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Quantitative Resilience Analysis through Control Design

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Outline

- Brief Controls Overview
  - Linear Quadratic Regulator (LQR) with Tracking

- Order Reduction
  - Proper Orthogonal Decomposition (POD)
  - Weak Galerkin Projection
  - Limitations

- Split-POD
  - Advantages and Examples

- Chemical Supply Chain Example

- Summary

Resilience

In a general sense, resilience is a measure of the robustness of a system to unexpected disturbance.

- Historically quantified by the departure from a desired operating condition.
- Typically doesn’t take into account the cost of recovery.

We seek an systematic resilience framework that:

- Provides automatic recovery from disturbance.
- Provides a measure of departure costs.
- Provides a measure of recovery costs.

Resilience through Control

Control: Driving system behavior to a desired state.

- Typically requires additional input to the system.

Feedback Control: Using system information to drive control inputs.

- Maintains or improves system performance.
- Adds robustness.
- Can be computationally expensive/intractable.

Feedback control uses information about the system to drive inputs aimed at altering system dynamics in a desirable way.

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Feedback Control Formulations

There are many feedback control formulations.

Example: Tracking **Linear** Quadratic Regulator (LQR)

Cost Function

$$\min_u \int_0^\infty \left[(X - X_{ref})^T Q (X - X_{ref}) + u^T R u \right] dt$$

subject to

$$\begin{bmatrix} \dot{X} \\ \dot{X}_{ref} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X \\ X_{ref} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u$$

$$= \bar{A}\bar{X} + \bar{B}u,$$

$$\bar{X}(0) = \begin{bmatrix} X_0 \\ X_{ref} \end{bmatrix}.$$

- X is the state
- X_{ref} is the target state
- X_0 is the state at t=0
- u is the control input
- Q, R are weighting matrices
- A is the state matrix
- B is the input matrix

Minimize a quadratic cost subject to linear system dynamics.

Optimal Control Solution

The tracking LQR problem has a unique optimal solution.

Unique Optimal Solution

$$
\begin{aligned}
u &= -K\overline{X} \\
&= -\begin{bmatrix} K_1 & K_2 \end{bmatrix}\begin{bmatrix} X \\ X_{ref} \end{bmatrix} \\
&= -\begin{bmatrix} R^{-1}B^{T}\Pi_{11} & R^{-1}B^{T}\Pi_{12} \end{bmatrix}\begin{bmatrix} X \\ X_{ref} \end{bmatrix}
\end{aligned}
$$

where

$$A^T\Pi_{11} + \Pi_{11}A - \Pi_{11}BR^{-1}B^T\Pi_{11} + Q = 0 \leftarrow \text{Algebraic Riccati Equation}$$

$$\left[A^T - \Pi_{11}BR^{-1}B^T\right]\Pi_{12} = Q$$

Many commercial packages (e.g. function LQR in Matlab) are available that can solve the Riccati Equation

Problem Dimension Problems

A large number of unknowns results in a HUGE control problem

Numerous control formulations require a dynamical model and a Riccati solution.

The Riccati matrix π is full, but symmetric.

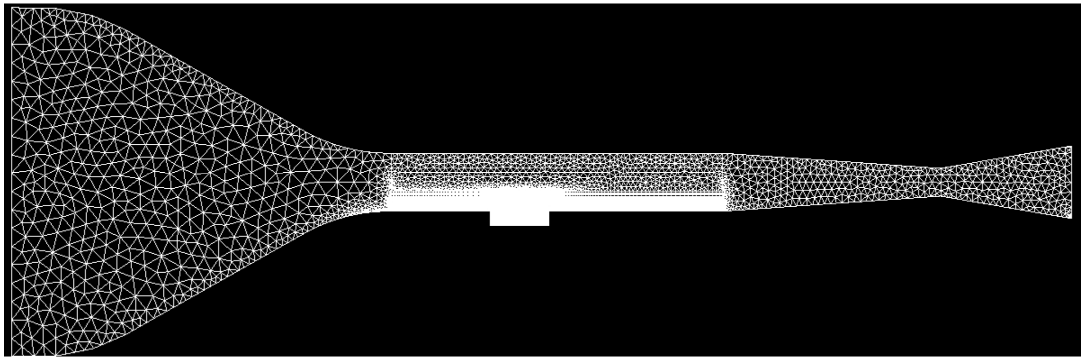
$$A^T\Pi + \Pi A - \Pi B R^{-1} B^T \Pi + Q = 0.$$

nonlinear matrix equation

Computation of Riccati solution becomes expensive/intractable as the system dimension increases.

Intractability

Fluid Flow Example



63,000 Cells
~10^8 Riccati unknowns

For turbulent 3D applications, discretized flow models describing >10^6 flow states are not uncommon → more than 10^{12} Riccati unknowns

Large model dimensions lead to computationally intractable control problems. We need a smaller model!

Order Reduction

Reduce then Design

System order reduction prior to control law design makes the control problem feasible.

Requirements:

- A model with greatly reduced dimension, i.e. a reduced-order model.
- A reduced-order model in a form suitable for control design.
- A reduced-order model with dynamics comparable to the “real” system.

Proper Orthogonal Decomposition (POD)

Proper Orthogonal Decomposition: Generates an optimal basis for a given data set.

Algorithm outline:

- Generate a set $\{S_i(\boldsymbol{x})\}_{i=1}^{N}$ of instantaneous model data (by experiment or simulation).
- Construct the NxN correlation matrix L, where $L_{i,j} = \langle S_i, S_j \rangle_{L^2(\Omega)}$
- The eigenvalues $\{\lambda_i\}_{i=1}^{N}$ of L are calculated and sorted in descending order.
- The eigenvalues of L provide a measure of the data variability captured in a POD basis consisting of M basis modes.

Proper Orthogonal Decomposition (POD)

For example, a POD basis with at least 99% of the total data variability would consist of M modes such that

$$100\left(\frac{\sum_{i=1}^{M}\lambda_i}{\sum_{i=1}^{N}\lambda_i}\right) \geq 99$$

- Find the eigenvectors $\{v_i\}_{i=1}^{M}$ and normalize so that $\| v_i \|_{L^2(\Omega)}^2 = \frac{1}{\lambda_i}$.
- Construct the POD basis set $\{\varphi_i(\boldsymbol{x})\}_{i=1}^{M}$ according to $\varphi_i = \sum_{j=1}^{N} v_{i,j} S_j$.

Proper Orthogonal Decomposition (POD)

- System evolution approximated as a linear combination of POD modes, i.e.,

$$w(t,\boldsymbol{x}) \cong \sum_{i=1}^{M} \alpha_i(t)\varphi_i(\boldsymbol{x})$$

<u>Properties</u>:

- The POD basis is optimal
- POD basis modes are orthonormal, i.e. $\langle \varphi_i, \varphi_j \rangle = \delta_{i,j}$
- Temporal coefficients can be found by projection

$$\alpha_i(t) = \int_{\Omega} w(t,\boldsymbol{x})\varphi_i(\boldsymbol{x})\boldsymbol{dx}$$

<u>POD</u>

- Provides a systematic way to reduce system order
- Provides an optimal basis
- Does **<u>not</u>** provide a dynamical model for temporal coefficients
- Does **<u>not</u>** provide a model for systematic control design

Weak Formulation

<u>Powerful machinery</u>: Weak formulation of the system

- Variational formulation of the problem → test functions
- Weakens regularity requirements on the problem solution
- Natural way to make control inputs explicit
  - Uses integration by parts $\int_a^b v''u\,dx = v'u\Big|_a^b - \int_a^b v'u'dx$

In many systems, the weak solution is the solution that "makes sense"

Example 1: Inviscid Burgers

$$\dot{w}(t,x) + w(t,x)w'(t,x) = 0$$

Example 2: Navier-Stokes

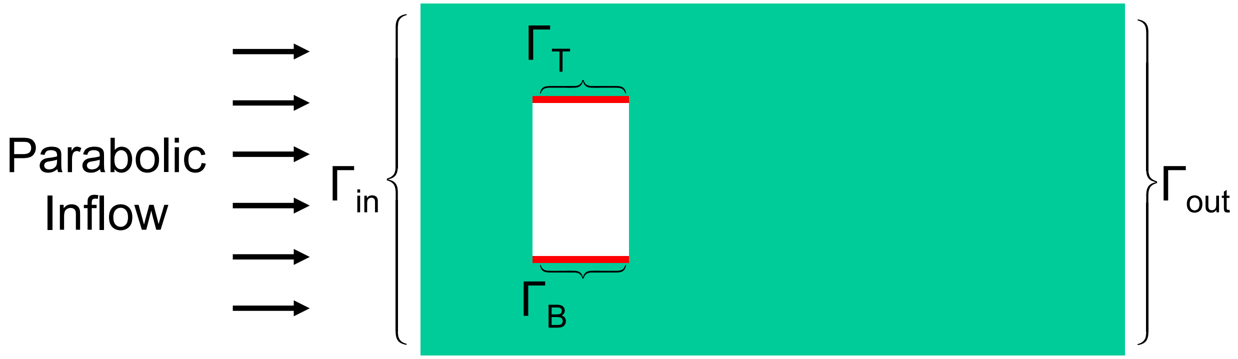
$$\vec{u}_t + (\vec{u}\cdot\nabla)\vec{u} + \nabla p = \frac{1}{\mathrm{Re}}\Delta\vec{u}$$

Under the right initial and boundary conditions, both of these examples can develop shock (discontinuous) solutions → spatial derivatives???

Approach: Use POD basis for test functions in weak formulation

Camphouse, R. C., "Boundary Feedback Control Using Proper Orthogonal Decomposition Models," *Journal of Guidance, Control, and Dynamics,* Vol. 28, No. 5, pp 931-938.

Example: Convection Over an Obstacle



Control on Γ_T and Γ_B to influence behavior downstream of obstacle

Governing Equation

$$w_t + \left(\frac{K_1}{2}w^2\right)_x + \left(\frac{K_2}{2}w^2\right)_y = \frac{1}{\text{Re}}\left(w_{xx} + w_{yy}\right),\ t > 0$$

Boundary Controls

$$w(t,\Gamma_B) = u_B(t)\Psi_B(x)$$

$$w(t,\Gamma_T) = u_T(t)\Psi_T(x)$$

$$w(t,\Gamma_U) = 0$$

Inflow/Outflow

$$w(t,\Gamma_{in}) = f(y)$$

$$w_x(t,\Gamma_{out}) = 0$$

$$w(0,x,y) = w_0(x,y)$$

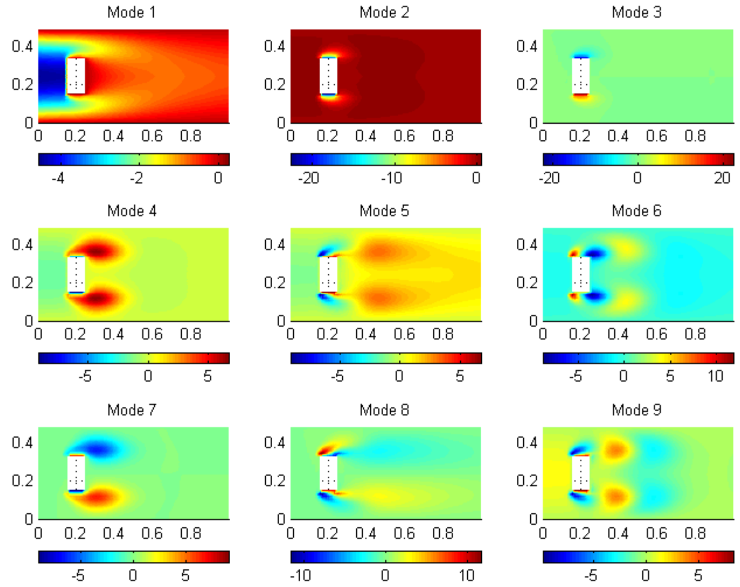
POD Basis and Reduced Model

- Data vector ensemble generated via finite-difference simulation
- Broad range of boundary inputs used during ensemble generation

$$\begin{array}{ll} u_B(t)=0 & u_T(t)=0 \\ u_B(t)=N\sin(0.25t^2) & u_T(t)=0 \\ u_B(t)=0 & u_T(t)=N\sin(0.25t^2) \\ u_B(t)=N\sin(0.25t^2) & u_T(t)=N\sin(0.25t^2) \\ \multicolumn{2}{c}{\textit{for } N = 1,2,3} \end{array}$$

- 25 modes capture 99.9% of the ensemble variability
- Weak formulation and projection onto POD basis yields the reduced order model

First 9 POD Modes



$$\dot{\alpha} = A\alpha + Bu + N(\alpha) + F,$$

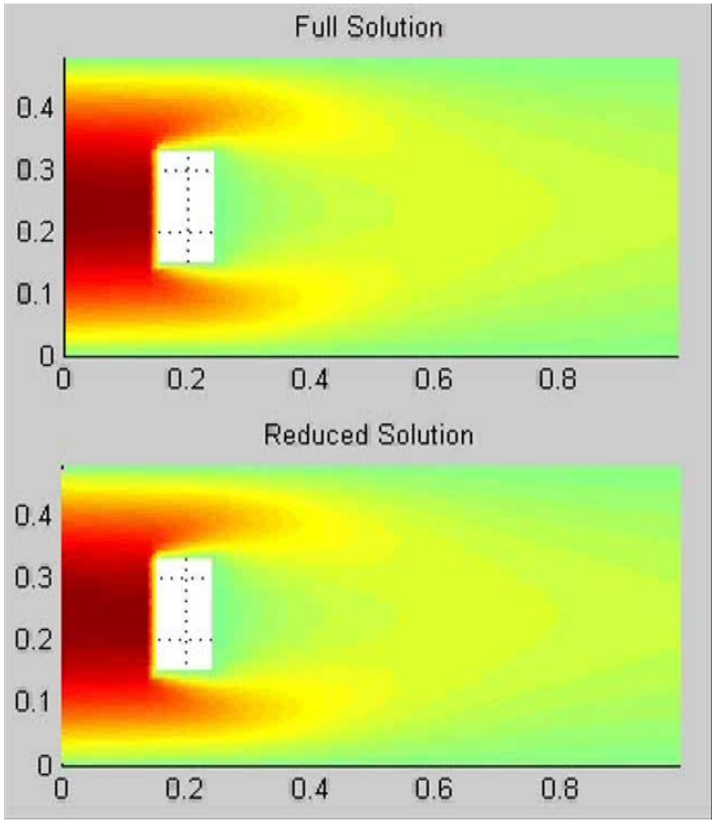
$$\alpha(0) = \alpha_0.$$

Explicit Control Input

Camphouse, R. C. and Myatt, J. H., "Reduced Order Modeling and Boundary Feedback Control of Nonlinear Convection," AIAA Paper 2005-5844, August 2005

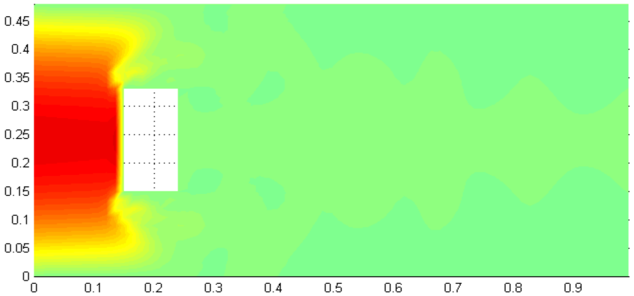
Model Accuracy and Control Effectiveness

Open-Loop Accuracy

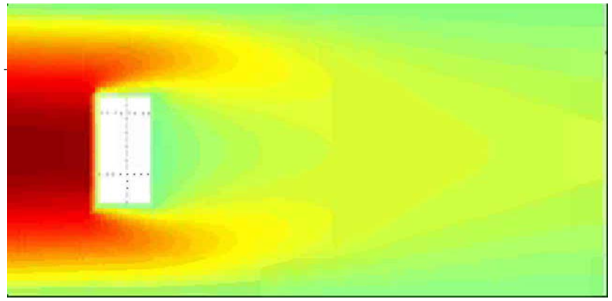


Control Inputs Must Be Carefully Addressed

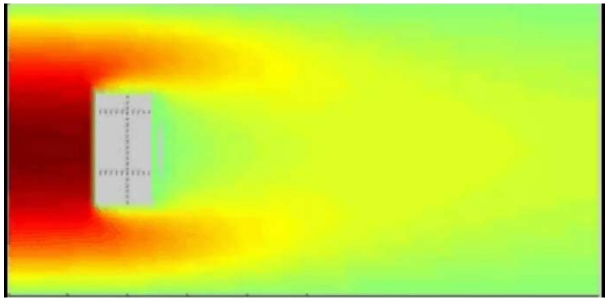
Controlled Target Profile



Controlled Reduced Model (99.9% Variability)



Controlled Reduced Model (99% Variability)



99% Solution Fails!

Extension: Split POD

POD variability argument can be inadequate for control applications.

- Boundary actuation energy small compared to baseline system energy → important data from controls standpoint discarded during order reduction
- Boundary input difficult to reconstruct from reduced model at off-design conditions (feedback boundary input not known a priori)

Split POD:

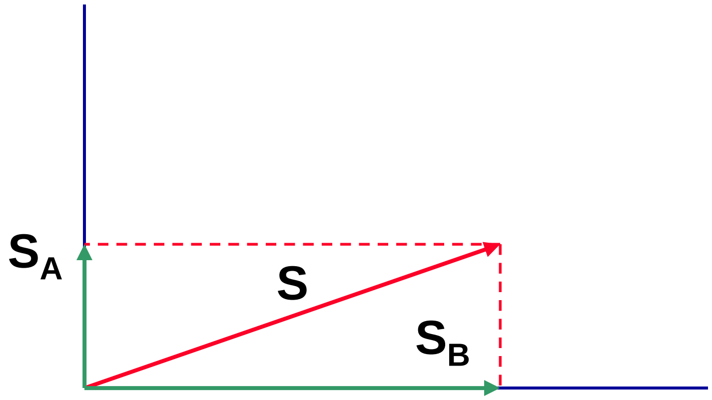
- Decompose each snapshot into a component spanned by a baseline POD basis and an orthogonal component due to control input.
- Perform POD on control input components to get a basis of "actuator modes".
- Determine variability requirements for baseline and actuator bases separately.
- Combine baseline and actuator bases into overall basis that spans baseline and actuated dynamics.

Basic Idea

Basic Idea:

Each actuated data vector can be decomposed into a component spanned by a baseline basis and an actuated component orthogonal to the baseline basis.

$$S = S_B + S_A$$



Details

Split POD Algorithm:

- Generate an ensemble of baseline data snapshots. From this data, construct a set of baseline POD modes $\{ \eta_j \}$, $1 \le j \le M_B$.

- Turn on actuation. Generate an ensemble of actuated data snapshots, $\{ T_i \}$, $1 \le i \le C$.

- Remove baseline information from each actuated snapshot.
  - ✓ Define b_{ij} by $b_{ij} = < T_i , \eta_j >$, i.e. the projection of T_i onto η_j. Then, $b_{ij}\,\eta_j$ is the component of T_i in the direction of η_j.
  - ✓ The linear combination $\sum_{j=1}^{M_B} b_{ij}\eta_j$ is the baseline component of T_i.

Details

Split POD Algorithm (cont):

✓ Remove the baseline information from each actuated snapshot, i.e. construct

$$\tilde{\mathrm{T}}_{\mathrm{i}} = \mathrm{T}_{\mathrm{i}} - \sum_{\mathrm{j=1}}^{\mathrm{M_B}} b_{ij}\eta_j, \; i = 1,2,\ldots,C.$$

- Generate an ensemble of actuation POD modes from the actuated data $\left\{\tilde{T}_j\right\}_{j=1}^{C}$. Denote the actuator modes by $\left\{\xi_j\right\}_{j=1}^{M_A}$.

Nice Properties

- Baseline and actuator modes are orthonormal – can be combined to form overall basis set for the baseline and actuated system

$$\{\varphi_i\}_{i=1}^{M_B+M_A} = \{\eta_1, \eta_2, \ldots, \eta_{M_B}, \xi_1, \xi_2, \ldots, \xi_{M_A}\}$$

- Allows system approximation by linear combination of baseline and actuated components

$$w(t,\mathbf{x}) \cong \sum_{i=1}^{M_B} a_i(t)\eta_i(\mathbf{x}) + \sum_{j=1}^{M_A} b_j(t)\xi_j(\mathbf{x})$$

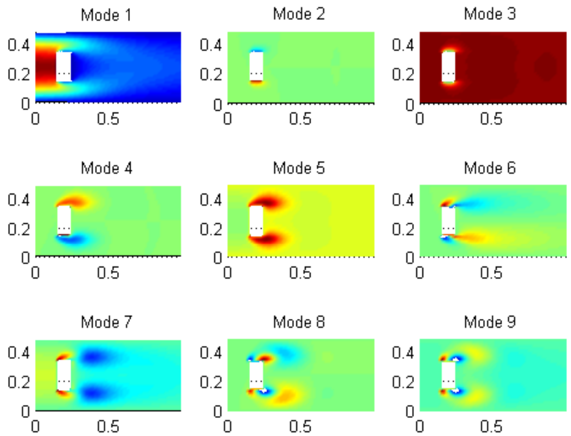
Baseline Mode Expansion

Actuator Mode Expansion

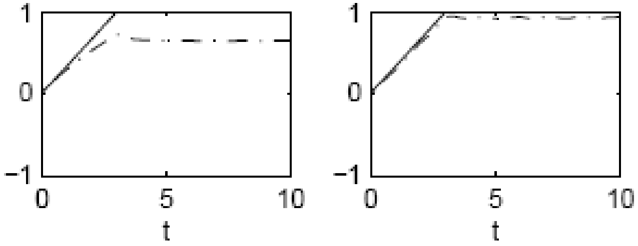
- Subtle but significant actuation effects explicitly accounted for during order reduction

Split POD Improvements

Baseline and Actuator Modes



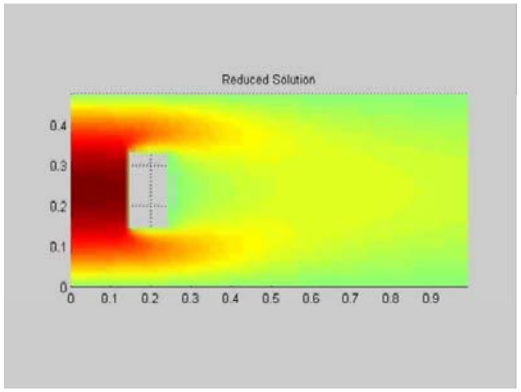
Off-design Boundary Condition Improvement



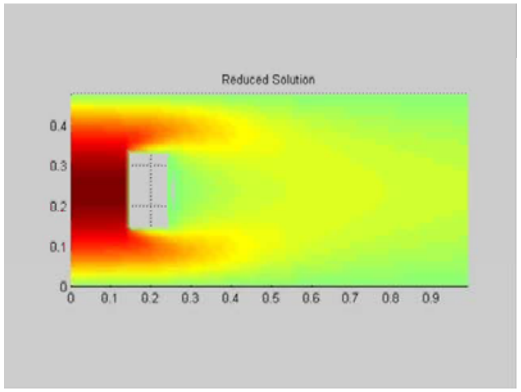
Identical data ensembles, control formulation,
POD basis variability requirements

Camphouse, R. C. et al, "A Snapshot Decomposition Method for Reduced Order Modeling and Boundary Feedback Control," AIAA Paper 2008-4195, June 2008

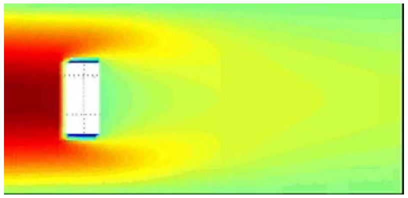
Split POD Not Used - Reduced



Split POD Used - Reduced



Split POD Used - Full



Order Reduction in Hardware

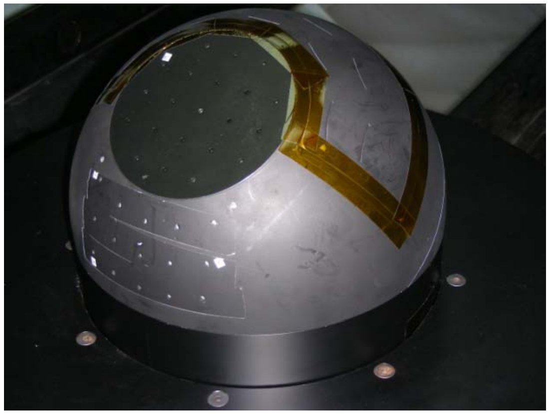
For some systems, there may not be a dynamical model available from which an optimal feedback control can be designed.

- System may be too complicated.
- No simulation results.
- “Build and Test”.
- Experimental approach may be the only feasible option.

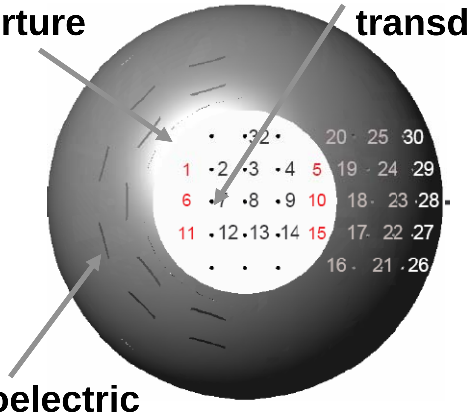
Order reduction techniques can help guide experimental hardware placement and can be used to develop “ad-hoc” controls.

Hemisphere Turret Experiment

Wind Tunnel Model



Hardware Layout

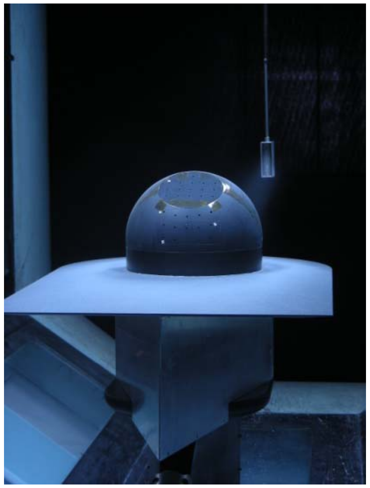


aperture

pressure transducers

piezoelectric synthetic jets

Tunnel Setup



Goal: Develop real-time controller to reduce/eliminate pressure fluctuation over the aperture.

Data Generation

Data Ensemble:

2000 baseline pressure data vectors
3900 actuated pressure data vectors
 - 13 actuated cases, 300 vectors each
Total: 5900 Data Vectors

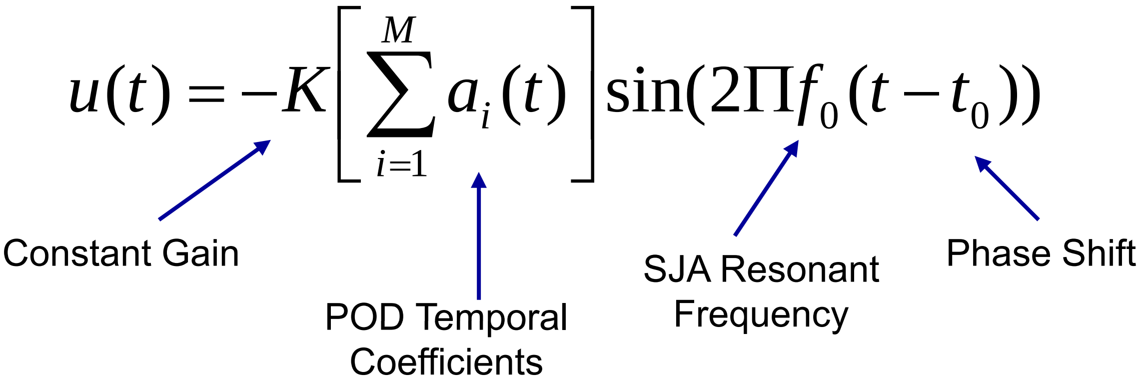
Approach 1: Use downstream pressure measurements to build Split POD basis.

Surprise Result: Downstream sensor locations provide no useful control information – actuated snapshots completely spanned by baseline basis. **Split POD provides design insight!**

Approach 2: Use sensor measurements over the aperture to construct the basis.

Proportional Feedback Controller

Proportional Controller:



$$u(t) = -K\left[\sum_{i=1}^{M} a_i(t)\right]\sin(2\Pi f_0(t-t_0))$$

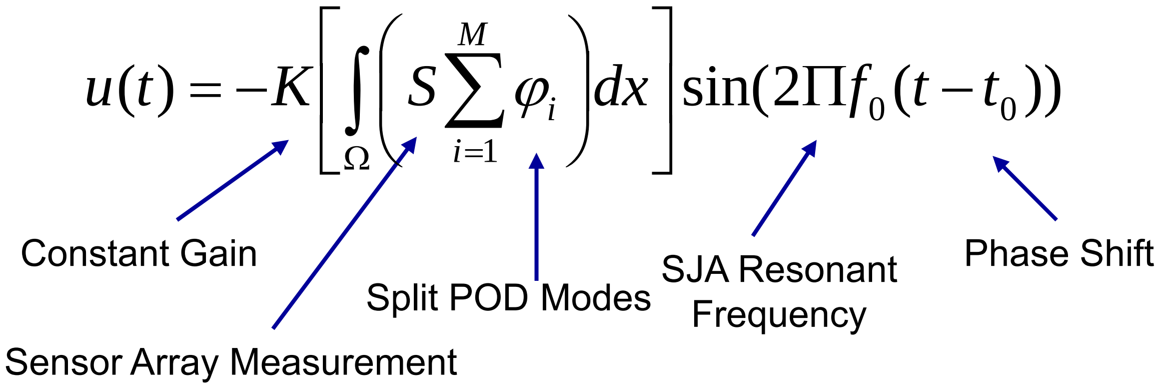
Constant Gain

POD Temporal Coefficients

SJA Resonant Frequency

Phase Shift

Real-Time Implementation:



$$u(t) = -K\left[\int_{\Omega}\left(S\sum_{i=1}^{M}\varphi_i\right)dx\right]\sin(2\Pi f_0(t-t_0))$$

Constant Gain

Sensor Array Measurement

Split POD Modes

SJA Resonant Frequency

Phase Shift

Split POD Implemented Easily in the Wind Tunnel

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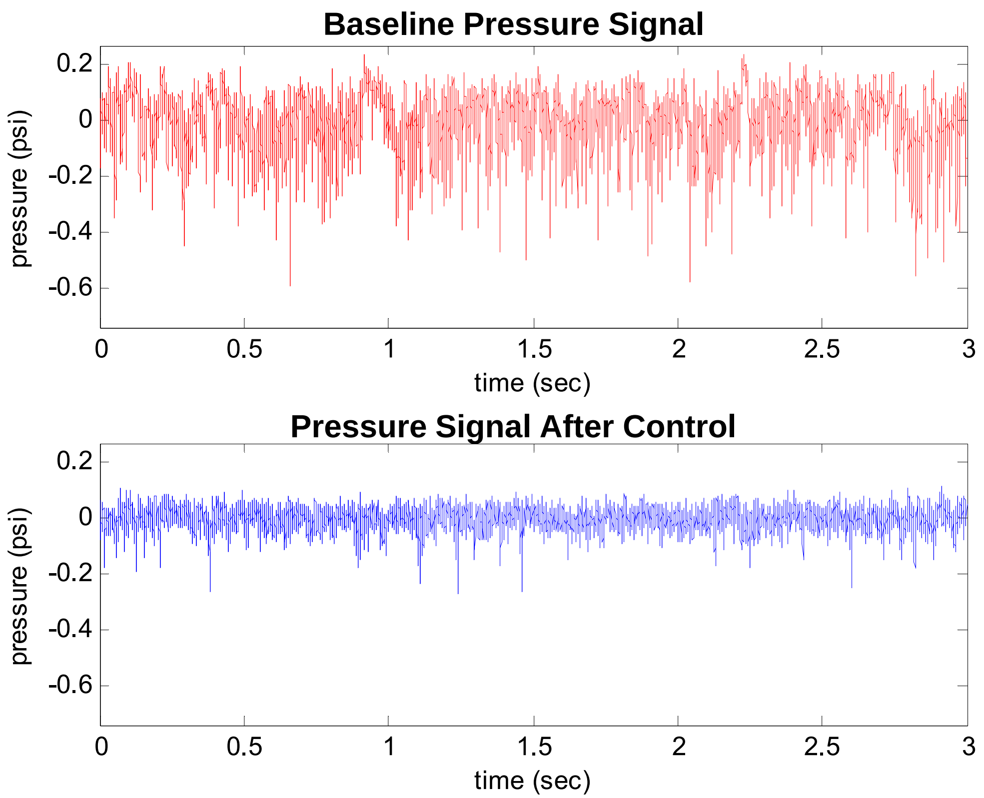
Results



Pressure RMS reduced by 65%!

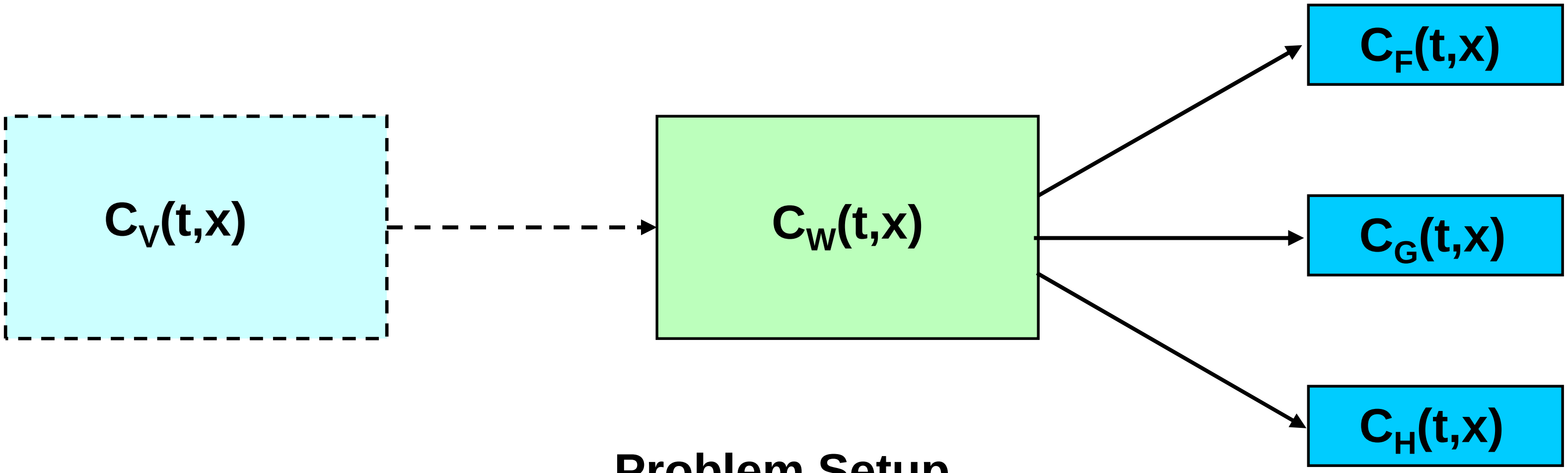
Wallace, R. D. et al, "Flow and Aero-Optics Around a Turret Part II: Surface Pressure Based Proportional Closed Loop Flow Control," AIAA Paper 2008-4217, June 2008

Control and Order Reduction Summary

- Systematic control design methods lend themselves nicely to a quantitative resilience framework.
- Optimal control calculation can be computationally expensive/intractable.
- System order reduction can make control calculation feasible for large systems. Careful treatment of control input is required.
- Order reduction methods can be used to provide insight into hardware placement, and can be used in the design of “ad-hoc” controllers.

Chemical Supply Chain

Supply Chain Configuration



Problem Setup

- Supply chain composed of 5 chemical species V, W, F, G, and H with concentrations $C_V(t,x)$, $C_W(t,x)$, $C_F(t,x)$, $C_G(t,x)$, and $C_H(t,x)$ defined over the interval [0,L].

- It is desired that daughter concentration $C_G(t,x)$ have a specific value at x=L. A nominal value of $C_W(t,0)$ provides the desired concentration of species G at x=L.

- If the availability of species W is disrupted, can species V automatically compensate to hold $C_G(t,L)$ at its desired value during the disruption?

Governing Equations

Temporal and Spatial Evolution of Network Concentrations Modeled by 5 Coupled Partial Differential Equations.

First Order Reaction Rates

k_V, k_W, k_F, k_G, k_H

Stoichiometric Yield Factors

y_W, y_F, y_G, y_H

Dispersion Coefficient

ε

Advective Velocity

μ

$$\frac{\partial}{\partial t}C_V(t,x)=\varepsilon\frac{\partial^2}{\partial x^2}C_V(t,x)-\mu\frac{\partial}{\partial x}C_V(t,x)-k_V C_V(t,x),$$

$$\frac{\partial}{\partial t}C_W(t,x)=\varepsilon\frac{\partial^2}{\partial x^2}C_W(t,x)-\mu\frac{\partial}{\partial x}C_W(t,x)+y_W k_V C_V(t,x)-k_W C_W(t,x),$$

$$\frac{\partial}{\partial t}C_F(t,x)=\varepsilon\frac{\partial^2}{\partial x^2}C_F(t,x)-\mu\frac{\partial}{\partial x}C_F(t,x)+y_F k_W C_W(t,x)-k_F C_F(t,x),$$

$$\frac{\partial}{\partial t}C_G(t,x)=\varepsilon\frac{\partial^2}{\partial x^2}C_G(t,x)-\mu\frac{\partial}{\partial x}C_G(t,x)+y_G k_W C_W(t,x)-k_G C_G(t,x),$$

$$\frac{\partial}{\partial t}C_H(t,x)=\varepsilon\frac{\partial^2}{\partial x^2}C_H(t,x)-\mu\frac{\partial}{\partial x}C_H(t,x)+y_H k_W C_W(t,x)-k_H C_H(t,x),$$

Boundary Conditions

- The five concentrations are prescribed a Robin boundary condition at *x=L*.

$$rC_V(t,L)+\frac{\partial}{\partial x}C_V(t,L)=0,$$

$$rC_W(t,L)+\frac{\partial}{\partial x}C_W(t,L)=0,$$

$$rC_F(t,L)+\frac{\partial}{\partial x}C_F(t,L)=0,$$

$$rC_G(t,L)+\frac{\partial}{\partial x}C_G(t,L)=0,$$

$$rC_H(t,L)+\frac{\partial}{\partial x}C_H(t,L)=0.$$

- Parent concentrations are given time-varying boundary conditions at *x=0*.

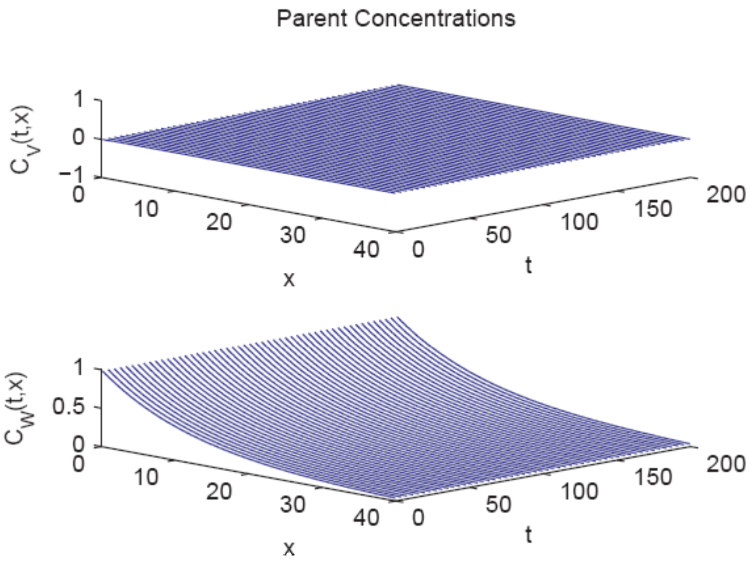
$$C_W(t,0)=f(t), \longleftarrow \text{Nominal Condition}$$

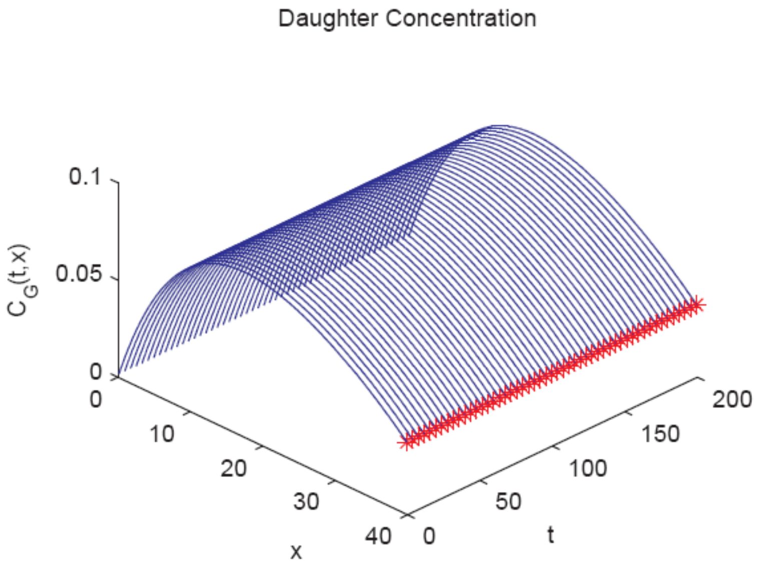
$$C_V(t,0)=u(t). \longleftarrow \text{Control Input}$$

- Daughter concentrations prescribed homogeneous Dirichlet conditions at *x=0*.

$$C_F(t,0)=C_G(t,0)=C_H(t,0)=0.$$

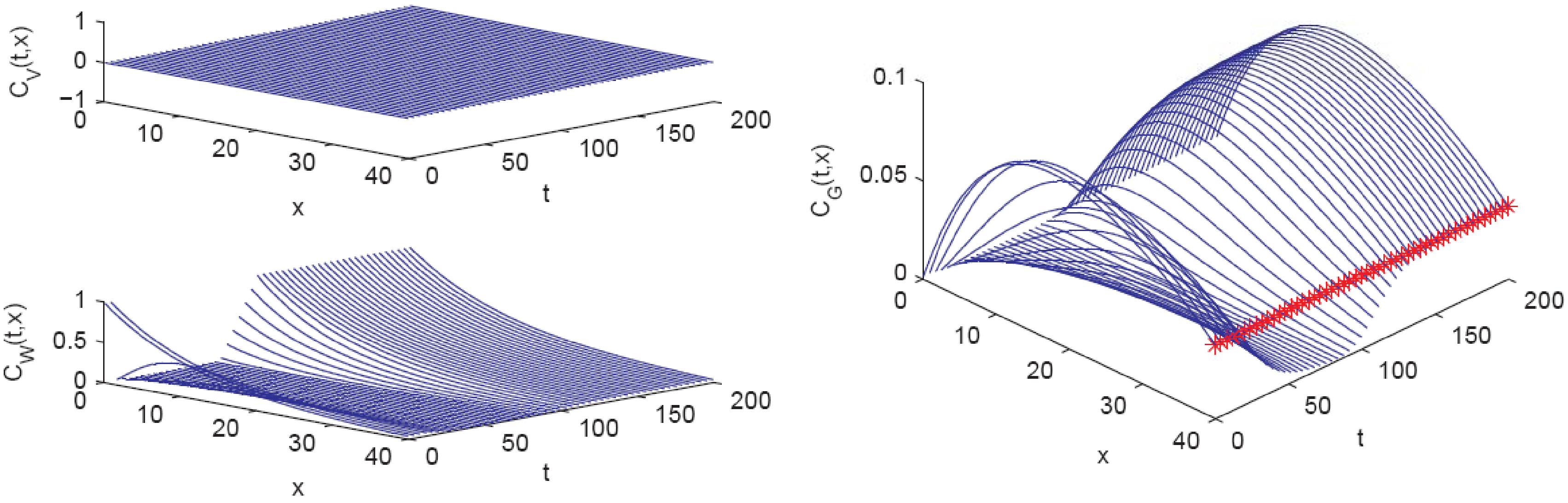
Nominal Dynamics





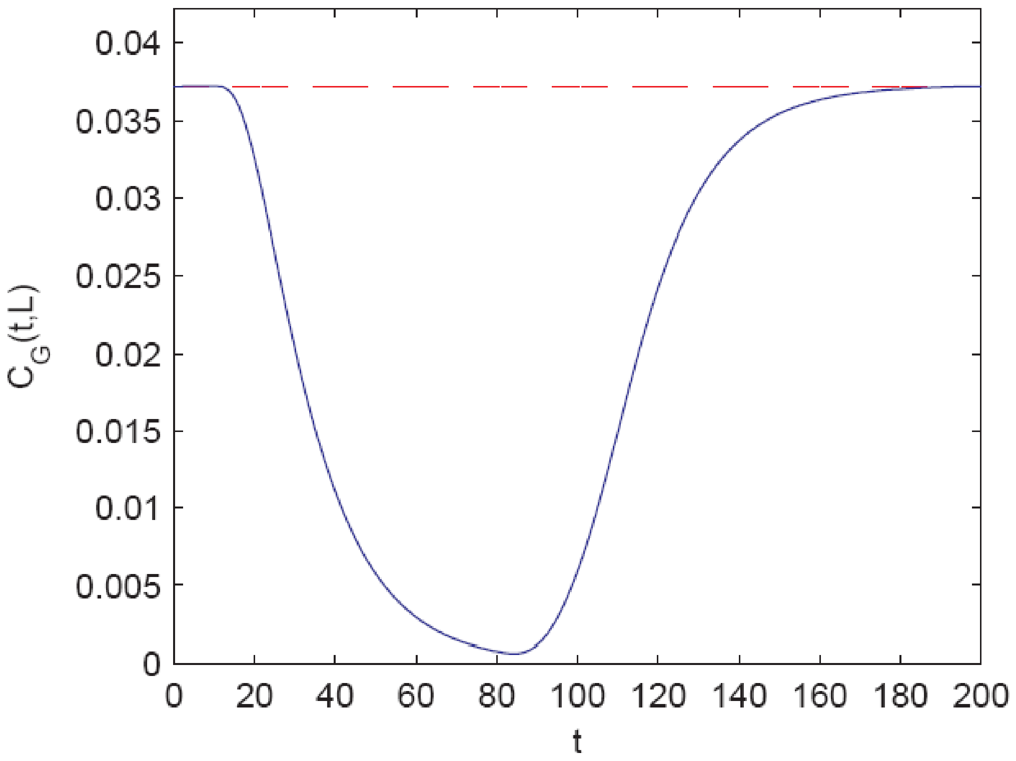
Under nominal conditions, $C_W(t,0) = 1$ yields the desired value of $C_G(t,L)$.

Disrupted Dynamics



Severe disruption in supply of species W

Departure from Target



Disruption in chemical supply causes severe departure from desired operating condition.

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Goals

- Generate a low-dimensional basis for the supply chain from observed data.
- Develop a reduced-order supply chain model using the low-dimensional basis.
- Validate the accuracy of the reduced-order supply chain model.
- Use the reduced-order model to design an optimal feedback control.
- Validate control effectiveness in the reduced and full-order systems.

Parameters Used
(Backup)

First Order Reaction Rates

$k_V = 0.2$

$k_W = 0.1$

$k_F = k_G = k_H = 0.02$

Stoichiometric Yield Factors

$y_W = 0.5$

$y_F = 0.3$

$y_G = 0.2$

$y_H = 0.1$

Dispersion Coefficient

$\varepsilon = 10$

Robin's BC Constant

$r = 0.1$

Advective Velocity

$\mu = 0.4$

Spatial Step-Size $h = 0.15$

Split POD Basis
(Baseline Modes)

A variety of initial concentration profiles are specified for $C_V(t,x)$.

<u>Initial Profiles</u>

$$C_V(0,x) \equiv 1,$$

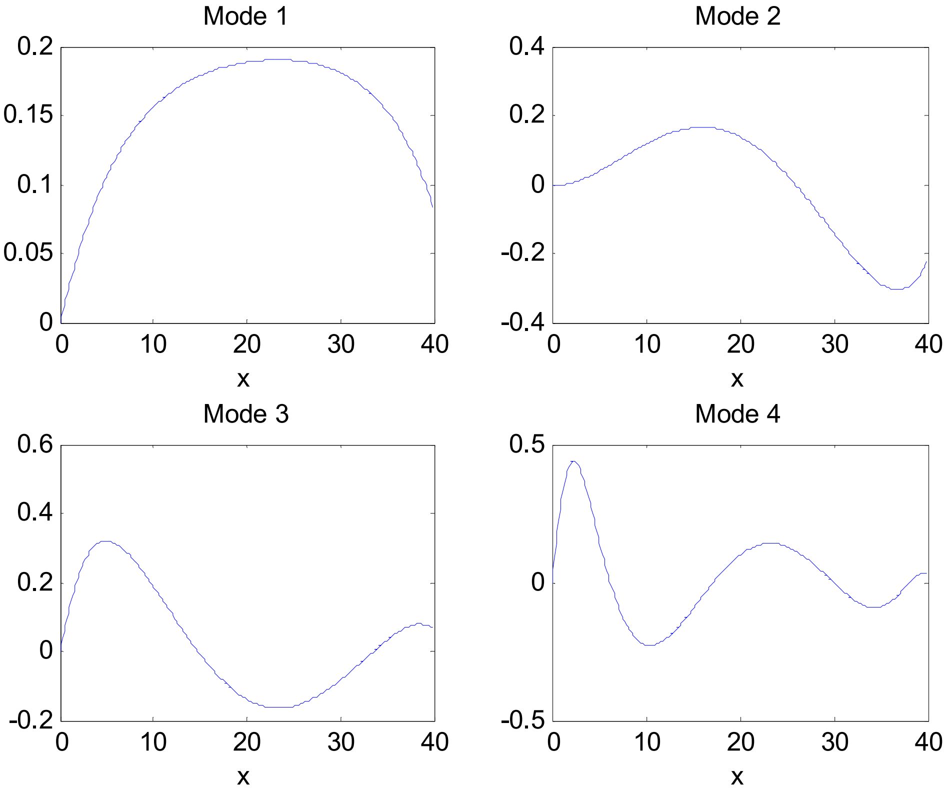
$$C_V(0,x) = \frac{x}{L},$$

$$C_V(0,x) = -\frac{x}{L},$$

$$C_V(0,x) = \sin\left(\frac{\pi x}{L}\right),$$

$$C_V(0,x) = \frac{1}{2}\left[\cos\left(\frac{2\pi x}{L}\right)+1\right]$$

- Remaining concentrations initially held at nominal conditions
- Data collected from t = 0.1 to t = 20 in increments of $\Delta t = 0.1$



4 baseline modes capture 99.9% of the baseline data variability

Split POD Basis
(Actuator Modes)

Chirp signals are specified for boundary inputs f(t) and u(t)

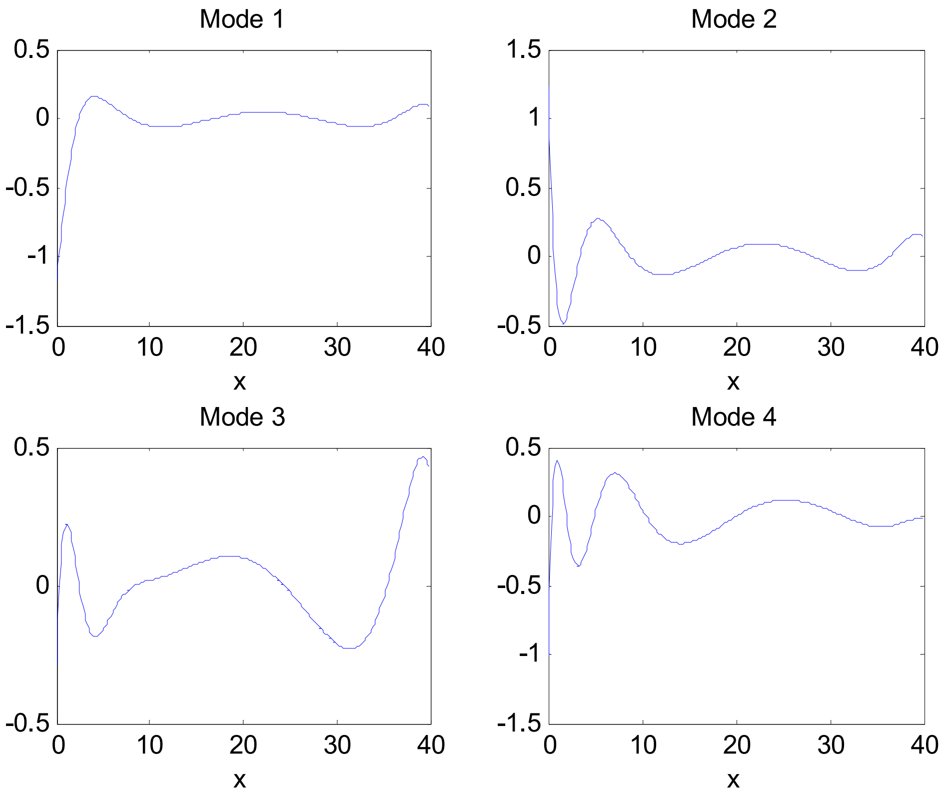
Inputs Specified

$$u(t) = \left|\sin\left(\frac{t^2}{10}\right)\right|,\ \ f(t) = 0$$

$$u(t) = 0,\ \ f(t) = \left|\sin\left(\frac{t^2}{10}\right)\right|$$

$$u(t) = f(t) = \left|\sin\left(\frac{t^2}{10}\right)\right|$$

- Each species concentration held at zero initially
- Data collected from t = 0.1 to t = 30 in increments of Δt = 0.1



4 actuator modes capture 99.9% of new variability induced by actuation

Basis Expansions

A single Split POD basis is used to construct mode expansions of the 5 chemical concentrations

Parent Concentrations

$$C_V(t,x) \approx \sum_{i=1}^{M_B+M_A} \alpha_i(t)\varphi_i(x),$$

$$C_W(t,x) \approx \sum_{i=1}^{M_B+M_A} \beta_i(t)\varphi_i(x),$$

$$\boxed{M_B = M_A = 4}$$

Daughter Concentrations

$$C_F(t,x) \approx \sum_{i=1}^{M_B} \varsigma_i(t)\varphi_i(x),$$

$$C_G(t,x) \approx \sum_{i=1}^{M_B} \eta_i(t)\varphi_i(x),$$

$$C_H(t,x) \approx \sum_{i=1}^{M_B} \theta_i(t)\varphi_i(x),$$

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Reduced-Order Model

The governing PDEs are written weakly and difference equations are used to make time-varying inputs explicit in the reduced model

For Concentration $C_V(t,x)$:

$$\int_0^L \frac{\partial}{\partial t} C_V(t,x)\varphi_i(x)dx = \int_0^L \left(\varepsilon C_V''(t,x) - \mu C_V'(t,x) - k_V C_V(t,x)\right)\varphi_i(x)dx$$

Integrating by Parts Yields

$$\begin{aligned}\int_0^L C_V''(t,x)\varphi_i(x)dx &= C_V'(t,L)\varphi_i(L) - C_V'(t,0)\varphi_i(0) - \int_0^L C_V'(t,x)\varphi_i'(x)dx \\ &\approx -rC_V(t,L)\varphi_i(L) - \left[\frac{C_V(t,h) - u(t)}{h}\right]\varphi_i(0) - \int_0^L C_V'(t,x)\varphi_i'(x)dx\end{aligned}$$

Similarly,

$$\int_0^L C_V'(t,x)\varphi_i(x)dx = C_V(t,L)\varphi_i(L) - u(t)\,\varphi_i(0) - \int_0^L C_V(t,x)\varphi_i'(x)dx$$

Galerkin System

Substituting and expanding $C_V(t,x)$ as a linear combination of POD modes results in a Galerkin system of the form

$$\frac{\partial}{\partial t}\alpha_i(t) = \sum_{j=1}^{M_B+M_A}\left[-\varphi_i(L)\varphi_j(L)\left(\varepsilon r+\mu\right)-\frac{\varepsilon}{h}\varphi_j(h)\varphi_i(0)\right]\alpha_j(t)$$

$$+\sum_{j=1}^{M_B+M_A}\left[\int_0^L\left\{\left(\mu\varphi_j(x)-\varepsilon\varphi_j'(x)\right)\varphi_i'(x)-k_V\varphi_j(x)\varphi_i(x)\right\}dx\right]\alpha_j(t)$$

$$+\left[\left(\mu+\frac{\varepsilon}{h}\right)\varphi_i(0)\right]u(t)\,,\qquad i=1,2,\ldots,M_B+M_A$$

The system of ODEs is written as a matrix equation of the form

$$\boxed{\frac{\partial}{\partial t}\alpha = A_\alpha\alpha + B_\alpha\,u(t)}$$

Reduced Model State Equation

The Galerkin systems for the remaining concentrations are found similarly.

Remaining Coefficient Equations

$$\frac{\partial}{\partial t}\beta = A_\beta \beta + F_\beta f(t) + I_\beta \alpha,$$

$$\frac{\partial}{\partial t}\varsigma = A_\varsigma \varsigma + I_\varsigma \beta,$$

$$\frac{\partial}{\partial t}\eta = A_\eta \eta + I_\eta \beta,$$

$$\frac{\partial}{\partial t}\theta = A_\theta \theta + I_\theta \beta.$$

Overall State Equation

$$\dot{\begin{bmatrix}\alpha\\ \beta\\ \xi\\ \eta\\ \theta\end{bmatrix}} = \begin{bmatrix}A_\alpha & 0 & 0 & 0 & 0\\ I_\beta & A_\beta & 0 & 0 & 0\\ 0 & I_\xi & A_\xi & 0 & 0\\ 0 & I_\eta & 0 & A_\eta & 0\\ 0 & I_\theta & 0 & 0 & A_\theta\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\\ \xi\\ \eta\\ \theta\end{bmatrix} + \begin{bmatrix}B_\alpha\\ 0\\ 0\\ 0\\ 0\end{bmatrix} u(t) + \begin{bmatrix}0\\ F_\beta\\ 0\\ 0\\ 0\end{bmatrix} f(t)$$

$$= A_r X + B_r u(t) + F_r f(t)$$

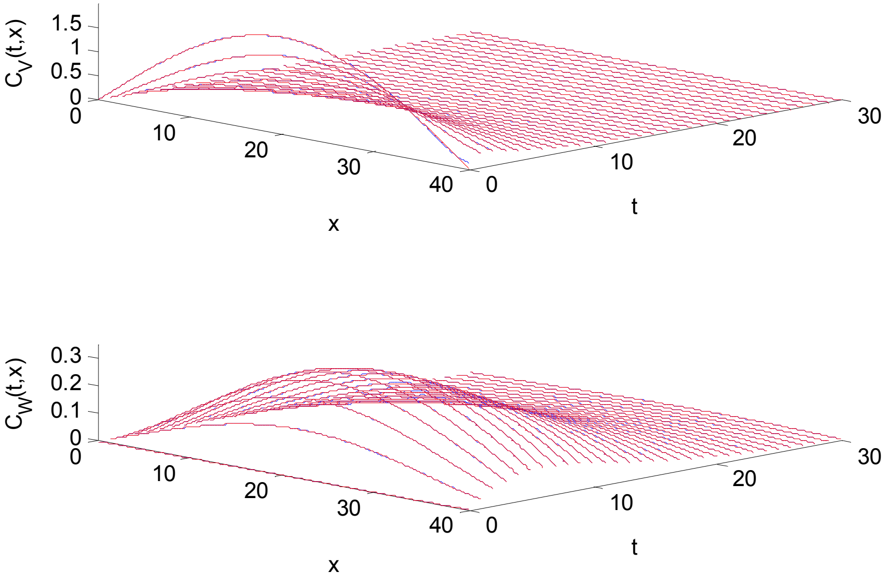
with initial condition

$$X(0) = X_0 = \begin{bmatrix}\alpha(0) & \beta(0) & \varsigma(0) & \eta(0) & \theta(0)\end{bmatrix}^T$$

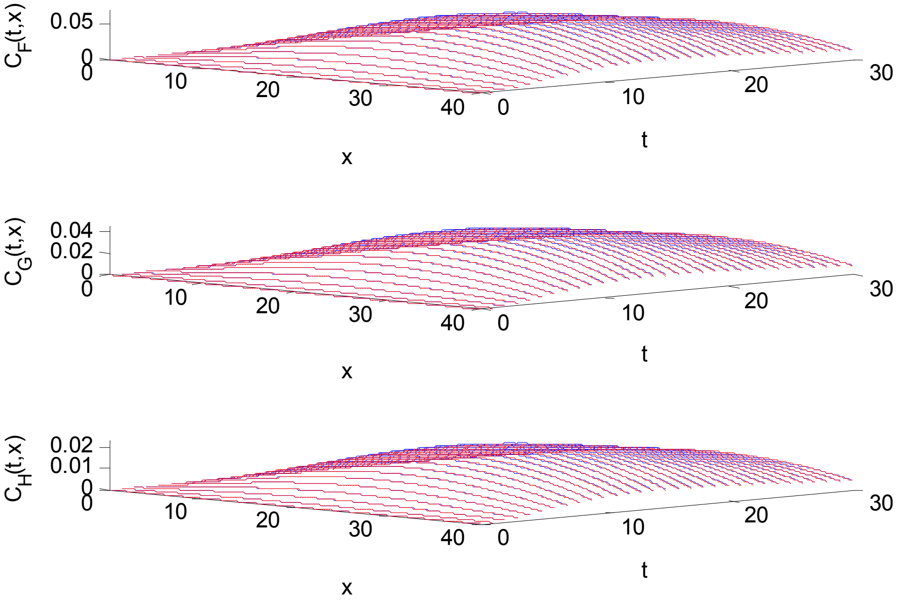
Baseline Validation

The reduced-order baseline concentrations are compared to their full-order counterparts

Parent Concentrations

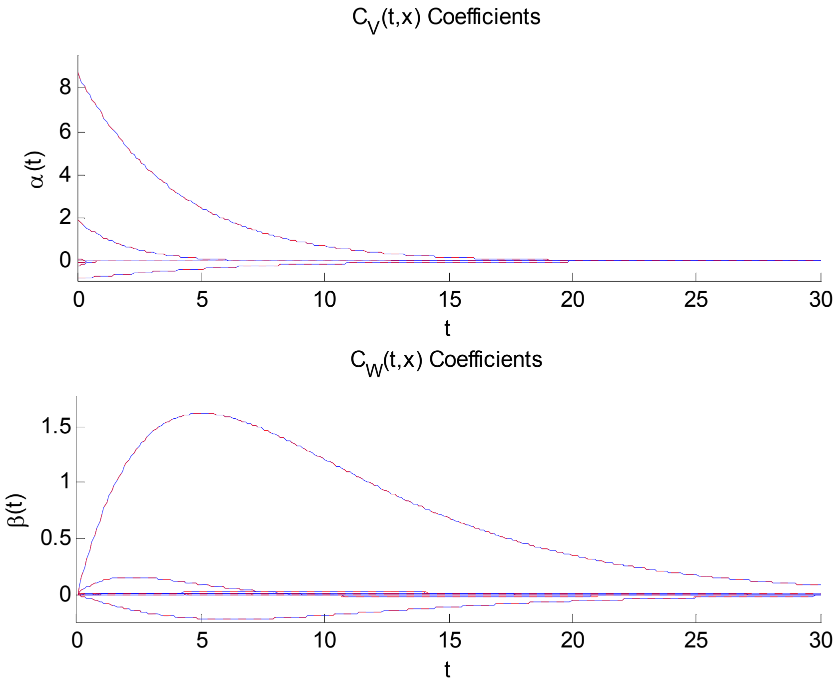


Daughter Concentrations

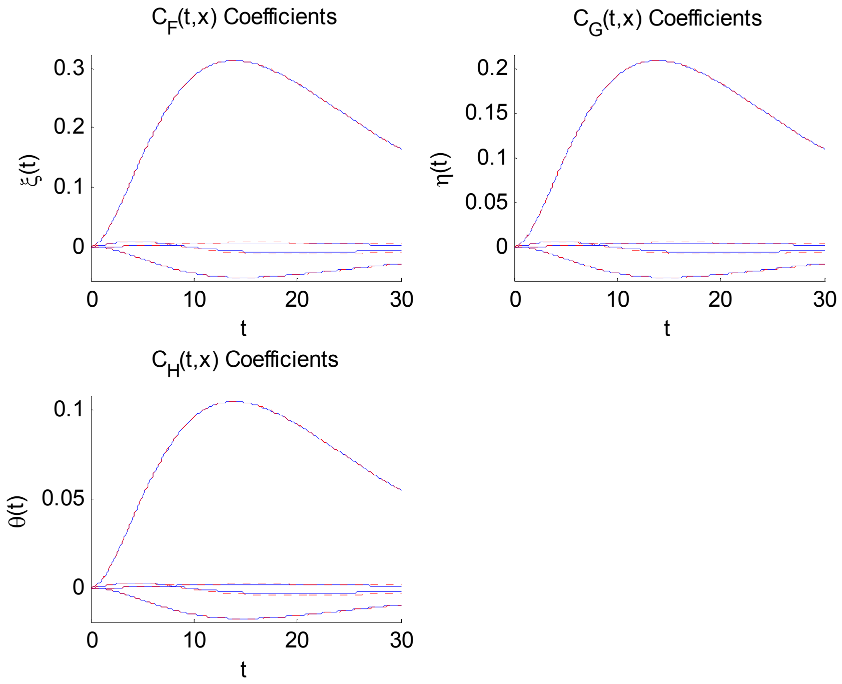


POD Coefficients vs Projections (Baseline)

Parent Coefficients



Daughter Coefficients



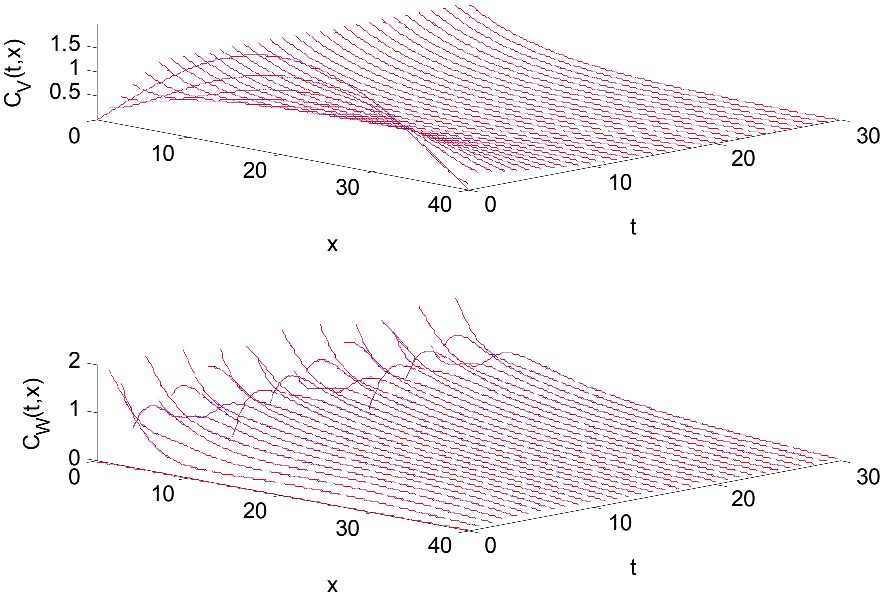
Excellent Agreement Between Full-Order Projections and Reduced Model Solutions

Actuated Validation

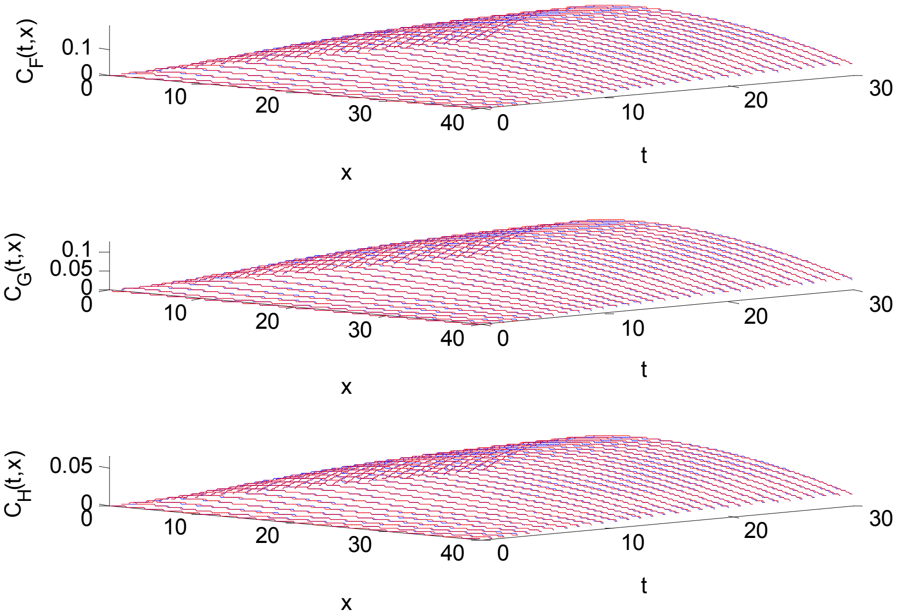
The reduced-order actuated concentrations are compared to their full-order counterparts

$$f(t) = 2\left|\sin(2t)\right|, \quad u(t) = \min\left(\frac{t}{5}, 1\right)$$

Parent Concentrations

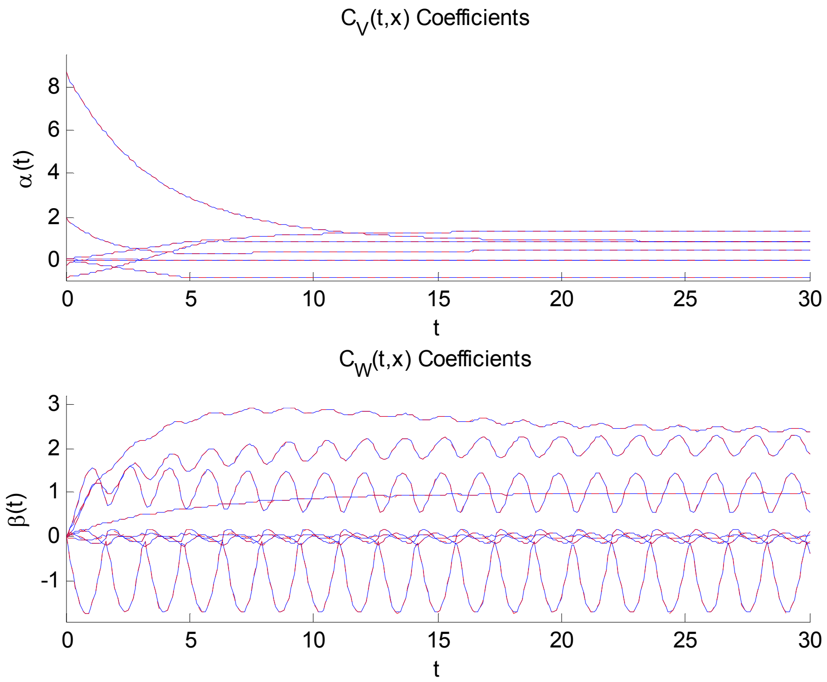


Daughter Concentrations

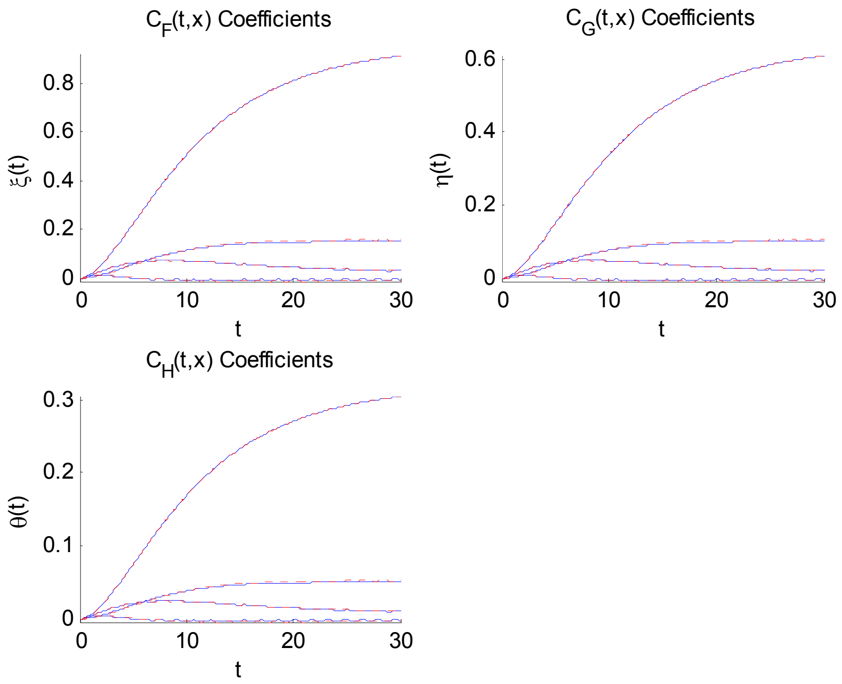


POD Coefficients vs Projections (Actuated)

Parent Coefficients



Daughter Coefficients

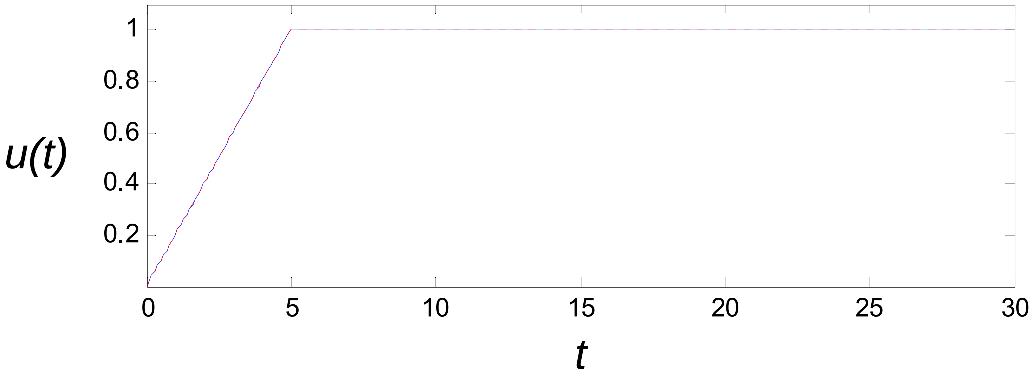


Excellent Agreement Between Full-Order Projections and Reduced Model Solutions

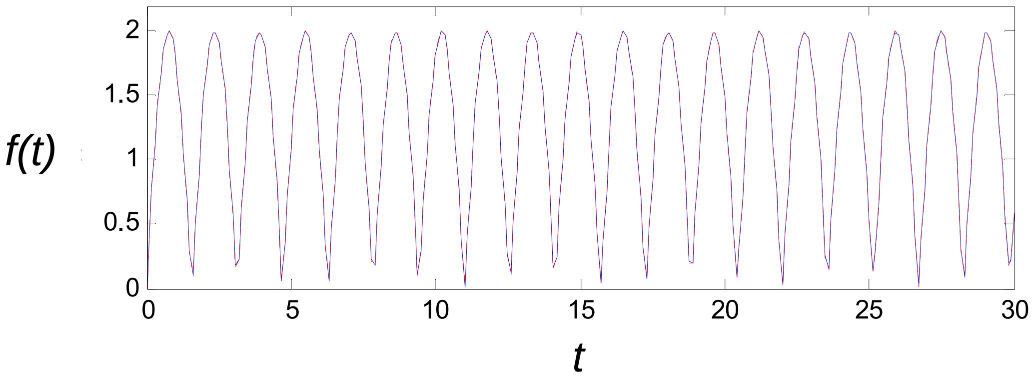
Boundary Condition Accuracy

The exact boundary conditions are compared to the linear combination of POD modes restricted to the boundary

$$\boxed{u(t) = C_V(t,0) \cong \sum_{i=1}^{8} \alpha_i(t)\varphi_i(0)}$$



$$\boxed{f(t) = C_W(t,0) \cong \sum_{i=1}^{8} \beta_i(t)\varphi_i(0)}$$



Exact Boundary Condition and Reduced Model Representation are Identical

Order Reduction and Savings

State Dimension Reduction

Concentration	Full	Reduced
$C_V(t,x)$	266	8
$C_W(t,x)$	266	8
$C_F(t,x)$	266	4
$C_G(t,x)$	266	4
$C_H(t,x)$	266	4
	1330 ⟶	28

Computation Time

Full	Reduced
7.7945 sec	0.8737 sec

Computation time decreased by roughly a factor of 9.

Tracking Formulation

A desired steady-state profile is specified for daughter concentration $C_G(t,x)$

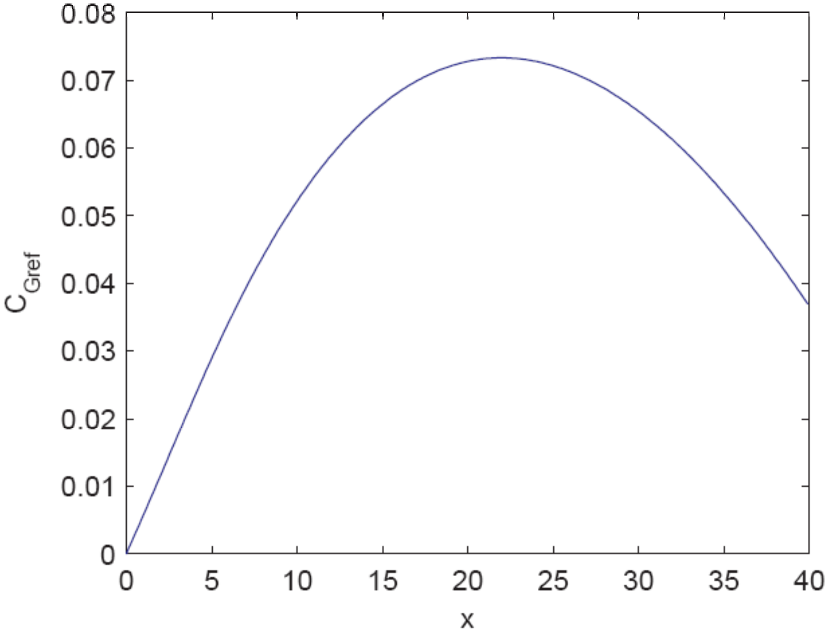
A feedback controller is designed by use of a tracking LQR formulation

$$\min_u \int_0^\infty \left[\left(X - X_{ref} \right)^T Q \left(X - X_{ref} \right) + u^T R u \right] dt$$

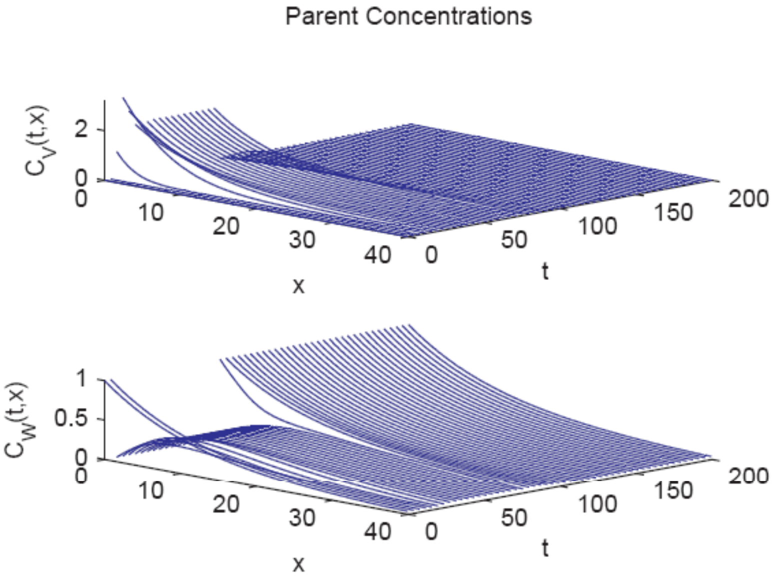
subject to

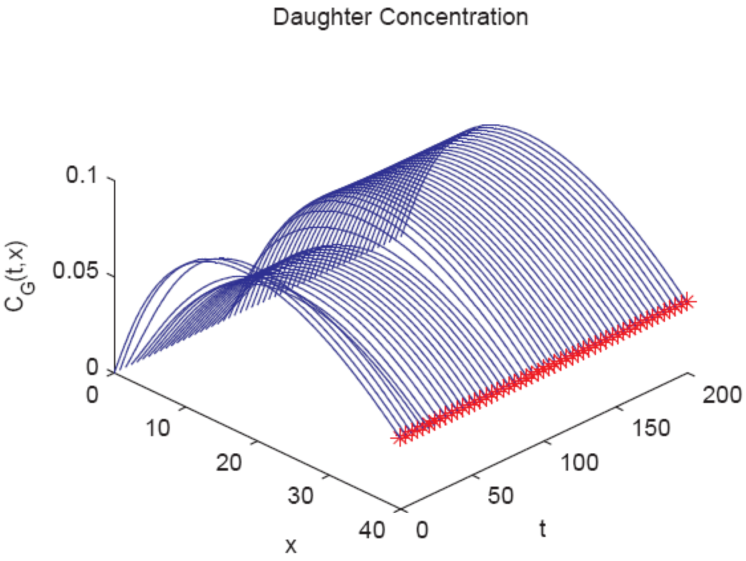
$$\frac{\partial}{\partial t} \begin{bmatrix} X \\ X_{ref} \end{bmatrix} = \begin{bmatrix} A_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X \\ X_{ref} \end{bmatrix} + \begin{bmatrix} B_r \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} F_r \\ 0 \end{bmatrix} f(t)$$

with
$$\begin{bmatrix} X(0) \\ X_{ref}(0) \end{bmatrix} = \begin{bmatrix} X_0 \\ X_{ref} \end{bmatrix}$$



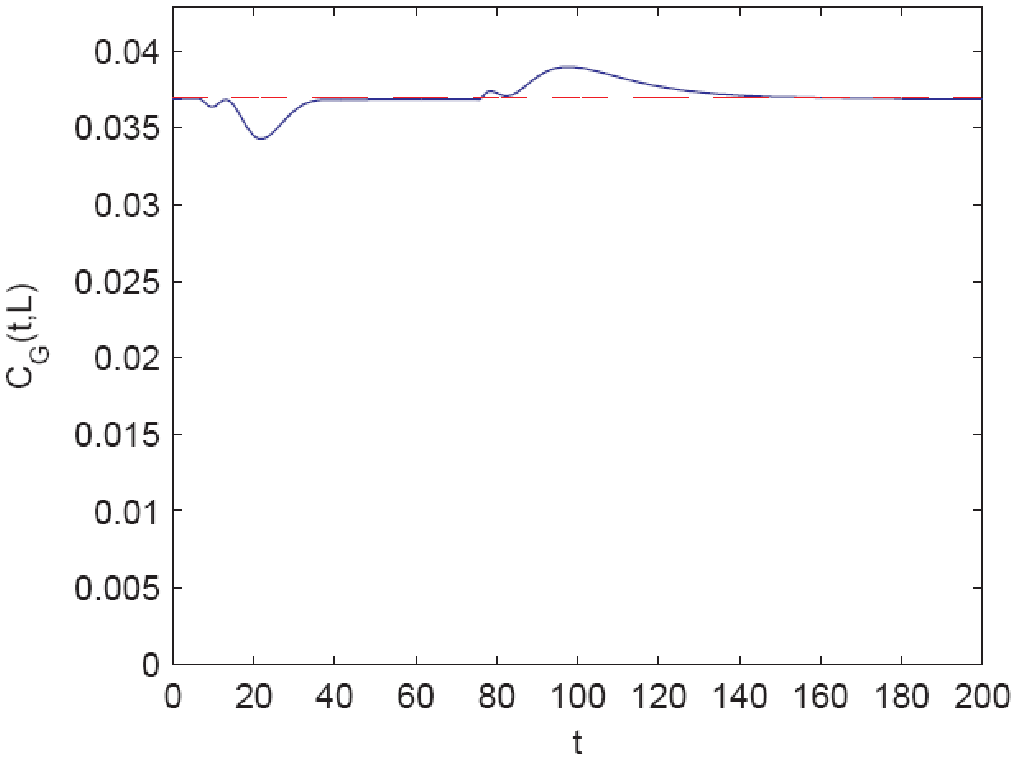
Reduced-Order Disrupted Dynamics (Controlled)


Parent Concentrations


Daughter Concentration

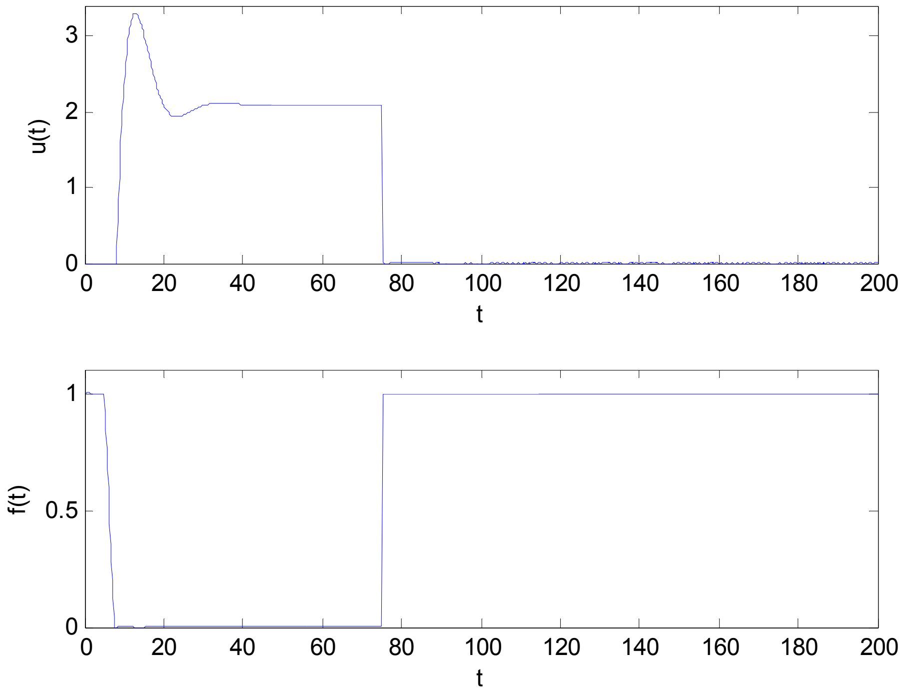
Target Value Maintained Much Better During Disruption

Reduced-Order Departure from Target (Controlled)



Target Value Maintained Much Better During Disruption

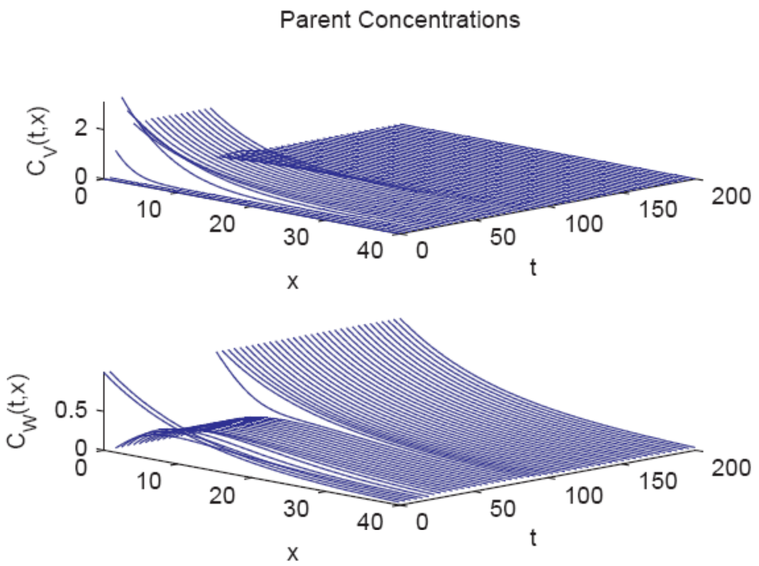
Reduced-Order System Inputs

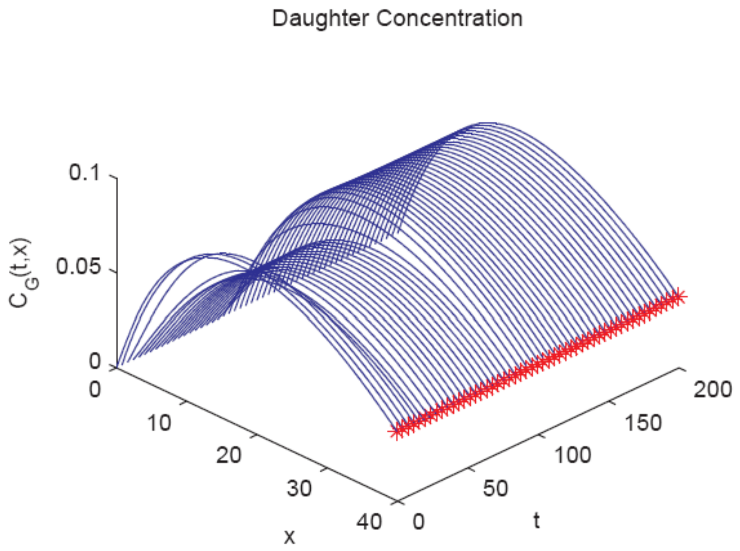


Control Input Provides a Measure of “Cost of Recovery”

Controlled Full-Order Model

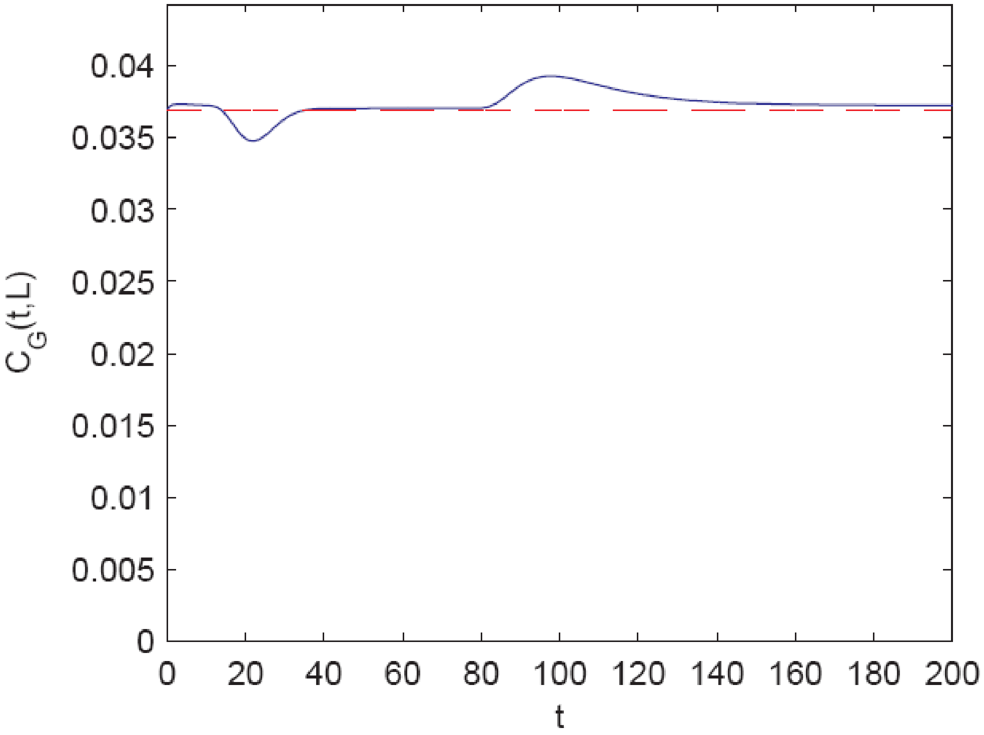
The reduced-order controller is implemented in the full-order chemical supply chain model.





Controlled Reduced and Full-Order Models are in Very Good Agreement

Controlled Full-Order Model



“Small” Controller is Capable of Maintaining Target Conditions in the Full-Order Model During Disruption

Savings in Control Calculation

Reduction of System Dimension Yields Significant Savings in Optimal Control Calculation

Time Required to Solve the Optimal Control Problem

Full-Order Model: 62.63 sec
Reduced-Order Model: 0.1977 sec

Computation Time Required for Calculation of Optimal Control was Reduced by Roughly a Factor of 317!

Conclusions

- Systematic feedback control design provides a quantitative framework for resilience analysis.

- Very large systems require a reduction of system order prior to control calculation.

- Control inputs must be carefully accounted for during order reduction.
  - Split POD explicitly accounts for control inputs, and can be used to guide hardware placement.

- Controls calculated from reduced-order models can be effective when implemented in full-order systems.

- Using reduced-order models to calculate optimal controls yields significant computational savings.