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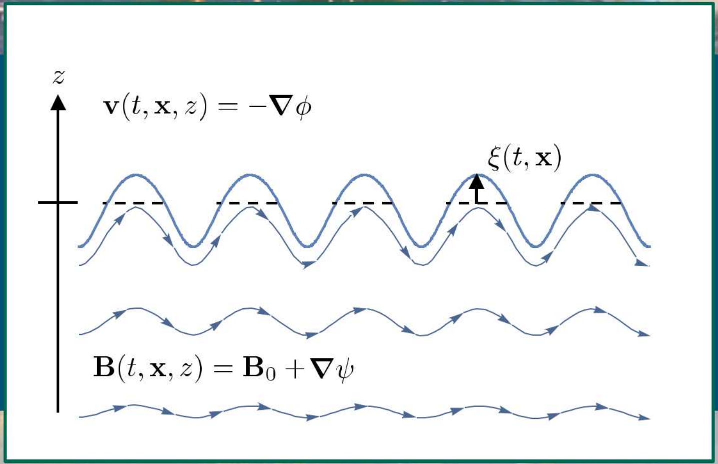

On a variational formulation of the weakly nonlinear magnetic-Rayleigh—Taylor instability

Daniel Ruiz

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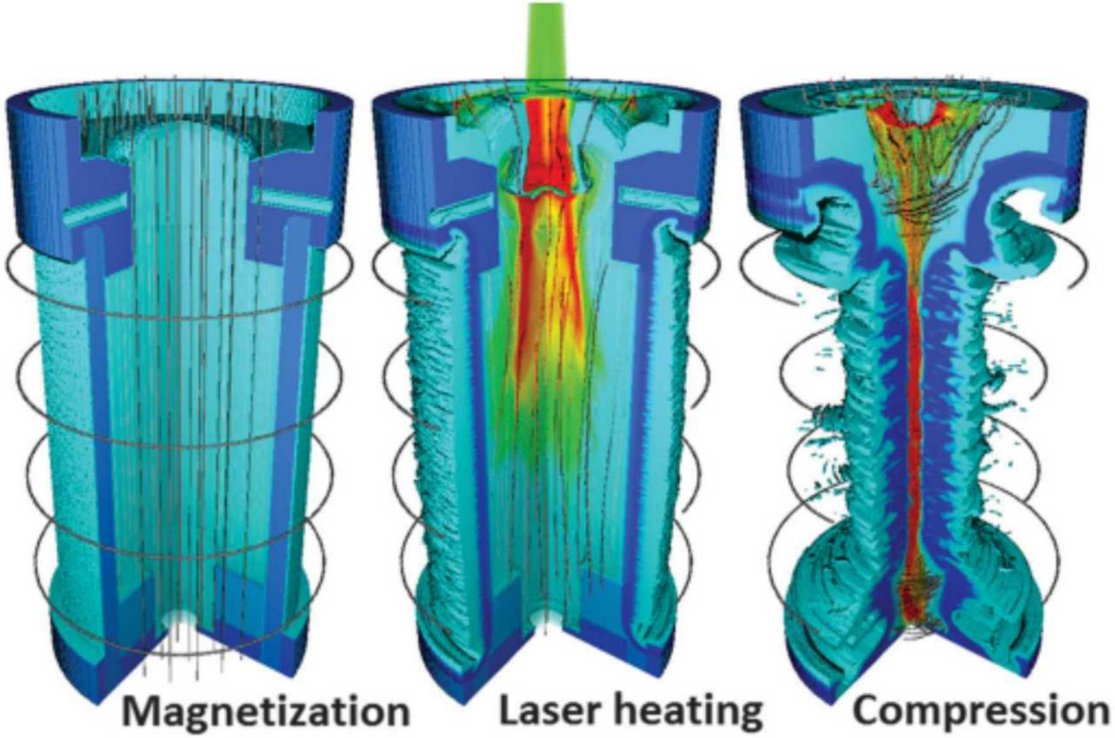
Naval Research Laboratory

The magnetic-Rayleigh—Taylor (MRT) instability is ubiquitous in Z-pinch implosions.

- In Z-pinch implosions, the $\mathbf{J} \times \mathbf{B}$ force is used to compress matter.
- Applications of Z-pinch implosions include fusion schemes in which the Z pinch compresses fuel to fusion-relevant conditions.
- As in the classical Rayleigh—Taylor (RT) instability, the driving magnetic pressure plays the role of a light fluid pushing on the liner, which acts as a heavy fluid.
- Target performance is highly dependent on the integrity of the liner, which can be broken up by the MRT instability.



Schematic of the MagLIF fusion concept.[1]

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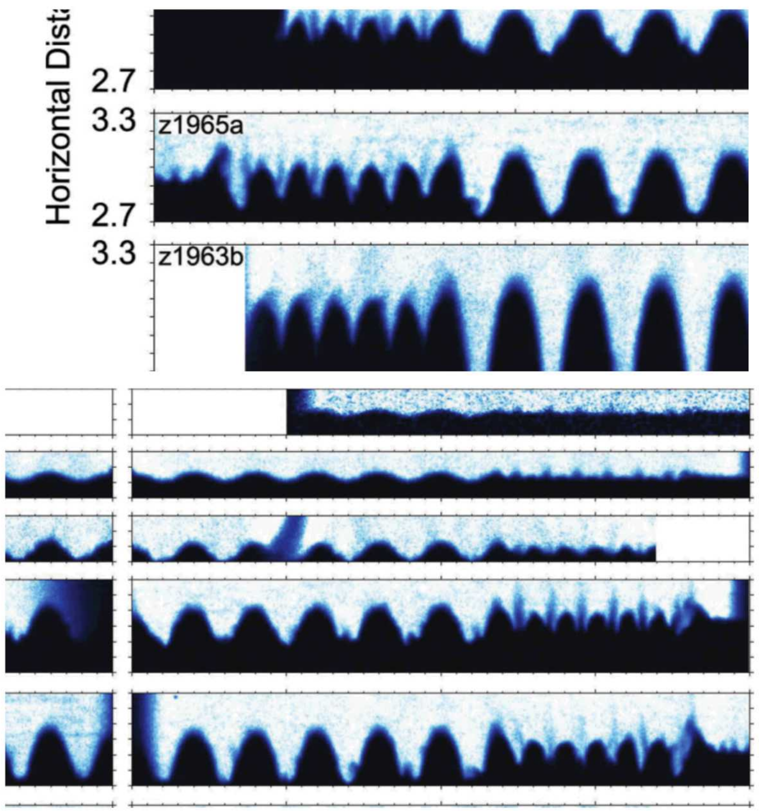
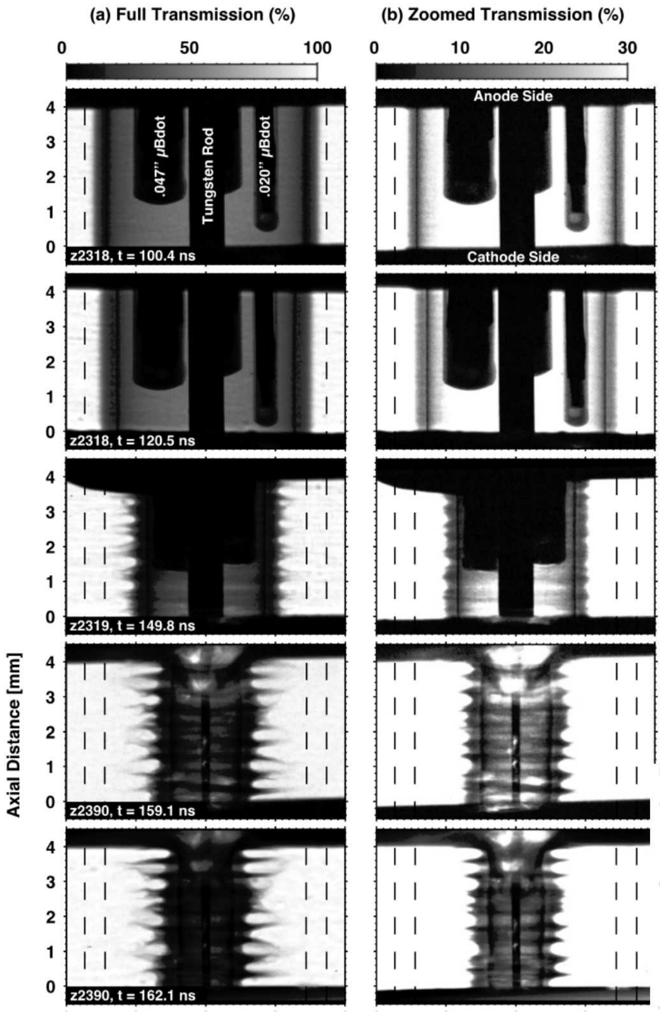
[1] M. R. Gomez, et al., Phys. Rev. Lett. **113**, 155003 (2014).

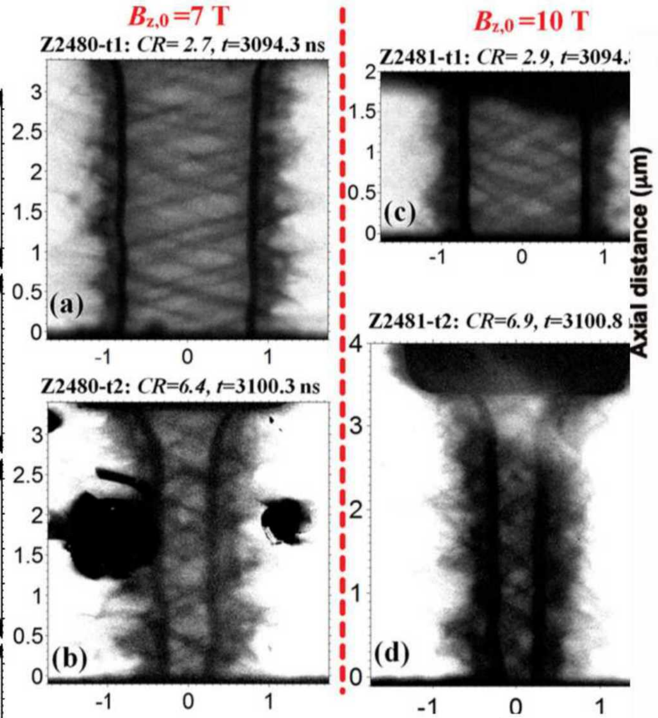
Due to its importance, the MRT instability has been thoroughly studied both experimentally and computationally at Sandia.



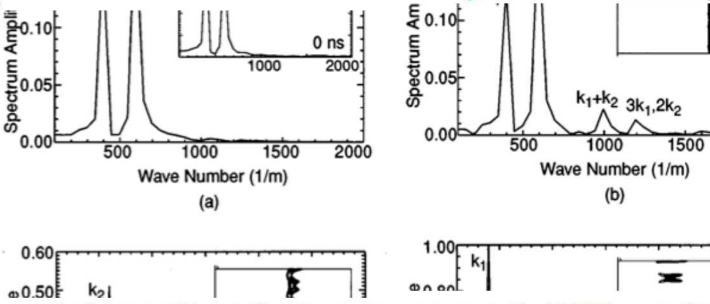
Machined-perturbation experiments[2]



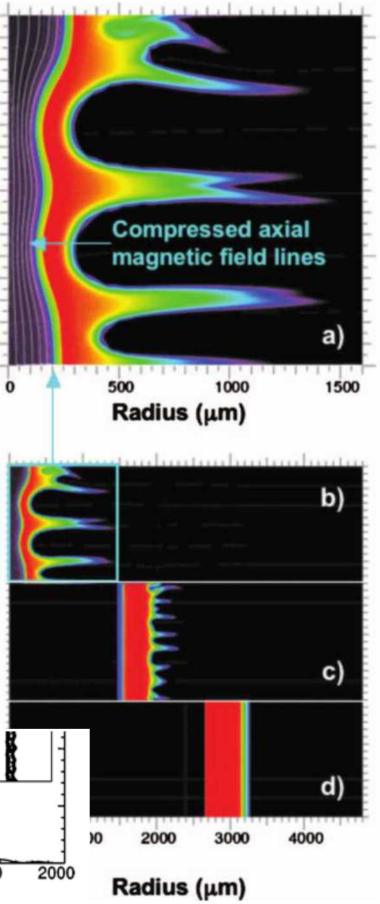
Beryllium liner experiments[3]



Helical instability[4]



Interaction between several MRT modes[5]



Liner temporal dynamics[6]

[2] D. B. Sinars, et al., Phys. Rev. Lett. **105**, 51 (2010).
[3] R. D. McBride, et al., Phys. Plasmas **20**, 056309 (2013).
[4] T. J. Awe, et al., Phys. Rev. Lett. **111**, 956 (2013).
[5] M. R. Douglas, C. Deeney, and N. F. Roderick, Phys. Plasmas **5**, 4183 (1998).
[6] S. A. Slutz, et al., Phys. Plasmas **17**, 056303 (2010).

From the theoretical standpoint, much work is left to be done.

- Due to effects coming from the magnetic-field tension, the MRT instability is rich in complexity compared to the classical RT instability.
- On the theoretical side, most works have focused on the linear phase of MRT while adding a variety of effects; e.g.,
    - slab and cylindrical geometry,[7,8,9]
    - Magnetization effects,[8]
    - Magnetic-shear effects,[10] and
    - Bell—Plesset effects.[11]
- There are a lot of pending questions regarding the nonlinear stages of MRT:[12]
    - What is the saturation amplitude for MRT?
    - Can we describe the nonlinear MRT observed in our experiments in Z with a simple model?
    - How will MRT scale for currents envisioned in a next-generation machine?

In this talk, I will report on some recent advances on understanding nonlinear MRT.

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[7] E. G. Harris, Phys. Fluids **5**, 1057 (1962).
[8] M. R. Weis, et al., Phys. Plasmas **21**, 122708 (2014).
[9] M. R. Weis, et al., Phys. Plasmas **22**, 032706 (2015).
[10] P. Zhang, et al., Phys. Plasmas **19**, 022703 (2012).
[11] A. L. Velikovich and P. F. Schmit, Phys. Plasmas **22**, 122711 (2015).
[12] D. D. Ryutov and M. A. Dorf, Phys. Plasmas **21**, 112704 (2014).

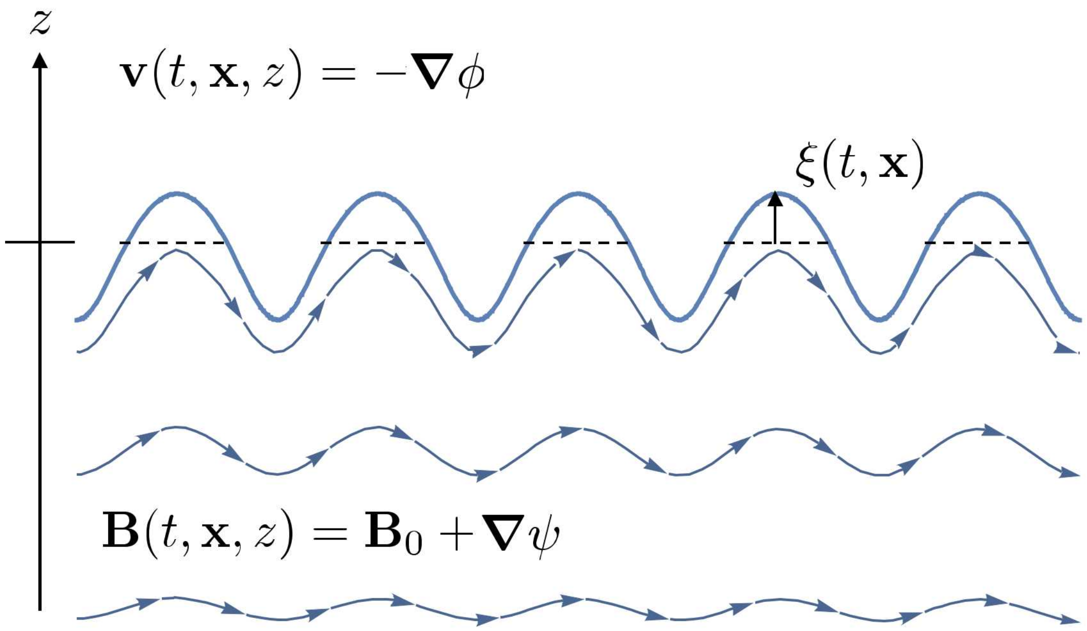
Program of today’s talk

1. Main motivation
2. Introduction to the problem considered
3. Hamiltonian theory of the weakly-MRT instability
4. Comparison of weakly nonlinear theory to experiment
5. Current work: Regularizing of the weakly nonlinear MRT theory
6. Conclusions and future work

2. *Introduction to the problem considered*

To begin, let’s consider the single-surface MRT problem.

- For this talk, I focus on a semi-infinite planar fluid slab under some gravitational field.[13,14]
- We assume that the fluid is
  - incompressible,
  - irrotational,
  - unmagnetized,
  - immiscible, and
  - perfectly conducting.
- The fluid slab interacts with a magnetic field **B** situated in the vacuum region.



Schematic of the problem considered in this talk.

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[13] Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability,* (Oxford University Press, London, 1961).
[14] M. Kruskal and M. Schwarzchild, Proc. R. Soc. London Ser. A **223**, 348 (1954).

The governing equations of the single-surface MRT are well known.

- The fluid is irrotational and incompressible.

$$\nabla \times \mathbf{v} = 0 \quad \Rightarrow \quad \mathbf{v} = -\nabla\phi$$

$$\nabla \cdot \mathbf{v} = 0 \quad \Rightarrow \quad \boxed{\nabla^2\phi = 0}$$

- The magnetic field $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$ is in vacuum and in contact with a perfectly-conducting surface.

$$\nabla \times \mathbf{B}_1 = 0 \quad \Rightarrow \quad \mathbf{B}_1 = \nabla\psi$$

$$\nabla \cdot \mathbf{B}_1 = 0 \quad \Rightarrow \quad \boxed{\nabla^2\psi = 0}$$

$$\boxed{\mathbf{B}|_{z=\xi} \cdot \nabla(z-\xi) = 0}$$

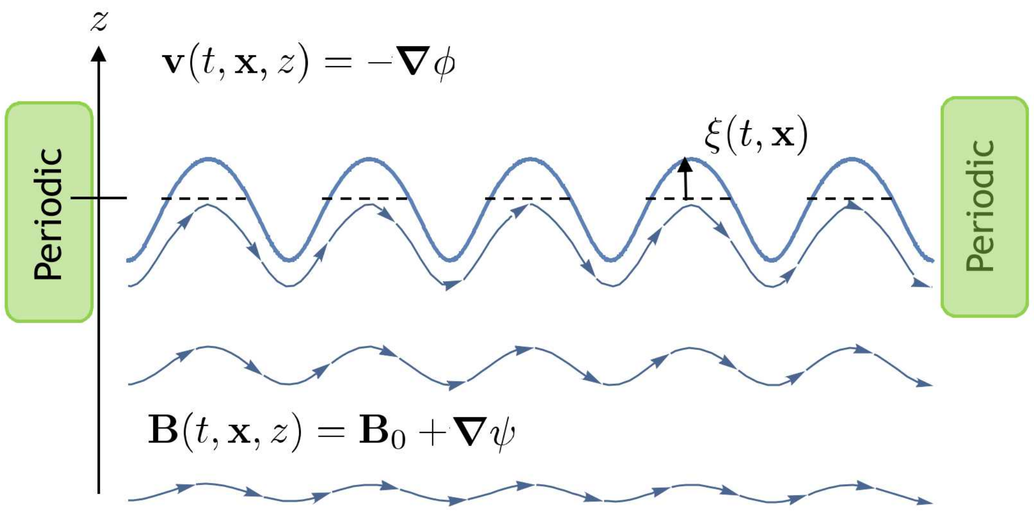
- The fluid—vacuum interface is self-consistently advected.

$$\boxed{\left[\partial_t\xi + \nabla\phi \cdot \nabla(z-\xi)\right]_{z=\xi} = 0}$$

- The fluid is immiscible and obeys the force-balance equation.

$$\rho\partial_t\mathbf{v} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} = -\rho g\mathbf{e}_z - \nabla P \quad \Rightarrow \quad \boxed{\rho\left[\frac{\partial}{\partial t}\phi - \frac{1}{2}(\nabla\phi)^2 - gz\right]_{z=\xi} = \overbrace{\frac{1}{8\pi}\left|\mathbf{B}_0 + \nabla\psi\right|^2_{z=\xi}}^{\textit{Magnetic pressure}}}$$

$$\boxed{\lim_{z\to+\infty} \mathbf{v} = \mathbf{0}}$$



$$\mathbf{v}(t,\mathbf{x},z) = -\nabla\phi$$

$$\mathbf{B}(t,\mathbf{x},z) = \mathbf{B}_0 + \nabla\psi$$

$$\boxed{\lim_{z\to-\infty} \mathbf{B}_1 = \mathbf{0}}$$

What has been the traditional approach to study weakly NL RT?[15-18]

- The Fourier representation is used to care of the constraints and the boundary conditions:

$$\xi(t,\mathbf{x}) = \sum_{n\in\mathbb{Z}^+} \epsilon^n\, \widehat{\xi}_n(t)\, \cos(n\mathbf{k}\cdot\mathbf{x}) \qquad \phi(t,\mathbf{x},z) = \sum_{n\in\mathbb{Z}^+} \epsilon^n \widehat{\phi}_n(t) \cos(n\mathbf{k}\cdot\mathbf{x}) e^{-nkz} \qquad \psi(t,\mathbf{x},z) = \sum_{n\in\mathbb{Z}^+} \epsilon^n \widehat{\psi}_n(t) \sin(n\mathbf{k}\cdot\mathbf{x}) e^{nkz}$$

- The small parameter $\epsilon \ll 1$ serves as an asymptotic parameter for the perturbation series.

- For MRT, the boundary condition for the fluid—vacuum interface provides an equation for $\widehat{\psi}_n(t)$.

$$\boxed{\mathbf{B}|_{z=\xi} \cdot \boldsymbol{\nabla}(z-\xi) = 0} \quad \Rightarrow \quad \widehat{\psi}_n(t) = \widehat{\psi}_n[\widehat{\xi}_n(t)]$$

- Once the magnetic pressure is obtained, we obtain the dynamical equations for $\widehat{\xi}_n(t)$ and $\widehat{\phi}_n(t)$.

$$\boxed{\left[\partial_t \xi + \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}(z-\xi)\right]_{z=\xi} = 0}$$

$$\boxed{\rho\left[\frac{\partial}{\partial t}\phi - \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 - gz\right]_{z=\xi} = \frac{1}{8\pi}\left|\mathbf{B}_0 + \boldsymbol{\nabla}\psi\right|^2_{z=\xi}}$$

$$\Rightarrow \quad \boxed{\begin{aligned} \frac{\partial}{\partial t}\widehat{\xi}_n(t) &= \dots \\ \frac{\partial}{\partial t}\widehat{\phi}_n(t) &= \dots \end{aligned}}$$

The resulting equations are then solved...

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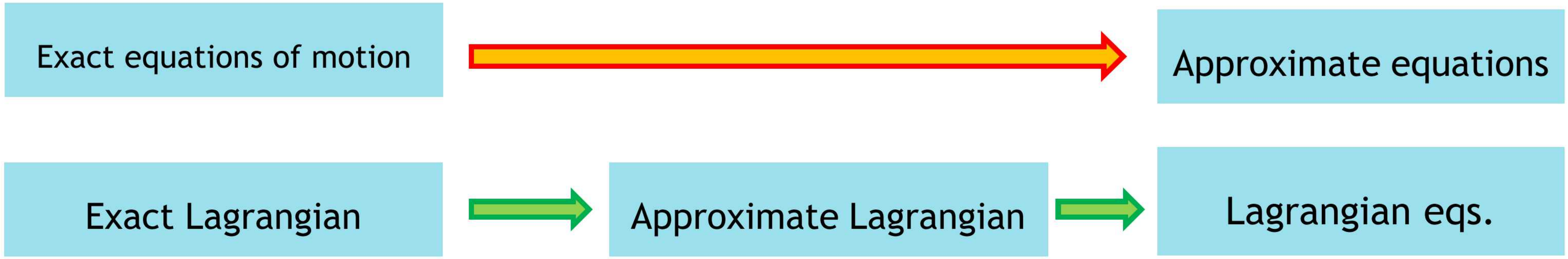
[15] J. W. Jacobs and I. Catton, J. Fluid Mech. **187**, 329 (1988).
[16] R. L. Ingraham, Proc. Phys. Soc. B **67**, 748 (2002).
[17] H.-Y. Guo, et al., Chinese Phys. Lett. **34**, 045201 (2017).
[18] L.-F. Wang, et al., Phys. Plasmas **21**, 122710 (2014).

Variational principles can provide an alternative procedure to study nonlinear MRT.

- **The traditional approach can lead to some problems.**
    - Approximating equations without care can break certain invariants of the system, e.g., energy conservation.
    - When lots of equations are obtained, it has hard to physically interpret the terms appearing in the equations.
- **One can gain insights to the problem by using variational principles.**[19]
    - All information about the physical system is contained in a single object: the **Lagrangian**.
    - Lagrangians have other good benefits: simpler calculations, good integrators,…

The two approaches can sometimes lead to different equations!



Traditional approach

Exact equations of motion → Approximate equations

Procedure based on variational principles

Exact Lagrangian → Approximate Lagrangian → Lagrangian eqs.

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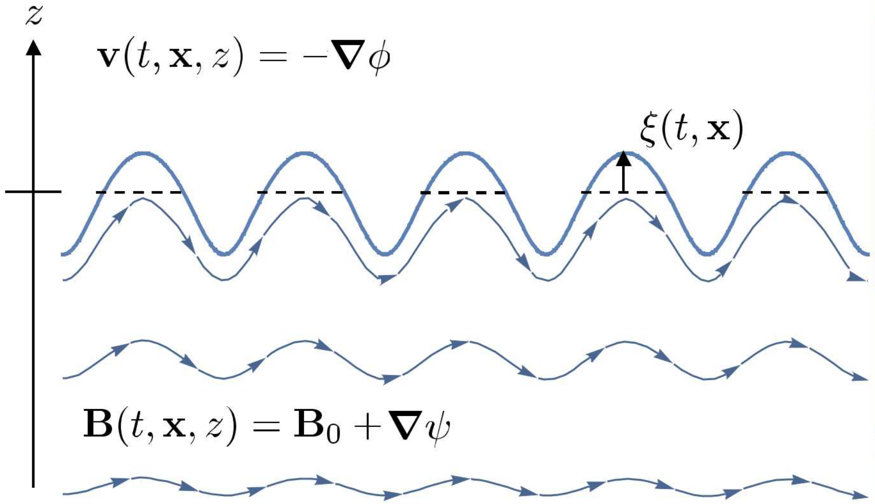
[19] D. E. Ruiz, *A Geometric Theory of Waves and Its Applications to Plasma Physics*, Princeton University, 2017.

3. *Hamiltonian theory for the weakly-nonlinear MRT instability*

The MRT instability is a Hamiltonian system !

- Based on a well-known variational principle (VP) in quantum hydrodynamics, I found a VP for the fully nonlinear single-surface MRT problem.

$$\Lambda = \int_{t_1}^{t_2} L[\xi, \phi, \psi] \, \mathrm{d}t \qquad\qquad L \doteq L_{\mathrm{fluid}}[\xi, \phi] + L_{\mathrm{B}}[\xi, \psi]$$



- The Lagrangian is separated into a fluid and a magnetic component.

$$L_{\mathrm{fluid}} \doteq \int_D \int_{\xi}^{+\infty} \rho \left[\frac{\partial}{\partial t}\phi - \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 - gz \right] \mathrm{d}z \, \mathrm{d}^2\mathbf{x}$$

$$L_{\mathrm{fluid}} \doteq \int_D \int_{-\infty}^{+\infty} \rho \left[\frac{\partial}{\partial t}\phi - \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 - gz \right] \Theta(z - \xi) \, \mathrm{d}z \, \mathrm{d}^2\mathbf{x}$$

$$L_{\mathrm{B}} \doteq \frac{1}{8\pi} \int_D \int_{-\infty}^{\xi} |\mathbf{B}_0 + \boldsymbol{\nabla}\psi|^2 \, \mathrm{d}z \, \mathrm{d}^2\mathbf{x}$$

$$L_{\mathrm{B}} \doteq \frac{1}{8\pi} \int_D \int_{-\infty}^{+\infty} |\mathbf{B}_0 + \boldsymbol{\nabla}\psi|^2 \, \Theta(\xi - z) \, \mathrm{d}z \, \mathrm{d}^2\mathbf{x}$$

- The coupling between the fluid and the magnetic field comes from the surface perturbation $\xi(t, \mathbf{x})$ appearing in the integration boundaries. This is unique to MRT.

This variational principle leads to the correct equations.

- Varying the action with respect to the flow potential leads to

$$\delta\phi: \quad \partial_t\Theta(z-\xi) - \nabla\cdot\left[\nabla\phi\,\Theta(z-\xi)\right] = 0$$

$$\boxed{\left[\partial_t\xi + \nabla\phi\cdot\nabla(z-\xi)\right]_{z=\xi} = 0}$$

$$\boxed{\nabla^2\phi = 0}$$

Inside the fluid

- Varying the action with respect to the magnetic potential gives

$$\delta\psi: \quad \nabla\cdot\left[(\mathbf{B}_0 + \nabla\psi)\,\Theta(\xi - z)\right] = 0$$

$$\boxed{\mathbf{B}|_{z=\xi}\cdot\nabla(z-\xi) = 0}$$

$$\boxed{\nabla^2\psi = 0}$$

In vacuum

- Finally varying the action with respect to the surface perturbation gives

$$\delta\xi: \quad 0 = -\int_{-\infty}^{\infty}\left\{\rho\left[\frac{\partial}{\partial t}\phi - \frac{1}{2}(\nabla\phi)^2 - gz\right] + \frac{1}{8\pi}\left|\mathbf{B}_0 + \nabla\psi\right|^2\right\}\delta(\xi - z)\mathrm{d}z \Rightarrow \boxed{\rho\left[\frac{\partial}{\partial t}\phi - \frac{1}{2}(\nabla\phi)^2 - gz\right]_{z=\xi} = \frac{1}{8\pi}\left|\mathbf{B}_0 + \nabla\psi\right|^2_{z=\xi}}$$

where we used $\delta\Theta(z-\xi) = -\delta(z-\xi)\delta\xi$.

All the information of the system is encapsulated inside the Lagrangian.

We can rewrite the problem as a Hamiltonian system.

- We integrate by parts the $\partial_t \phi$ term in the action. We can rewrite the VP as a Hamiltonian system:[19-21]

$$L \doteq \int_D \Phi \, \partial_t \xi \, \mathrm{d}^2\mathbf{x} - H$$

where $\Phi(t, \mathbf{x}) \doteq \phi(t, \mathbf{x}, z = \xi(t, \mathbf{x}))$ and

Total Hamiltonian	$H \doteq H_{\mathrm{kin}} + H_{\mathrm{g}} + H_{\mathrm{B}},$
Kinetic Hamiltonian	$H_{\mathrm{kin}} \doteq \frac{1}{2} \int_D \int_{-\infty}^{\infty} (\boldsymbol{\nabla}\phi)^2 \Theta(z - \xi) \, \mathrm{d}z \, \mathrm{d}^2\mathbf{x},$
Gravitational Hamiltonian	$H_{\mathrm{g}} \doteq \int_D \int_{-\infty}^{\infty} gz \, \Theta(z - \xi) \, \mathrm{d}z \, \mathrm{d}^2\mathbf{x},$
Magnetic Hamiltonian	$H_{\mathrm{B}} \doteq -\frac{1}{8\pi\rho} \int_D \int_{-\infty}^{\infty} \lvert \mathbf{B}_0 + \boldsymbol{\nabla}\psi \rvert^2 \, \Theta(\xi - z) \, \mathrm{d}z \, \mathrm{d}^2\mathbf{x}.$

In classical mechanics, the phase-space Lagrangian for a point-particle is

$$L = \mathbf{P} \cdot \dot{\mathbf{X}} - H(t, \mathbf{X}, \mathbf{P}).$$

Question 1: Can we draw analogies between point particles and MRT?

Question 2: Do Rayleightons exist?

- **Remember:** only $\Phi(t, x)$ and $\xi(t, x)$ are dynamical variables. $\psi(t, x)$ is only a constraint.

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[19] V. E. Zakharov, Sov. Phys. JETP **9**, 86 (1968).

[20] E. A. Kuznetsov and P. M. Lushnikov, JETP **81**, 332 (1995).

[21] M. Berning and A. M. Rubenchik, Phys. Fluids **10**, 1564 (1998).

To obtain a VP for weakly NL MRT, we write the fields in the Fourier representation and insert them into the Lagrangian.

- Taking into account the required boundary conditions, we write the fields in terms of Fourier components:

$$\xi(t,\mathbf{x}) = \sum_{n\in\mathbb{Z}^+} \epsilon^n\, \widehat{\xi}_n(t)\ \cos(n\mathbf{k}\cdot\mathbf{x}), \qquad \Phi(t,\mathbf{x}) = \sum_{n\in\mathbb{Z}^+} \epsilon^n\, \widehat{\Phi}_n(t)\ \cos(n\mathbf{k}\cdot\mathbf{x}),$$

$$\phi(t,\mathbf{x},z) = \sum_{n\in\mathbb{Z}^+} \epsilon^n \widehat{\phi}_n(t)\cos(n\mathbf{k}\cdot\mathbf{x})e^{-nkz}, \qquad \psi(t,\mathbf{x},z) = \sum_{n\in\mathbb{Z}^+} \epsilon^n \widehat{\psi}_n(t)\sin(n\mathbf{k}\cdot\mathbf{x})e^{nkz}.$$

- When inserting these into the Lagrangian and integrating, only the non-oscillating terms survive. Easy!

- The general expression that one obtains is

$$\boxed{L = \sum_{n\in\mathbb{Z}^+} \epsilon^{2n}\, \widehat{\Phi}_n\, \frac{\mathrm{d}\widehat{\xi}_n}{\mathrm{d}t} - H\left(t, \widehat{\xi}_n, \widehat{\Phi}_n\right).}$$

- The Fourier components $(\widehat{\xi}_n, \widehat{\Phi}_n)$ act as phase-space coordinates of the Rayleighton.

For weakly NL MRT, it all boils down to the order of accuracy in which one can calculate the Hamiltonian.

Single-harmonic linear MRT theory

- To obtain the linear theory, one needs to calculate the Hamiltonian up to $O(\varepsilon^2)$. The result is the following.

$$L_{\text{linear}} = \epsilon^2 \widehat{\Phi}_1 \frac{\mathrm{d}\widehat{\xi}_1}{\mathrm{d}t} - \epsilon^2 H[\widehat{\xi}_1, \widehat{\Phi}_1]$$

$$H[\widehat{\xi}_1, \widehat{\Phi}_1] = \underbrace{\frac{k}{2}\widehat{\Phi}_1^2}_{\textit{Kinetic term}\ \ \frac{\mathbf{P}^2}{2m}} \underbrace{- \frac{1}{2}\left(g - \frac{(\mathbf{k}\cdot\mathbf{v}_A)^2}{k}\right)\widehat{\xi}_1^2}_{\textit{Potential energy}\ \ V(t,\mathbf{X})}$$

- Here $\mathbf{v}_A \doteq \mathbf{B}_0/\sqrt{4\pi\rho}$ is the Alfvén velocity.

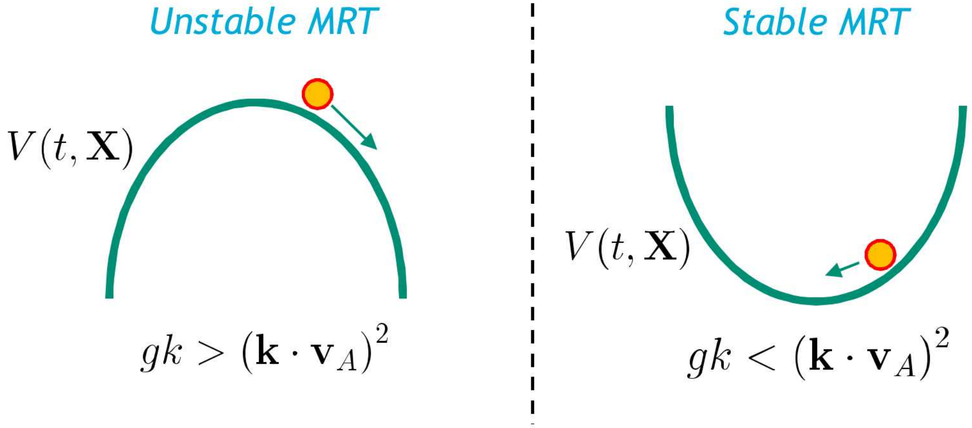
- The equations of motion are simply Hamilton's equations.

$$\delta\widehat{\Phi}_1 : \quad \frac{\mathrm{d}}{\mathrm{d}t}\widehat{\xi}_1 = k\widehat{\Phi}_1$$

$$\delta\widehat{\xi}_1 : \quad \frac{\mathrm{d}}{\mathrm{d}t}\widehat{\Phi}_1 = \left(g - \frac{(\mathbf{k}\cdot\mathbf{v}_A)^2}{k}\right)\widehat{\xi}_1$$

$$\boxed{\frac{\mathrm{d}^2}{\mathrm{d}t^2}\widehat{\xi}_1 - \underbrace{\left[gk - (\mathbf{k}\cdot\mathbf{v}_A)^2\right]^2}_{\gamma_{2\mathbf{k}}^2(t)}\widehat{\xi}_1 = 0}$$

Well-known equation for linear MRT

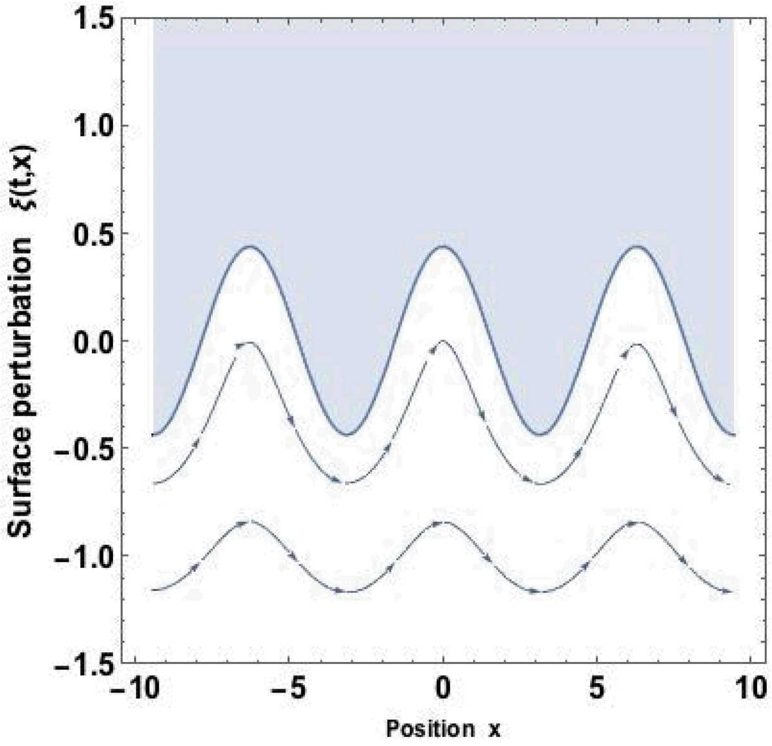


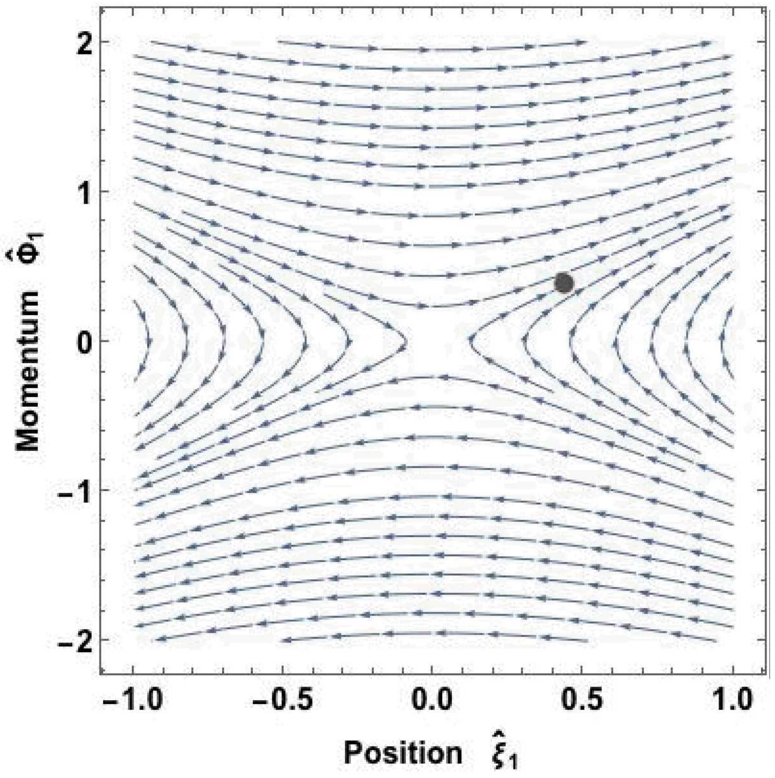
The sign of the potential determines whether the motion is stable or unstable. The topology of the phase-space plots changes.

- As a reminder, the linear Hamiltonian is $H[\widehat{\xi}_1, \widehat{\Phi}_1] = \frac{k}{2}\widehat{\Phi}_1^2 - \frac{1}{2}\left(g - \frac{(\mathbf{k}\cdot\mathbf{v}_A)^2}{k}\right)\widehat{\xi}_1^2$.

$$\sigma \doteq \frac{(\mathbf{k}\cdot\mathbf{v}_A)^2}{gk} < 1$$

Unstable MRT



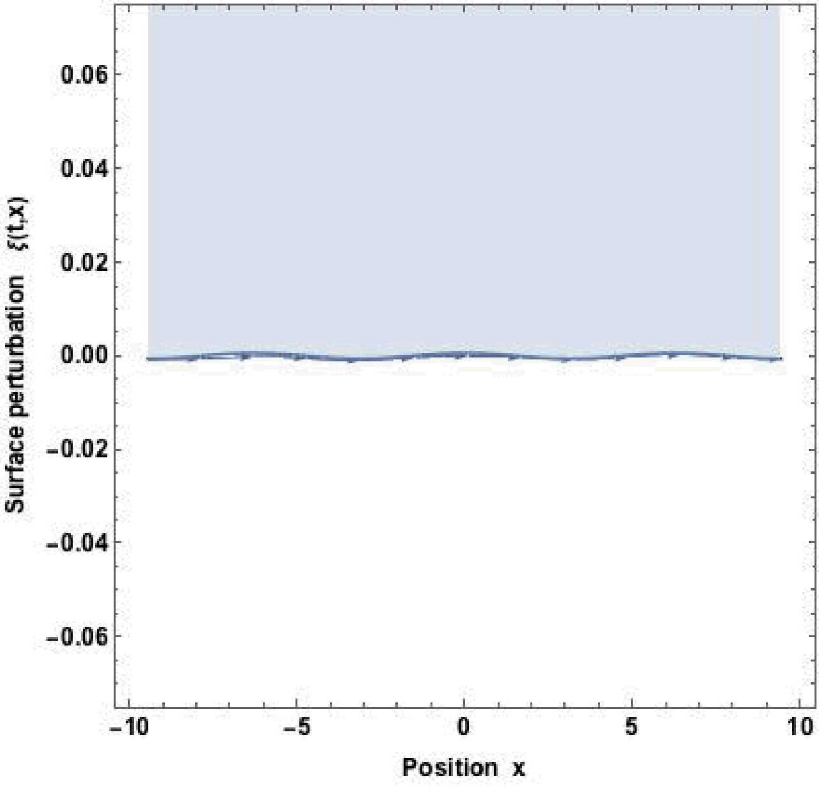


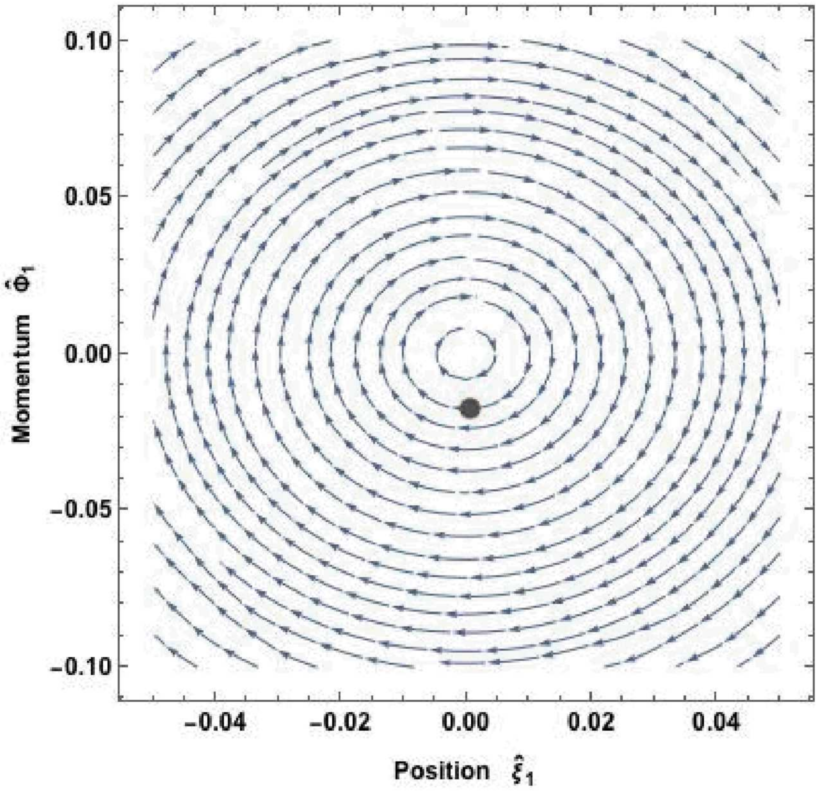
The sign of the potential determines whether the motion is stable or unstable.

- As a reminder, the linear Hamiltonian is $H[\widehat{\xi}_1, \widehat{\Phi}_1] = \frac{k}{2}\widehat{\Phi}_1^2 - \frac{1}{2}\left(g - \frac{(\mathbf{k}\cdot\mathbf{v}_A)^2}{k}\right)\widehat{\xi}_1^2$.

$$\sigma \doteq \frac{(\mathbf{k}\cdot\mathbf{v}_A)^2}{gk} > 1$$

Stable MRT





Double-harmonic weakly nonlinear MRT theory

- To account for two harmonics, one needs to calculate the Hamiltonian up to $O(\varepsilon^4)$. The end result is[22]

$$L = \sum_{n=1}^{2} \epsilon^{2n}\, \widehat{\Phi}_n \frac{\mathrm{d}\widehat{\xi}_n}{\mathrm{d}t} - H\big(t, \widehat{\xi}_1, \widehat{\xi}_2, \widehat{\Phi}_1, \widehat{\Phi}_2\big),$$

$$H = \sum_{n=1}^{2} \epsilon^{2n} \left(\frac{nk}{2} \widehat{\Phi}_n^2 - \frac{\gamma_{n\mathbf{k}}^2}{2nk}\, \widehat{\xi}_n^2 \right) - \epsilon^4 \frac{k^3}{8}\, \widehat{\Phi}_1^2\, \widehat{\xi}_1^2 + \epsilon^4 \frac{k^2}{2}\, \widehat{\Phi}_1^2\, \widehat{\xi}_2 - \frac{\epsilon^4}{2} |\mathbf{k}\cdot\mathbf{v}_A|^2\, \widehat{\xi}_1^2 \widehat{\xi}_2 - \epsilon^4 \frac{k}{8} |\mathbf{k}\cdot\mathbf{v}_A|^2\, \widehat{\xi}_1^4.$$

- The resulting equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\xi}_1 = k\widehat{\Phi}_1 - \epsilon^2 \frac{k^3}{4} \widehat{\Phi}_1 \widehat{\xi}_1^2 + \epsilon^2 k^2 \widehat{\Phi}_1 \widehat{\xi}_2,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\Phi}_1 = \frac{\gamma_{\mathbf{k}}^2(t)}{k} \widehat{\xi}_1 + \epsilon^2 \frac{k^3}{4} \widehat{\xi}_1 \widehat{\Phi}_1^2 + \epsilon^2 |\mathbf{k}\cdot\mathbf{v}_A|^2 \widehat{\xi}_1 \widehat{\xi}_2 + \epsilon^2 \frac{k}{2} (\mathbf{k}\cdot\mathbf{v}_A)^2 \widehat{\xi}_1^3,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\xi}_2 = 2k\widehat{\Phi}_2,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\Phi}_2 = \frac{\gamma_{2\mathbf{k}}^2(t)}{2k} \widehat{\xi}_2 - \frac{k^2}{2} \widehat{\Phi}_1^2 + \frac{1}{2} (\mathbf{k}\cdot\mathbf{v}_A)^2 \widehat{\xi}_1^2.$$

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[22] D. E. Ruiz, "On a variational formulation of the weakly nonlinear magnetic Rayleigh—Taylor instability," in preparation.

One can learn a lot about the temporal dynamics by simply looking at the Hamiltonian.

- To account for two harmonics, one needs to calculate the Hamiltonian up to $O(\varepsilon^4)$.

$$H = \underbrace{\sum_{n=1}^{2} \epsilon^{2n} \left(\frac{nk}{2} \widehat{\Phi}_n^2 - \frac{\gamma_{n\mathbf{k}}^2}{2nk} \widehat{\xi}_n^2 \right)}_{\text{Linear dynamics}} \underbrace{- \epsilon^4 \frac{k^3}{8} \widehat{\Phi}_1^2 \widehat{\xi}_1^2}_{\text{Nonlinear self-coupling of 1st harmonic}} \underbrace{+ \epsilon^4 \frac{k^2}{2} \widehat{\Phi}_1^2 \widehat{\xi}_2}_{\text{Nonlinear minetic coupling}} \underbrace{- \frac{\epsilon^4}{2} |\mathbf{k} \cdot \mathbf{v}_A|^2 \widehat{\xi}_1^2 \widehat{\xi}_2}_{\text{Nonlinear magnetic coupling}} \underbrace{- \epsilon^4 \frac{k}{8} |\mathbf{k} \cdot \mathbf{v}_A|^2 \widehat{\xi}_1^4 .}_{\text{Nonlinear self-coupling of the 1st harmonic}}$$

- The equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\xi}_1 = \boxed{k\widehat{\Phi}_1} \boxed{- \epsilon^2 \frac{k^3}{4} \widehat{\Phi}_1 \widehat{\xi}_1^2} + \boxed{\epsilon^2 k^2 \widehat{\Phi}_1 \widehat{\xi}_2 ,}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\xi}_2 = \boxed{2k\widehat{\Phi}_2 ,}$$

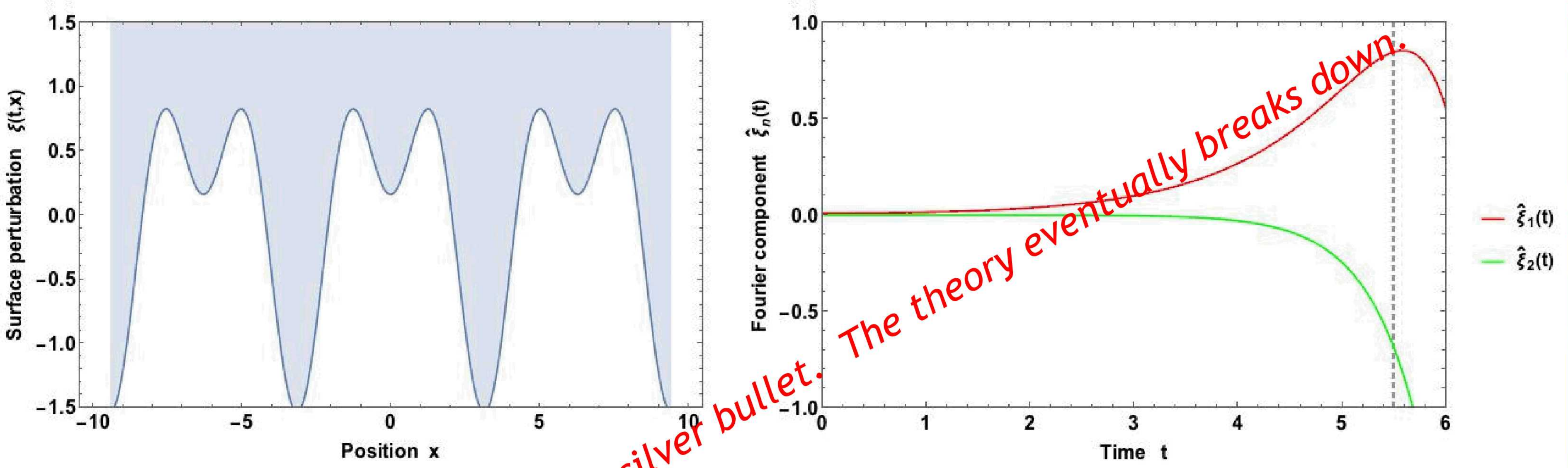
$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\Phi}_1 = \boxed{\frac{\gamma_{\mathbf{k}}^2(t)}{k} \widehat{\xi}_1} + \boxed{\epsilon^2 \frac{k^3}{4} \widehat{\xi}_1 \widehat{\Phi}_1^2} + \boxed{\epsilon^2 |\mathbf{k} \cdot \mathbf{v}_A|^2 \widehat{\xi}_1 \widehat{\xi}_2} + \boxed{\epsilon^2 \frac{k}{2} (\mathbf{k} \cdot \mathbf{v}_A)^2 \widehat{\xi}_1^3 ,}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\Phi}_2 = \boxed{\frac{\gamma_{2\mathbf{k}}^2(t)}{2k} \widehat{\xi}_2} \boxed{- \frac{k^2}{2} \widehat{\Phi}_1^2} + \boxed{\frac{1}{2} (\mathbf{k} \cdot \mathbf{v}_A)^2 \widehat{\xi}_1^2 .}$$

Examples of obtained dynamics using the double harmonic weakly nonlinear MRT theory

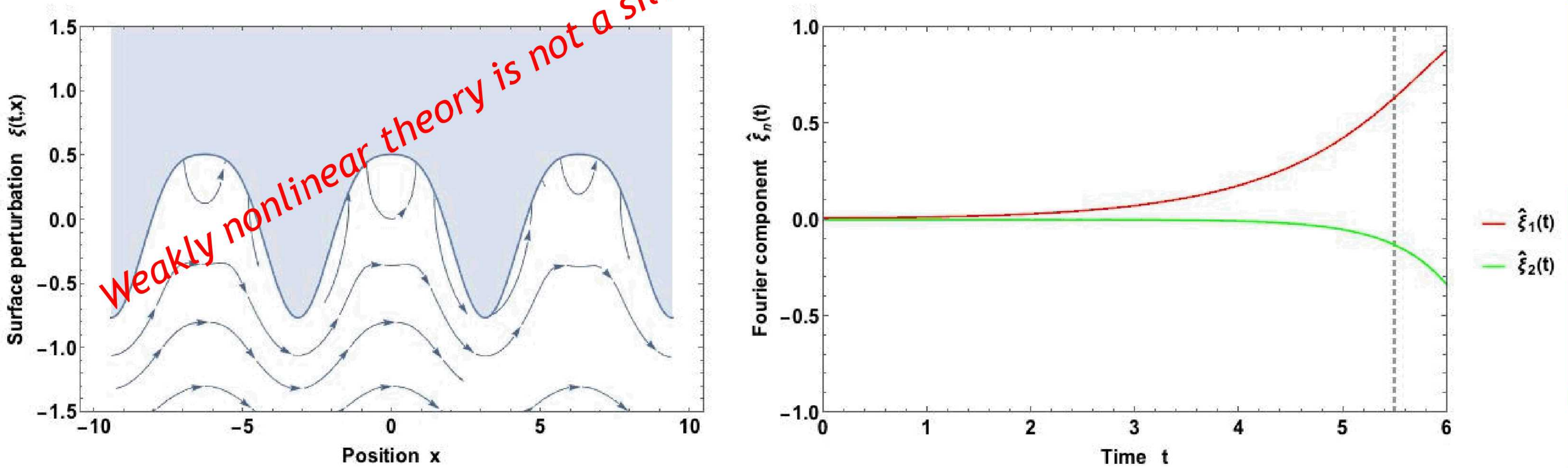
Without magnetic field

$$\sigma \doteq \frac{(\mathbf{k}\cdot\mathbf{v}_A)^2}{gk} = 0$$



With magnetic field

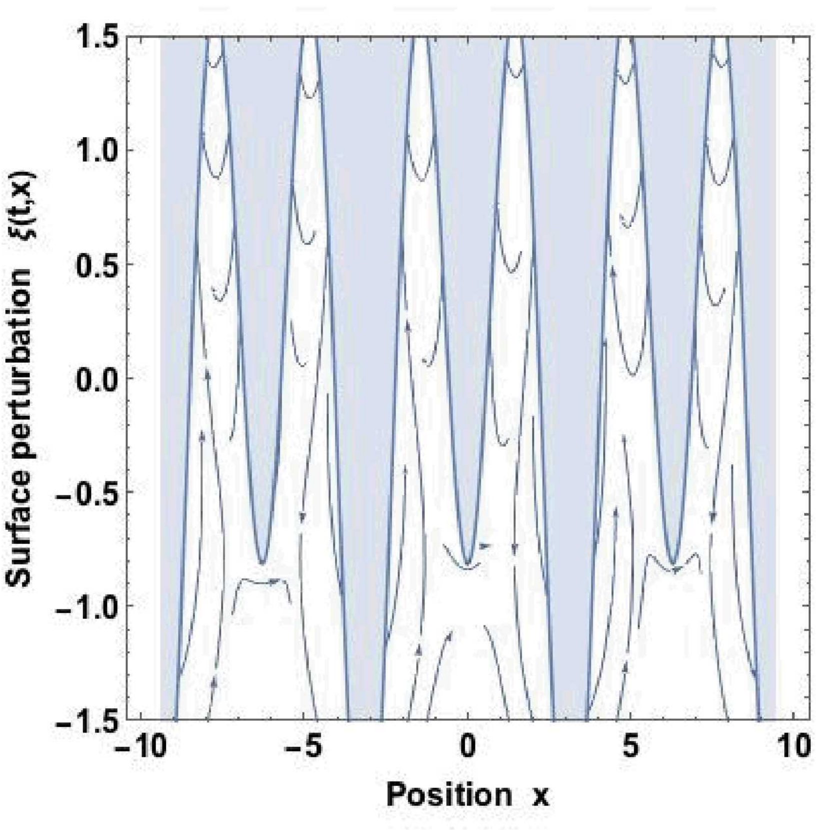
$$\sigma \doteq \frac{(\mathbf{k}\cdot\mathbf{v}_A)^2}{gk} = 0.2$$

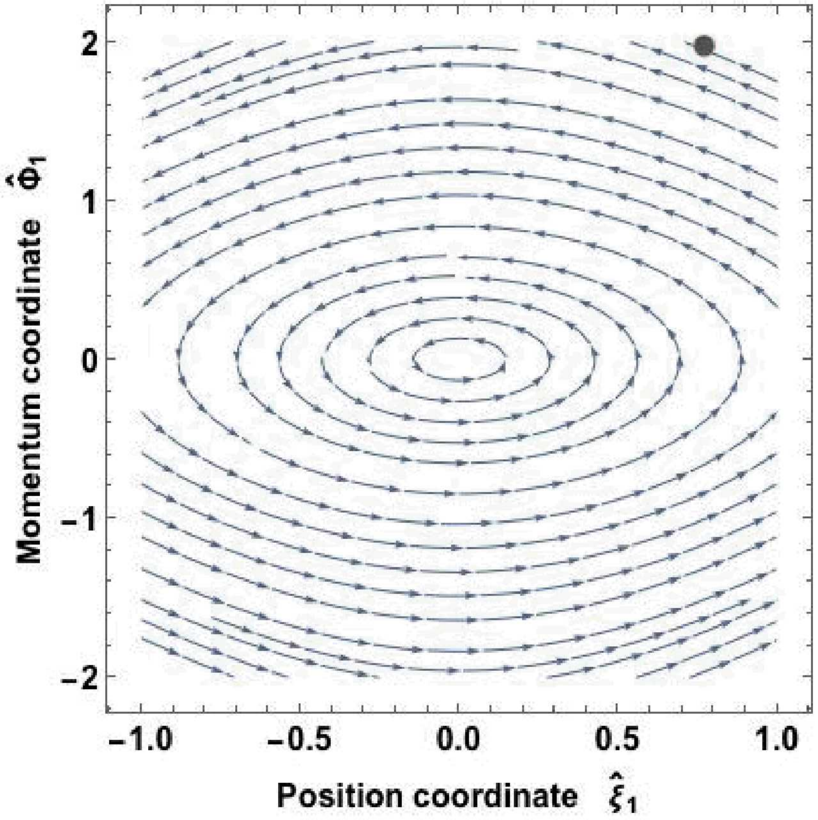


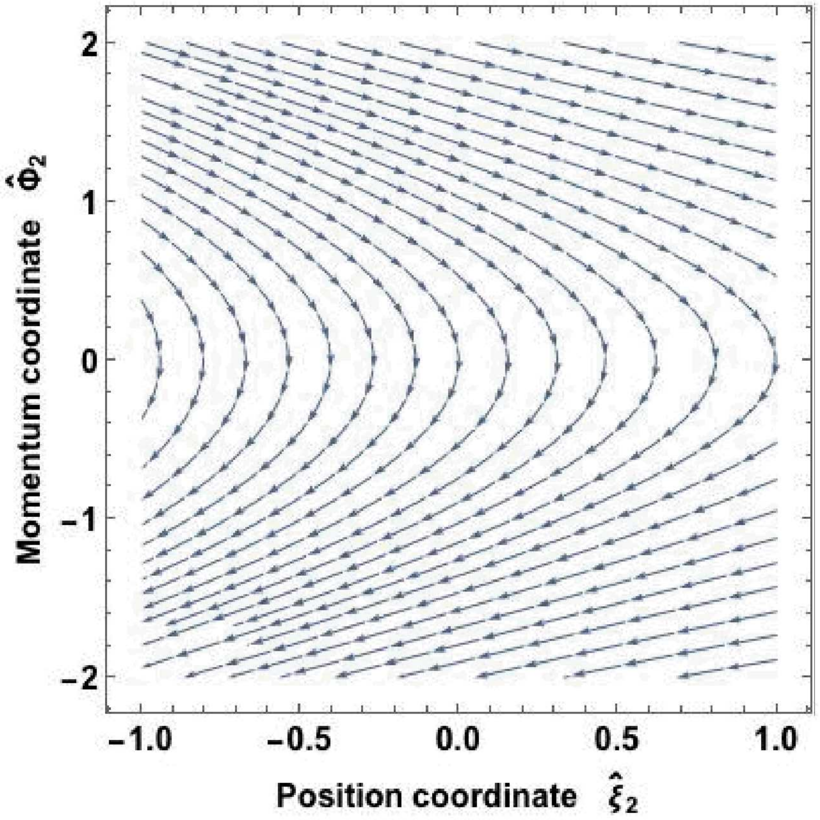
Weakly nonlinear theory is not a silver bullet. The theory eventually breaks down.

When the theory breaks down, the topology of the phase-space plots changes radically.

With magnetic field $\sigma \doteq \frac{(\mathbf{k}\cdot\mathbf{v}_A)^2}{gk} = 0.2$







Within the range of validity of the theory, we can learn about the long-term behavior of the solutions.

- The governing equations are the following:

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\xi}_1 = k\widehat{\Phi}_1 - \epsilon^2\frac{k^3}{4}\widehat{\Phi}_1\widehat{\xi}_1^2 + \epsilon^2 k^2\widehat{\Phi}_1\widehat{\xi}_2, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\widehat{\Phi}_1 = \frac{\gamma_{\mathbf{k}}^2(t)}{k}\widehat{\xi}_1 + \epsilon^2\frac{k^3}{4}\widehat{\xi}_1\widehat{\Phi}_1^2 + \epsilon^2|\mathbf{k}\cdot\mathbf{v}_A|^2\widehat{\xi}_1\widehat{\xi}_2 + \epsilon^2\frac{k}{2}(\mathbf{k}\cdot\mathbf{v}_A)^2\widehat{\xi}_1^3,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\xi}_2 = 2k\widehat{\Phi}_2, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\widehat{\Phi}_2 = \frac{\gamma_{2\mathbf{k}}^2(t)}{2k}\widehat{\xi}_2 - \frac{k^2}{2}\widehat{\Phi}_1^2 + \frac{1}{2}(\mathbf{k}\cdot\mathbf{v}_A)^2\widehat{\xi}_1^2.$$

- **Key idea:** At large times, the behavior of the solutions is dominated by the growth of the 1st harmonic.

$$\widehat{\xi}_1(t) \sim a_0 e^{\gamma_{\mathbf{k}}t} + \epsilon^2 a_2 e^{3\gamma_{\mathbf{k}}t} + \ldots, \qquad \widehat{\xi}_2(t) \sim b_0 e^{2\gamma_{\mathbf{k}}t} + \ldots \qquad \left(e^{\gamma_{2\mathbf{k}}t} \ll e^{2\gamma_{\mathbf{k}}t}\right)$$

At sufficiently large times

- Using this ansatz, we solve a system of algebraic equations. We obtain

First harmonic

$$\widehat{\xi}_1(t) \sim \widehat{\xi}_{1,\mathrm{lin}}(t)\left[1 - \epsilon^2\frac{k(kg+\gamma_{\mathbf{k}}^2)\gamma_{2\mathbf{k}}^2}{16g\gamma_{\mathbf{k}}^2}\left[\widehat{\xi}_{1,\mathrm{lin}}(t)\right]^2\right],$$

Second harmonic

$$\widehat{\xi}_2(t) \sim -\frac{\gamma_{2\mathbf{k}}^2}{4g}\left[\widehat{\xi}_{1,\mathrm{lin}}(t)\right]^2,$$

This is how the 2nd harmonic grows!

$$\widehat{\xi}_{1,\mathrm{lin}}(t) \sim \frac{1}{2}\widehat{\xi}_1(0)e^{\gamma_{\mathbf{k}}t}$$

Asymptotically dominant term of the linear solution

The asymptotic solutions compare well to numerical solutions.

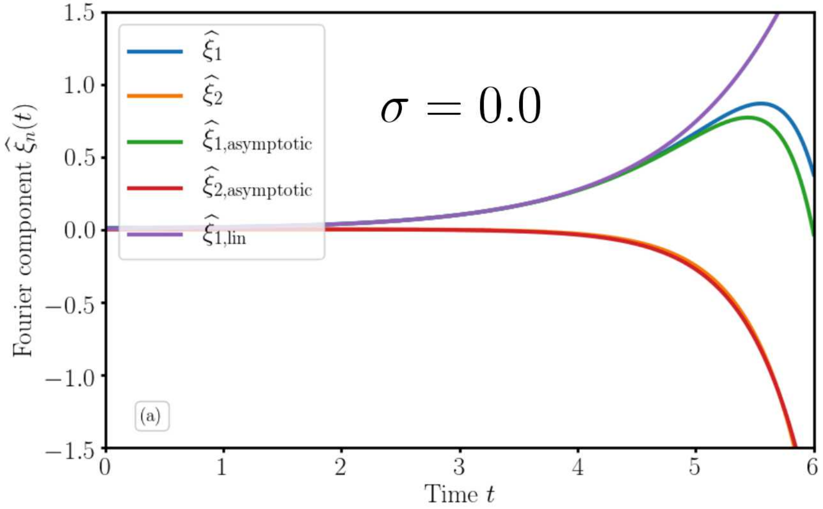
First harmonic $$\widehat{\xi}_1(t) \sim \widehat{\xi}_{1,\mathrm{lin}}(t)\left[1-\epsilon^2\frac{k(kg+\gamma_{\mathbf{k}}^2)\gamma_{2\mathbf{k}}^2}{16g\gamma_{\mathbf{k}}^2}\left[\widehat{\xi}_{1,\mathrm{lin}}(t)\right]^2\right],$$

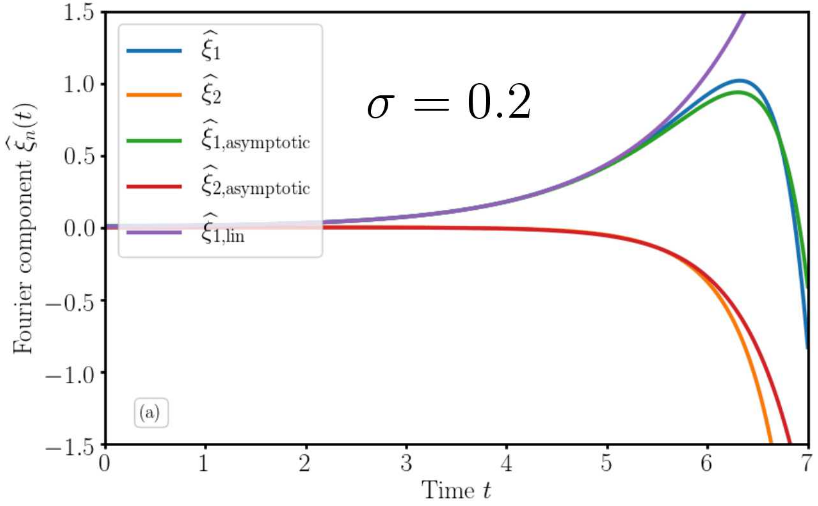
Second harmonic $$\widehat{\xi}_2(t) \sim -\frac{\gamma_{2\mathbf{k}}^2}{4g}\left[\widehat{\xi}_{1,\mathrm{lin}}(t)\right]^2,$$

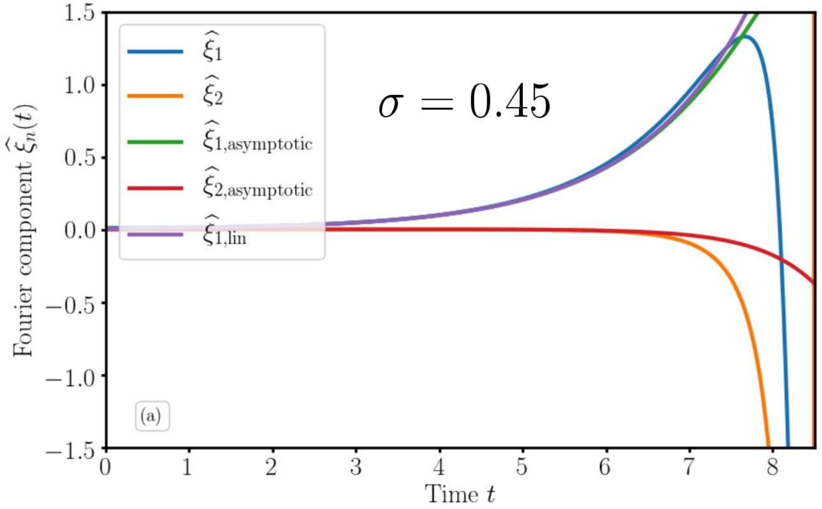
$$\widehat{\xi}_{1,\mathrm{lin}}(t) \sim \frac{1}{2}\widehat{\xi}_1(0)e^{\gamma_{\mathbf{k}}t}$$

Asymptotically dominant term of linear solution

$$\sigma \doteq \frac{(\mathbf{k}\cdot\mathbf{v}_A)^2}{gk}$$







- When $\sigma \approx 0.5$, $\gamma_{2\mathbf{k}} \approx 0$. Thus, the calculated terms for the second harmonic and the correction of the first harmonic are no longer dominant. Hence, the asymptotic solutions are not valid in this regime.

Another observation: The presence of a magnetic field can increase the saturation amplitude of the linear-growth phase.

- The saturation amplitude of the 1st harmonic can be defined as the amplitude when the fundamental mode is reduced by 10% in comparison of the linear solution.

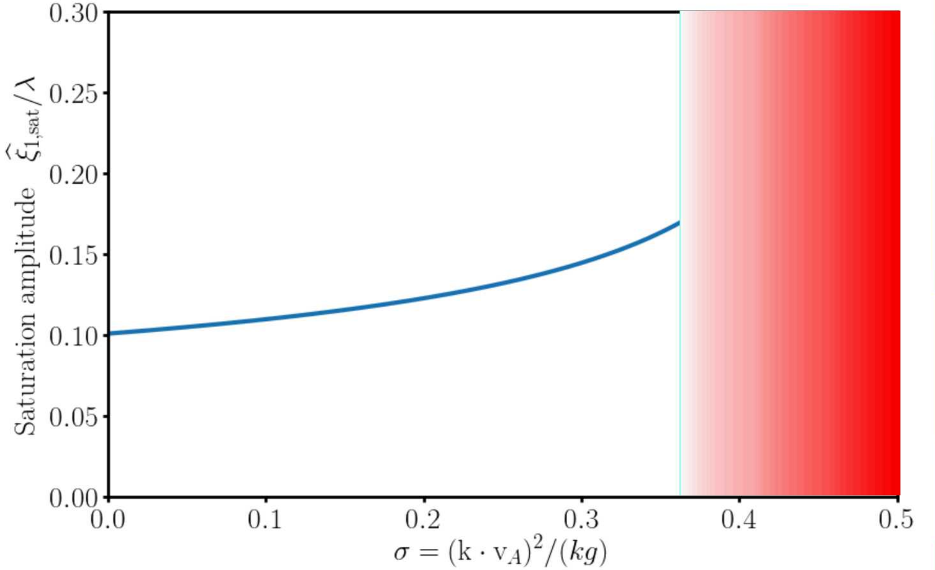
$$\frac{\widehat{\xi}_{1,\text{lin}} - \widehat{\xi}_1}{\widehat{\xi}_{1,\text{lin}}} = 0.10$$

- Using the obtained asymptotic solution of the first harmonic leads to

$$\frac{\widehat{\xi}_{1,\text{sat}}}{\lambda} = \frac{1}{\pi}\sqrt{\frac{1}{10}}\, f(\sigma) \simeq \underbrace{0.1}_{\text{Saturation value in classical RT}} f(\sigma). \qquad f(\sigma) \doteq \sqrt{\frac{1-\sigma}{(1-2\sigma)(1-\sigma/2)}}.$$

The saturation amplitude increases as a function of the magnetic field.

No longer valid



The saturation amplitude increases as a function of the magnetic field.

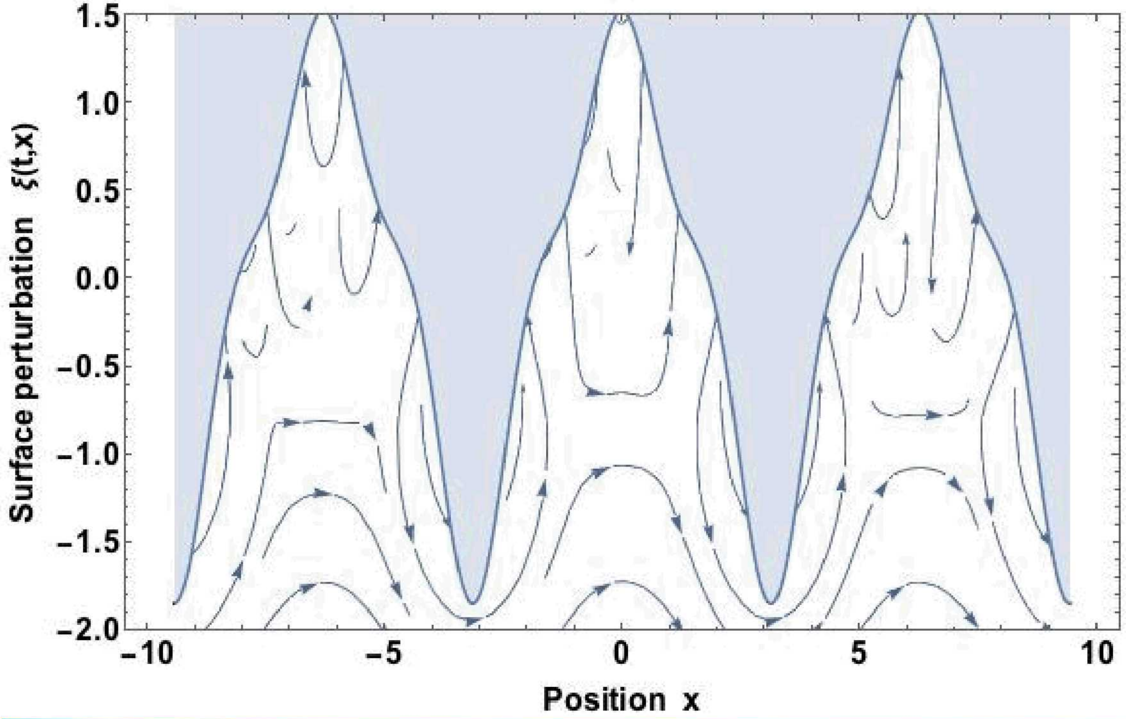
$$\sigma \doteq \frac{(\mathbf{k}\cdot\mathbf{v}_A)^2}{gk}$$

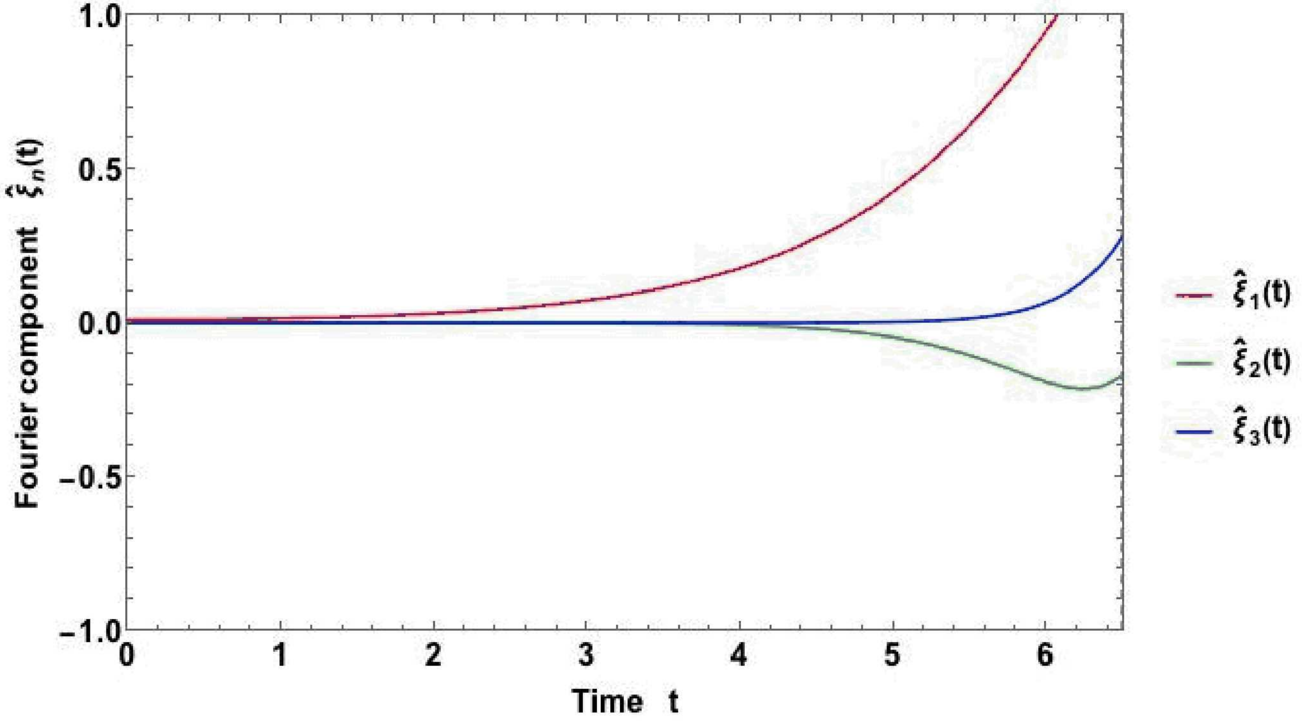
- In the limit of no magnetic fields, we recover the classical result for the RT instability.

Triple-harmonic weakly nonlinear MRT theory

- One can extend the results to account for three harmonics. The resulting $O(\varepsilon^6)$ Hamiltonian is[23]

$$
\begin{aligned}
H = \sum_{n=1}^{3} \epsilon^{2n} \left(\frac{nk}{2} \widehat{\Phi}_n^2 - \frac{\gamma_{n\mathbf{k}}^2}{2nk}\, \widehat{\xi}_n^2 \right) - \epsilon^4 \frac{k^3}{8}\, \widehat{\Phi}_1^2\, \widehat{\xi}_1^2 + \epsilon^4 \frac{k^2}{2}\, \widehat{\Phi}_1^2\, \widehat{\xi}_2 - \frac{\epsilon^4}{2} (\mathbf{k}\cdot\mathbf{v}_A)^2\, \widehat{\xi}_1^2 \widehat{\xi}_2 - \epsilon^4 \frac{k}{8} (\mathbf{k}\cdot\mathbf{v}_A)^2\, \widehat{\xi}_1^4 \\
+ 2\epsilon^6 k^2 \widehat{\xi}_3 \widehat{\Phi}_1 \widehat{\Phi}_2 + \epsilon^6 \frac{k^3}{4} \left(\widehat{\xi}_2^2 \widehat{\Phi}_1^2 + \widehat{\xi}_1 \widehat{\xi}_3 \widehat{\Phi}_1^2 - 4 \widehat{\xi}_1 \widehat{\xi}_2 \widehat{\Phi}_1 \widehat{\Phi}_2 \right) + \epsilon^6 \frac{k^4}{4} \left(\frac{1}{3} \widehat{\xi}_1^3 \widehat{\Phi}_1 \widehat{\Phi}_2 - \frac{3}{2} \widehat{\xi}_1^2 \widehat{\xi}_2 \widehat{\Phi}_1^2 \right) + \epsilon^6 \frac{11k^5}{192} \widehat{\xi}_1^4 \widehat{\Phi}_1^2 \\
- 2\epsilon^6 (\mathbf{k}\cdot\mathbf{v}_A)^2 \widehat{\xi}_1 \widehat{\xi}_2 \widehat{\xi}_3 + \epsilon^6 \frac{k}{4} (\mathbf{k}\cdot\mathbf{v}_A)^2 \left(\widehat{\xi}_1^3 \widehat{\xi}_3 - 3 \widehat{\xi}_1^2 \widehat{\xi}_2^2 \right) + \epsilon^6 \frac{7k^2}{24} (\mathbf{k}\cdot\mathbf{v}_A)^2 \widehat{\xi}_1^4 \widehat{\xi}_2 + \epsilon^6 \frac{7k^3}{96} (\mathbf{k}\cdot\mathbf{v}_A)^2 \widehat{\xi}_1^6 .
\end{aligned}
$$





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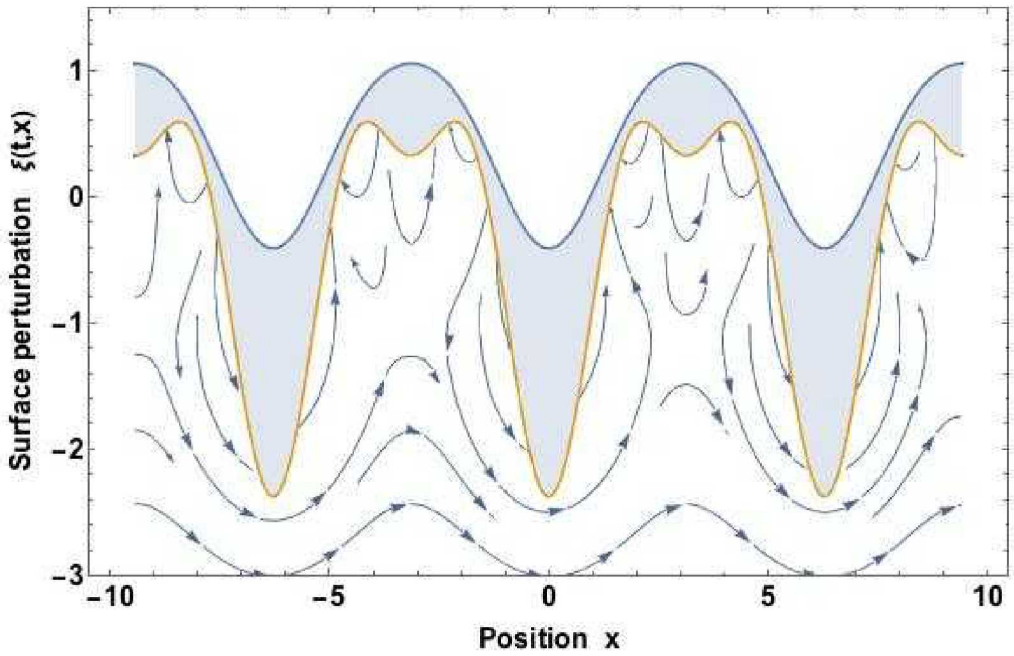
[23] Calculation done with Mathematica. S. Wolfram, The Mathematica Book (Wolfram Media/Cambridge University Press, Cambridge, United Kingdom, 2003), 5th ed.

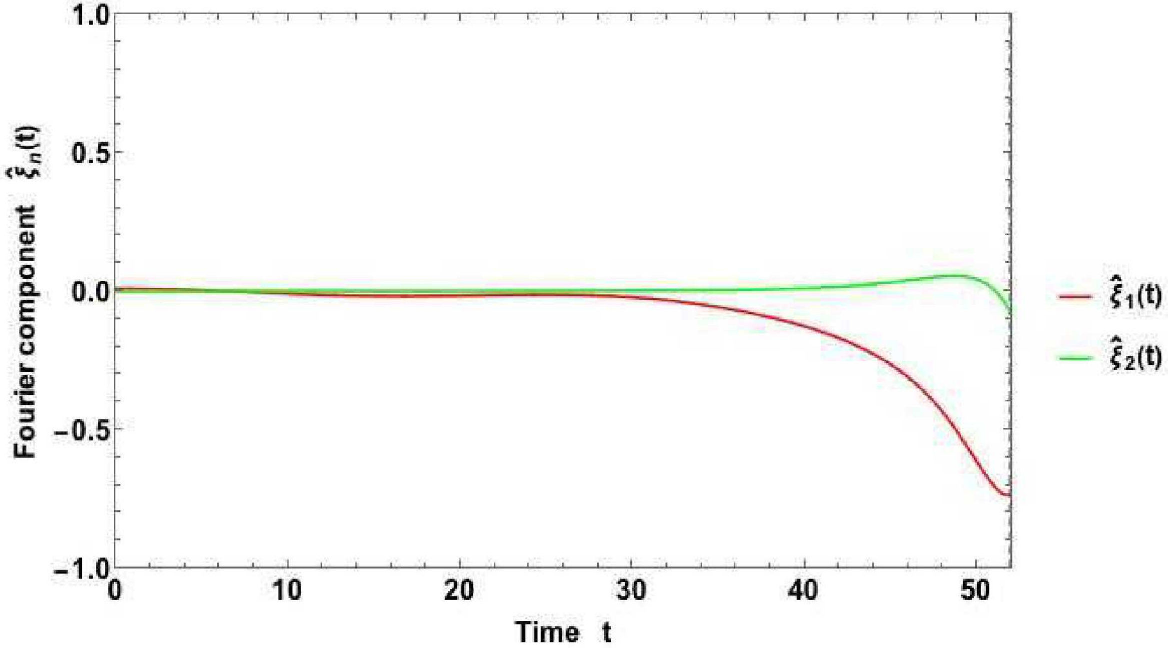
The theory can be easily extended to a finite-width planar slab.

- For a finite-width slab, the corresponding VP is given by

$$L = \int_D \int_\xi^\eta \rho \left[\frac{\partial}{\partial t}\phi - \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 - gz \right] \mathrm{d}z\, \mathrm{d}^2\mathbf{x} + \frac{1}{8\pi} \int_D \int_{-\infty}^{\xi} |\mathbf{B}_0 + \boldsymbol{\nabla}\psi|^2 \, \mathrm{d}z\, \mathrm{d}^2\mathbf{x}.$$

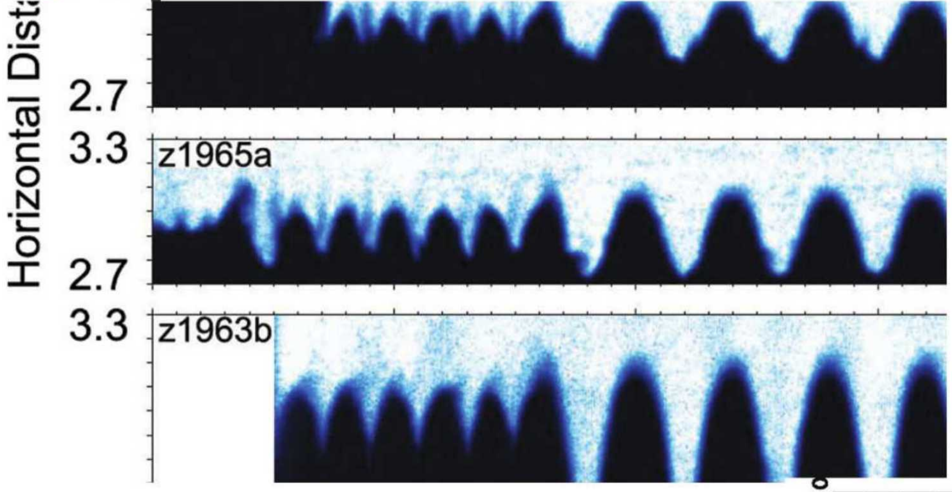
- One can repeat the same procedure to obtain the governing weakly nonlinear equations.

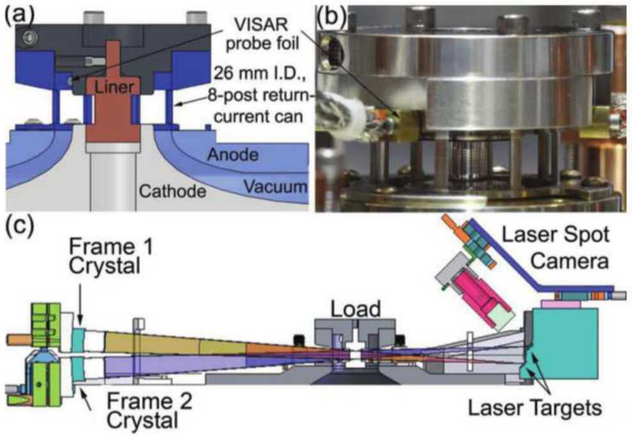


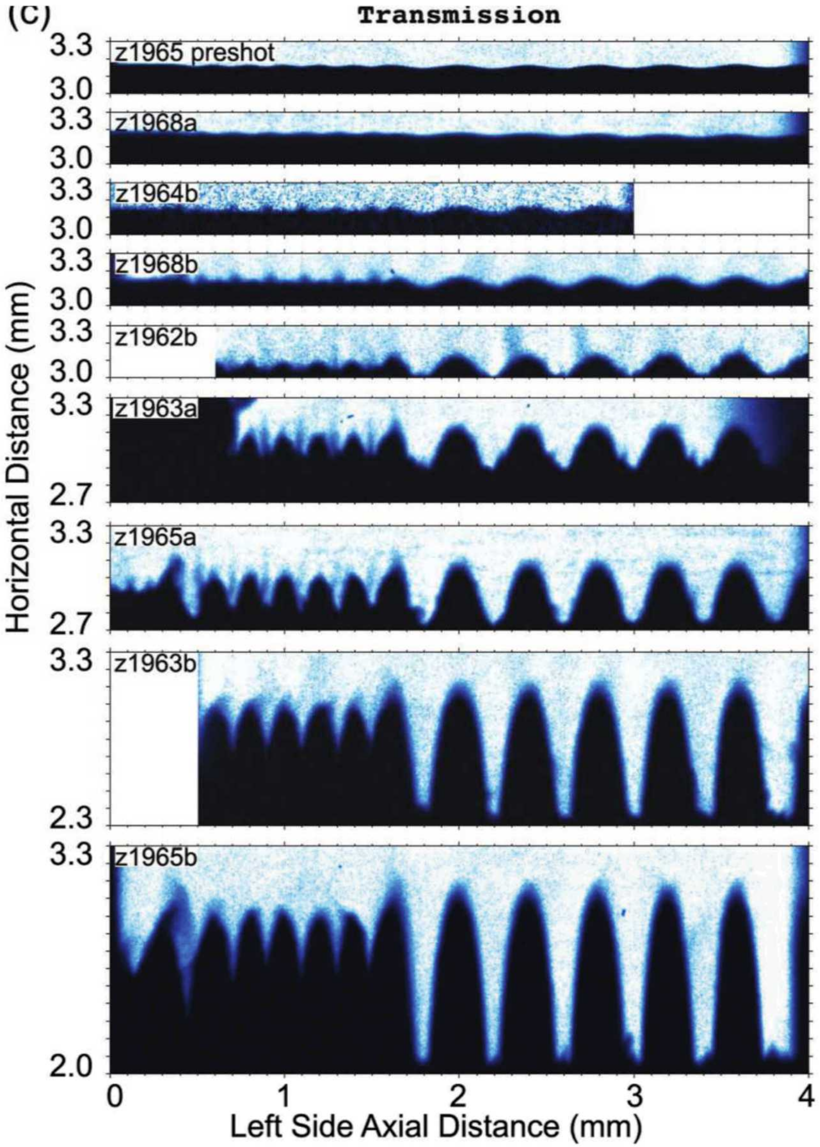


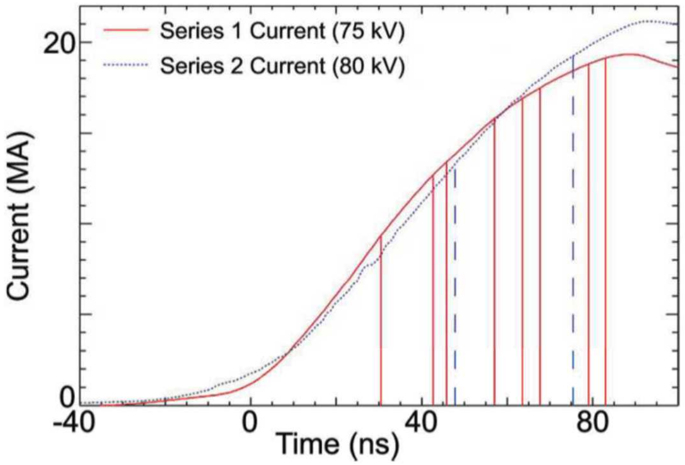
4. Comparison of weakly nonlinear theory to experiment

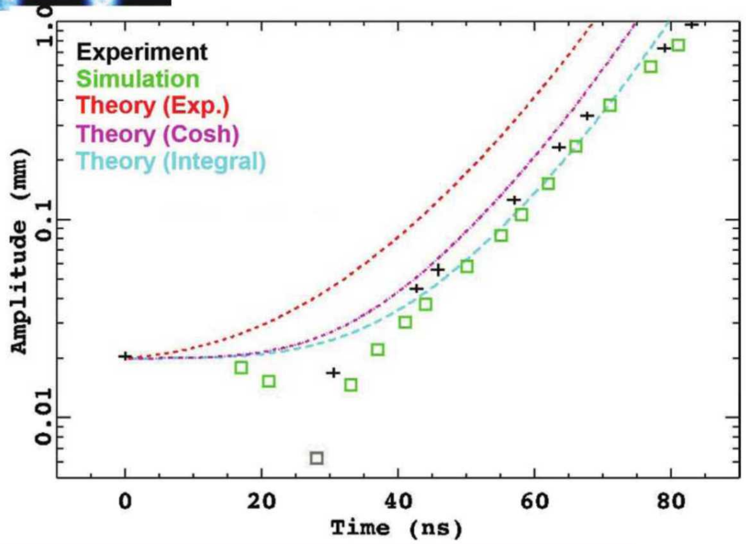
Experiments at Sandia have provided valuable data on nonlinear MRT.[2]











Question: Why does linear theory predict well the observed peak-to-valley amplitudes when the data shows nonlinear MRT behavior?

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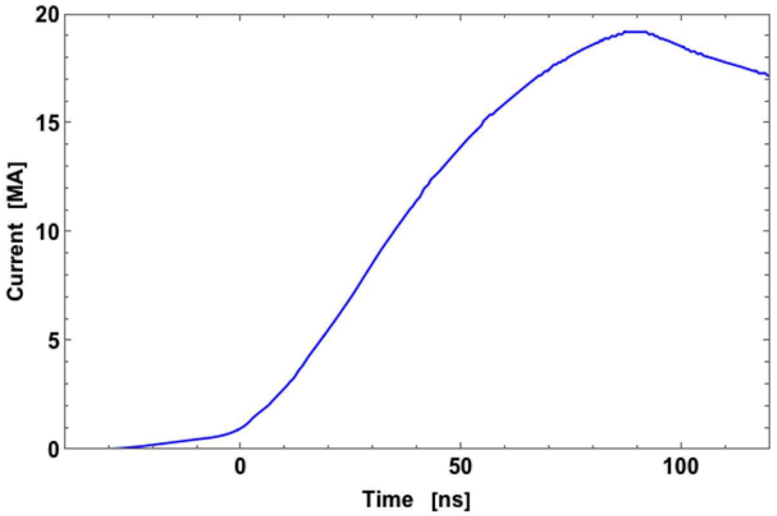
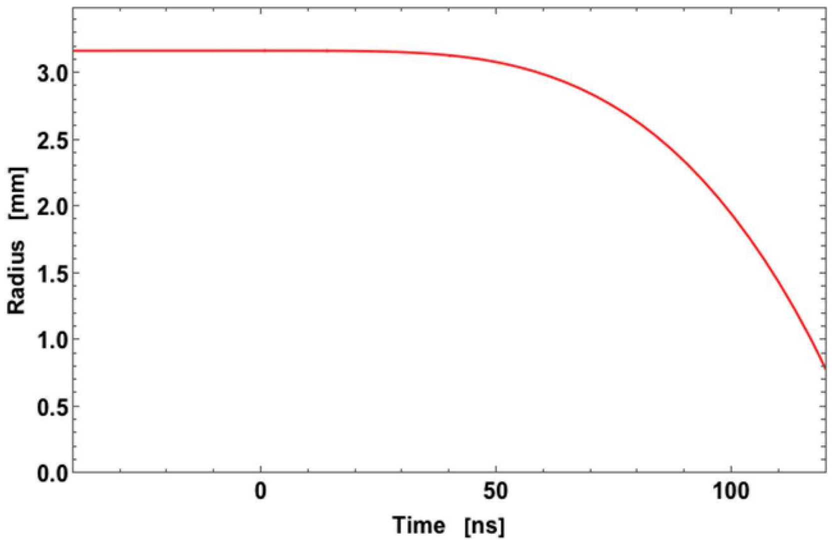
[2] D. B. Sinars, et al., Phys. Rev. Lett. **105**, 51 (2010).

To compare to experiment, we used the weakly NL, semi-infinite slab mode

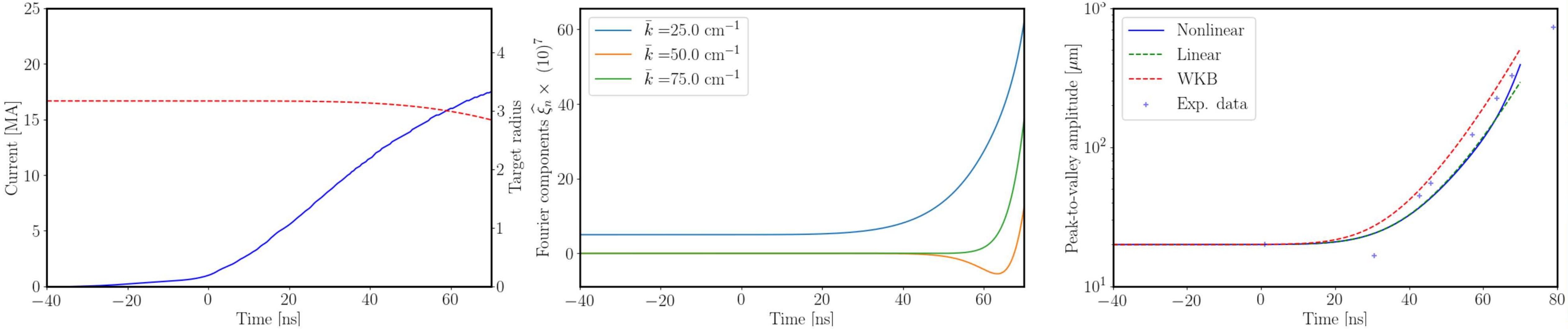
- Why can we do this? Initially, one has
  - $\dfrac{1}{kR} = \dfrac{1}{2\pi}\left(\dfrac{400\ \mu\mathrm{m}}{3.17\ \mathrm{mm}}\right) \simeq 0.02 \ll 1$ *Cylindrical effects are small. Planar is good.*
  - $\dfrac{1}{k\Delta} = \dfrac{1}{2\pi}\left(\dfrac{400\ \mu\mathrm{m}}{292\ \mu\mathrm{m}}\right) \simeq 0.21 \ll 1$ *Feedthrough is negligible. Semi-infinite slab model is good.*
  - $\dfrac{\xi}{R} = \dfrac{20\ \mu\mathrm{m}}{3.17\ \mathrm{mm}} \simeq 0.006 \ll 1$ *Perturbation incisions are small. Planar is good.*
- To obtain a time history for the liner acceleration, we took the measured current traces and fed them to a thin-shell model to obtain liner trajectory.



$$\hat{m}\frac{\mathrm{d}^2 R}{\mathrm{d}t^2} = -(2\pi R)P_{\mathrm{ext}} = -\frac{\mu_0 I^2(t)}{4\pi R}$$



The theory breaks down at a mid-time during the implosion.



- The first harmonic begins to grow at $t \approx 20$ ns when the liner begins to move.
- The second and third harmonic begin to grow at $t \approx 50$ ns and at $t \approx 60$ ns, respectively.
- The peak-to-valley amplitude observed by experiment is reproduced by the linear and weakly-nonlinear theories.
- The theory breaks down at $t \approx 65$ ns.

The theory breaks down before the data shows a strong nonlinear structure.

Preshot

T=30.5 ns

T=42.7 ns

T=45.8 ns

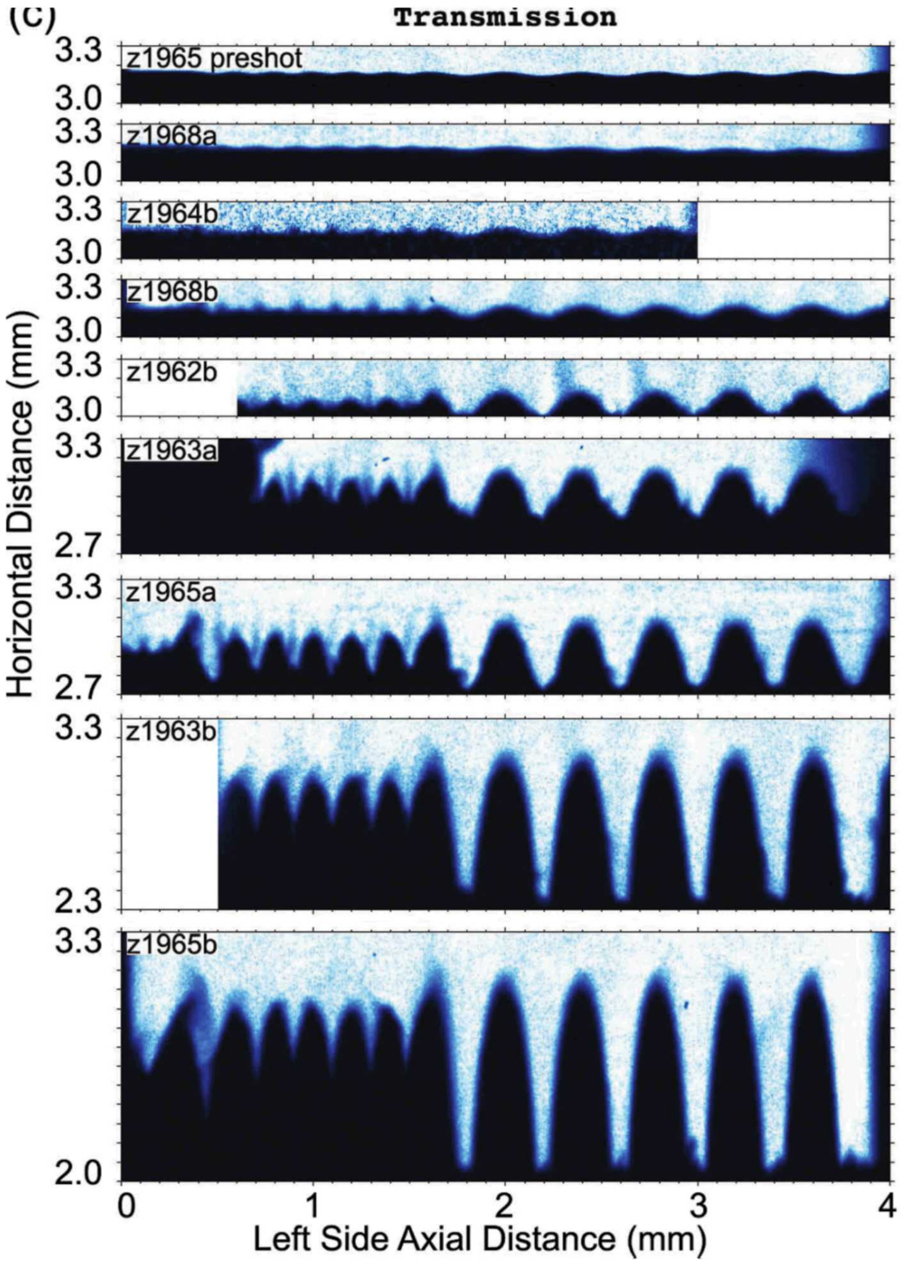
T=57.0 ns

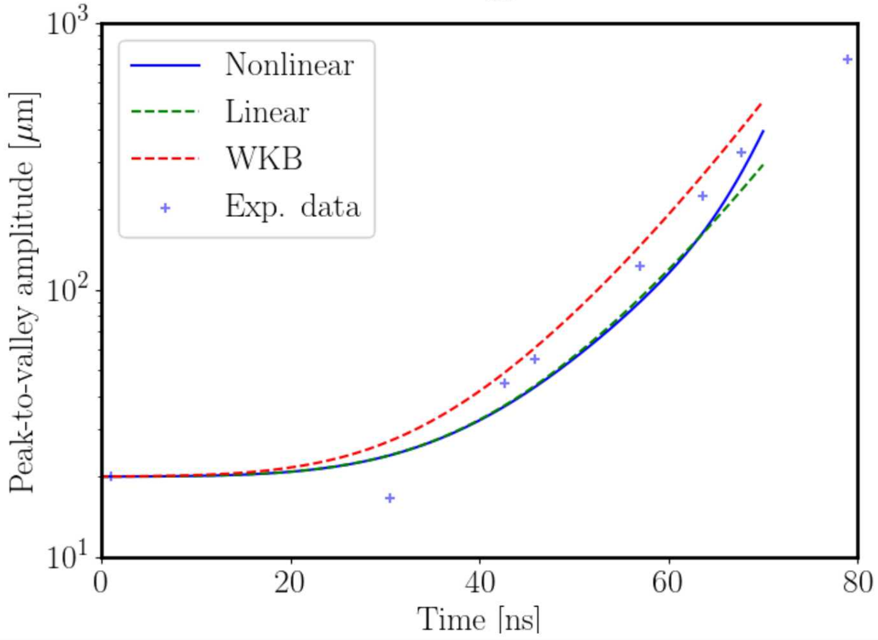
T=63.6 ns

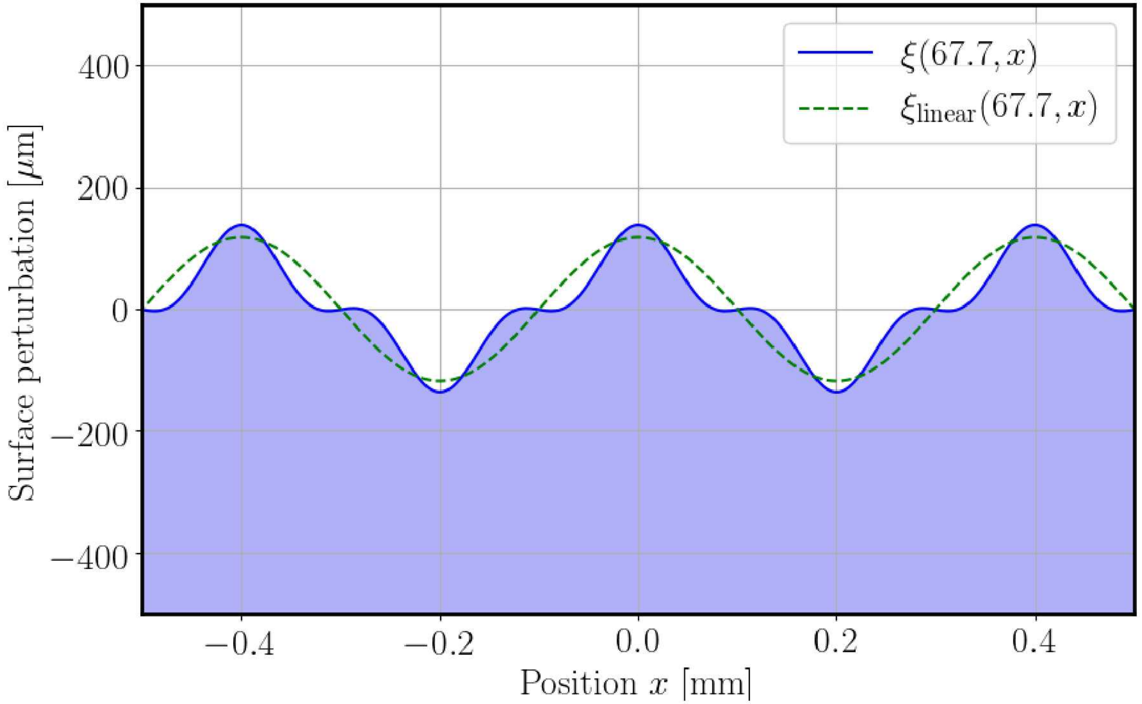
T=67.7 ns

T=79.0 ns

T=83.0 ns







5. *Current work: Regularizing the weakly nonlinear MRT theory*

Weakly nonlinear theory can be fixed by regularizing the Hamiltonian.

- When perturbation amplitudes grow, some terms in the Hamiltonian become too large and lead to “bad” behavior.

$$H = \frac{1}{2}\left(\epsilon^2 k\widehat{\Phi}_1^2 + 2\epsilon^4 k\widehat{\Phi}_2^2\right) - \frac{g}{2}\left(\epsilon^2\widehat{\xi}_1^2 + \epsilon^4\widehat{\xi}_2^2\right) - \epsilon^4\frac{k^3}{8}\,\widehat{\Phi}_1^2\,\widehat{\xi}_1^2 + \epsilon^4\frac{k^2}{2}\,\widehat{\Phi}_1^2\,\widehat{\xi}_2$$

Classical RT Hamiltonian

- The Padé approximant has been used before to regularize solutions for the Richtmyer—Meshkov instability.[24,25]

- What if we regularize the Hamiltonian by representing it as a Padé approximant?

$$H \simeq \frac{1}{2}\left(\epsilon^2 k\widehat{\Phi}_1^2 + 2\epsilon^4 k\widehat{\Phi}_2^2\right) + \frac{g}{2}\left(\frac{2}{5k^2}\right)\frac{1}{1 + \frac{5\epsilon^2}{2}k^2\widehat{\xi}_1^2 + \frac{25\epsilon^4}{4}k^4\widehat{\xi}_1^4 + \frac{5\epsilon^4}{2}k^2\widehat{\xi}_2^2 + \frac{5\epsilon^4}{8g}k^5\widehat{\xi}_1^2\widehat{\Phi}_1^2 - \frac{5\epsilon^4}{2g}k^4\widehat{\xi}_2\widehat{\Phi}_1^2}$$

$$H \simeq \frac{1}{2}\left(\epsilon^2 k\widehat{\Phi}_1^2 + 2\epsilon^4 k\widehat{\Phi}_2^2\right) + \frac{\frac{g}{2}\left(\widehat{\xi}_{1,\mathrm{sat}}\right)^2}{1 + \frac{5\epsilon^2}{2}k^2\widehat{\xi}_1^2 + \frac{25\epsilon^4}{4}k^4\widehat{\xi}_1^4 + \frac{5\epsilon^4}{2}k^2\widehat{\xi}_2^2 + \frac{5\epsilon^4}{8g}k^5\widehat{\xi}_1^2\widehat{\Phi}_1^2 - \frac{5\epsilon^4}{2g}k^4\widehat{\xi}_2\widehat{\Phi}_1^2}$$

$$H = \frac{1}{2}\left(\epsilon^2 k\widehat{\Phi}_1^2 + 2\epsilon^4 k\widehat{\Phi}_2^2\right) - \frac{g}{2}\left(\epsilon^2\widehat{\xi}_1^2 + \epsilon^4\widehat{\xi}_2^2\right) - \epsilon^4\frac{k^3}{8}\,\widehat{\Phi}_1^2\,\widehat{\xi}_1^2 + \epsilon^4\frac{k^2}{2}\,\widehat{\Phi}_1^2\,\widehat{\xi}_2$$

$$\widehat{\xi}_{1,\mathrm{sat}} \simeq \frac{1}{k}\sqrt{\frac{2}{5}}$$

- The Padé-approximated Hamiltonian leads to similar behavior at small amplitudes.

Question: Can it lead to better behavior at large amplitudes?

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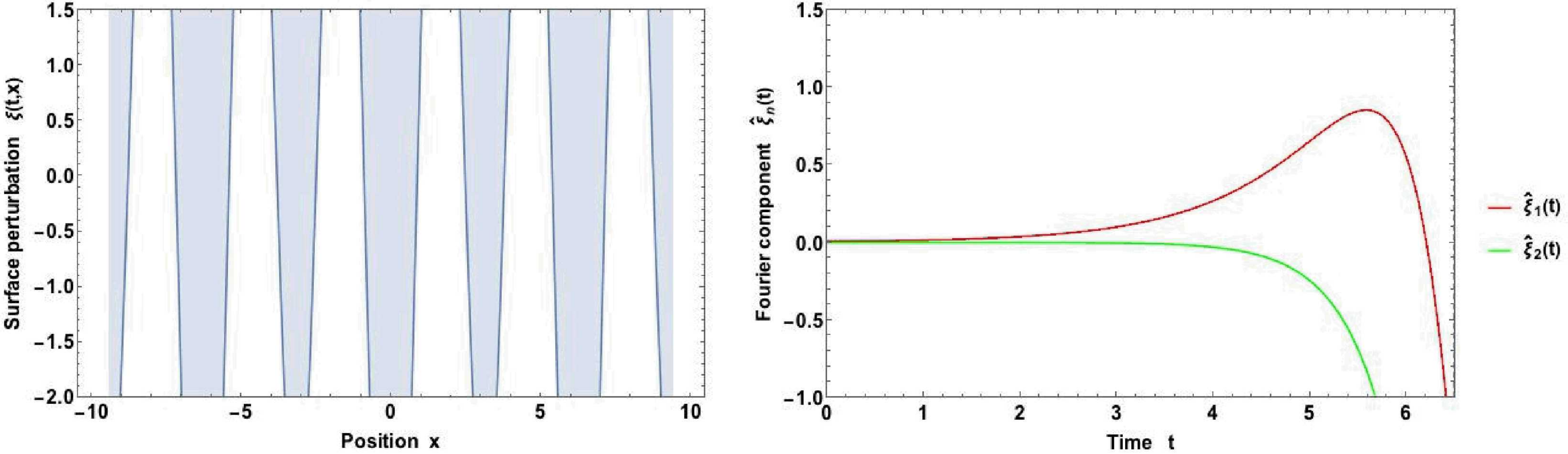
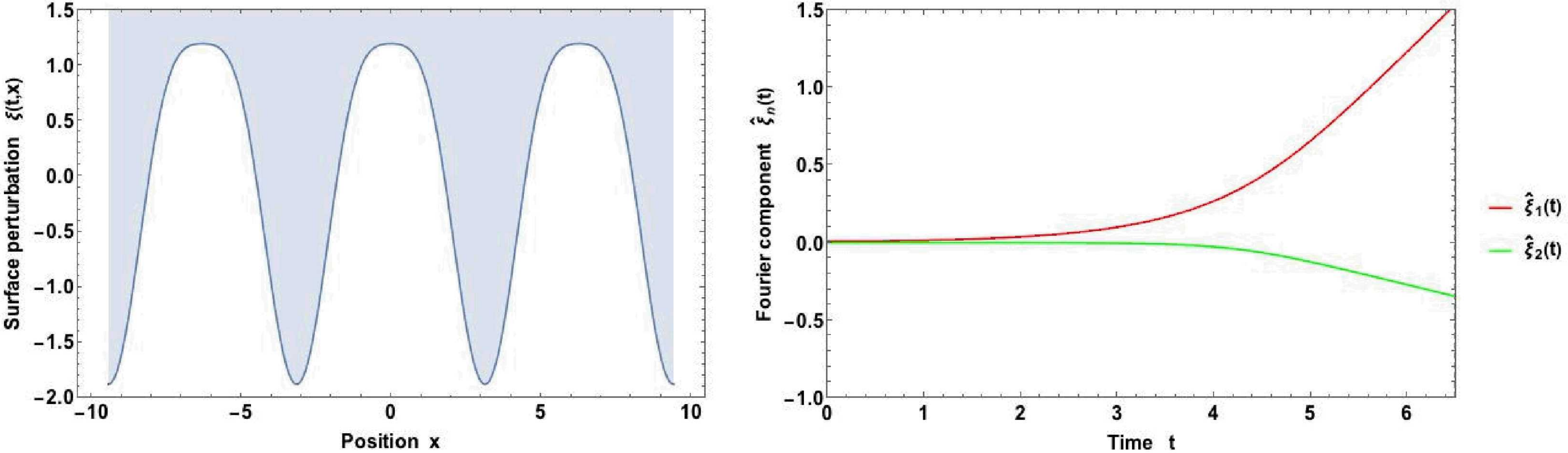
[24] A. L. Velikovich and G. Dimonte, Phys. Rev. Lett. **76**, 3112 (1996);
[25] A. L. Velikovich, M. Hermann, and S. I. Abarzhi, J. Fluid. Mech. 751, 432, (1994).

The regularized theory leads to ballistic behavior for the Fourier components at large times. This agrees with nonlinear RT theory.

Original Hamiltonian

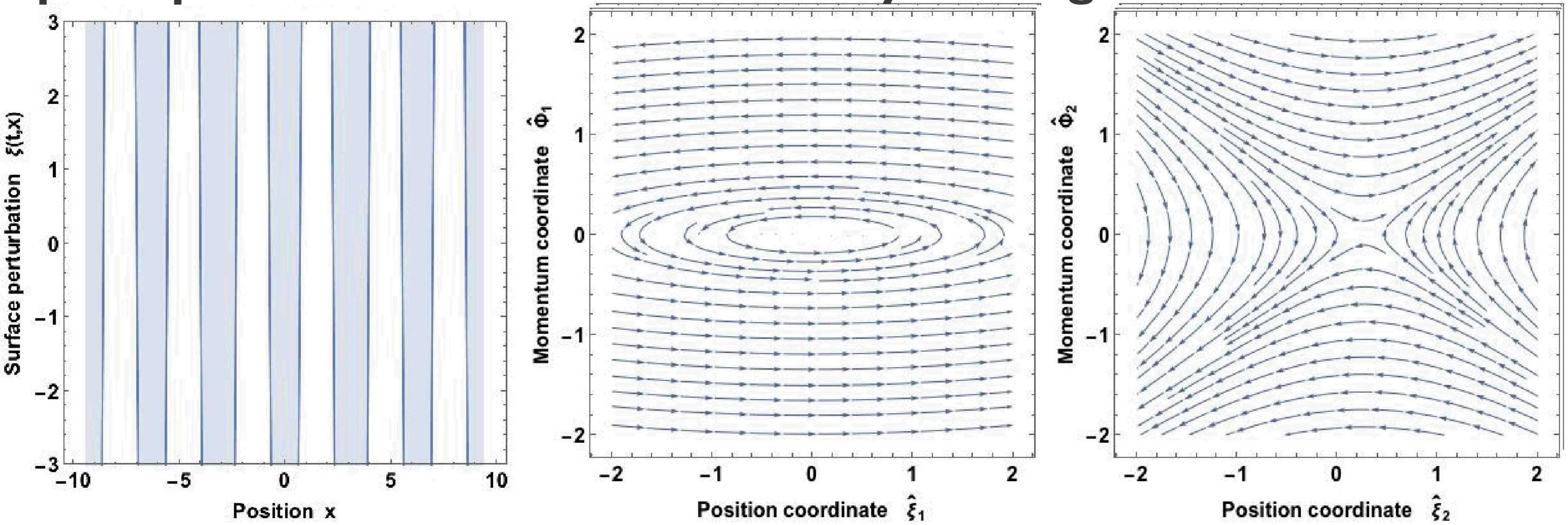


Padé approximated Hamiltonian

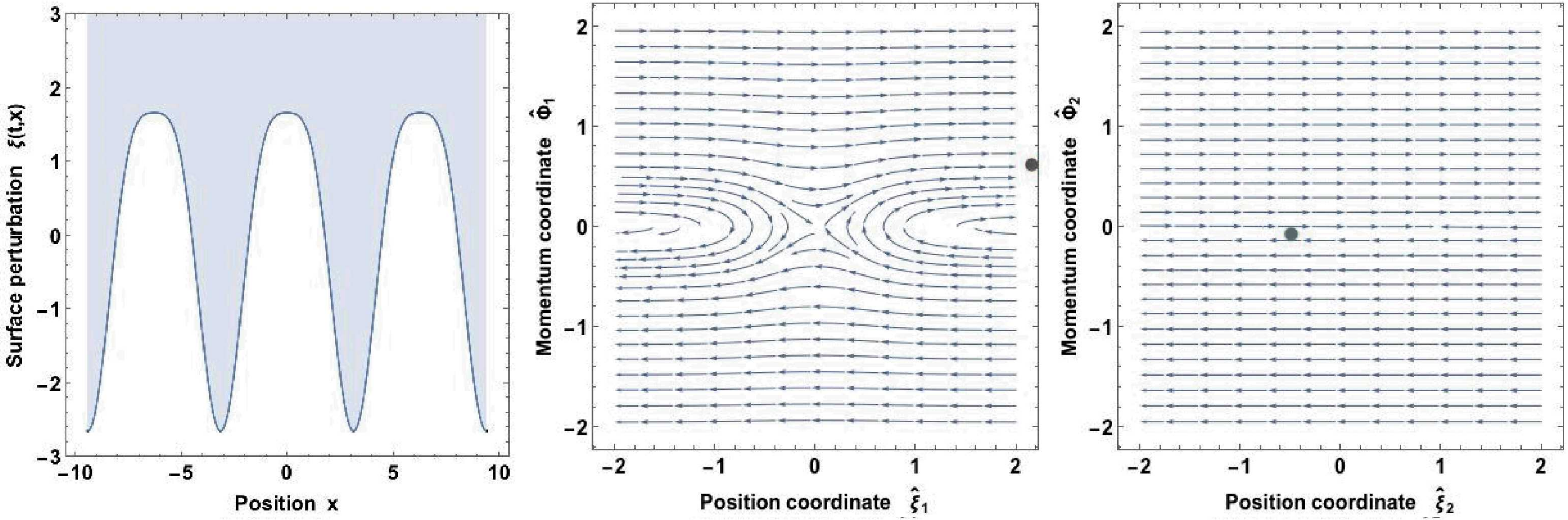


Phase-space plots of renormalized theory show good behavior.

Original Hamiltonian



Padé approximated Hamiltonian



6. Conclusions and future work

Conclusions

- A **variational principle** was found for the MRT instability. It was used to construct an asymptotic theory for the weakly-nonlinear MRT instability.
- In Fourier space, the MRT instability can be interpreted as the temporal dynamics of a particle **(Rayleighton)** in phase space. Much insights about the dynamics can be learned in this manner.
- The resulting theory captures **harmonic generation** and gives the **saturation amplitude** for MRT.
- When compared to experiments at Sandia, the theory breaks down relatively early during the implosion.

Future work

1. More work needs to be done to fully scope out the potential of Hamiltonian methods for understanding the MRT instability.
   - Can we regularize the Hamiltonian and “fix” the dynamics for large amplitudes? Can the Padé approximation do the trick?
2. The weakly nonlinear theory needs to be modified to cylindrical geometry.
   - Weakly nonlinear, cylindrical MRT
   - Weakly nonlinear MRT with Bell—Plesset effects and “ponderomotive” effects
   - Weakly nonlinear sausage and kink instabilities.
3. The current theory or any extensions to it need to be further scrutinized and compared to numerical simulations and experiments.