have a curve $\mathbf{x}(s)$ such that $\mathbf{x}(s_0) = \mathbf{x}_0$ and $\mathbf{x}(s_1) = \mathbf{x}_1$. The time to get from the point $\mathbf{x}_0$ to the point $\mathbf{x}_1$ may be written as

$$T = \int_{s_0}^{s_1} \sqrt{\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}}\,\frac{1}{c(\mathbf{x})}\,ds. \tag{220}$$

The first variation of this integral may be written as

$$\delta T = \int_{s_0}^{s_1} \frac{\delta\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}}{c(\mathbf{x})\sqrt{\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}}} - |\dot{\mathbf{x}}|\frac{\nabla c\cdot\delta\mathbf{x}}{c^2(\mathbf{x})}\,ds. \tag{221}$$

If we integrate by parts to get rid of the derivative with respect to $\delta\mathbf{x}$, and if we require that $\delta\mathbf{x}$ vanish at the end points, we find that

$$\delta T = -\int_{s_0}^{s_1}\left(\frac{d}{ds}\left(\frac{\dot{\mathbf{x}}}{|\dot{\mathbf{x}}|\, c(\mathbf{x})}\right) + |\dot{\mathbf{x}}|\frac{\nabla c}{c^2(\mathbf{x})}\right)\cdot\delta\mathbf{x}\,ds. \tag{222}$$

If the path is stationary, then this integral must vanish for all functions $\delta\mathbf{x}$, and hence $\mathbf{x}$ must satisfy the equation

$$\frac{1}{c(\mathbf{x})|\dot{\mathbf{x}}|}\frac{d}{ds}\left(\frac{\dot{\mathbf{x}}}{|\dot{\mathbf{x}}|c(\mathbf{x})}\right)=-\frac{\nabla c}{c^3}. \tag{223}$$

This is identical to the Eq. (219) which we derived using the eikonol equation, and requiring that the equation be invariant under a change of parameterization.

F. Fermat's Principle for Reflecting Surfaces

In our analysis of beam shaping systems we will not consider any cases where the rays reflect off of mirrors. However, since it may sometimes be desirable to use reflecting surfaces in beam shaping systems, we now consider Fermat's principle for reflecting surfaces. Suppose we have a surface S defined parametrically by $\mathbf{x} = \mathbf{f}(\xi_1, \xi_2)$. Suppose that a ray goes from the point $\mathbf{x}_0$ to the point $\mathbf{x}_l$, but first bounces off the surface S. The theory of waves shows that at the point where the rays get reflected by the surface, the following conditions hold:

- The normal to the surface, the incident ray, and the reflected ray all lie in the same plane.
- The incident and reflected rays make the same angle with respect to the normal to the surface.

We now show that these conditions can be derived by assuming that the path from $\mathbf{x}_0$ to $\mathbf{x}_1$ that touches the surface S is stationary with respect to travel time. Suppose we have a path that goes from $\mathbf{x}_0$ to $\mathbf{x}_1$ after first touching some point $\mathbf{q} = \mathbf{f}(\xi_1, \xi_{,2})$ on the surface S. Clearly the paths from $\mathbf{x}_0$ to $\mathbf{q}$ and from $\mathbf{q}$ to $\mathbf{x}_1$ must themselves be stationary. It follows that in order to determine the true path we need only determine the point $\mathbf{q}$ on the surface S. In particular, suppose $\phi(\mathbf{x}, \mathbf{z})$ gives the travel time to get from the point $\mathbf{x}$ to the point $\mathbf{z}$. We have shown that the travel time $\phi(\mathbf{x}, \mathbf{z})$ is in fact a solution to the eikonol equation. The total travel time from $\mathbf{x}_0$ to $\mathbf{x}_l$ is given by

$$T=\phi(\mathbf{x}_0,\mathbf{f}(\xi_1,\xi_2))+\phi(\mathbf{x}_1,\mathbf{f}(\xi_1,\xi_2)). \tag{224}$$

If this travel time is stationary, then we must have

$$\frac{\partial \mathbf{f}}{\partial \xi_k}\cdot(\mathbf{p}_{\mathrm{i}}+\mathbf{p}_r)=0 \text{ for } k=1,2, \tag{225}$$

where

$$\mathbf{p}_i=\frac{\partial\phi(\mathbf{x}_0,\mathbf{q})}{\partial\mathbf{q}} \tag{226a}$$

and

$$\mathbf{p}_r=\frac{\partial\phi(\mathbf{x}_1,\mathbf{q})}{\partial\mathbf{q}} \tag{226b}$$

are the incident and the reflected ray vectors.

Note that the vectors $\mathbf{p}_i$ and $\mathbf{p}_r$ must satisfy $|\mathbf{p}| = 1/c(\mathbf{q})$, and hence we must have $|\mathbf{p}_i| = |\mathbf{p}_r|$. Let $\mathbf{n}$ be the normal to the surface S at $\mathbf{f}(\xi_1, \xi_2)$, and let $\mathbf{t}_1$ and $\mathbf{t}_2$ be two independent tangent vectors to the surface. These vectors can be written as linear combinations of the vectors $\partial\mathbf{f}/\partial\xi_1$ and $\partial\mathbf{f}/\partial\xi_2$, and hence Eq. (225) shows that the tangential components of $\mathbf{p}_i$ and $\mathbf{p}_r$ must be negatives of each other. That is, if

$$\mathbf{p}_i=a_i\mathbf{n}+b_i\mathbf{t}_1+c_i\mathbf{t}_2, \tag{227}$$

then

$$\mathbf{p}_r=a_r\mathbf{n}-b_i\mathbf{t}_1-c_i\mathbf{t}_2. \tag{228}$$

Furthermore, in order for $\mathbf{p}_r$ and $\mathbf{p}_i$ to have the same magnitude, we must have $a_i = a_r$.This shows that $\mathbf{p}_i$, $\mathbf{p}_r$, and $\mathbf{n}$ all lie in the same plane. Furthermore,

$$\mathbf{p}_i\cdot\mathbf{n}=\mathbf{p}_r\cdot\mathbf{n}, \tag{229}$$

and hence the vectors $\mathbf{p}_i$ and $\mathbf{p}_r$ make the same angle with respect to $\mathbf{n}$. This is precisely what we wanted to prove.

VI. FOURIER OPTICS AND DIFFRACTION THEORY

A. Fresnel Diffraction Theory

Fresnel diffraction theory plays an important role in the theory of beam shaping since it allows us to access the validity of the geometrical optics approximation. Through the theory of Fresnel diffraction we will be able to turn our physical optics problem into a mathematical problem concerning Fourier transforms. After giving a derivation of the Fresnel approximation, we will outline the conditions necessary for it to be a good approximation.

The Fresnel approximation is concerned with the wave field for $z>0$ produced when an incoming wave passes through an aperture at $z = 0$. In general the aperture may contain an optical element that changes the amplitude or phase of the incoming wave. Elementary theories of diffraction usually are concerned with the field far from the aperture, and in a narrow solid angle normal to the aperture. The theory of Fresnel diffraction can be outlined in three basic steps.

- Write down an exact expression for determining the wave field for all values of $z>0$ provided one knows the wavefield at the plane of the aperture $z = 0$.
- Use a paraxial approximation that simplifies this expression assuming the observation point is near the axis.
- Compute the wavefield away from the aperture by using the first two steps along with a very simple assumption concerning the field in the plane of the aperture. The assumption is that at the aperture the wavefield is equal to the undisturbed incoming wavefield (modified by any optical element inside the aperture), and zero everywhere else.

The first of these steps can be carried out rigorously. The second step can be justified quite well using simple asymptotics. The third step is by far the hardest to justify, but it can be argued that it is plausible provided the aperture is large compared to the wavelength of the incoming light.

We begin with a discussion of the Fresnel approximation for the scalar wave equation. Physically we can think of this equation arising from the equations of acoustics. When we present the vector theory of diffraction we will see that this theory can be used to determine the various components of the electric field, but a slight error occurs in the component of the field normal to the aperture. This error is not big as long as we are near the axis.

B. A Fourier Approach to Diffraction Theory

We suppose that function $u(x,y,z)$ satisfies the Helmholtz equation.

$$\nabla^2 u + k^2 u = 0, \tag{230a}$$

where

$$k = \frac{2\pi}{\lambda}, \tag{230b}$$

and λ is the wavelength. The function u must also satisfy the boundary condition

$$u(x,y,0) = f(x,y) \tag{230c}$$

and

$$u(x,\ y,\ z) \text{ has no incoming waves as } z \to \infty\ . \tag{230d}$$

This last boundary condition, sometimes referred to as the Sommerfeld radiation condition, is a somewhat subtle condition, but it is quite straightforward to implement when doing analytical work. It requires that as $z\to\infty$ all of the waves will be traveling away from $z = 0$, not towards it.

We choose to solve these equations by spatially Fourier transforming the function $u(x,y,z)$ in the x and y directions. Let

$$U(k_x,k_y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(k_x x + k_y y)} u(x,y,z)\,dxdy, \tag{231}$$

be the Fourier transform of u, and $F(k_x,k_y)$ be the Fourier transform of $f(x,y)$. The function U must satisfy the equations

$$\frac{d^2U}{dz^2} + (k^2 - k_x^2 - k_y^2)U = 0, \tag{232a}$$

$$U(k_x,k_y,0) = F(k_x,k_y), \tag{232b}$$

$$U(k_x,k_y,z) \text{ has only outgoing waves as } z \to \infty. \tag{232c}$$

The solution to this set of equations can be written as

$$U(k_x,k_y,z)=F(k_x,k_y)e^{iz}\sqrt{k^2-k_x^2-k_y^2}. \tag{233}$$

The sign of the square root must be chosen so that the field decays as $z\rightarrow\infty$, and so that there are no incoming waves from infinity. In order to ensure this we must choose the positive square root for $k^2-k_x^2-k_y^2>0$, and choose it so that $i\sqrt{k^2-k_x^2-k_y^2}<0$ for $k^2-k_x^2-k_y^2<0$.

We can now inverse Fourier transform this to get

$$u(x,y,z)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i(k_x x+k_y y)}F(k_x,k_y)e^{ikz\sqrt{1-k_x^2/k^2-k_y^2/k^2}}dk_x dk_y. \tag{234}$$

We have now accomplished the first step in deriving the Fresnel approximation; we have derived an exact expression for the field u in terms of its value at $z = 0$.

This form for the field is sometimes used in diffraction theory. However, both analytical and numerical work is usually much simpler if the square root is approximated by

$$\sqrt{1-k_x^2/k^2-k_y^2/k^2}\approx 1-\frac{k_x^2+k_y^2}{2k^2}. \tag{235}$$

This approximation is referred to as the paraxial or Fresnel approximation. Using this approximation we can write

$$U(k_x,k_y,z)=F(k_x,k_y)e^{ikz}e^{-iz(k_x^2+k_y^2/2k)}. \tag{236}$$

Assuming the paraxial approximation, the Fourier convolution theorem tells us that the field $u(x,y,z)$ can be written as the convolution of $f(x,y)$ with the inverse Fourier transform of $e^{ikz}e^{-iz(k_x^2+k_y^2)/2k}$ The inverse Fourier transform $e^{ikz}e^{-iz(k_x^2+k_y^2)/2k}$ is $i(k/2\pi z)e^{ikz}e^{i(k/2z)(x^2+y^2)}$. It follows that we can write

$$u(x,y,z)=i\frac{k}{2\pi z}e^{ikz}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\xi,\eta)e^{ik((x-\xi)^2+(y-\eta)^2/2z}d\xi d\eta. \tag{237}$$

It is often convenient to write this as

$$\begin{aligned}u(x,y,z)=&\frac{ik}{2\pi z}e^{ikz}e^{ik(x^2+y^2)/2z}\\&\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\xi,\eta)e^{ik(\xi^2+\eta^2)/2z}e^{-ik((x\xi+y\eta)/z}d\xi d\eta.\end{aligned} \tag{238}$$

This is usually referred to as the Fresnel approximation (16). This approximation can greatly simplify both analytical and numerical calculations.

C. Fourier Optics

We will now consider what happens when the beam passes through a lens of focal length f at the aperture. We claim that modifying the field at the aperture by the phase factor $e^{-ik(x^2+y^2)/2f}$ is equivalent to passing the beam through a lens with focal length f. In order to see this note that if we had a beam of light coming in from infinity, then the field of the incoming light would be constant over the aperture.

$$f(x,k)=Ae^{i\psi}. \tag{239}$$

At the plane $z = f$, the field would be given by

$$u(x,y,f)=\frac{ik}{2\pi f}Ae^{i\psi}e^{ikf}e^{ik(x^2+y^2)/2f}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-ik((x\xi+y\eta)/f}d\xi d\eta, \tag{240}$$

which can be written as

$$u(x,y,f)=\frac{i2\pi k}{f}e^{ikf}e^{ik(x^2+y^2)/2f}Ae^{i\psi}\delta(kx/f,ky/f). \tag{241}$$

This formula assumes that the aperture is infinitely large, and hence goes beyond the limits of validity of the Fresnel approximation. However, we could consider the case of an aperture of finite diameter, and we would get a more complicated but similar result, namely that the field at $z = f$ is all concentrated near the origin $(x,y) = (0,0)$. This is exactly what a lens of focal length f would do to an incoming field of this sort.

We now consider the case where the incoming beam is not necessarily constant at the aperture, but is equal to $f(x,y)$. We assume that at the aperture $z = 0$ we have a lens with focal length f, which modifies the phase of the incoming beam by a factor $e^{-ik(x^2+y^2)/2f}$. In this case the output will be given by

$$u(x,y,z)=\frac{ik}{2\pi z}e^{ikz}e^{ik(x^2+y^2)/2z} \times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)e^{ik(\xi^2+\eta^2)/2z}e^{-ik((\xi^2+\eta^2)/2f}e^{-ik((x\xi+y\eta)/z}d\xi d\eta. \quad (242)$$

The output at the focal plane is given by

$$u(x,y,z)=\frac{ik}{2\pi f}e^{ikz}e^{ik(x^2+y^2)/2f}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)e^{-ik((x\xi+y\eta)/f}d\xi d\eta, \quad (243)$$

which can be written as

$$\frac{ik}{2\pi f}e^{ikf}e^{ik(x^2+y^2)/2f}F(kx/f,ky/f). \quad (244)$$

where $F(\omega_x, \omega_y)$ is the Fourier transform of the function $f(x,y)$. We see that except for the term outside of the integral, the field distribution is given by the Fourier transform of the incoming field distribution. Note that the x and y dependence of the term outside of the integral has only a phase dependence. It follows that if we are only concerned with the irradiance distribution, then we can in fact ignore the terms outside of the integral.

D. Limits of Validity of the Fresnel Approximation

We now comment on the errors introduced by making the Fresnel approximation. We should emphasize that we are only considering the errors introduced in the problem of approximating the field $u(x, y, z)$ assuming we know the field at $z = 0$. In a real diffraction problem we do not know the field at $z = 0$, but approximate it as being the incoming wavefield.

We will now briefly summarize the conditions under which the Fresnel approximation can be assumed to be valid. In what follows R will be the effective dimension of the aperture and λ will be the wavelength of the light. We assume that the aperture lies in the plane $z = 0$, and that we are evaluating the field at a point $(d\cos(\theta), d\sin(\theta), z)$.

The Fresnel approximation always assumes that

$$R \gg \lambda. \quad (245)$$

Assuming that this restriction holds, the following is a summary of the conditions for the validity of the Fresnel approximation.

- The Fresnel approximation will be valid for all values of d if

$$N_F=\frac{2\pi R^2}{\lambda z} \text{ is not small} \quad (246)$$

- The amplitude of the wave predicted by the Fresnel approximation will be valid even if $N_F \ll 1$, provided

$$d/z \ll 1. \quad (247)$$

- Both the phase and amplitude predicted by the Fresnel approximation will be valid when $N_F \ll 1$, if

$$\frac{d^4}{z^4} \ll \frac{1}{kz}. \quad (248)$$

It should be noted that for the most part we are only concerned with the irradiance of the field, so the phase errors introduced by the Fresnel approximation for large values of z will not be important to us. For this reason we will be justified in using the Fresnel approximation provided $R \gg \lambda$, and that $d/z \ll 1$.

We will analyze the two-dimensional case where the aperture and field is independent of the y coordinate. The three-dimensional case is conceptually no more difficult, but the notation and the algebraic manipulations are simpler in two dimensions. We will assume that the incoming wave field is equal to $f(x)$ at the aperture $z = 0$. If $F(k_x)$ is the Fourier transform of $f(x)$, then the field is given by

$$u(x,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik_x x} F(k_x) e^{ikz\sqrt{1-k_x^2/k^2}}\, dk_x. \tag{249}$$

In analyzing the Fresnel approximation we find it fruitful to consider a family of problems where the form of the function $f(x)$ stays the same, but the scaling of the function changes. In particular we will set

$$f(x) = g(x/R). \tag{250}$$

This includes the situation where the function $f(x)$ is equal to 1 inside the aperture, and 0 elsewhere. In this case the parameter R would be the characteristic dimension of the aperture. The Fourier transform of $f(x)$ can be written as

$$F(k_x) = RG(k_x R), \tag{251}$$

where $G(\alpha)$ is the Fourier transform of $g(x)$. The field for $z>0$ can be written as

$$u(x,z) = R\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik_x x} G(Rk_x) e^{ikz\sqrt{1-k_x^2/k^2}}\, dk_x. \tag{252}$$

If we make the change of variables

$$\xi = Rk_x, \tag{253}$$

we can write this integral as

$$u(x,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\xi x/R)} G(\xi) e^{ikz\sqrt{1-\xi^2/(Rk)^2}}\, d\xi. \tag{254}$$

If $g(\xi)$ is a well-behaved function, the Fourier transform $G(\xi)$ goes to zero as $|\xi|\to\infty$. It follows that our answers will not be very sensitive to how we approximate the term $\sqrt{1-\xi^2/(kR)^2}$ when ξ is large. This means that we only need to approximate this well for $\xi = O(1)$. We now make the approximation that

$$kR \gg 1. \tag{255}$$

This is the first approximation that will be made when doing the Fresnel approximation. This is equivalent to assuming that the aperture is much bigger than the wavelength, an assumption that will have to hold in order to carry out the general plan of diffraction theory. Under this assumption it is reasonable to expand the square root in a Taylor series:

$$kz\sqrt{1-\xi^2/(kR)^2} = kz\left(1 - \frac{\xi^2}{2(kR)^2} + \frac{\xi^4}{8(kR)^4} + \cdots\right). \tag{256}$$

If we ignore the third term and all of the remaining terms in the Taylor series, we will end up with the Fresnel approximation. We will now see when we can ignore these terms, and what sorts of errors we will make when we ignore them. Note that assuming that $kR \gg 14$ and $\xi = O(1)$, these terms will always be small compared to the first two terms. However, it is possible that if $kz \gg 1$, then they will not necessarily be small. For simplicity we will now keep the first three terms, and see when we can ignore the third term. The conditions for ignoring this term will be the same as for ignoring all of the remaining terms. If we keep the first three terms in the Taylor series we get

$$u(x,y) = \frac{1}{2\pi}e^{ikz}\int_{-\infty}^{\infty} G(\xi) e^{-i(1/N_F)(-\xi x^* + 1/2\xi^2 - (\xi^4/8(Rk)^2))}\, d\xi, \tag{257}$$

where

$$N_F = \frac{kR^2}{z}, \tag{258}$$

is known as the Fresnel number and

$$x^* = \frac{xkR}{z}. \tag{259}$$

We see that if

$$\frac{1}{N_F k^2 R^2} = \frac{z}{R}\frac{1}{k^3 R^3} \ll 1, \tag{260}$$

then we can ignore the third term in the Taylor series. This means that if we are close to the aperture then the Fresnel approximation will be valid (assuming $kR \gg 1$). In this case there is no restriction on the value of x. If $1/N_F$ is not large, then the Fresnel approximation will hold. This case is not that interesting, because it is essentially the case when the geometrical optics approximation holds and diffraction effects are unimportant.

We now consider the much more interesting case when N_F is small. In this case the phase in the integrand is multiplied by the large parameter $1/N_F$, and we can apply the method of stationary phase to the integral.

The phase will be stationary at the point ξ_0 satisfying

$$-x^* + \xi_0 - \frac{\xi_0^3}{2(kR)^2} = 0 \tag{261}$$

The method of stationary phase predicts that the field will be given by

$$u(x,z) \approx \frac{1}{2\pi} e^{ikz} e^{i\pi/4} \sqrt{\frac{2N_F \pi}{\psi_0''}} G(\xi_0) e^{-i(1/N_F)\psi_0}, \tag{262}$$

where

$$\psi_0 = -x^* \xi_0 + \frac{\xi_0^2}{2} - \frac{\xi_0^4}{8k^2 R^2}, \tag{263}$$

and

$$\psi_0'' = 1 - \frac{3\xi_0^2}{2k^2 R^2}. \tag{264}$$

This is the result predicted by the method of stationary phase assuming that $N_F \ll 1$ when we keep the first three terms in the Taylor series expansion of $\sqrt{1 - k_x^2 / k^2}$. We would like to know how this compares to the answer we would get if we only kept the first two terms (the Fresnel approximation).

In the Fresnel approximation we would have

$$\xi_0 = x^*. \tag{265}$$

We will have been justified in ignoring the cubic term in the equation for ξ_0 provided

$$\frac{x^{*2}}{k^2 R^2} \ll 1. \tag{266}$$

This is equivalent to requiring that

$$\frac{x^2}{z^2} \ll 1. \tag{267}$$

This means that the stationary point when we include the higher-order term will be nearly the same as the stationary point for the Fresnel approximation provided the opening angle from the midpoint of the aperture to the point (x,z) is small.

From the form of the answer in Eq. (262) we see that if we are not concerned with the phase errors, our answer will be accurate provided we have approximated ξ_0 well. This means that the amplitude of the Fresnel approximation will agree with the amplitude of the answer obtained by keeping three terms in the Taylor expansion provided $x^2 / z^2 \ll 1$, and $N_F \ll 1$. However, in order for the phase of the answer predicted by the Fresnel approximation to agree with the more refined answer, it is necessary that we also approximate ψ_0/N_F well. The value ψ_0/N_F predicted by the Fresnel approximation is

$$\psi_0 / N_F \approx \frac{1}{N_F}(-x^* \xi_0 + \xi_0^2 / 2). \tag{268}$$

This will be a good approximation to the more refined answer provided that

$$\frac{1}{N_F}\frac{x^{*4}}{k^2R^2} \ll 1. \tag{269}$$

This can be written as

$$\frac{x^4}{z^4} \ll \frac{1}{kz}. \tag{270}$$

This agrees with the results we have already summarized concerning the errors in the Fresnel approximation.

E. The Vector Theory of Diffraction

The theory we have presented so far is limited to the scalar wave equation. In optics we are concerned with vector fields, the electric and magnetic fields. We begin by outlining the most naive, but nearly correct, approach to the vector theory. We know that each component of the electric and magnetic fields satisfies the scalar wave equation. Just as in the scalar theory we can assume that the field in the aperture is the same as the incoming field, and that the fields vanish elsewhere in the plane of the aperture. Using the scalar theory, we could compute each component of the electric and magnetic fields.

What are some possible difficulties with this approach? Just because each individual component of the field satisfies the wave equation does not mean that the vector field satisfies Maxwell's equations. If each component of the field were chosen exactly right at the plane of the aperture, then this would be the case. However, the assumption that we have made for the fields at the aperture are not necessarily consistant with the correct fields. For this reason we may end up getting inconsistant fields in the far field.

As an example of an inconsistancy, suppose that $z = 0$ is the plane of the aperture, and that the incoming field is a plane wave propagating in the z direction. The naive approach to vector difraction theory would imply that the z components of **E** and **B** vanish at the aperture, and hence vanish everywhere. A thorough analysis of this situation shows that the z components of the fields do not vanish identically.

This last example merely shows that the results of scalar diffraction theory cannot be exactly right. However, the theory was never intended to give exact answers. Just because the fields are inconsistent does not necessarily mean that they are worse approximations than a theory where the fields satisfy Maxwell's equations. However, the theory that takes into account the vector nature of the fields is in fact more accurate for large angles.

A more consistent approach to the vector theory can be obtained by noting that it is not possible to arbitrarily specify all three components of **E** and **B** at the aperture. It is only necessary to specify the tangential electric fields at the aperture. We now argue that once these fields are known, we know E_x and E_y for $z>0$, and we can then determine E_z and **B**.

Clearly both E_x and E_y satisfy the scalar wave equation. It follows that if we know these components at $z = 0$ then we can determine them everywhere for $z>0$. Once we know E_x and E_y we can use the equation

$$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}=0, \tag{271}$$

to determine E_z up to an arbitrary additive function $f(x,y)$. Assuming that we have a finite sized aperture, the field **E**, and in particular the function E_z, must approach zero as $z\to\infty$. This fact allows us to determine this arbitrary function $f(x,y)$. It follows that we can determine E_z. We can now determine **B** by taking the curl of **E** and using Faraday's law. It follows that we can determine all the components of both **E** and **B** once we specify the tangential components of **E** at the aperture.

The vector theory of diffraction (17) approximates the tangential components of the electric field using scalar diffraction, but then computes the z component based on these fields. We will restrict ourselves to the case where the incoming wave has no z component of the electric field. In this case the scalar theory of diffraction predicts that the diffracted field will also have no z component of the electric field. We will now show that in this situation the z component of the electric field can be ignored provided we are only interested in small angles, a condition that we have already assumed in making the Fresnel approximation.

Suppose that at the plane of the aperture the tangential components of the electric field are given by

$$(E_x(x,y,0), E_y(x,y,0)) = (g_x(x/R, y/R), g_y(x/R, y/R)). \tag{272}$$

The x and y components of the electric field each satisfy the scalar wave equation. By Fourier transforming the wave equation we can conclude that

$$\begin{aligned} E_x(x,y,z) = \frac{1}{4\pi^2} R^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(k_x x + k_y y)} G_x(k_x R, k_y R) \\ \times e^{iz\sqrt{k^2-k_x^2-k_y^2}} dk_x dk_y, \end{aligned} \tag{273}$$

and

$$\begin{aligned} E_y(x,y,z) = \frac{1}{4\pi^2} R^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(k_x x + k_y y)} G_y(k_x R, k_y R) \\ \times e^{iz\sqrt{k^2-k_x^2-k_y^2}} dk_x dk_y, \end{aligned} \tag{274}$$

where $G_x(k_x,k_y)$ and $G_y(k_x,k_y)$ are the Fourier transforms of $g_x(x,y)$ and $g_y(x,y)$.

Using the fact that $\nabla \cdot \mathbf{E} = 0$ we can write the field E_z as

$$E_z(x,y,z) = \frac{1}{4\pi^2} R^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(k_x x + k_y y)} \Gamma(k_x, k_y) e^{iz\sqrt{k^2-k_x^2-k_y^2}} dk_x dk_y, \tag{275}$$

where

$$\Gamma(k_x, k_y) = \frac{1}{\sqrt{k^2 - k_x^2 - k_y^2}} (k_x G_x(k_x R, k_y R) + k_y G_y(k_x R, k_y R)). \tag{276}$$

These are exact expressions assuming that we know the tangential electric field at the plane of the aperture.

The expression for E_z is very similar to the expressions for E_x and E_y except that it has $k_x / \sqrt{k^2 - k_x^2 - k_y^2}$ multiplying G_x, and $k_y / \sqrt{k^2 - k_x^2 - k_y^2}$ multiplying G_y. Under the conditions for the Fresnel approximation we can make the approximation

$$\frac{k_x}{\sqrt{k^2 - k_x^2 - k_y^2}} = \frac{k_x}{k}, \tag{277}$$

and

$$\frac{k_y}{\sqrt{k^2 - k_x^2 - k_y^2}} = \frac{k_y}{k}. \tag{278}$$

The Fresnel approximation is based on the assumption that k_x/k and k_y/k are both small in the region of interest. It follows that the factors multiplying G_x and G_y will always make the term E_z negligible compared to E_x and E_y.

For example, if the Fresnel number is small, then we can evaluate these integrals using the method of stationary phase. We could put these integrals in dimensionless form and arrange things so that there was a large parameter multiplying the phase. However, we can take a short cut and note that in the Fresnel approximation, the phases of the integrands are given by

$$\phi = k_x x + k_y y - z \frac{k_x^2 + k_y^2}{2k}. \tag{279}$$

The phase will be stationary when

$$x - k_x z / k = 0, \tag{280}$$

and

$$y - k_y z / k = 0. \tag{281}$$

This shows that when we apply the method of stationary phase, the *z* component of the electric field can be related to the other two components by

$$E_z(x,y,z) \approx \frac{x}{z} E_x(x,y,z) + \frac{y}{z} E_y(x,y,z). \tag{282}$$

This shows that provided $|x/y| \ll 1$, and $|y/z| \ll 1$, the *z* component of the electric field will be negligible compared to the tangential components. This was based on the assumption that the Fresnel number was small. If the Fresnel number is order 1, we can show that the *z* component will be small provided only that $kR \gg 1$.

VII. GEOMETRICAL THEORY OF BEAM SHAPING

A. One-dimensional Theory

In this section we present a theory of beam shaping based on geometrical optics. Special cases of this theory may be found in the literature on geometrical beam shaping (18). The theory we present is not the most general one using geometrical optics since we assume that the rays are moved around continuously, and in a very orderly manner. In the geometrical optics limit it is possible to accomplish the same goal by moving the rays around in a discontinuous and less orderly manner, but when we analyze beam shaping using diffraction theory we will see that this is very undesirable. We believe that it is very difficult to improve on a beam shaping system designed by the techniques described in this section. However, some systems designed this way will work very well, while others will work very poorly. One must go beyond the geometrical theory and use diffraction theory in order to understand why this is so. That will be the subject of the next section.

We begin by considering the beam shaping problem in one dimension. This theory is directly applicable to cases where the incoming beam has an irradiance distribution that is the direct product of two one-dimensional distributions. A two-dimensional function $f(x,y)$ is the direct product of one-dimensional functions if

$$f(x,y) = f_1(x) f_2(y). \tag{283}$$

If both the input and the desired output can be written as a direct product, then the problem can be decomposed into two one-dimensional beam shaping problems. This is the case when we try to turn a Gaussian beam into a rectangular flat top beam.

We suppose that an incoming parallel beam of light has an irradiance distribution of $I(x/R)$, and at the plane $z = 0$ the beam passes through a phase element that refracts the beam. We would like to determine the phase element such that the irradiance distribution at the plane $z = f$ is given by $(AR/D)Q(x/D)$, where A is a constant chosen so that the energy of our light beam is conserved.

In our analysis we assume that the aperture contains a lens of focal length f, plus an additional optical element that allows us to shape the beam. In practice these two optical elements can be combined into a single optical element, but this may not be a desirable feature if one wants to use the same element to shape the beam at several different focal planes. We suppose that our beam shaping element introduces a phase shift of $(RD/fc)\phi(x/R)$ at the plane $z = 0$ (c is the speed of light). The goal of our analysis is to determine the function ϕ such that the beam at the plane $z = f$ has the desired shape. This analysis is carried out in three steps.

- Determine the constant A that determines the irradiance of the output beam. This is accomplished by requiring that the total energy of the output beam is the same as the energy of the incoming beam.
- Determine a function that maps rays at the plane of the aperture into rays at the focal plane. In particular, we determine a function $\alpha(\xi)$ such that a ray that passes through the aperture at $x = R\xi$ passes through the focal plane at $x = D\alpha(\xi)$. This step can be carried out by requiring that the energy of any bundle of rays that enters the aperture is the same as the energy of the same bundle of rays as they pass through the focal plane.

- Determine the function $\phi(\xi)$ that gives us the phase shift introduced by our beam shaping element. Once we know the function $\alpha(\xi)$ this step can be carried out by requiring that the time for a ray to get from $z = -\infty$ to the focal plane is consistent with Fermat's principle.

At this point the reader may feel annoyed by our introduction of the lengths R and D. For example, it would be simpler if we said that the input beam had the irradiance $I(x)$ rather than $I(x/R)$. However, the lengths R and D have been included in the definition of our irradiance profiles, our normalization constant A, and our phase shift ϕ in order to bring out certain scaling properties of beam shaping. These scaling properties will be especially important in the next section when we discuss diffraction effects.

In order to carry out the first step in this process we note that the energy of the incoming beam can be written as

$$E_{in} = \int_{-\infty}^{\infty} I(s/R)ds = R\int_{-\infty}^{\infty} I(s)ds. \tag{284}$$

The energy of the outgoing beam can be written as

$$E_{out} = \frac{AR}{D}\int_{-\infty}^{\infty} Q(s/D)ds = AR\int_{-\infty}^{\infty} Q(s)ds. \tag{285}$$

If we equate these two expressions we arrive at the result

$$A = \frac{\int_{-\infty}^{\infty} I(s)ds}{\int_{-\infty}^{\infty} Q(s)ds}. \tag{286}$$

We have accomplished the first of our three steps. We now determine the function $\alpha(\xi)$ using the conservation of energy.

$$\int_{-\infty}^{R\xi} I(s/R)ds = A\frac{R}{D}\int_{-\infty}^{D\alpha(\xi)} Q(s/D)ds. \tag{287}$$

This is a mathematical statement of the fact that the energy of all of the rays with initial x coordinates less than $R\xi$ must have the same energy as all of the rays at the focal plane that have x coordinate less than $D\alpha(\xi)$. A simple change of variables gives us the equation

$$\int_{-\infty}^{\xi} I(s)ds = A\int_{-\infty}^{\alpha(\xi)} Q(s)ds. \tag{288}$$

As long as the functions $I(s)$ and $Q(s)$ are both positive, it is clear that the function $\alpha(\xi)$ is uniquely determined by this equation. This follows from the fact that for a given value of ξ we can increase the value α until the integral on the right equals the integral on the left. Since $Q(\xi)>0$, it is clear that for any value of ξ there is only one value of a such that the two integrals will be equal.

The functions $I(s)$ and $Q(s)$ are both non-negative, but it is possible that they could vanish on certain intervals. This would be the case if we were trying to transform a beam into a beam that had a core of zero irradiance (such as an annulus). In this case we could have a whole interval of points α that are assigned to the same point ξ. This degenerate case can be thought of as a limiting case of when the functions $I(s)$ and $Q(s)$ are both positive.

Equation (288) determines the functions $\alpha(\xi)$. However, there are a few motivations for differentiating this equation to get

$$AQ(\alpha)\frac{d\alpha}{d\xi} = I(\xi). \tag{289}$$

This gives us a differential equation for the function $\alpha(\xi)$. One way of solving this differential equation is to integrate this equation once to get back to Eq. (288). However, if one needs to solve the equation numerically, it may be more convenient to solve the differential equation than to solve Eq. (288). Another motivation for writing down the differential equation is that when we make the stationary phase approximation to diffraction theory we end up with this differential equation. Yet another motivation comes from the fact when we consider problems that are neither one dimensional nor radially summetric we must revert to a differential equation that is analogous to (289).

In the energy equation (288) we have assumed that the orientation of the incoming rays is the same as the orientation of the rays at the focal plane $z = f$. By this we mean that incoming rays with ξ □ 0 get mapped into rays with α □ 0 at the focal plane, and incoming rays with ξ □ 0 get mapped into rays with α □ 0 at the focal plane. It is possible to reverse the orientation of the rays so that incoming rays with ξ □ 0 end up at the focal plane with α □ 0, and vice versa. In this case the energy equation can be written as

$$\int_{-\infty}^{R\xi} I(s/R)ds = A\frac{R}{D}\int_{D\alpha(\xi)}^{\infty} Q(s/D)ds. \tag{290}$$

Changing variables in the integrals gives us the equation

$$\int_{-\infty}^{\xi} I(s)ds = A\int_{\alpha(\xi)}^{\infty} Q(s)ds. \tag{291}$$

If we differentiate this equation we get

$$AQ(\alpha)\frac{d\alpha}{d\xi} = -I(\xi). \tag{292}$$

These two solutions will give identical irradiance distributions as long as we evaluate the irradiance at the plane $z = f$. However, as we move away from the plane $z = f$ these two solutions have very different properties. When we apply the method of stationary phase, the two different types of solutions appear by choosing different signs of the phase function. These will also be discussed in Sec. IV.E of Chapter 3.

The solutions derived using Eq. (288) or Eq. (291) are the only ones that allow us to shape the beam so that the rays are moved around in a continuous fashion, and so that the function $\xi(\alpha)$ (the inverse of $\alpha(\xi)$) is single valued. When we study the effects of diffraction we will see that beam shaping systems that do not satisfy these requirements will suffer much more from the effects of diffraction than ones that do.

We have now completed the first two steps in our analysis, and we are ready to determine the function $\phi(\xi)$. We assume that the rays that enter the aperture are coming in parallel. For our purposes it is simpler to assume that rays are coming from a distant point source at (0, –L), and we will then let $L\to\infty$. The travel time for a ray to get from the point source to a point ($D\alpha$,f) consists of three parts:

- The time $t_L(\xi)$ to get from the source at (0, –L) to a point ($R\xi$, 0) on the aperture;
- The time $t_{\text{delay}}(\xi)$ that it takes to get through the Fourier transform lense, and the beam shaping element at ($R\xi$, 0);
- The time $t_f(\xi, \alpha)$ that it takes to get from a point ($R\xi$, 0) on the aperture, to a point ($D\alpha$,f) at the focal plane.

The total travel time is given by

$$t(\xi,\alpha) = t_L(\xi) + t_{\text{delay}}(\xi) + t_f(\xi,\alpha). \tag{293}$$

Fermat's principle requires that the travel time of a ray that starts out at (0, –L), passes through the aperture at ($R\xi$, 0), and ends up at ($D\alpha$,f) must be stationary. This means that it must be stationary compared with the travel time of any nearby ray. In particular it will be stationary with respect to the travel time of a ray that goes from (0, –L), passes through the aperture at ($R\xi$ + $Rd\xi$, 0) and then goes straight to the point ($D\alpha$, f). In order for this to be so we must have

$$\frac{\partial t(\xi,\alpha)}{\partial\xi} = 0. \tag{294}$$

We will now see that this equation allows us to determine $\phi(\xi)$.

The travel time t_L is given by

$$t_L(\xi) = \frac{1}{c}\sqrt{L^2+\xi^2} \approx \frac{1}{c}\left(L + \frac{\xi^2}{2L}\right). \tag{295}$$

In the limit as $L\to\infty$ we end up with the equation

$$\frac{\partial}{\partial\xi}t_L(\xi) = 0. \tag{296}$$

The travel time $t_{\text{delay}}(\xi)$ is given by

$$t_{\text{delay}}(\xi) = -\xi^2 \frac{R^2}{2fc} + \phi(\xi)\frac{RD}{fc}. \tag{297}$$

The first term on the right gives the time delay introduced by the transform lens, and the second term gives the time delay introduced by the beam shaping element.

Taking the derivative of this we get

$$\frac{\partial}{\partial\xi} t_{\text{delay}}(\xi) = -\xi\frac{R^2}{fc} + \frac{RD}{fc}\frac{\partial}{\partial\xi}\phi(\xi). \tag{298}$$

The travel time $t_f(\xi, \alpha)$ is given by

$$t_f(\xi,\alpha) = \frac{1}{c}\sqrt{f^2 + (R\xi - D\alpha)^2}. \tag{299}$$

The paraxial approximation assumes that $D^2\alpha^2/f^2 \ll 1$, and $R^2\xi^2/f^2 \ll 1$,so that we can make the approximation

$$\sqrt{(\varepsilon^2 + f^2)} \approx f + \frac{\varepsilon^2}{2f}. \tag{300}$$

In this approximation we get

$$t_f(\xi,\alpha) = \frac{f}{c} + \frac{(D\alpha - R\xi)^2}{2fc}, \tag{301}$$

and hence

$$\frac{\partial}{\partial\xi} t_f(\xi,\alpha) = \frac{R}{fc}(R\xi - D\alpha). \tag{302}$$

Combining our expressions for $\partial/\partial\xi(t_L + t_{\text{delay}} + t_f)$ we end up with the very simple equation

$$\frac{d\phi}{d\xi} = \alpha(\xi). \tag{303}$$

Assuming we know the function $\alpha(\xi)$, the function $\phi(\xi)$ can be determined by quadrature.

We now collect our beam shaping equations into a single set of equations. Given the functions $I(s)$ and $Q(s)$, the phase function $\phi(\xi)$ is determined by first calculating the constant A

$$A = \frac{\int_{-\infty}^{\infty} I(s)ds}{\int_{-\infty}^{\infty} Q(s)ds}, \tag{304a}$$

then solving the differential equation

$$AQ(\alpha)\frac{d\alpha}{d\xi} = \pm I(\xi) \tag{304b}$$

in order to determine $\alpha(\xi)$. The sign in this equation depends on whether or not we have reversed the orientation of the rays or not. Finally the function $\phi(\xi)$ is obtained by solving the differential equation

$$\frac{d\phi}{d\xi} = \alpha(\xi). \tag{304c}$$

A very simple scaling property of these equations will now be pointed out. If we determine a beam shaping system for the lengths D and f, then we can use the same phase function $(RD/fc)\phi(\xi)$ for a new beam shaping system with lengths D_1 and f_1, provided $D_1/f_1 = D/f$. This means that we can change the scale of our system by merely using a different quadratic lens, without changing the optical element determined by ϕ. This follows from the fact that the function $\alpha(\xi)$ is independent of the D, f, and R. It follows that the function $\phi(\xi)$ is also independent of these quantities. Clearly the function $(RD/fc)\phi(\xi)$ will not change as long as we keep the ratio D/f fixed.

B. Direct Product Distributions

We would once again like to emphasize that the theory of the last section can be applied when both the input and the desired output can be written as direct products. That is, we can use the theory of the last section if we can write

$$I(x,y) = I_1(x)I_2(y),$$

and

$$Q(x,y) = Q_1(x)Q_2(y).$$

In this case the phase function of the beamshaping element can also be written as a direct product.

$$\phi(x,y) = \phi_1(x)\phi_2(y)$$

One very important example of this is when the input is a circular Gaussian,

$$I(x,y) = e^{-(x^2+y^2)/2}$$

and the output is a rectangular flat-top beam:

$$Q(x,y) = \mathrm{Rect}(x/A)\mathrm{Rect}(y/B)$$

where

$$\mathrm{Rect}(x) = 1 \, |x| \leq 1$$
$$\mathrm{Rect}(x) = 0 \, |x| > 1$$

C. Radially Symmetric Problems

We now derive a geometrical theory of beam shaping that applies when we are trying to convert a radially symmetric beam with irradiance profile $I(r/R)$ into a radially symmetric beam with irradiance profile that is proportional to $Q(r/D)$. We assume that the desired output beam has the irradiance $(AR^2/D^2)Q(r/D)$. As in the one-dimensional case we begin by computing the normalization constant A. The total energy of the incoming beam is given by

$$E_{\mathrm{in}} = 2\pi\int_0^\infty I(s/R)sds = 2\pi R^2\int_0^\infty sI(s)ds. \tag{305}$$

The energy of the output beam is given by

$$E_{\mathrm{out}} = \frac{AR^2}{D^2}\int_0^\infty sQ(s/D)ds = AR^2\int_0^\infty sQ(s)ds. \tag{306}$$

If we require that the energy of the incoming beam is the same as the outgoing beam we must have

$$A = \frac{\int_0^\infty sI(s)ds}{\int_0^\infty sQ(s)ds}. \tag{307}$$

We now determine the function $\alpha(\xi)$ such that a ray that encounter our optical element at $(R\xi,0)$ ends up at $(D\alpha, f)$.

The conservation of energy now implies that

$$\int_{R\xi}^\infty I(s/R)sds = \frac{AR^2}{D^2}\int_{D\alpha(\xi)}^\infty sQ(s/D)ds. \tag{308}$$

This equation is a mathematical statement of the fact that the energy of the rays that encounter the plane z = 0 with $r>R\xi$ is the same as the energy of the rays that encounter the focal plane with $r>D\alpha(\xi)$.

A simple change of variables gives us the equations

$$\int_\xi^\infty I(s)sdz = A\int_{\alpha(\xi)}^\infty sQ(s)ds. \tag{309}$$

Just as in the one-dimensional case we can argue that the Eq. (309) uniquely determines the function $\alpha(\xi)$. As in the one-dimensional case it may be convenient to differentiate this equation to get a differential equation for $\alpha(\xi)$.

$$A\alpha Q(\alpha)\frac{d\alpha}{d\xi} = \xi I(\xi). \tag{310}$$

This equation assumes that the ray that starts at the axis of symmetry ends up at the axis of symmetry at $z = f$. In analogy to the one-dimensional case we could also consider the case where the ray that started on the axis is sent out infinitely far from the axis when $z = f$. We could devise an optical element that did this, but it would necessarily be quite degenerate and suffer from diffraction effects.

Now that we know the function $\alpha(\xi)$ we can use Fermat's principle to determine the optical thickness $\phi(r/R)$ that can actually accomplish this beam shaping.

Once again, let $-r^2/2fc + RD\phi(r/R)/fc$ be the time delay introduced by our optical element, and $z = f$ be the imaging plane. Fermat's principle requires that

$$\frac{d\phi}{\partial \xi} = \alpha(\xi). \tag{311}$$

This is exactly the same equation we used in the one-dimensional case. Since we know the function $\alpha(\xi)$ we can determine the function $\phi(\xi)$ by quadrature.

Once again we can argue that the function ϕ is independent of the parameters D and f, and hence the time delay $(RD/fc)\phi(r/R)$ depends on D and f only through the ratio D/f.

D. More General Distributions

So far we have considered one-dimensional (applicable to direct product profiles) and radially symmetric beam shaping. In this section we outline how one would determine an optical element that turns an incoming irradiance profile $I(x/R, y/R)$ into an irradiance distribution that is proportional to $Q(x/D, y/D)$ at the image plane f.

The solution to this problem is much more difficult than the ones we have already encountered. We do not have any first hand experience in actually doing this, but feel that it is worth writing down the equations that would allow one to solve this problem.

We begin by assuming that the irradiance distribution at the focal plane f is equal to $(AR^2/D^2)Q(x/D, y/D)$. In order for the energy of input beam to be the same as the output beam we must have

$$A = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I(s,t)\,ds\,dt}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q(s,t)\,ds\,dt}. \tag{312}$$

We now write down an equation for the conservation of energy of any bundle of rays. Suppose rays that encounter the optical element at $(s, t, 0)$ end up at $(x(s, t), y(s, t), f)$. In order to conserve energy we must have

$$I(s,t) = \pm AQ(x(s,t), y(s,t))J(s,t), \tag{313a}$$

where

$$J(s,t) = \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}. \tag{313b}$$

This is the generalization of the differential form of the energy equations that we have written down previously. It can be justified by noting that the rays in the area $s < x < s + ds$, $t < y < t + dt$ get mapped into a region with area $J(s, t)ds\,dt$ at the focal plane.

If the time delay produced by our beam shaping element is given by $(RD/fc)\phi(x/R, y/R)$, then Fermat's principle shows us that the function $\phi(s, t)$ must satisfy

$$\frac{\partial \phi}{\partial s} = x(s,t), \tag{314a}$$

$$\frac{\partial \phi}{\partial t} = y(s,t). \tag{314b}$$

These two equations can be derived almost identically to the one-dimensional and radial cases. We need two equations because we need to guarantee that the path is stationary with respect to changes in both the x and y directions. Using this last set of equations we can write our energy equation as

$$I(s,t) = \pm AQ\left(\frac{\partial\phi}{\partial s}, \frac{\partial\phi}{\partial t}\right)\left(\frac{\partial^2\phi}{\partial s^2}\frac{\partial^2\phi}{\partial t^2} - \left(\frac{\partial^2\phi}{\partial s\partial t}\right)^2\right). \tag{315}$$

This is a nonlinear partial differential equation for the function $\phi(s, t)$. For the special cases where the profiles are radially symmetric, or can be written as direct products, we end up with our previous results. In general it is not clear that this equation is enough to determine the function $\phi(s, t)$. In order to get a feel for this equation we consider a linearized version of this equation. We will see that the linearized equations end up giving us an equation that is very similar to Poisson's equation. We will see that the linearized equations give us a well-posed mathematical problem, indicating that the same will likely be true of the full non-linear equations.

In order to get a linearized system of equations we suppose that the function $I(s, t)$ is almost identical to the function $Q(x, y)$. This would imply that the function $(x(s, t), y(s, i))$ is very nearly equal to (s, t), and hence

$$\phi(s,t) \approx \frac{s^2+t^2}{2}. \tag{316}$$

This means that the function ϕ is merely reversing the phase difference caused by the lens that focuses the beam at $z = f$. We will now assume that

$$Q(x,y) = I(x,y) + \delta P(x,y), \tag{317}$$

where δ is a very small number. We also assume that

$$A = 1 + \delta a, \tag{318}$$

and

$$\phi(s,t) = \frac{s^2+t^2}{2} + \delta\psi(s,t). \tag{319}$$

To first order in δ we can write

$$Q\left(\frac{\partial\phi}{\partial s}, \frac{\partial\phi}{\partial t}\right) = I(s,t) + \delta(P(s,t) + \nabla\psi(s,t)\cdot\nabla I(s,t)), \tag{320}$$

$$\frac{\partial^2\phi}{\partial s^2}, \frac{\partial^2\phi}{\partial t^2} - \left(\frac{\partial^2\phi}{\partial s\partial t}\right)^2 = I + \delta\left(\frac{\partial^2\psi}{\partial s^2} + \frac{\partial^2\psi}{\partial t^2}\right). \tag{321}$$

If we expand Eq. (315) to first order in δ we end up with the equation

$$P(s,t) + aI(s,t)\left(\frac{\partial^2\psi}{\partial s^2} + \frac{\partial^2\psi}{\partial t^2}\right) + \nabla\psi\cdot\nabla I = 0, \tag{322}$$

which can be written as

$$\nabla\cdot(I(s,t)\nabla\psi) = -P(s,t) - aI(s,t). \tag{323}$$

If we integrate these equations over the xy plane, we find that the left-hand side vanishes (assuming ψ vanishes at ∞), and hence the constant a must be chosen so that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(s,t)ds\,dt + a\int_{-\infty}^{\infty} I(s,t)ds\,dt = 0. \tag{324}$$

Once we have chosen a in this way, we can uniquely solve for ψ if we require that ψ vanishes at ∞.

The fact that we can solve the linearized equations is an excellent sign that the nonlinear equations (315) will uniquely determine the function ϕ.

E. Examples

We will now present some concrete examples from the geometrical theory of beam shaping. Some of these examples are important for their own sake, but other examples are presented to illustrate some of the difficulties that can arise when applying the geometrical theory. The difficulties will not appear until we analyze them using diffraction theory.

Example 9 *Turning a Gaussian into a Flat-top Beam—I* Let

$$I(s) = e^{-s^2}, \tag{325}$$

and

$$Q(s) = 1 \text{ for } |s| < 1, \tag{326}$$

$$Q(s) = 0 \text{ for } |s| > 1. \tag{327}$$

The normalization of the energy requires that

$$\int_{-\infty}^{\infty} e^{-s^2} ds = 2A, \tag{328}$$

or

$$A = \frac{\sqrt{\pi}}{2}. \tag{329}$$

The function $\alpha(\xi)$ must satisfy

$$Q(\alpha)\frac{d\alpha}{d\xi} = \frac{2}{\sqrt{\pi}} e^{-\xi^2}. \tag{330}$$

As long as $|\alpha|<1$ this can be written as

$$\frac{d\alpha}{d\xi} = \frac{2}{\sqrt{\pi}} e^{-\xi^2}. \tag{331}$$

The solution to this equation can be written as

$$\alpha(\xi)=\text{erf}(\xi), \tag{332}$$

where

$$\text{erf}(\xi) = \frac{2}{\sqrt{\pi}} \int_0^{\xi} e^{-s^2} ds. \tag{333}$$

Since $|\alpha|<1$ for $-\infty<\xi<\infty$, we conclude that we do not need to consider the case where $Q(\alpha) = 0$.

We now use the equation

$$\frac{d\phi}{d\xi} = \text{erf}(\xi), \tag{334}$$

to find the solution

$$\phi(\xi) = \frac{2}{\sqrt{\pi}} \left(\xi \frac{\sqrt{\pi}}{2} \text{erf}(\xi) + \tfrac{1}{2} e^{-\xi^2} - \tfrac{1}{2} \right). \tag{335}$$

This example has been presented without any reference to the scalings R and D. If we were trying to turn a beam with the initial distribution $I(x/R)$ into a beam with distribution $Q(x/D)$ at the focal plane f, then our beam shaping element would need to introduce a phase delay of $(RD/fc)\phi(x/R)$.

Example 10 *Turning a Gaussian into a Flat-top Beam—II* We consider the same problem as in the previous example. However, this time we present a solution that reverses the order of the rays.

The function $\alpha(\xi)$ must satisfy

$$Q(\alpha)\frac{d\alpha}{d\xi} = -\frac{2}{\sqrt{\pi}} e^{-\xi^2}. \tag{336}$$

As long as $|\alpha|<1$ this can be written as

$$\frac{d\alpha}{d\xi} = -\frac{2}{\sqrt{\pi}} e^{-\xi^2}. \tag{337}$$

The solution to this equation can be written as

$$\alpha(\xi) = -\text{erf}(\xi). \tag{338}$$

We now use the equation

$$\frac{d\phi}{d\xi} = -\text{erf}(\xi), \tag{339}$$

to find the solution

$$\phi(\xi) = -\frac{2}{\sqrt{\pi}}\left(\xi\frac{\sqrt{\pi}}{2}\operatorname{erf}(\xi) + \tfrac{1}{2}e^{-\xi^2} - \tfrac{1}{2}\right). \qquad (340)$$

Example 11 Turning a *Radial Gaussian into a Radial Flat top* We now consider the problem of turning a radial Gaussian into a radial flat top. In particular suppose $I(s) = e^{-s^2}$, and

$$Q(s) = 1 \text{ if } s < 1, \qquad (341)$$

$$Q(s) = 0 \text{ if } s > 1. \qquad (342)$$

In this case we must choose the constant A so that

$$A\pi = \int_0^\infty s e^{-s^2} ds. \qquad (343)$$

It follows that

$$A = \frac{1}{2\pi}. \qquad (344)$$

Equation (310) implies

$$\alpha\frac{d\alpha}{d\xi} = 2\pi\xi e^{-\xi^2}. \qquad (345)$$

If we require that $\alpha(0) = 0$, this equation implies that

$$\alpha(\xi) = \sqrt{2\pi}\sqrt{1 - e^{-\xi^2}}. \qquad (346)$$

Equation (311) for ϕ now implies that

$$\phi(\xi) = \sqrt{2\pi}\int_0^\xi \sqrt{1 - e^{-s^2}} ds. \qquad (347)$$

Example 12 *Turning a Gaussian into a Stairstep* We consider the case where the input beam is a Gaussian

$$I(s) = e^{-s^2}, \qquad (348)$$

and the desired output beam is a stair step function.

$$Q(s) = \gamma \;|s| < \alpha_0, \qquad (349)$$

$$Q(s) = 1 \;\alpha_0 < |s| < 1, \qquad (350)$$

$$Q(s) = 0 \;|s| > 1. \qquad (351)$$

This situation clearly is symmetrical, so that the phase function $\phi(\xi) = \phi(-\xi)$, and $\alpha(-\xi) = -\alpha(-\xi)$. For this reason we will only concern ourselves with finding ϕ and α for $\xi>0$.

The normalization condition requires that

$$A = \frac{\sqrt{\pi}}{2\alpha_0(\gamma - 1) + 2}. \qquad (352)$$

There will be a point ξ_0 that separates the rays that get sent into the first step from those that get sent into the second step. We do not know this point ahead of time, but must calculate its value given the parmeters γ and α_0. The function $\alpha(\xi)$ must satisfy

$$\frac{d\alpha}{d\xi} = \frac{1}{A\gamma}e^{-\xi^2} \text{ for } \xi < \xi_0. \qquad (353)$$

This equation is valid for $\alpha<\alpha_0$.We also have

$$\frac{d\alpha}{d\xi} = \frac{1}{A}e^{-\xi^2} \text{ for } \xi > \xi_0 \qquad (354)$$

This equation is valid for $\alpha_0<\alpha<1$.

The first of these equations can be integrated from 0 to ξ_0 to give

$$\operatorname{erf}(\xi_0) = \frac{2}{\sqrt{\pi}}A\gamma\alpha_0. \qquad (355)$$

This is not an explicit expression for ξ_0, but it can very quickly be determined using an iterative method such as Newton's method. Once we have determined ξ_0 and A we have explicit expressions for $\alpha(\xi)$. We can now determine the function $\phi(\xi)$ by solving the equation

$$\frac{d^2\phi}{d\xi^2} = \frac{1}{A\gamma} e^{-\xi^2} \text{ for } \xi < \xi_0, \tag{356}$$

$$\frac{d^2\phi}{d\xi^2} = \frac{1}{A} e^{-\xi^2} \text{ for } \xi < \xi_0, \tag{357}$$

along with the requirements

$$\phi(0) = 0, \tag{358}$$

and the requirement that ϕ and its derivative are continuous at ξ_0. These equations are almost identical for those of turning a Gaussian into a flat top. Let $\phi_0(\xi)$ be given by

$$\phi_0(\xi) = \xi \frac{\sqrt{\pi}}{2} \operatorname{erf}(\xi) + \tfrac{1}{2} e^{-\xi^2} - \tfrac{1}{2}. \tag{359a}$$

Then the phase function for the stair step can be written as

$$\phi(\xi) = \frac{1}{A\gamma} \phi_0(\xi) \text{ for } \xi < \xi_0, \tag{359b}$$

and

$$\phi(\xi) = \frac{1}{A\gamma} \left(\gamma\phi_0(\xi) + (1-\gamma)\phi_0(\xi_0) + (\xi - \xi_0)(1-\gamma)\operatorname{erf}(\xi_0) \frac{\sqrt{\pi}}{2} \right) \text{ for } \xi\,..\,\xi_0. \tag{359c}$$

Example 13 *Numerical Solutions for Symmetrical Profiles* There are many situations where it is very cumbersome, or impossible to obtain closed form analytical solutions for the function $\phi(\xi)$. However, it is not difficult to write a computer code that solves for ϕ. We now consider how to write a code for the special case where both $I(s)$ and $Q(s)$ are symmetric with respect to reflections in s. That is

$$I(s) = I(-s), \tag{360a}$$

and

$$Q(s) = Q(-s). \tag{360b}$$

In this case we can argue that

$$\alpha(-\xi) = -\alpha(\xi), \tag{361a}$$

and

$$\phi(-\xi) = \phi(\xi). \tag{361b}$$

This means that we can solve for α and ϕ on the interval $\xi>0$, and this will allow us to determine these functions everywhere.

We now outline how one can use an ODE (ordinary differental equation) solver to determine the function ϕ, given the functions $Q(s)$ and $I(s)$. In order to do this we first determine the constant A.

$$A = \frac{\int_{-\infty}^{\infty} I(s)ds}{\int_{-\infty}^{\infty} Q(s)ds}. \tag{362}$$

In many situations, this constant can be determine analytically, even when the function $\phi(\xi)$ cannot. In these situations, one can analytically compute A. In general, one can use the ODE solver to compute the integrals in both the numerator and the denominator. Once the constant A has been determined, we use the ODE solver to solve the following intial value problem.

$$\frac{d\alpha}{d\xi} = \pm \frac{1}{AQ(\alpha)} I(\xi), \tag{363a}$$

$$\frac{d\phi}{d\xi} = \alpha(\xi). \tag{363b}$$

Either sign can be taken in the first of these equations. As we have already mentioned, each sign corresponds to a physically different solution.

These initial conditions for these equations can be written as

$$\alpha(0) = 0, \tag{364a}$$

$$\phi(0) = 0. \tag{364b}$$

These equations can now be integrated out to any value of ξ that you want. A plot of $\phi(\xi)$ can be made by outputting the values as the integration procedes.

VIII. DIFFRACTIVE THEORY OF LOSSLESS BEAM SHAPING

A. Scaling Properties

We now present a theory of lossless beam shaping that is based on diffraction theory (19). In the geometrical theory of beam shaping it is possible to turn a beam with one irradiance distribution into a beam with any desired irradiance distribution, provided only that the energies of the incoming and outgoing beams are the same. However, when diffraction effects are taken into account, this is no longer possible. The geometrical theory is valid provided the wavelength is small. The major goal of this section is to quantify what we mean by a small wavelength. As in our discussion of geometrical beam shaping, we are interested in turning a beam with an incoming irradiance distribution of $I(x/R,y/R)$ at the plane $z = 0$ into a beam with an irradiance distribution of $Q(x/D,y/D)$ at the plane $z = f$. We will see that the parameter

$$\beta = \frac{2\pi RD}{f\lambda} \tag{365}$$

is a dimensionless measure of how small the wavelength λ is. If this parameter is large, then the results from the geometrical theory of beam shaping should be valid. If it is small, then diffractive effects will be important. The parameter β is one, but not the only, measure of how difficult our beam shaping problem is. We will see that the smoothness properties of our input and output beam is another important measure of the difficulty of the beam shaping problem.

Suppose that at the plane $z = 0$ the incoming wave field is given by $g(x/R, y/R)$, and we have an aperture that has a lens with focal length f along with an additional phase element $\psi(x/R, y/R)$. The theory of Fourier optics shows us that the wavefield at $z = f$ is given by

$$\begin{aligned} U(x_f, y_f, f) = {} & \frac{1}{i\lambda f} e^{ikf} e^{ik(x_f^2+y_f^2)/2f} \\ & \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x/R, y/R) e^{i\psi(x/R,y/R)} e^{-ik(x_f x + y_f y)/f} \, dx\, dy. \end{aligned} \tag{366}$$

We would like to determine a function ψ such that the output $U(x_f, y_f)$ satisfies

$$|U(x_f, y_f)|^2 = A\frac{R^2}{D^2} Q(x_f/D, y_f/D), \tag{367}$$

where the function Q determines the shape of the desired irradiance distribution, D determines the scale of the desired irradiance distribution, and A is a scaling factor that guarantees that the energy of the output beam is the same as that of the incoming beam. At this point our problem has the parameters $\lambda = 2\pi/k$, f, R, and D, and it is not clear what we mean when we say the wavelength is small. We can collect all of our parameters into a single parameter by introducing dimensionless coordinates. In particular, assuming we could choose ψ so that our desired output had exactly the right shape, we would have

$$|G(\omega_x, \omega_y)|^2 = \frac{4\pi^2 A}{\beta^2} Q(\omega_x/\beta, \omega_y/\beta), \tag{368a}$$

where

$$G(\omega_x, \omega_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta) e^{-i(\omega_x \xi + \omega_y \eta)} e^{i\beta\phi(\xi,\eta)} d\xi\, d\eta, \tag{368b}$$

where we have introduced the variables

$$\xi = x / R, \tag{369a}$$

$$\eta = y / R, \tag{369b}$$

$$\omega_x = x_f Rk / f, \tag{369c}$$

$$\omega_y = y_f Rk / f, \tag{369d}$$

and

$$\psi(\xi,\eta) = \beta\phi(\xi,\eta). \tag{369e}$$

We have chosen to write the phase as $\beta\phi$ rather than as ψ. This will be convenient when we are doing the large β approximation. We will refer to the Eqs. (368) as the dimensionless beam shaping equations. Given the function g, the function Q, and the parameter β our goal is to determine a constant A and a function $\phi(\xi, \eta)$ such that Eqs. (368) are satisfied. This statement of the beam shaping problem is very nice because we have collected all of our parameters into the single parameter β.

B. One-dimensional Beam Shaping

As in the theory of geometrical beam shaping, we now consider problems where the incoming beam $g(\xi, \eta)$ and the desired output $Q(s, t)$ can be written as a direct product. This allows us to separate the beam shaping problem into two one-dimensional problems. In particular we are trying to find a function ϕ and a constant A such that for a given $g(\xi)$, $Q(s)$ and β we have

$$|G(\omega)|^2 = A\frac{2\pi}{\beta} Q(\omega / \beta), \tag{370a}$$

where

$$G(\omega) = \int_{-\infty}^{\infty} g(\xi) e^{-i\omega\xi} e^{i\beta\phi(\xi)} d\xi. \tag{370b}$$

In general it is not possible to choose ϕ so that Eqs. (370) are satisfied exactly. For example, if β is small, then we would need the Fourier transform of $g(\xi)e^{i\phi(\xi)}$ to be very concentrated around the origin. This would contradict the uncertainty principle. To make this statement more precise, we can apply the uncertainty principle to the function $g(\xi)e^{i\beta\phi(\xi)}$ and its desired Fourier transform to get

$$\Delta_g \Delta_G \geq \tfrac{1}{4}, \tag{371}$$

where

$$\Delta_G = \frac{\int_{-\infty}^{\infty} \omega^2 |G(\omega)|^2 \, d\omega}{\int_{-\infty}^{\infty} |G(\omega)|^2 \, d\omega}, \tag{372}$$

and

$$\Delta_g = \frac{\int_{-\infty}^{\infty} \xi^2 |g(\xi)|^2 \, d\xi}{\int_{-\infty}^{\infty} |g(\xi)|^2 \, d\xi}. \tag{373}$$

If we could choose ϕ so that we accomplished our beam shaping exactly, we would have

$$|G(\omega)|^2 = \frac{A2\pi}{\beta} Q(\omega / \beta). \tag{374}$$

This would imply that

$$\Delta_G = \beta^2 \Delta_Q, \tag{375}$$

where

$$\Delta_Q = \frac{\int_{-\infty}^{\infty} \omega^2 \,|\, Q(\omega) \,|^2 \; d\omega}{\int_{-\infty}^{\infty} |\, Q(\omega) \,|^2 \; d\omega}, \tag{376}$$

and hence,

$$\beta^2 \Delta_g \Delta_Q \geq \tfrac{1}{4}. \tag{377}$$

This inequality cannot be satisfied if β is too small. It should be evident that if β is very small, then it will not even be possible to turn the beam into a profile that is even near the desired profile. This shows that it is not possible to do a good job of beam shaping if the parameter β is small. We now consider the case where β is large, and show that in this case if we choose ϕ to be the function obtained from using geometrical beam shaping, then this will nearly satisfy our beam shaping problem.

We begin our analysis of the beam shaping problem by commenting on our decision to write the phase delay as $\beta\phi(\xi)$. This scaling will allow us to use the method of stationary phase to determine the behavior for large values of β. It should be noted that this scaling predicts that the phase function grows linearly with the frequency of light that we are using, a result that would hold if we designed a lens based on geometrical optics, and kept the same lens for all frequencies of light.

If we use the variable

$$\alpha = \frac{\omega}{\beta}, \tag{378}$$

our beam shaping problem can be written as follows: given the function g, the function Q, and the parameter β, try to determine the constant A, and the function ϕ such that

$$G(\alpha) = \int_{-\infty}^{\infty} g(\xi) e^{i\beta(\phi(\xi)-\alpha\xi)} d\xi, \tag{379a}$$

$$|\, G(\alpha) \,|^2 = \frac{2\pi A}{\beta} Q(\alpha). \tag{379b}$$

The integral in Eq. (2.379a) is in a form that can be evaluated using the method of stationary phase. To lowest order in β, the method of stationary phase shows us that the integral is given by

$$G(\alpha) \approx e^{i\pi/4} e^{i\beta(\phi(\xi(\alpha))-\xi(\alpha))} \sqrt{2\pi}\, \frac{g(\xi(\alpha))}{\sqrt{\beta\phi''(\xi(\alpha))}}, \tag{380}$$

where the function $\xi(\alpha)$ is determined implicitly by the equation

$$\frac{d}{d\xi}\phi(\xi(\alpha)) - \alpha = 0. \tag{381a}$$

If we have chosen ϕ so that the beam has the desired output, then we have

$$AQ(\alpha) = \frac{g^2(\xi(\alpha))}{\phi''(\xi(\alpha))}. \tag{381b}$$

With a little bit of manipulation we can make these equations identical to the equations for geometrical beam shaping. In order to do this we begin by differentiating the first of Eqs. (381) with respect to α. This gives us

$$\frac{d^2\phi(\xi(\alpha))}{d\xi^2} \frac{d\xi(\alpha)}{d\alpha} = 1. \tag{382}$$

Using this equation, the Eq. (381b) can be written as

$$\frac{d\xi(\alpha)}{d\alpha} g^2(\xi) = AQ(\alpha). \tag{383}$$

If we use the fact that the irradiance of the incoming beam is given by
$|g(\xi)|^2 = I(\xi)$, we §et the system of equations

$$\frac{d\xi}{d\alpha} I(\xi) = AQ(\alpha), \tag{384a}$$

$$\frac{d}{d\xi}\phi(\xi(\alpha)) - \alpha = 0. \tag{384b}$$

If we integrate the first of these equations from $-\infty$ to ∞ we find the normalization condition

$$A = \frac{\int_{-\infty}^{\infty} I(\xi) d\xi}{\int_{-\infty}^{\infty} Q(\alpha) d\alpha}. \tag{384c}$$

These equations are identical to Eqs. (304) derived using the geometrical theory of beam shaping.

C. Two-dimensional Beam Shaping

We will now quickly summarize how our results can be extended to apply to arbitrary beam shape problems, that is, ones that are not separable. In general we want to find a function $\phi(\xi, \eta)$ such that

$$G(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi) e^{i\beta(\phi(\xi,\eta) - x\xi - y\eta)} d\xi\, d\eta, \tag{385a}$$

$$|G(x, y)|^2 = \frac{4\pi^2 A}{\beta^2} Q(x, y). \tag{385b}$$

An argument almost identical to that used in the separable case shows that the uncertainty principle requires that

$$\beta^2 \Delta_g \Delta_Q \geq 1, \tag{386a}$$

where

$$\Delta_g = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\xi^2 + \eta^2) |g(\xi,\eta)|^2\, d\xi\, d\eta}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |g(\xi,\eta)|^2\, d\xi\, d\eta}, \tag{386b}$$

and

$$\Delta_Q = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\omega_x^2 + \omega_y^2) |Q(\omega_x, \omega_y)|^2\, d\omega_x d\omega_y}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Q(\omega_x, \omega_y)|^2\, d\omega_x\, d\omega_y}. \tag{386c}$$

As in the separable case, this inequality cannot be satisfied if β is too small. We now consider the limit of the integral in (385a) as $\beta \to \infty$. Using the two-dimensional method of stationary phase we find that

$$|G(x, y)|^2 \approx \frac{4\pi^2}{\beta^2 J(\xi_0, \eta_0)} |g(\xi_0, \eta_0)|^2, \tag{387}$$

where (ξ_0, η_0) are determined implicitly by the stationarity conditions

$$\frac{\partial}{\partial\xi}\phi(\xi_0, \eta_0) = x, \tag{388a}$$

$$\frac{\partial}{\partial\eta}\phi(\xi_0, \eta_0) = y. \tag{388b}$$

and the function J is defined by

$$J(\xi_0, \eta_0) = \frac{\partial^2\phi(\xi_0, \eta_0)}{\partial\xi^2}\frac{\partial^2\phi(\xi_0, \eta_0)}{\partial\eta^2} - \left(\frac{\partial^2\phi(\xi_0, \eta_0)}{\partial\xi\partial\eta}\right)^2. \tag{389}$$

If we use the stationarity conditions, we can write the function J as

$$J(\xi_0, \eta_0) = \frac{\partial x(\xi_0, \eta_0)}{\partial\xi}\frac{\partial y(\xi_0, \eta_0)}{\partial\eta} - \frac{\partial x(\xi_0, \eta_0)}{\partial\eta}\frac{\partial y(\xi_0, \eta_0)}{\partial\xi}. \tag{390a}$$

If we require that the function $|G(x, y)|^2$ has the desired output we arrive at the equation

$$I(\xi_0, \eta_0) = AQ(x, y) J(\xi_0, \eta_0). \tag{390b}$$

These last two equations along with the stationarity conditions in Eq. (388) are identical to the two-dimensional equations that we derived using geometrical optics.

D. Radially Symmetric Problems

In our section on geometrical beam shaping we considered problems that have radial symmetry. We now consider how to analyze these problems for the effect of diffraction. Problems with radial symmetry can be considered as a special case of the general theory of two-dimensional beam shaping. These problems are important enough that they deserve some special attention. Suppose both the input beam g and the desired output beam Q have radial symmetry. In this case the phase function ϕ will also have radial symmetry, and we can replace our two-dimensional Fourier transforms with Hankel transforms (see Sec. II.C in this chapter). The theory of Hankel transforms shows that our beam shaping problems can be phrased as follows.

Given a function $g(\xi)$, a function $Q(\alpha)$, and a parameter β, find a function $\phi(\xi)$ such that

$$G(\alpha) = 2\pi \int_0^\infty g(\xi)\xi e^{i\beta\phi(\xi)} J_0(\alpha\xi) d\xi \tag{391a}$$

satisfies

$$|G(\alpha)|^2 = \frac{4\pi^2 A}{\beta^2} Q(\alpha / \beta). \tag{391b}$$

We already know that a lens designed using the first-order term in the stationary phase approximation gives the same lens as one designed using geometrical optics. Since radially symmetric problems are special cases of the two-dimensional case, if we design a radially symmetric lens using the large β limit, we should get the same lens as when we design it using geometrical optics. We conclude that the function $\phi(\xi)$ can be obtained by using the techniques described in our section on the geometrical theory of beam shaping. Once we have obtained this function, we can use Eqs. (391) to see how our system performs with a finite value of β. To carry this out in practice, we have used ODE solvers in order to compute the function ϕ, and to perform the integration in the definition of the Hankel transform.

E. The Continutity of ϕ

We have seen that the first term in the method of stationary phase is identical to the results obtained using geometrical optics. In order for us to know how well the geometrical optics approximation is working, it is necessary to understand the next order term in the stationary phase approximation. We discussed the higher-order terms in the method of stationary phase in Sec. III. There we saw that if the functions ϕ and g are infinitely differentiate, then the next order term in the method of stationary phase is $1/\beta$ times the size of the first term. However, if the third derivative of ϕ (or g) are discontinuous, then the next order term will only be $1/\sqrt{\beta}$ times smaller than the first-order term. If ϕ has a discontinuity in a lower derivative, we get even worse convergence.

We now consider what class of functions $Q(a)$ will lead to discontinuities in the phase function $\phi(\xi)$ designed by using geometrical optics. We will assume that the function $I(\xi)$ is smooth (such as a Gaussian). Equations (384) show that the derivative of ϕ has the same continuity properties as the function $\alpha(\xi)$ If we take the derivative of the first of Eqs. (384) with respect to ξ we find

$$A\left(\frac{dQ}{d\alpha}\left(\frac{d\alpha}{d\xi}\right)^2 + Q(\alpha)\frac{d^2\alpha}{d\xi^2}\right) = \frac{dI}{d\xi}. \tag{392}$$

We see that if the function $Q(\alpha)$ has a discontinuous derivative at a point $\alpha = \alpha(\xi_0)$ where $I(\xi_0)\neq 0$, then this will lead to a discontinuity in the second derivative of α with respect to ξ, and hence to a discontinuity in the third derivative of ϕ. It follows that discontinuities in the derivatives of Q or I will slow down the convergence towards the geometrical optics limit.

Note that we excluded the case where the discontinuity in Q occurs at a point where I vanishes. In this case we must have $d\alpha/d\xi = 0$, and when we look at our expression for the second derivative of a we find that it does not have a discontinuity. Similar arguments hold for the case where Q itself is discontinuous at a point where I vanishes. A very important example of this is the case where one turns a Gaussian profile into a flat-top beam. In that case the phase function is infinitely differentiable, even though the function $Q(\alpha)$ has a discontinuity in it. This is because the discontinuity in Q occurs as $\xi \to \infty$, and hence at a point where $I(\xi) = 0$.

For the case where the incoming distribution $I(\xi)$ is a Gaussian, we see that discontinuities in the first derivative of Q will lead to discontinuities in the third derivative of ϕ, unless the discontinuity in Q occurs at an extremity. By an extremity we mean a point where the rays reaching this point have come from points infinitely far off the axis.

F. One-Dimensional Examples

In order to illustrate the principles of beam shaping a computer code was written that allows us to compute the function ϕ as well as the effects of using a finite value of β. In these examples we calculate

$$G(\alpha) = \int_{-\infty}^{\infty} g(\xi) e^{i\beta(\phi(\xi)-\alpha\xi)} d\xi, \tag{393}$$

by using an ODE integrator. When an analytical expression for ϕ cannot be found, we compute ϕ with the ODE integrator as we are computing the integral. We output the quantity

$$\Gamma(\alpha,\beta) = \frac{2\pi A}{\beta} |G(\alpha)|^2, \tag{394}$$

where

$$A = \frac{\int_{-\infty}^{\infty} I(\xi) d\xi}{\int_{-\infty}^{\infty} Q(\alpha) d\alpha}. \tag{395}$$

If the effects of diffraction are negligible, the function $\Gamma(\alpha,\beta)$ should be very close to $Q(\alpha)$.

We could have used a code that computed the function ϕ using the technique described in the section on geometrical beam shaping, and then fed this input into an FFT for computing the effects of a finite value of β.

In all of the examples we present we will use the function

$$g(\xi) = e^{-\xi^2/2}, \tag{396}$$

and hence

$$I(\xi) = e^{-\xi^2}. \tag{397}$$

Example 14 *Turning a Gaussian Into a Flat top* We want to turn the output beam into a flat top with

$$Q(\alpha) = 1 \text{ for } |\alpha| < 1 \tag{398}$$

$$Q(\alpha) = 0 \text{ for } |\alpha| > 1. \tag{399}$$

We have already considered this example in our section on geometrical beam shaping, where it was shown that the function ϕ is given by

$$\phi(\xi) = \frac{2}{\sqrt{\pi}} \left(\xi \frac{\sqrt{\pi}}{2} \operatorname{erf}(\xi) + \tfrac{1}{2} e^{-\xi^2} - \tfrac{1}{2} \right). \tag{400}$$

We will be able to see the effects of having a finite value of β. Figure 5a shows plots of $\Gamma(\alpha,\beta)$ for various values of β. We see that for $\beta = 2$ the answer does not look at all like a square pulse, while for $\beta = 32$ the answer is starting to look very good.

Figure 5b shows a plot of the function $\phi(\xi)$.

Example 15 *A Polynomial Output—I* We will now let the output beam be a polynomial that has a hump in it.

$$Q(\alpha) = (1-\alpha^2)(\alpha^2 + \delta) \text{ for } |\alpha| < 1, \tag{401}$$

$$Q(\alpha) = 0 \text{ for } |\alpha| > 1. \tag{402}$$

The constant *A* is easily computed to be

$$A = \frac{15\sqrt{\pi}}{4 + 20\delta} \cdot \tag{403}$$

We will choose δ = 1, for this example. Once we know the constant *A*, we use the ODE solver to compute the function ϕ and the function Γ(*a*, β) for various values of β. Figure 6a shows plots of *Γ*(*a*, β) for various values of β. Once again, the results are not good for β. = 2, but get progressively better as we increase the value of β. A careful analysis of the data shows that the relative error

$$e(\alpha,\beta) = \frac{\Gamma(\alpha,\beta) - Q(\alpha)}{Q(\alpha)}, \tag{404}$$

is going to zero like $1/\beta$ everywhere except right at the endpoints $\alpha = \pm 1$.

Figure 6b shows a plot of the function $\phi(\xi)$.

Example 16 *A Polynomial Output—II* This example is the same as the last example except that we have chosen a value of $\delta = 0.25$ in the function $Q(\alpha)$. This causes the function *Q* to have two humps in it. Figure 7a shows plots of $\Gamma(\alpha,\beta)$ for varous values of β, and figure 7b shows a plot of the function ϕ. The relative error is dying down faster than $1/\beta^2$ almost everywhere. Once again right at the ends ($\alpha = \pm 1$), we do not get this behavior, and in the middle ($\alpha = 0$) the convergence is somewhat slower than $1/\beta^2$. The slow convergence at this point does not appear to be illustrating any fundamental principle, but appears to go away if we choose a large enough value of β.

Example 17 *A Triangle Function*

$$Q(\alpha) = 1 - |\alpha| \text{ for } |\alpha| < 1, \tag{405}$$

$$Q(\alpha) = 0 \text{ for } |\alpha| > 1. \tag{406}$$

This discontinuity in the derivative of the function $Q(\alpha)$ at α = 0 causes the function ϕ to have a discontinuity in its third derivative. Figure 8a shows plots of the function $\Gamma(\alpha, \beta)$ for various values of β. At the point $\alpha = 0$ the convergence towards the function $Q(\alpha)$ can be seen to be going like $1/\sqrt{\beta}$. Figure 8b shows a plot of the function ϕ.

Example 18 *A Stairstep Function—I* We now consider the case where $Q(\alpha)$ is a stair step function.

$$Q(\alpha) = \gamma \text{ for } |a| < 1/2, \tag{407a}$$

$$Q(\alpha) = 1 \text{ for } 1/2 < |\alpha| < 1, \tag{407b}$$

$$Q(\alpha) = 0 \text{ for } |\alpha| > 1. \tag{407c}$$

In this example we choose $\gamma = 3/4$. The discontinuity in the function *Q* at $\alpha = \pm 1/2$ causes the function ϕ to have a discontinuity in its second derivative. Figure 9a shows plots of the function $\Gamma(\alpha,\beta)$. The convergences towards the solution $Q(\alpha)$ is extremely slow. Figure 9b shows a plot of the function ϕ.

Example 19 *A Stairstep function—II* This is the same as in the last example except we choose the parameter γ in the function *Q* to be equal to zero. This causes the function *Q* to have a discontinuity in the first derivative. Figure 10a shows plots of the function $\Gamma(\alpha,\beta)$. We see that the convergence towards $Q(\alpha)$ is extremely slow in this case. Figure 10b shows a plot of ϕ.

G. An Axisymmetric Example

In our section on geometric beam shaping we considered the problem of turning a circular Gaussian beam into an axisymmetric flat-top beam. In this case the input beam $g(\xi, \eta)$ is given by

$$g(\xi,\eta) = e^{-\xi^2 - \eta^2}, \tag{408}$$

and the desired output is given by

$$Q(x,y) = 1 \text{ for } x^2 + y^2 < 1, \tag{409}$$

$$Q(x,y) = 0 \text{ for } x^2 + y^2 > 1. \tag{410}$$

The radially symmetric beam shaping equations give us the normalization constant

$$A = 1. \tag{411}$$

The phase function is given by

$$\phi(r) = \int_0^r \sqrt{1 - e^{-\xi^2}}\, d\xi, \tag{412}$$

where $r^2 = \xi^2 + \eta^2$. In order to analyze the effects of diffraction we compute the radially symmetric Fourier transform. In our section on mathematical preliminaries, we showed that this can be done using the Hankel transform.

$$G(\alpha) = 2\pi \int_0^\infty e^{i\beta\phi(r)} r J_0(\alpha\beta r) g(r) dr. \tag{413}$$

We are interested in the normalized irradiance of this function.

$$\Gamma(\alpha, \beta) = \frac{4\pi^2}{\beta^2} |G(\alpha)|^2 . \tag{414}$$

If the effects of diffraction are negligible, then the function $\Gamma(\alpha, \beta)$ should be nearly equal to $Q(\alpha)$.

Figure 11a shows a plot of $\Gamma(\alpha, \beta)$ for various values of β. We see that the results are quite similar to the one dimensional case. Figure 11b shows a plot of the function ϕ.

Sandia National Laboratories is a multi-program laboratory managed and operated by Sandia Corporation, a wholly owned subsidiary of Lockheed Martin Corporation, for the U.S. Department of Energy National Nuclear Security Administration under contract DE-AC04-94AL85000.

REFERENCES

1. GK Batchelor. An Introduction to Fluid Mechanics. Cambridge: Cambridge University Press, 1967.
2. PW Bridgeman. Dimensional Analysis. New Haven: Yale University Press, 1922.
3. RN Bracewell. The Fourier Transform and its Applications. New York: McGraw-Hill, 1978.
4. RV Churchill. Operational Mathematics. New York: McGraw-Hill, 1972.
5. LE Franks. Signal Theory, Englewood Cliffs, NJ: Prentice Hall, 1969.
6. A Messiah. Quantum Mechanics. Amsterdam: North Holland, 1958.
7. A Sommerfeld. Partial Differential Equations in Physics. New York: Academic Press, 1949.
8. N Bleistein and RA Handelsman. Asymptotic Expansions of Integrals. New York: Dover, 1986.
9. MJ Lighthill. Waves in Fluids. Cambridge: Cambridge University Press, 1978.
10. GB Whitham. Linear and Nonlinear Waves. New York: John Wiley, 1974.
11. M Born, E Wolf. Principles of Optics. London: Permagon Press, 1964.
12. J Lighthill. Fourier Analysis and Generalized Functions. London: Cambridge University Press, 1958.
13. JD Jackson. Classical Electrodynamics. New York: John Wiley, 1962.
14. PR Garabedian. Partial Differential Equations. New York: John Wiley, 1964.
15. N Bleistein. Mathematical Methods for Wave Propagation. Orlando, FL: Academic Press, 1984.
16. JW Goodman. An introduction to Fourier Optics. New York: McGraw-Hill, 1968.
17. JA Stratton. Elecrromagnetic Theory. New York: McGraw-Hill, 1941.
18. CC Aleksoff, KK Ellis, BD Neagle. Holographic conversion of a Gaussian beam to a near-field uniform beam. Opt Eng 30:537–543, 1991.
19. LA Romero, FM Dickey, Lossless laser beam shaping. J Opt Soc Amer A 13(4):751–760, 1996.

Figure 1 This is a schematic of the basic beam shaping system. A parallel beam of light enters the aperture where it encounters a Fourier transform lens, and then the beam shaping element. We would like to choose the beam shaping element so that the output at the focal plane has the desired intensity distribution.

Figure 2 Three examples of desired output distributions. The outputs (a), (b), and (c) get progressively harder to achieve when diffraction effects are taken into account.

Figure 3 A plot of the function $|\mathrm{Fr}(x)|^2$ that models the intensity when light gets diffracted by a plane.

Figure 4 A schematic of light being reflected by a circular mirror. In the shaded region, there are two rays that reach each point by a single reflection off of the mirror. Outside of this region, there is only one ray that reaches each point by a single reflection.

Figure 5 (a) The intensity distribution for different values of β for the problem of turning a one-dimensional Gaussian into a flat-top beam (Example 14). (b) The function $\phi(\xi)$ that accomplishes this exactly in the geometrical optics limit.

Figure 6 (a) The intensity distribution for different values of β for the problem of turning a Gaussian into the output $Q(\alpha) = (1-\alpha^2)(l + \alpha^2)$ for $|\alpha|<1$, $Q(\alpha) = 0$ for $|\alpha| > 1$. (b) The function $\phi(\xi)$ that accomplishes this in the geometrical optics limit.

Figure 7 (a) The intensity distribution for different values of β for the problem of turning a Gaussian into the output $Q(\alpha) = (1 - \alpha^2)(l/4 + \alpha^2)$ for $|\alpha|<1$, $Q(\alpha) = 0$ for $|\alpha| > 1$. (b) The function $\phi(\xi)$ that accomplishes this in the geometrical optics limit.

Figure 8 (a) The intensity distribution for different values of β for the problem of turning a Gaussian into a triangle function $Q(\alpha) = 1 - |\alpha|$ for $|\alpha|<1$, $Q(\alpha) = 0$ for $|\alpha| > 1$. (b) The function $\phi(\xi)$ that accomplishes this in the geometrical optics limit.

Figure 9 (a) The intensity distribution for different values of β for the problem of turning a Gaussian into a step function $Q(\alpha) = 3/4$ for $|\alpha|<1/2$, $Q(\alpha) = 1$ for $1/2<|\alpha|<1$, $Q(\alpha) = 0$ for $|\alpha| > 1$. (b) The function $\phi(\xi)$ that accomplishes this in the geometrical optics limit.

Figure 10 (a) The intensity distribution for different values of β for the problem of turning a Gaussian into a step function $Q(\alpha) = 0$ for $|\alpha|<1/2$, $Q(\alpha) = 1$ for $1/2<|\alpha|<1$, $Q(\alpha) = 0$ for $|\alpha| > 1$. (b) The function $\phi(\xi)$ that accomplishes this in the geometrical optics limit.

Figure 11 (a) The intensity distribution for different values of β for the problem of turning a radially symmetric Gaussian into a radially symmetric flat-top beam, (b) The function $\phi(\xi)$ that accomplishes this in the