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Mimetic Least Squares Methods with Preconditioners for Darcy Flow

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Outline

- **Introduction to Darcy Flow & Least Squares**
    - **Background Theory**
    - **Advantages of Mimetic Least Squares (MLS)**
- Preconditioners for Mimetic Least Squares (MLS)
- Numerical Examples
- Conclusions & Future Work

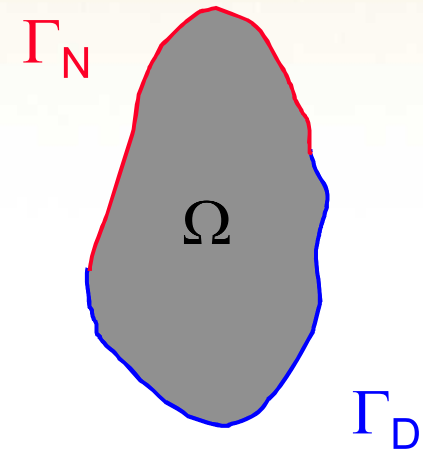
A Model Problem

$$
\begin{aligned}
-\nabla\cdot\mathbf{A}\nabla\phi+\gamma\phi &= f \quad \text{in } \Omega\\
\phi &= g \quad \text{on } \Gamma_D\\
\mathbf{n}\cdot\mathbf{A}\nabla\phi &= h \quad \text{on } \Gamma_N
\end{aligned}
$$

$$\gamma\in L^{\infty}(\Omega)\ \rightarrow\ \begin{cases}\gamma\equiv 0\\ \gamma\ge\gamma_0>0\end{cases}$$

$$f\in L^2(\Omega)$$

$$\mathbf{A}\in\mathbf{R}^{d\times d}\ \rightarrow\ \frac{1}{\alpha}|\xi|^2\le\xi^T\mathbf{A}\xi\le\alpha|\xi|^2$$



$$\begin{cases}\nabla\cdot\mathbf{u}+\gamma\phi=f\\ \mathbf{A}^{-1}\mathbf{u}+\nabla\phi=0\end{cases}\quad \text{in } \Omega$$

Equivalent first-order system

Artificial "energy" principle

$$\left.\begin{aligned}\nabla\cdot\mathbf{u}+\gamma\phi&=f\\ \mathbf{A}^{-1}\mathbf{u}+\nabla\phi&=0\end{aligned}\right\}\Leftrightarrow J(\mathbf{u},\phi;f)=\frac{1}{2}\left(\|\nabla\cdot\mathbf{u}+\gamma\phi-f\|_0^2+\|\mathbf{A}^{-1/2}(\mathbf{u}+\mathbf{A}\nabla\phi)\|_0^2\right)=0$$

Optimization problem

$$\min_{\mathbf{v}\in H_N(\Omega,div);\psi\in H_D^1(\Omega)} J(\mathbf{v},\psi;f)$$

Optimality system

$$
\begin{aligned}
&(\nabla\cdot\mathbf{u}+\gamma\phi,\nabla\cdot\mathbf{v})+\left(\mathbf{A}^{-1/2}\mathbf{u}+\mathbf{A}^{1/2}\nabla\phi,\mathbf{A}^{-1/2}\mathbf{v}\right)=(f,\nabla\cdot\mathbf{v}) \quad \forall\mathbf{v}\in H_N(\Omega,\text{div})\\
&(\nabla\cdot\mathbf{u}+\gamma\phi,\gamma\psi)+\left(\mathbf{A}^{-1/2}\mathbf{u}+\mathbf{A}^{1/2}\nabla\phi,\mathbf{A}^{1/2}\nabla\psi\right)=(f,\gamma\psi) \quad \forall\psi\in H_D^1(\Omega)
\end{aligned}
$$

Least-Squares Theory

Artificial "energy" norm

$$J(\mathbf{u},\phi;0)=\frac{1}{2}\left(\|\nabla\cdot\mathbf{u}+\gamma\phi\|_0^2+\|\mathbf{A}^{-1/2}(\mathbf{u}+\mathbf{A}\nabla\phi)\|_0^2\right)=|||\,(\mathbf{u},\phi)\,|||^2$$

Norm equivalence

$$C_1\left(\|\mathbf{u}\|_{div}^2+\|\phi\|_1^2\right)\le|||\,(\mathbf{u},\phi)\,|||^2\le C_2\left(\|\mathbf{u}\|_{div}^2+\|\phi\|_1^2\right)$$

Bilinear form

$$Q_{LS}(\mathbf{u},\phi;\mathbf{v},\psi)=(\nabla\cdot\mathbf{u}+\gamma\phi,\nabla\cdot\mathbf{v}+\gamma\psi)+\left(\mathbf{A}^{-1/2}\mathbf{u}+\mathbf{A}^{1/2}\nabla\phi,\mathbf{A}^{-1/2}\mathbf{v}+\mathbf{A}^{1/2}\nabla\psi\right)$$

Inner-product equivalence

$$Q_{LS}(\mathbf{u},\phi;\mathbf{v},\psi)=\langle(\mathbf{u},\phi);(\mathbf{v},\psi)\rangle$$ and $$Q_{LS}(\mathbf{u},\phi;\mathbf{u},\phi)=|||\,(\mathbf{u},\phi)\,|||^2$$

Stability

$$C_1\left(\|\mathbf{u}\|_{div}^2+\|\phi\|_1^2\right)\le Q_{LS}(\mathbf{u},\phi;\mathbf{u},\phi)$$ ← **coercivity**

continuity → $$Q_{LS}(\mathbf{u},\phi;\mathbf{v},\psi)\le C_2\left(\|\mathbf{u}\|_{div}^2+\|\phi\|_1^2\right)^{1/2}\left(\|\mathbf{v}\|_{div}^2+\|\psi\|_1^2\right)^{1/2}$$

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Mimetic Least Squares (MLS)

To use least squares:

- ☺ **Using C^0 elements**
- ☺ **No inf-sup condition**
- ☺ **Solving SPD systems**

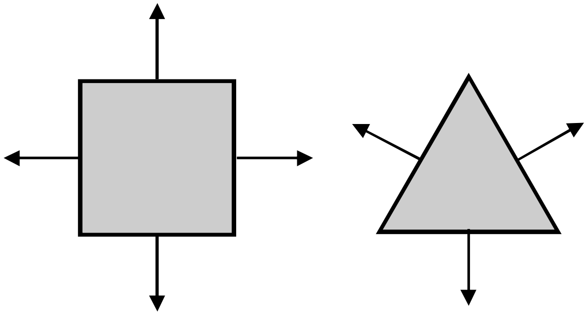
Top 3 reasons

Not to use least squares:

- ☹ **Conservation**
- ☹ **Conservation**
- ☹ **Conservation**

- C^0 elements give us everything on the left, but leave us with a conservation problem.
- What if we used a **Mimetic** (Face Element) discretization instead? (This should give us better conservation)

RT(k) spaces $k \geq 0$



Advantages of MLS

It has been shown that:

- Using a **Mimetic** approximation to *H(div)* gives optimal accuracy [1].

For $\phi_h \in P_k$ *and* $\mathbf{u}_h \in RT_k$

$$\|\phi - \phi_h\|_0 + \|\mathbf{u} - \mathbf{u}_h\|_0 = O\left(h^k\right)$$

$$\|\phi - \phi_h\|_1 + \|\mathbf{u} - \mathbf{u}_h\|_{div} = O\left(h^k\right)$$

- If $\gamma>0$, **MLS** gives same **pressure** as **Ritz-Galerkin** method and same **flux** as **Mixed Galerkin** method and is thus conservative [2].

We will show that:

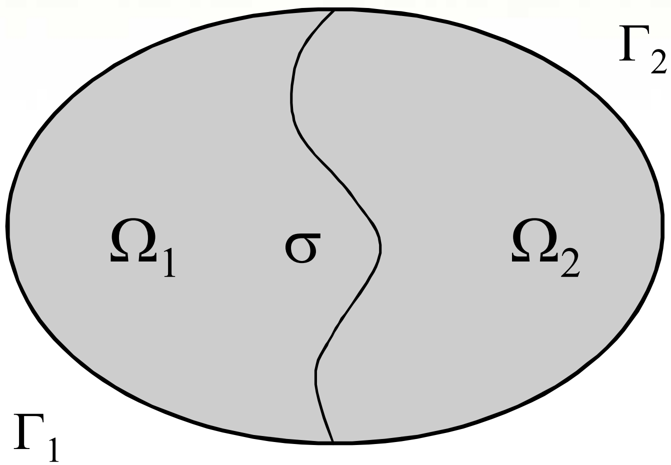
➪ There is more to LSFEM than **C^0 elements**. Moving to **MLS** has real advantages.

➪ We can actually solve the linear system from **MLS** (contrast this with **C^0 elements**).

[1] Bochev and Gunzberger. SIAM J. Numer. Anal., 2005.
[2] Bochev and Gunzburger, *Least-squares finite element methods*, Springer, 2009.

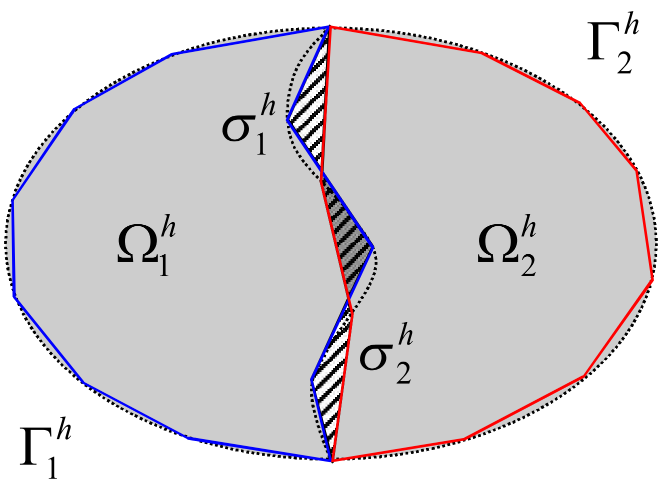
Bonus: Least-Squares for Mesh-Tying

Transmission problem



$$\begin{cases} \nabla\cdot \mathbf{u}_i + \gamma_i \phi_i = f & \text{in } \Omega_i \\ \mathbf{A}_i^{-1}\mathbf{u}_i + \nabla \phi_i = 0 & \text{in } \Omega_i \\ \phi_i = 0 & \text{on } \Gamma_{D,i} \\ \mathbf{n}_i \cdot \mathbf{u}_i = 0 & \text{on } \Gamma_{N,i} \end{cases} + \begin{cases} \phi_1 = \phi_2 \\ \mathbf{n}_1 \cdot \mathbf{u}_1 = \mathbf{n}_2 \cdot \mathbf{u}_2 \end{cases} \text{ on } \sigma$$

Discrete version



Approximation of curved interfaces leads to non-matching discrete interfaces and many problems:

- ➪ traditional mortars not appropriate: duplicate interface
- ➪ typically project values to one of the interfaces (master-slave)
- ➪ issues with counting physical energy in gaps/voids
- ➪ at best passes linear patch test (recovers linear pressure)
- ➪ Non-matching interfaces remain a tough challenge for most traditional methods

Least-Squares Offer a Surprisingly Simple and Effective Solution

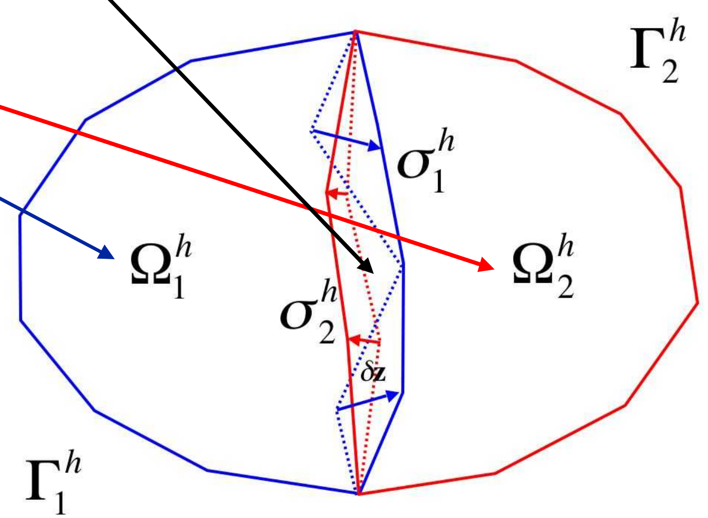
The reason:

- ➪ LS are based on minimization of **artificial residual energy**, not physical energy
- ➪ Minimization of **residual energy** allows to measure energy **redundantly**
- ➪ **All that is needed is elimination of the void regions to create sufficient overlap:**
  - **– Can be done by interface perturbation or by simply extending the domains**

$$\min \frac{1}{2}\sum_i \underbrace{\left(\left\|\nabla\phi_i + \mathbf{v}_i\right\|^2_{0,\Omega_i} + \left\|\nabla\cdot\mathbf{v}_i - f\right\|^2_{0,\Omega_i}\right)}_{\text{residual energy}} + \underbrace{\left\|\phi_1-\phi_2\right\|^2_{1,\Sigma} + \left\|\mathbf{v}_1-\mathbf{v}_2\right\|^2_{div,\Sigma}}_{\text{coupling term}}$$

Advantages

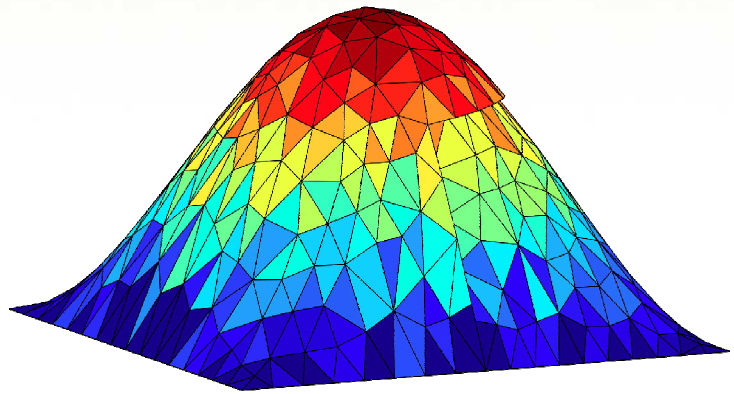
- ✓ Provably **stable** (coercive formulation)
- ✓ Provable **optimal convergence** rate
- ✓ Can pass an **arbitrary order** patch test
- ✓ No **mesh-dependant** tunable parameters
- ✓ No complications for **floating** domains
- ✓ Does not require extensive **mesh intersections**

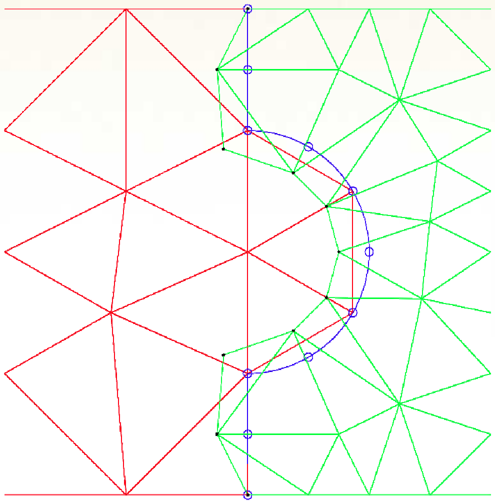


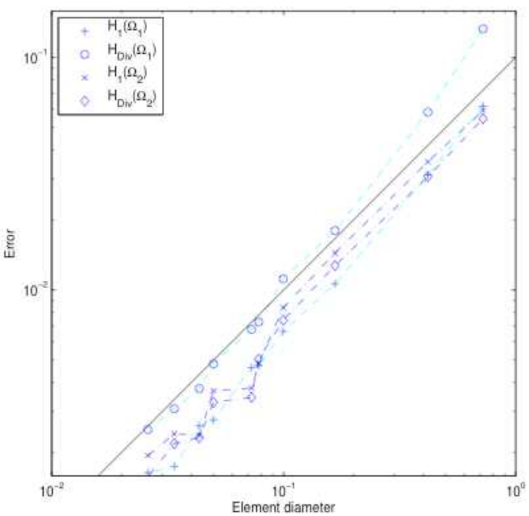
Proof Of Concept: Doughnut & Circular Cut

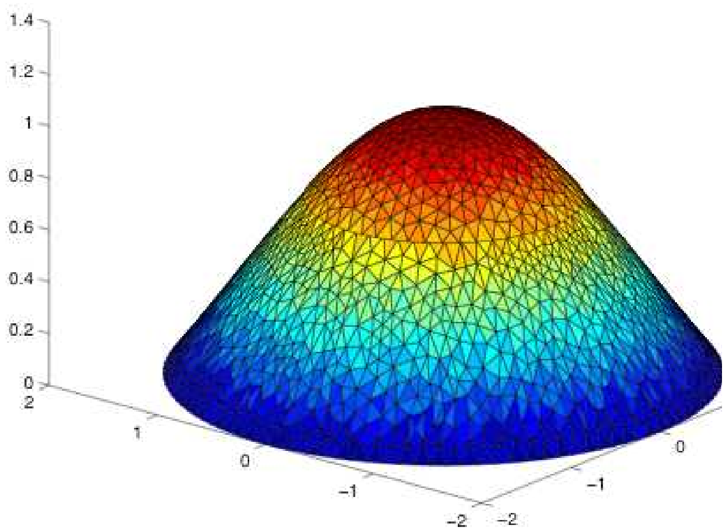
Unstructured triangles

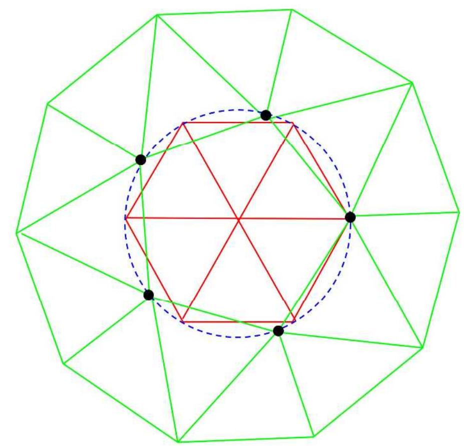
- ✓ theoretical rates achieved on both subdomains
- ✓ no ringing at interface nodes
- ✓ no complication from the floating domain!











Bochev, Day, JCAM 2007

Outline

- Introduction to Darcy Flow & Least Squares
- **Preconditioners for Mimetic Least Squares (MLS)**
  - **Block Preconditioning**
  - **Face Element AMG**
  - **Why Nodal LS is Problematic**
- Numerical Examples
- Conclusions & Future Work

Mimetic LS Linear System

$$
\begin{array}{lcccc}
<\mathbf{v},\mathbf{u}> + <\nabla\cdot\mathbf{v},\nabla\cdot\mathbf{u}> & + & <\mathbf{v},\nabla\phi> & = & <\mathbf{v},f_u> \\
<\psi,\nabla\cdot\mathbf{u}> & + & <\psi,\phi> + <\nabla\psi,\nabla\phi> & = & <\psi,f_p>
\end{array}
$$

- Pressure = lowest order nodal elements.
- Velocity = lowest order face elements.
- Block diagonal is now a Grad-Div + Mass and a Laplacian + Mass.

- How to precondition? Remember Norm Equivalence.

$$C_1\left(\|\mathbf{u}\|_{div}^2 + \|\phi\|_1^2\right) \le |||\,(\mathbf{u},\phi)\,|||^2 \le C_2\left(\|\mathbf{u}\|_{div}^2 + \|\phi\|_1^2\right)$$

- We will precondition block-diagonally!

Block Preconditioners

$$
\begin{aligned}
< \mathbf{v}, \mathbf{u} > + < \nabla \cdot \mathbf{v}, \nabla \cdot \mathbf{u} > \quad &+ \quad < \mathbf{v}, \nabla \phi > &= \quad < \mathbf{v}, f_u > \\
< \psi, \nabla \cdot \mathbf{u} > \quad &+ \quad < \psi, \phi > + < \nabla \psi, \nabla \phi > &= \quad < \psi, f_p >
\end{aligned}
$$

- For the BLUE block, use standard AMG.
- For the RED block, follow the approach of *Bochev, et al.* [1].
  - Smooth the RED block directly.
  - Smooth the associated *edge* problem.
  - Transfer both edge face problems to coarse nodal problems.
  - Apply standard (vector) AMG to the coarse nodal problems.

[1] P. Bochev, C. Siefert, R. Tuminaro, J. Xu and Y. Zhu. *Compatible Gauge Approaches for H(div) Equations.* Technical Report SAND 2007-5384P, August 2007.

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Face-Element AMG Review

1. $x =$StandardRelaxation(A_F, x, b).
2. $r = b - A_F x$.
3. $b_F - P_F^T r$.
4. $x_F =$StandardAMG$(P_F^T A_F P_F, b_E)$.
5. $b_E = D^T r$.
6. $x_E -$StandardRelaxation$(A_E, 0, b_E)$.
7. $r_E = b_E - A_E x_E$.
8. $b_E = P_E^T r_E$.
9. $x_{E2} =$StandardAMG$(P_E^T A_E P_E, b_E)$.
10. $x = x + P_F x_F + D(x_E + P_E x_{E2})$

$$
\begin{aligned}
A_F &= <\mathbf{v}, \mathbf{u}> + <\nabla \cdot \mathbf{v}, \nabla \cdot \mathbf{u}> \\
A_E &= <\nabla \times \mathbf{z}, \nabla \times \mathbf{w}> \\
D &= \text{Face-Edge Incidence Matrix} \\
P_F &= \text{Face-to-Node Coarsening} \\
P_E &= \text{Edge-to-Node Coarsening}
\end{aligned}
$$

Note: This can also be done symmetrically.

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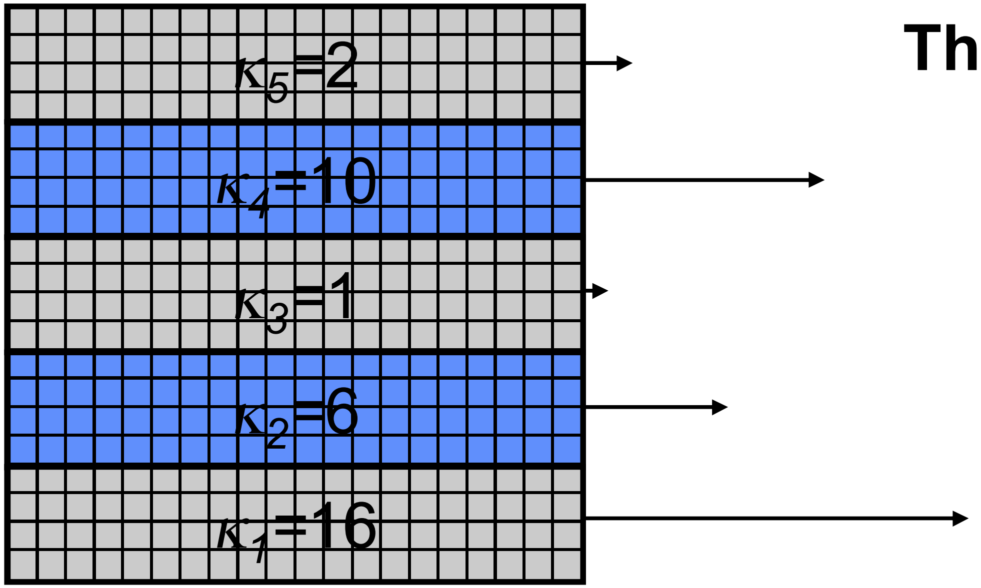
Why not Nodal LS?

- Edge/Face problems hard to solve… Why not go nodal?
  - Question #1: Solution quality.
  - Question #2: Linear system conditioning / solver performance.



The 5 Strip Problem

Exact solution

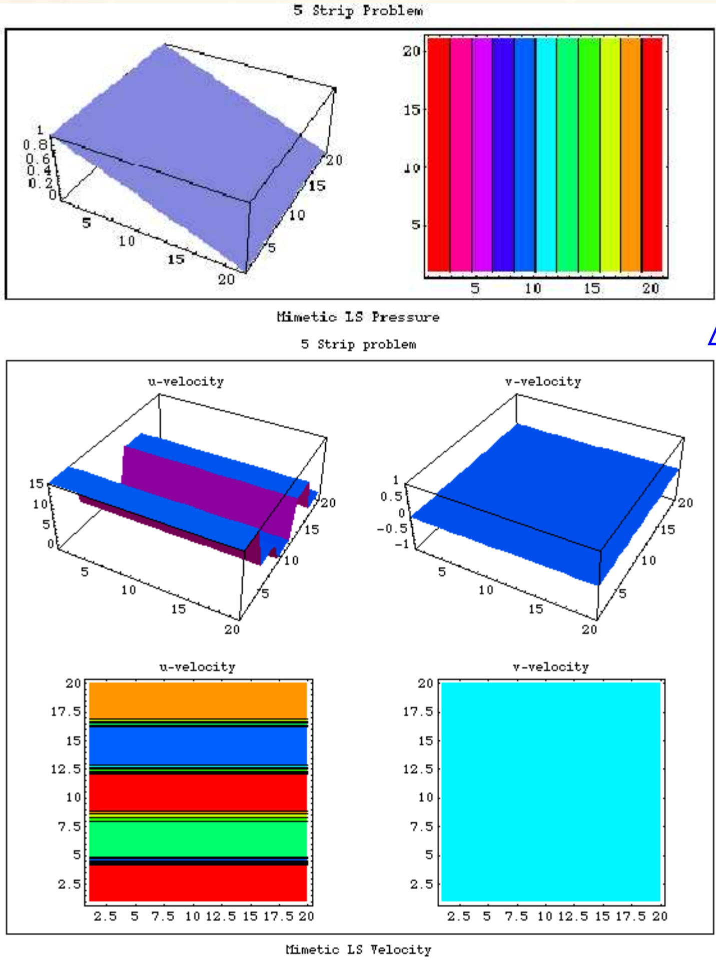
$$\phi = 1 - x;$$

$$\mathbf{u} = \begin{pmatrix} \kappa_i \\ 0 \end{pmatrix} \text{ in strip } i$$

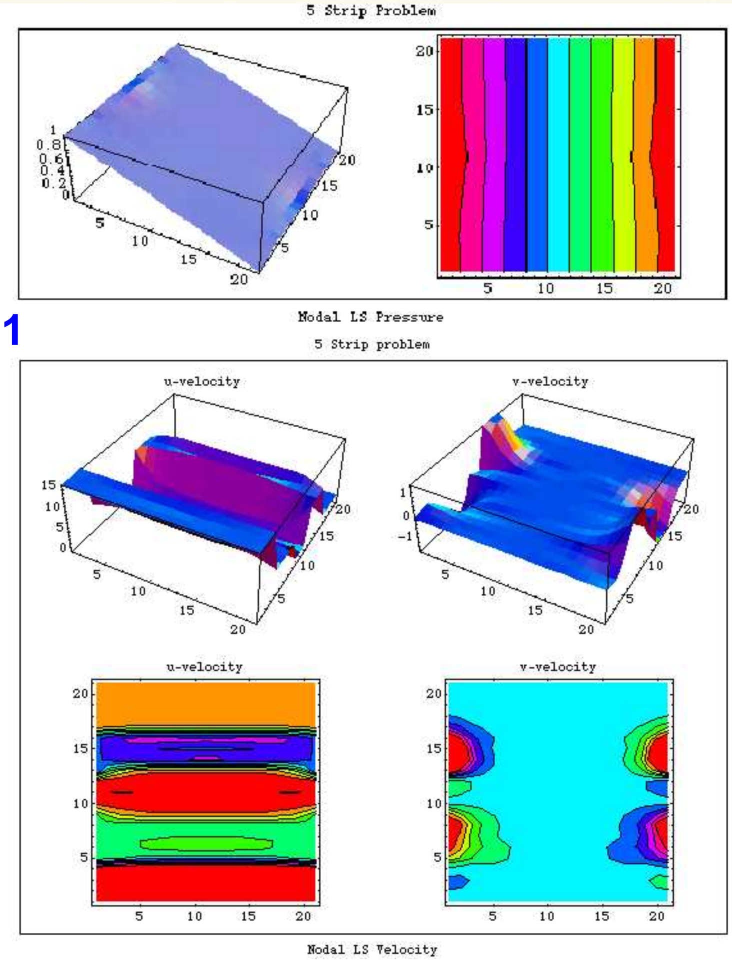
From: *T. Hughes, A. Masud and J. Wan, A stabilized mixed DG method for Darcy flow.*

Mimetic vs. Nodal Least Squares

Mimetic LS



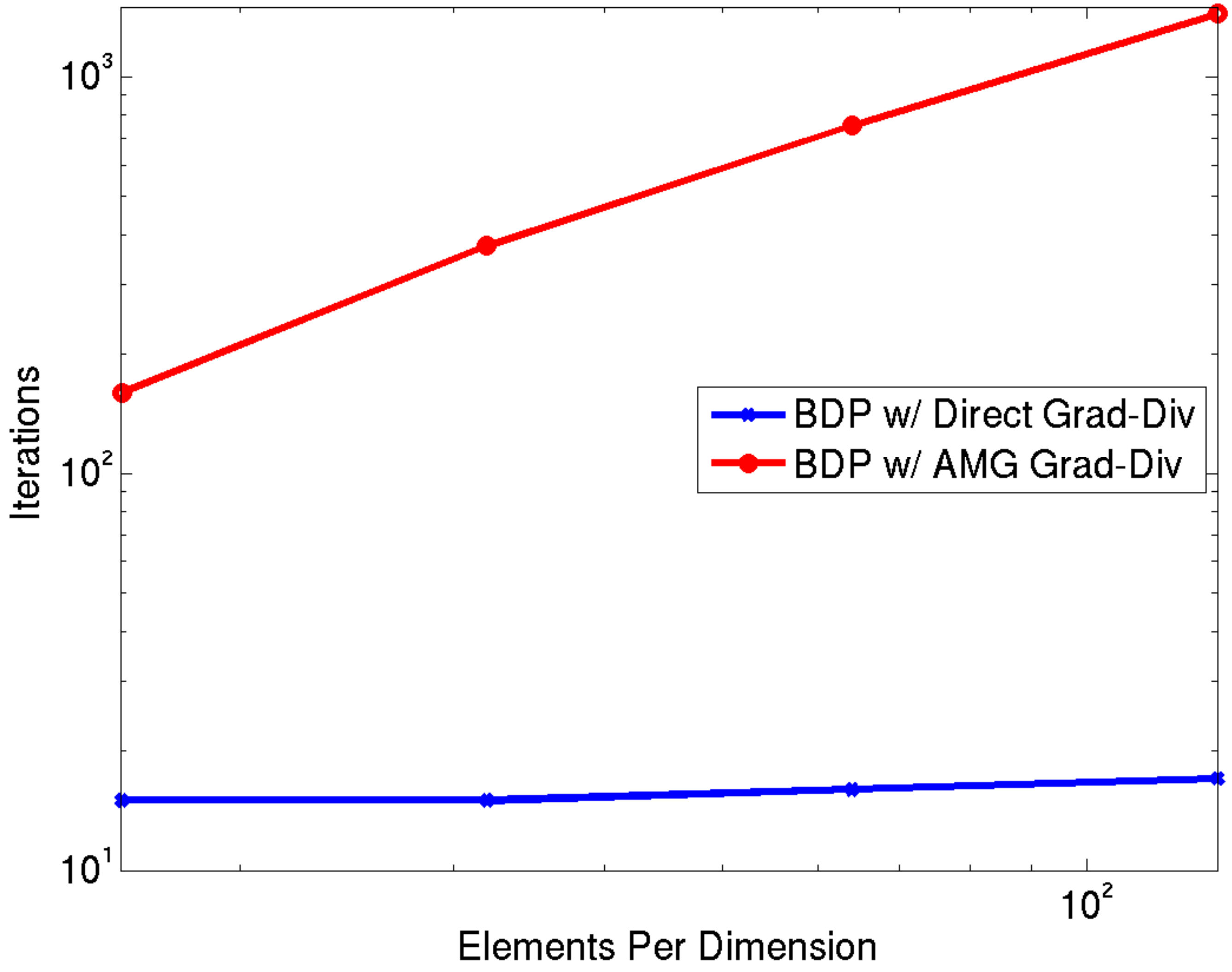
Δt=0.01



Nodal LS

Method	L2 Flux	H(div) Flux	L2 Scalar	H1 Scalar
Mimetic LS	0.1670E-08	0.9839E-13	0.4553E-11	0.3041E-13
Nodal LS	0.1759E+01	0.7470E+02	0.8926E-02	0.1425E+00

5 Strip (Nodal LS)



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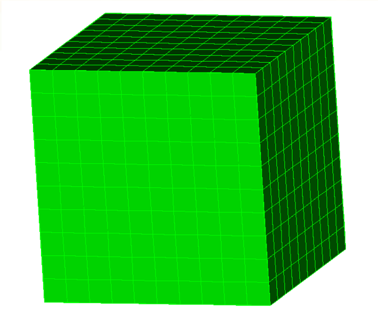
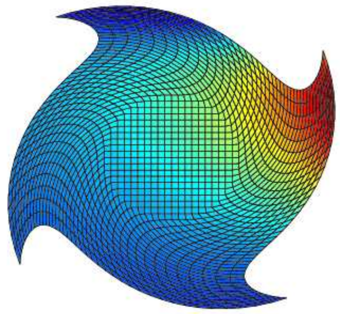
3D Scaling Studies



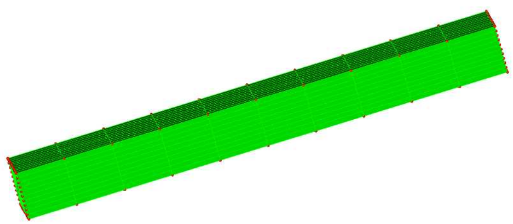
• Consider Three 3D Scaling Problems

1. Cube w/ Orthogonal Mesh



2. "Distorted" Mesh

3. Rectangular Solid w/ 10:1 mesh stretch

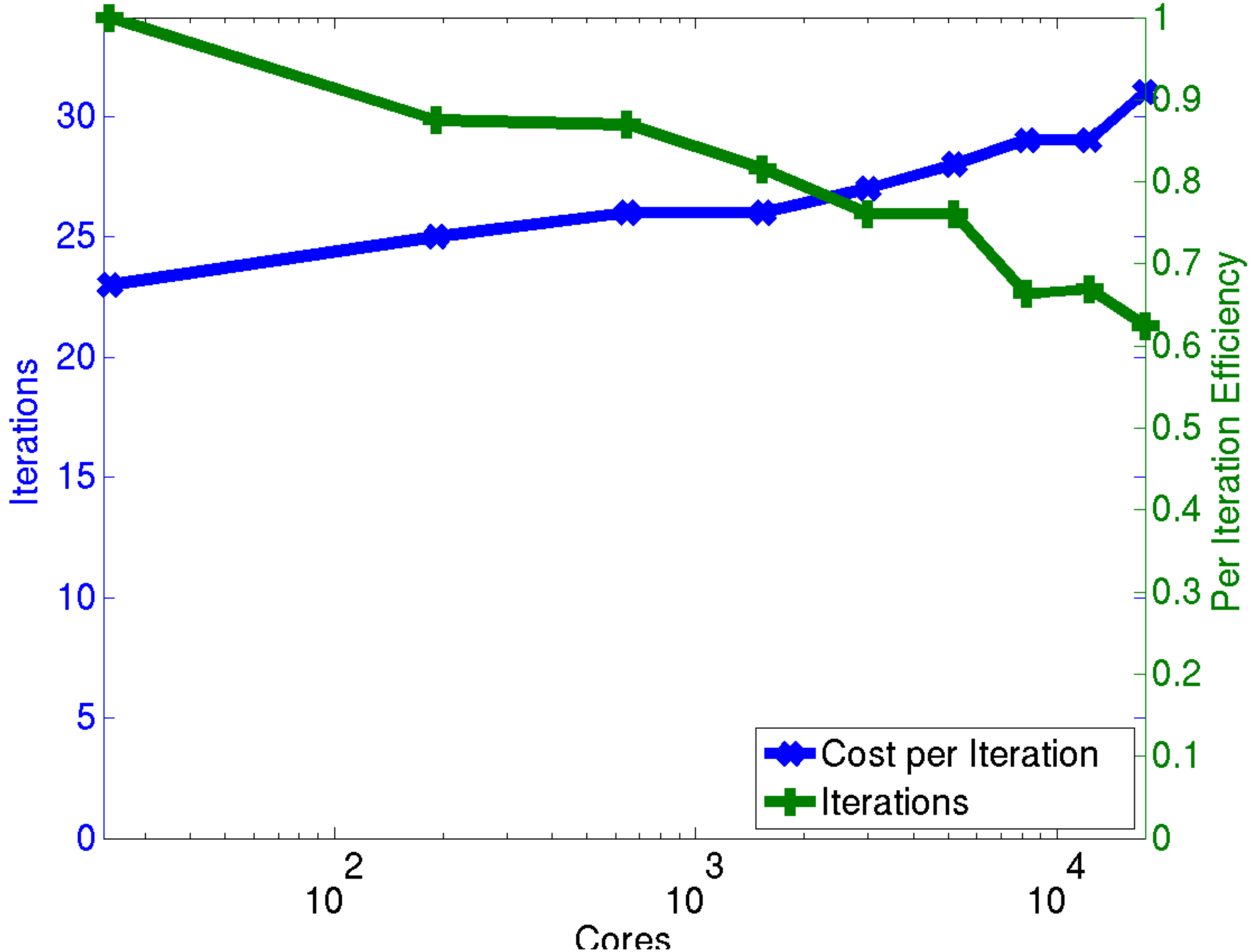


• CG (1e-10 tolerance) w/ block preconditioner.

• Two (pre & post) steps of Chebyshev for smoothing at each level.

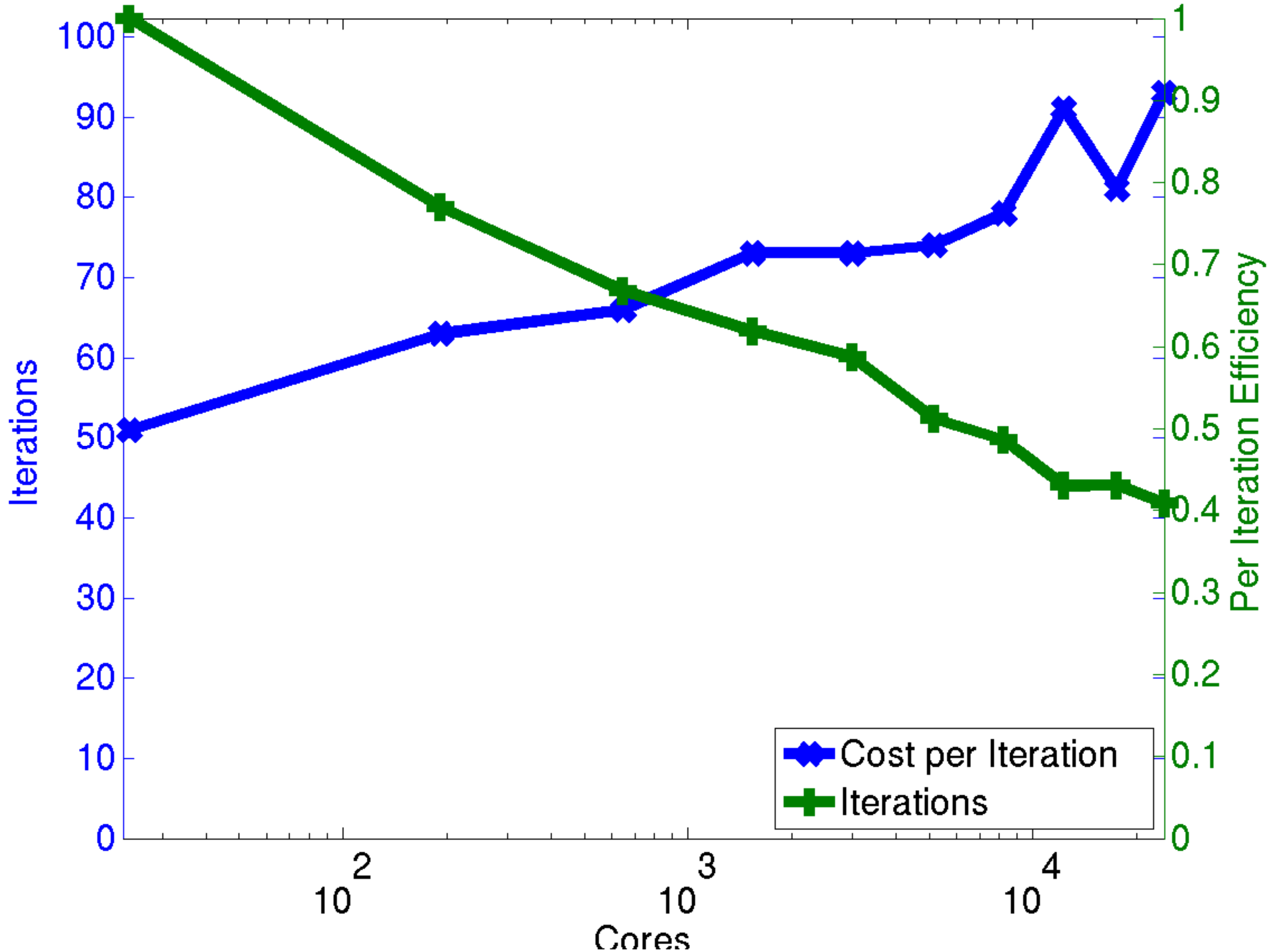
• Weak scaling to 630M nodes / 24k cores on NERSC's Hopper (~26k nodes/core).

3D Cube

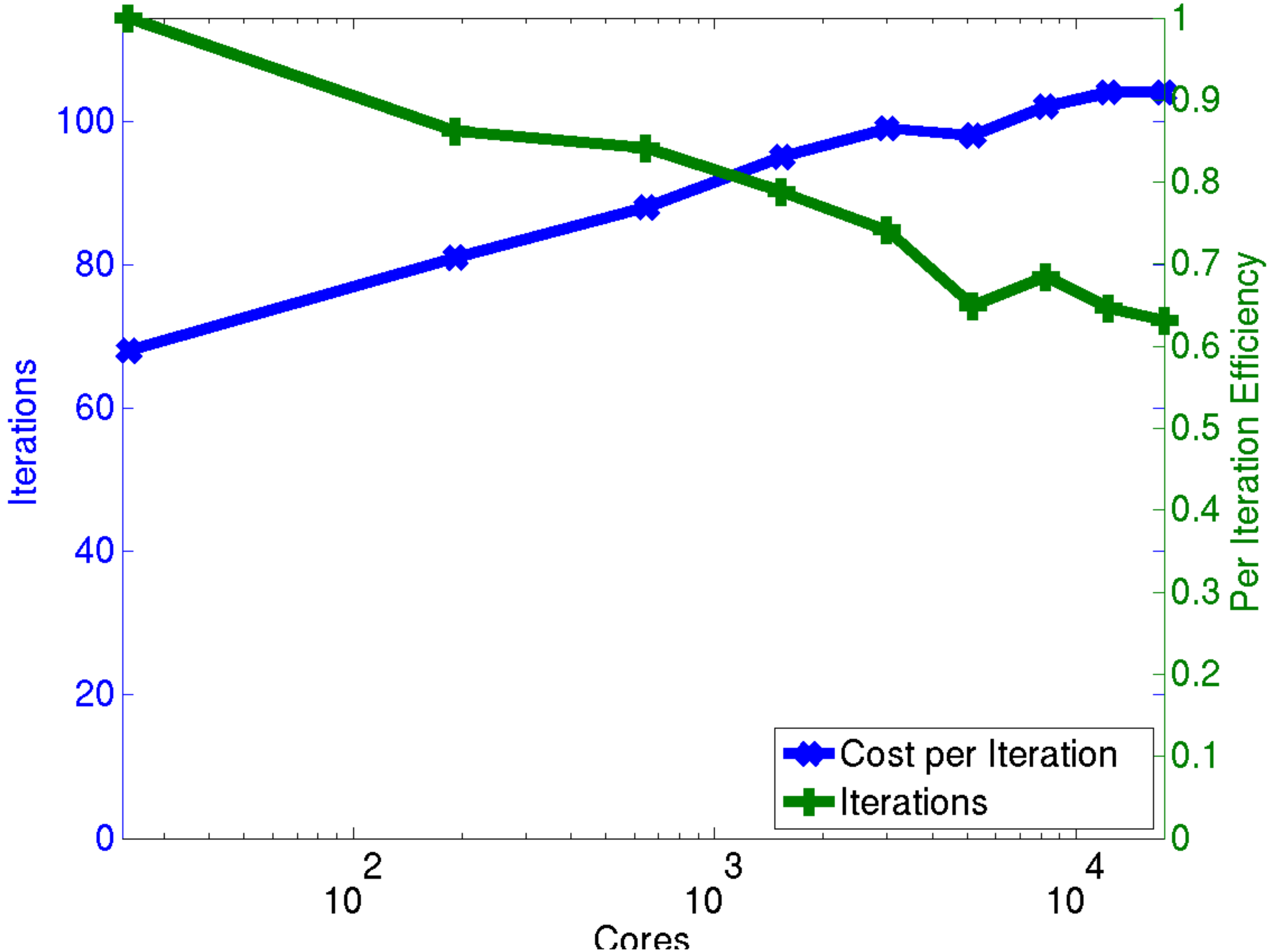


Iterations
0
5
10
15
20
25
30
Per Iteration Efficiency
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
10^2
10^3
10^4
Cores
Cost per Iteration
Iterations

3D Distorted Mesh



3D Box w/ 10:1 Stretch



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Conclusions

- **MLS** inherit the **best** properties of Galerkin and Mixed methods:
  - **Galerkin** → Optimal accuracy in the **pressure** variable
  - **Mixed** → Optimal accuracy in the **flux** variable
  - Better solutions than Nodal LS…
- Not obvious how to precondition nodal LS.
- **MLS** leads to SPD systems w/ block-diagonal preconditioners.
  - Good scalability for non-stretched meshes…
  - Needs work for mesh stretching…