

A New Approach to Nonlocal Advection

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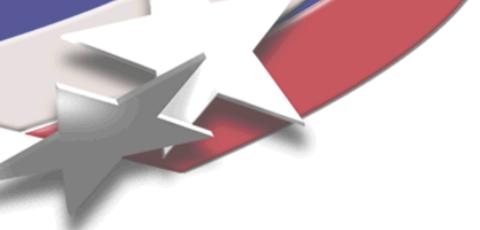
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Classical (Local) Advection

- Classical (local) advection is well-understood. Many, many papers, textbooks, etc.
- Conservation law (f is flux function):

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0$$

- Simple examples:
 - $f(u) = cu \rightarrow$ linear advection
 - $f(u) = u^2/2 \rightarrow$ Burgers' equation
- Complex, detailed, well-studied solution structure
- Many others have discussed various forms of nonlocal advection
 - Nonlocal wavespeed, integral operators (Hilbert transform), fractional differential operators, generalized flux, nonlocal regularization, nonlocal convection diffusion (Ignat & Rossi)*
- Can we develop a peridynamic-inspired approach for nonlocal advection that captures “shock-like” behavior?
 - Classical PDEs do not hold on discontinuities (shocks)
 - Explore local & nonlocal Burgers' equation: shocks, etc

* L. I. Ignat and J. D. Rossi. A nonlocal convection-diffusion equation. J. Funct. Anal., 251:399-437, 2007.



Outline

- Peridynamic-inspired model for nonlocal advection
- Gradient-free nonlocal flux
- Numerical methods for nonlocal models: Nonlocal Lax-Friedrichs method
- Nonlocal Burgers Equation
 - Theory
 - Computational results
- Relation to dispersive shock waves
- Conclusions



Nonlocal Advection

- Posit the integro-differential equation

$$\frac{\partial u}{\partial t} + \int_{x-\varepsilon}^{x+\varepsilon} \psi \left(\frac{u(y, t) + u(x, t)}{2} \right) \varphi(y, x) dy = 0 \quad (x, t) \in \mathbf{R} \times (0, \infty)$$
$$u(x, 0) = g(x) \quad x \in \mathbf{R}$$

- Points (x, y) interact directly and nonlocally.
- Maximum interaction distance ε (peridynamic horizon)
- Kernel (*micromodulus*) is **antisymmetric**:

$$\varphi(y, x) = -\varphi(x, y)$$

- Contrast with peridynamic models of solids, where kernel is *symmetric*
- The kernel is (usually) translation invariant:

$$\varphi(y, x) = \varphi(y - x) = \varphi(\xi)$$

- Requirement for consistency with classical (local) advection equation:

$$\lim_{\xi \rightarrow 0} \varphi(\xi) = -\frac{\partial \delta}{\partial x}$$

(in distributional sense).

- Contrast with peridynamic models of solids, where this limit (usually) gives δ

Relation to Local Advection

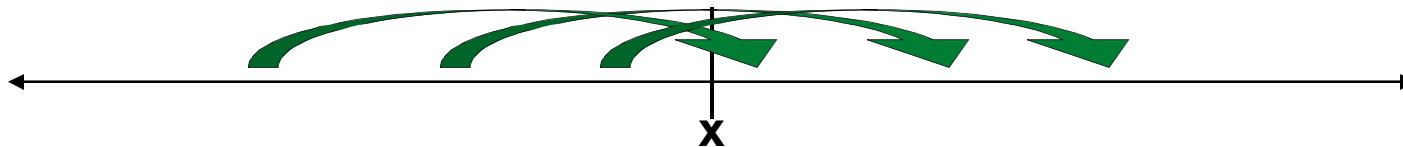
□ Compare forms of equations

$$\frac{\partial u}{\partial t} + \int_{x-\varepsilon}^{x+\varepsilon} \psi \left(\frac{u(y, t) + u(x, t)}{2} \right) \varphi(y, x) dy = 0$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0$$

“gradient of flux”

□ What is the nonlocal flux through a surface? (e.g., through x ?)



$$f(x) = \int_0^\varepsilon \int_0^\varepsilon \psi \left(\frac{u(x+y, t) + u(x-z, t)}{2} \right) \varphi(x+y, x-z) dy dz$$

- The flux is carried by an infinite number of nonlocal bonds passing through x .
 - Many have derived this expression before; See [1,2,3].
 - Under assumptions, Noll's lemma [4] can be used to derive flux function.
 - Under assumptions, nonlocal equation converges to local equation as $\varepsilon \rightarrow 0$.

¹ Silling, Zimmerman, and Abeyaratne, Deformation of a Peridynamic Bar, J. Elasticity. 73:173-190, 2003.

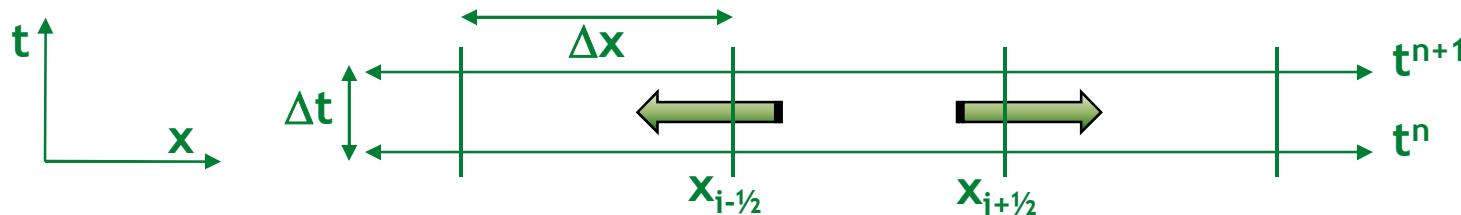
² Bobaru and Duangpanya, The peridynamic formulation for transient heat conduction, Int. J. Heat Mass Transfer. 53: 4047-4059, 2010.

³ Lehoucq and Silling, Force flux and the peridynamic stress tensor, J. Mech. Phys. Solids. 56:1566-1577, 2008.

⁴ Noll, Die herleitung der grundgleichungen der thermomechanik der kontinua aus der statistischen mechanik. J. Ration. Mech. Anal. 4:627-646, 1955.

Nonlocal Lax-Friedrichs Discretization (1)

- Discretize space & time into cells $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ and intervals $[t^n, t^{n+1}]$
- The flux out of cell i is the flux out of $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ through $x_{i-\frac{1}{2}}$ and $x_{i+\frac{1}{2}}$



- Flux out of $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ through $x_{i+\frac{1}{2}}$ (expression for $x_{i-\frac{1}{2}}$ similar)

$$f_{[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]}(x_{i+\frac{1}{2}}, t) = \int_0^{\Delta x} \int_0^{\varepsilon} \psi \left(\frac{u(x+y, t) + u(x-z, t)}{2} \right) \phi(x+y, x-z) dy dz$$

- Conservative numerical scheme: Change in u over cell $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ in time interval $[t^n, t^{n+1}]$ must be balanced by flux over that cell over that time interval

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (u(x, t^{n+1}) - u(x, t^n)) dx + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (f_{[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]}(x_{i-\frac{1}{2}}, t) + f_{[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]}(x_{i+\frac{1}{2}}, t)) dt = 0$$



Nonlocal Lax-Friedrichs Discretization (2)

□ Define

$$\bar{u}_i^n \approx \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x, t^n) dx$$

$$\bar{F}_{i+\frac{1}{2}}^n \approx \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} f_{[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]}(x_{i+\frac{1}{2}}, t) dt$$

$$\bar{F}_{i-\frac{1}{2}}^n \approx \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} f_{[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]}(x_{i-\frac{1}{2}}, t) dt$$

□ This gives

$$\bar{u}_i^{n+1} = \bar{u}_i^n - \frac{\Delta t}{\Delta x} \left(\bar{F}_{i+\frac{1}{2}}^n + \bar{F}_{i-\frac{1}{2}}^n \right)$$

□ This is the nonlocal equivalent of FTCS (unconditionally unstable)

□ Stabilization produces nonlocal Lax-Friedrichs:

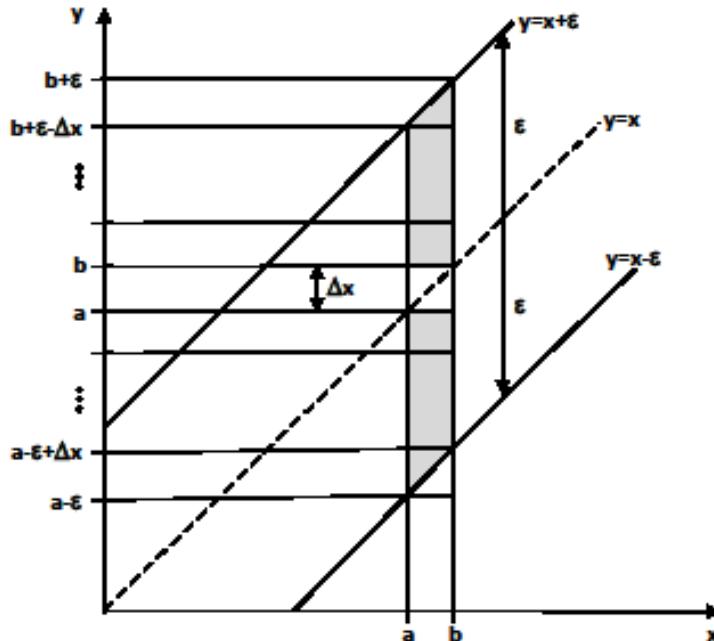
$$\bar{u}_i^{n+1} = \frac{\bar{u}_{i+1}^n + \bar{u}_{i-1}^n}{2} - \frac{\Delta t}{\Delta x} \left(\bar{F}_{i+\frac{1}{2}}^n + \bar{F}_{i-\frac{1}{2}}^n \right)$$

Nonlocal Lax-Friedrichs Discretization (3)

□ Suppose (for convenience) that $r \Delta x = \varepsilon$, r an integer.

□ Quadrature:

$$\left(f_{[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]}(x_{i-\frac{1}{2}}, t) + f_{[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]}(x_{i+\frac{1}{2}}, t) \right) = \sum_{j=-r}^{j=r} \omega_j \psi \left(\frac{u(x_{i+j}, t) + u(x_i, t)}{2} \right) \varphi(x_{i+j}, x_i) (\Delta x)^2$$



$$\omega_j = \begin{cases} 0 & j = 0 \\ 1 & j = \pm 1, \dots, \pm(r-1) \\ \frac{1}{2} & j = -r, r \end{cases}$$

□ Stability analysis: Let $\psi(u) = u$, $\varphi(y, x) = \frac{1}{\varepsilon^2} \begin{cases} 1 & y > x \\ 0 & y = x \\ -1 & y < x \end{cases}$

□ Then $\Delta t < \frac{2r}{r+1} \Delta x$

$$\varphi(y, x) = \frac{1}{\varepsilon^2} \begin{cases} 1 & y > x \\ 0 & y = x \\ -1 & y < x \end{cases}$$



Nonlocal Burgers: Theory

- Nonlocal Burgers: $\psi(u) = u^2/2$

- Conservation:

$$\frac{d}{dt} \int u(x, t) dx = 0$$

$$\frac{d}{dt} \int u^2(x, t) dx \neq 0$$

- Well-Posedness:

- Assume $\square \in L^1(-\varepsilon, \varepsilon)$, $g \in H^1(\square)$. Then, there exists a time interval $(0, T)$ such that the nonlocal Burgers equation has a unique solution. Moreover, let $(0, T)$ be the maximum time interval on which such a solution exists. Then, $\limsup(t \rightarrow T) \|u(., t)\|_{L^\infty} = \infty$.

- Consequences:

- If we start with smooth data, solution maintains H^1 regularity so long as it is pointwise bounded in space and time. Moreover, only finite-time blow-up can cause loss of H^1 regularity of the solution.
- If $\square \in L^1$ and initial data smooth, solution maintains H^1 regularity for positive horizon. **There is no shock formation with an L^1 kernel!**
- Contrast with local Burgers, where initial smooth data can lead to shock
 - Addition of viscosity forbids shock in local case



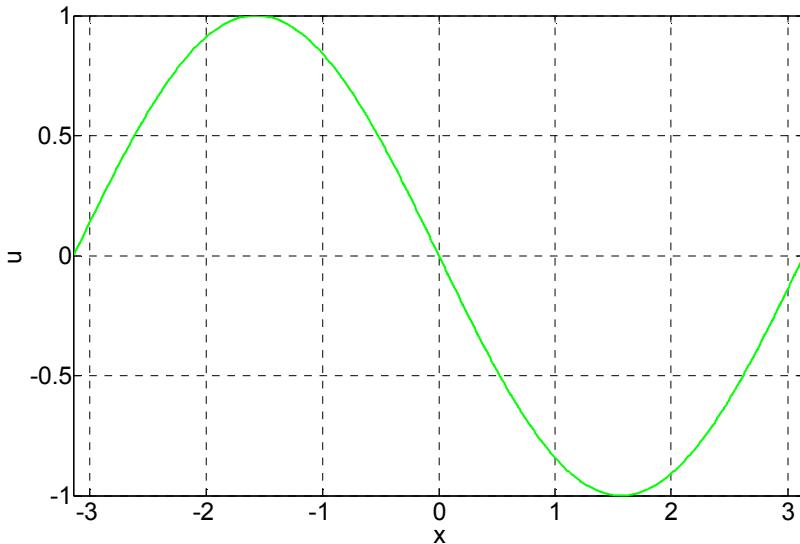
Nonlocal Burgers: Computational Results

- There are two primary nondimensional length scales:
 - ε/L Ratio of PD length scale to problem length scale
 - $\varepsilon/\Delta x$ Ratio of PD horizon to cell size
- So, perform two independent studies: ε -refinement and Δx refinement
- Let $\Delta t/\Delta x = 2/c$ fixed, $c = 80$. (i.e., artificial viscosity same for all experiments)
 - $\min(\varepsilon/L) \approx 0.004$ (small horizon)
 - $\max(\varepsilon/L) \approx 0.1$ (large horizon)
 - $\min(\varepsilon/\Delta x) \approx 16$
 - $\max(\varepsilon/\Delta x) \approx 256$
- The horizon is typically $3\times$ the mesh spacing in PD solid mechanics. In these numerical experiments, nonlocality is well-resolved.

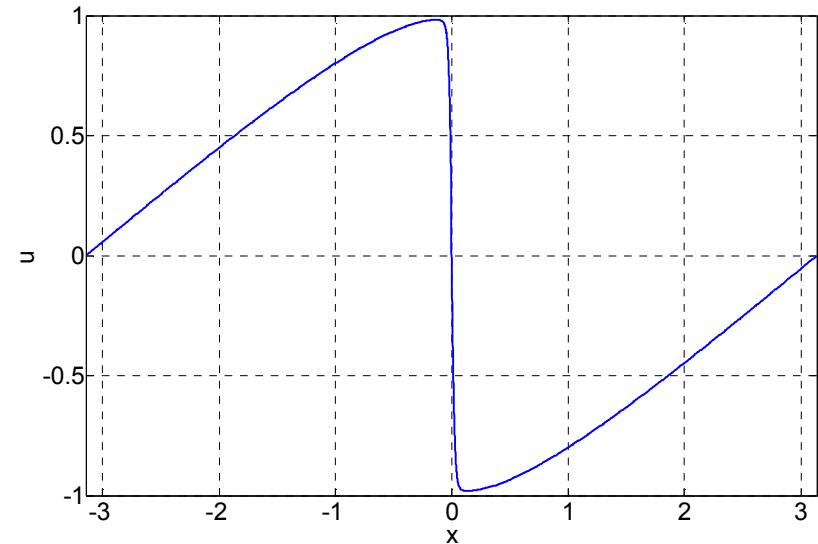
N	Δx -refinement study					ε -refinement study			
	2000	4000	8000	16000	32000	10000	10000	10000	10000
Δx	3.14e-3	1.57e-3	7.86e-4	3.93e-4	1.97e-4	6.28e-4	6.28e-4	6.28e-4	6.28e-4
ε	5.02e-2	5.02e-2	5.02e-2	5.02e-2	5.02e-2	1.26e-2	6.28e-2	1.57e-1	3.14e-1
ε/L	1.60e-2	1.60e-2	1.60e-2	1.60e-2	1.59e-2	4.00e-3	2.00e-2	5.00e-2	1.00e-1
$\varepsilon/\Delta x$	16	32	64	128	256	20	100	250	500

Nonlocal Burgers: Sine IC

- Domain: $-\pi \leq x < \pi$; N cells with $\Delta x = L/N$; $L = \pi$.
- Boundary conditions: $u(x + kL, t) = u(x, t)$; $k \in \mathbb{Z}$
- Initial condition: $u_0 = -\sin(x)$



Initial Condition

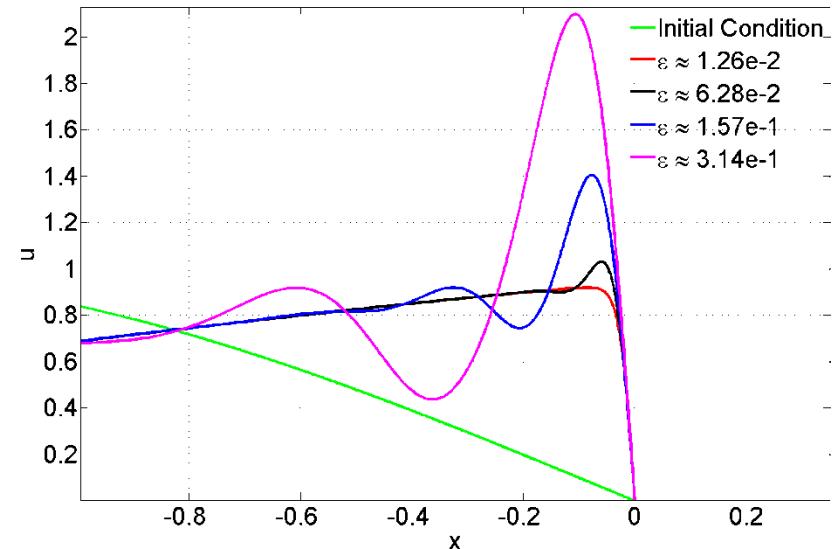
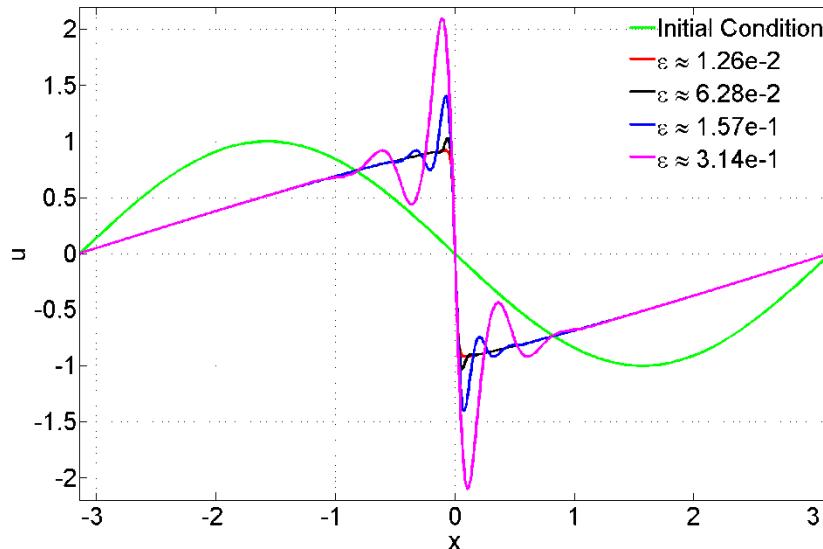


Local Lax-Friedrichs at $t=1.5$

- Compare with analytical and numerical for local Burgers' equation
 - Sinusoid IC leads to shock formation at $x=0$, $t=1$. N-wave develops as $t \rightarrow \infty$.

Nonlocal Burgers: Sine IC

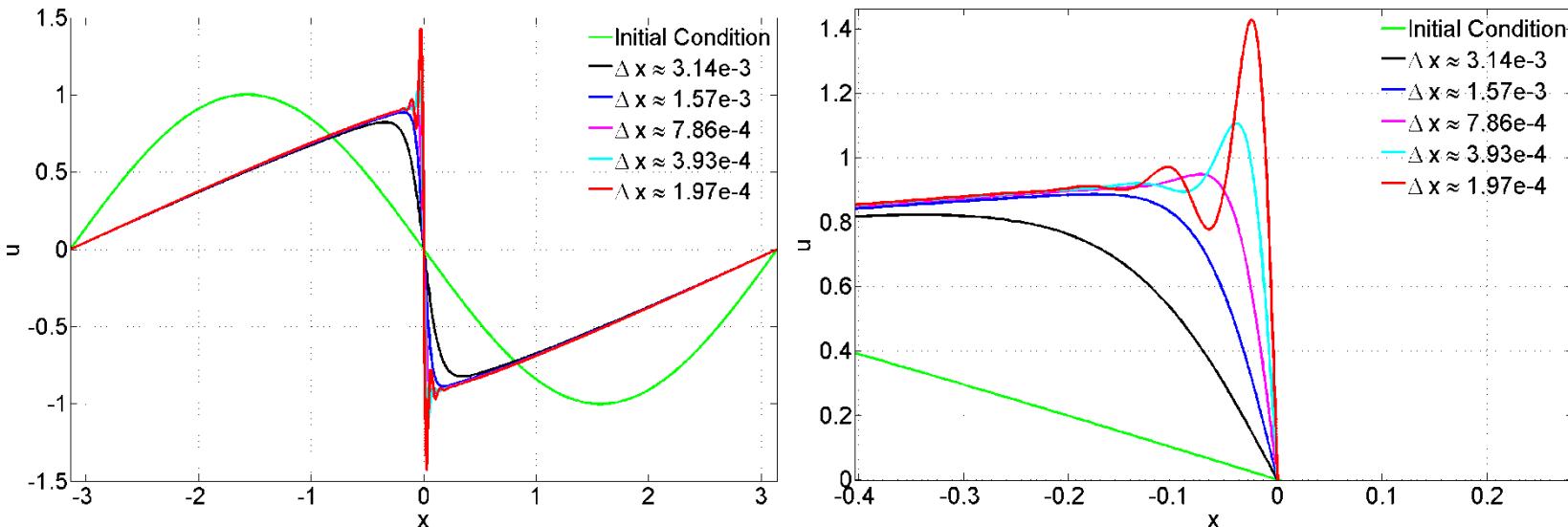
- ☐ ε -refinement study (for $\Delta x \approx 6.28e-4$) (fine mesh)



- ☐ For small ε , results qualitatively similar to N-wave
- ☐ Additional oscillations for larger ε
- ☐ $\int u \, dx$ conserved; Numerical method is conservative (not shown here)
- ☐ $\int u^2 \, dx$ not conserved (artificial viscosity)

Nonlocal Burgers: Sine IC

- Δx - refinement study (for $\varepsilon \approx 0.05$; $\varepsilon/L \approx 1.59e-2$) (small ε)

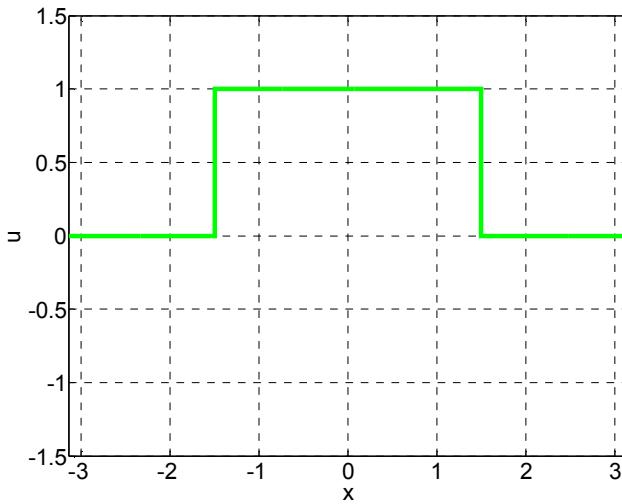


- Gibbs-like oscillations around shock-like feature

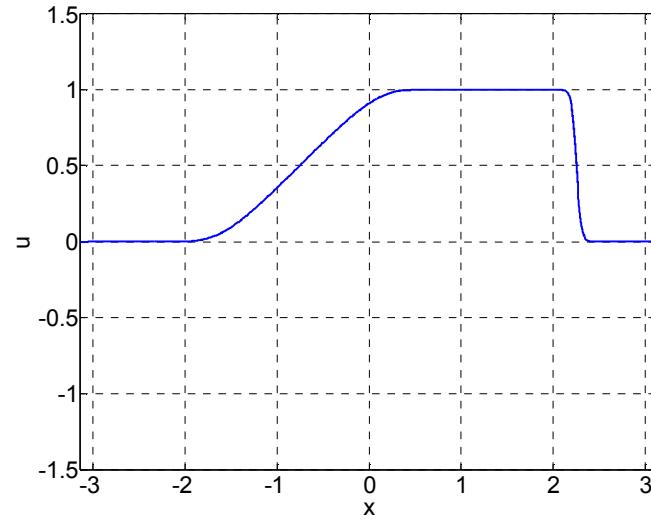
- $\int u \, dx$ conserved; Numerical method is conservative (not shown here)
- $\int u^2 \, dx$ not conserved (artificial viscosity)

Nonlocal Burgers: “Top Hat” IC

- Domain: $-\pi \leq x < \pi$; N cells with $\Delta x = L/N$; $L = \pi$.
- Boundary conditions: $u(x + 2kL, t) = u(x, t)$; $k \in \mathbb{Z}$
- Initial condition: $u_0 = 1$ if $-1.5 \leq x < 1.5$; 0 elsewhere



Initial Condition

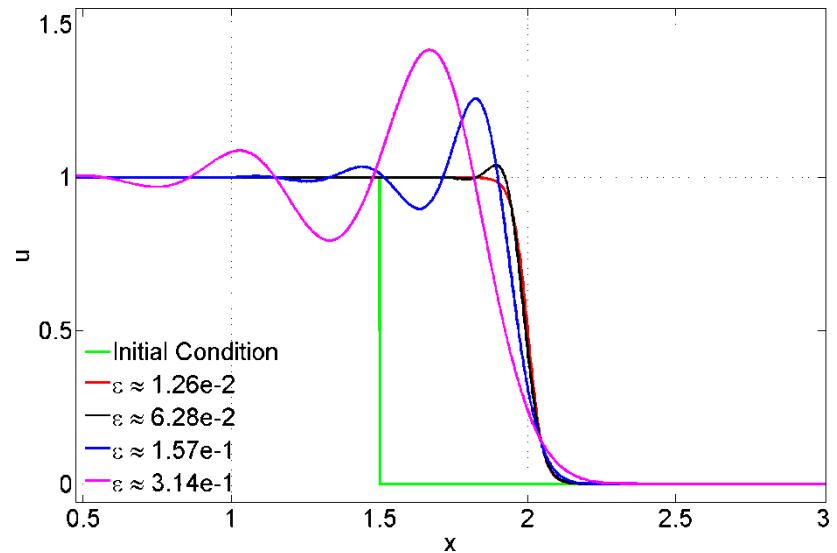
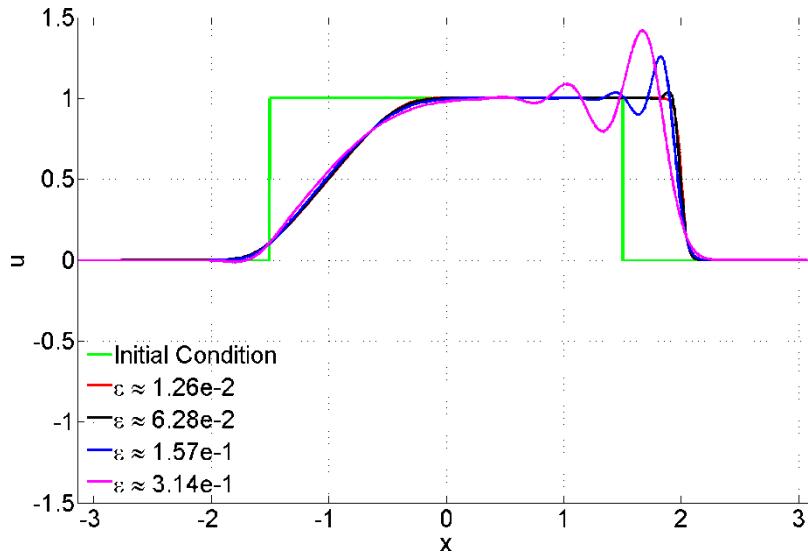


Local Lax-Friedrichs at $t=1.5$

- Compare with analytical and numerical for local Burgers' equation
 - “Top Hat” IC leads to rarefaction (left) plus shock (right)

Nonlocal Burgers: “Top Hat” IC

- ☐ ε -refinement study (for $\Delta x \approx 6.28e-4$) (fine mesh)

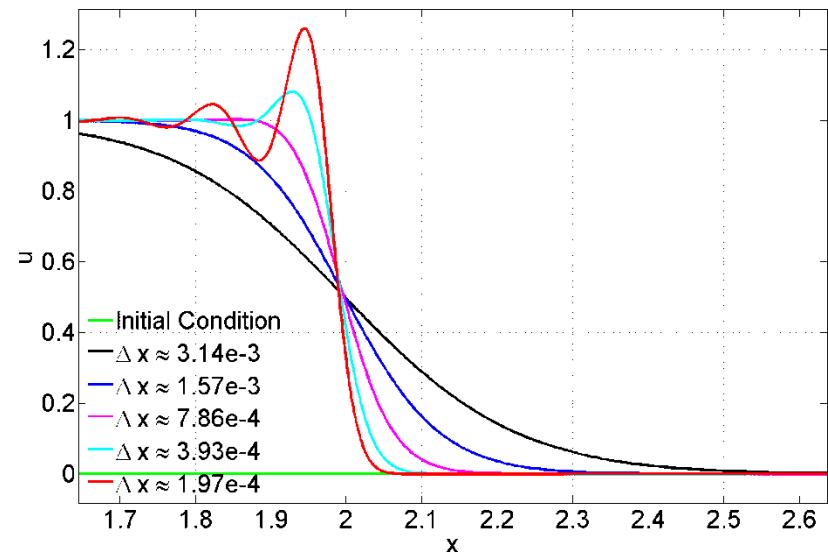
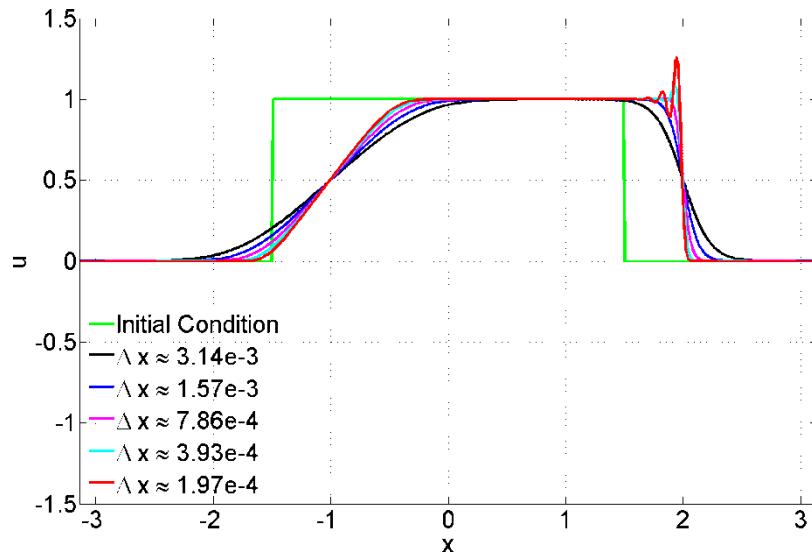


- ☐ For small ε , results qualitatively similar to classical results
- ☐ Additional Gibbs-like oscillations around shock-like feature for larger ε

- ☐ $\int u \, dx$ conserved; Numerical method is conservative (not shown here)
- ☐ $\int u^2 \, dx$ not conserved (artificial viscosity)

Nonlocal Burgers: “Top Hat” IC

- Δx - refinement study (for $\varepsilon \approx 0.05$; $\varepsilon/L \approx 1.59e-2$) (small ε)



- Gibbs-like oscillations around shock-like feature

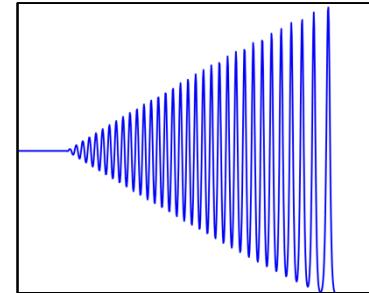
- $\int u \, dx$ conserved; Numerical method is conservative (not shown here)
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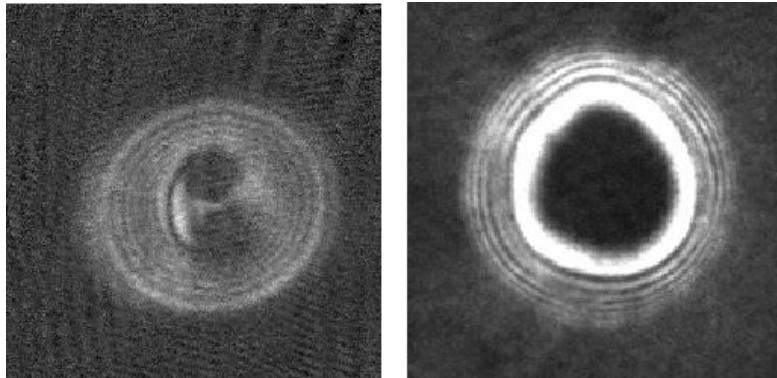
Relation to Dispersive Shocks (1)

- The Korteweg–de Vries equation produces *dispersive* shocks¹

$$\frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$



- Dispersive shocks appear (for example) in
 - Rotating Bose-Einstein condensate
 - Collisionless ion-acoustic shock waves observed from interaction of two plasmas
 - Optical wave breaking observed in propagation of light through nonlinear fiber
 - Propagation of intense electromagnetic wave through photorefractive medium



Experimental absorption images of Bose-Einstein condensate blast wave [2].
The oscillatory ring structures correspond to dispersive shock waves.

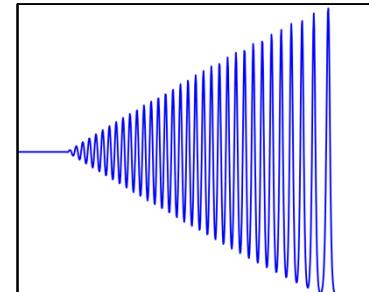
¹ Hoefer and Ablowitz, Dispersive Shock Waves, Scholarpedia, 4(11):5562, 2009.

2 Hoefer, Ablowitz, Coddington, Cornell, Engels, and Schweikhard, Dispersive and classical shock waves in Bose-Einstein condensates and gas dynamics, Phys. Rev. A, 74:023623, 2006.

Relation to Dispersive Shocks (2)

- The Korteweg–de Vries equation produces *dispersive* shocks¹

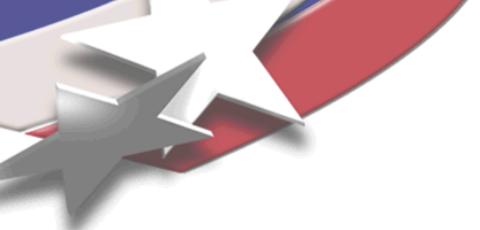
$$\frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$



- Oscillatory solution reminiscent of solutions to nonlocal Burgers equation
- Compare with leading terms of Taylor series of nonlocal Burgers equation:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{60\epsilon^2}{720} \frac{\partial^3 u}{\partial x^3} + \frac{90\epsilon^2}{720} \left(\frac{\partial u}{\partial x} \right) \left(\frac{\partial^2 u}{\partial x^2} \right) + \dots = 0$$

- Leading terms match (up to scaling) with KdV
- Reduces to local Burgers equation in limit as $\epsilon \rightarrow 0$
- In nonlocal Burgers, no shocks possible for $\epsilon > 0$ and $u \in L^1$
- Nonlocal Burgers provides additional regularity beyond KdV



Summary

- Proposed peridynamic-inspired model for nonlocal advection
 - Nonlocal flux
 - Nonlocal Lax-Friedrichs method
- Nonlocal Burgers equation
 - Shocks not possible for $\varepsilon > 0$ and $\square \in L^1$ for smooth data
 - Computational results (sine IC, “top hat” IC)
- Relation to dispersive shock waves

- Papers, codes
 - www.sandia.gov/~mlparks; mlparks@sandia.gov