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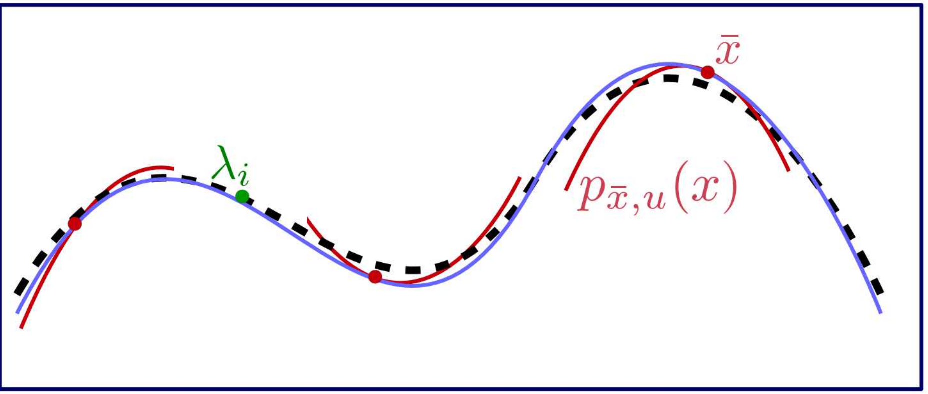

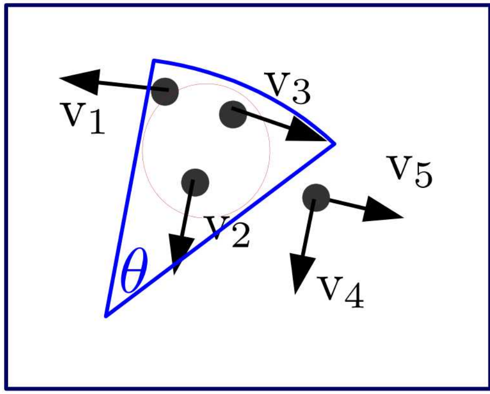

Generalized Moving Least Squares: Approximation Theory and Applications

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CCR Center for Computing Research

Humboldt University, Berlin, Germany, June 12, 2019

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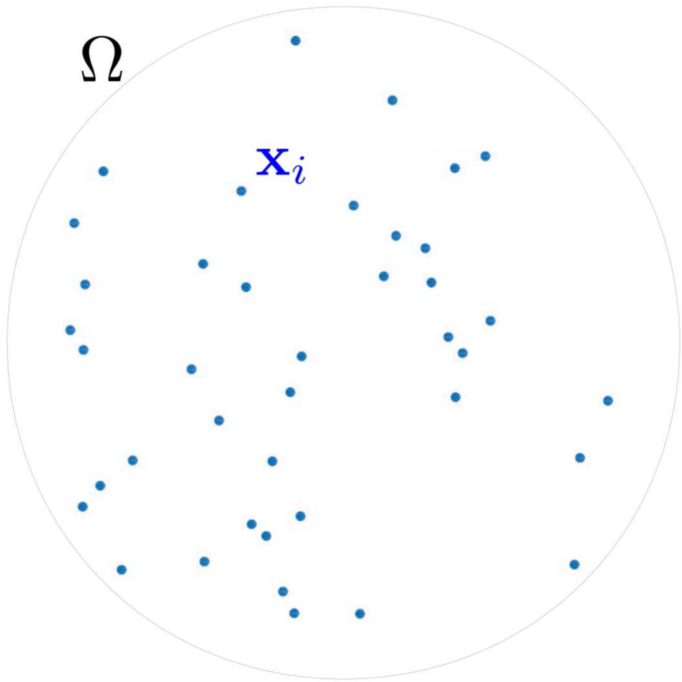
Motivation:

- GMLS allows to reconstruct functions/functionals given samples on scattered data points
- It can be use for data transfer between different meshes / point clouds
- It can be use for meshless discretizations of differential problems

Outline:

- Intro to (Classic) Moving least Square
- Intro to Generalized Moving least Square
- GMLS approximation theory / tools
- Meshless discretization
  - A collocation method
  - A Galerkin method

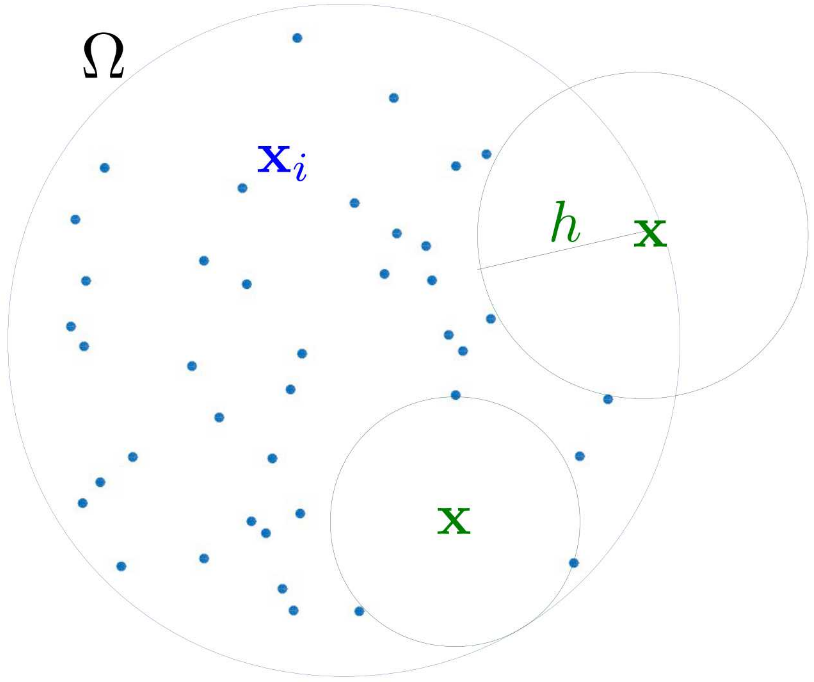
Moving Least Squares



point cloud:

$$X^h = \{\mathbf{x}_i\}_{i=1}^{N_h} \subset \Omega$$

Moving Least Squares



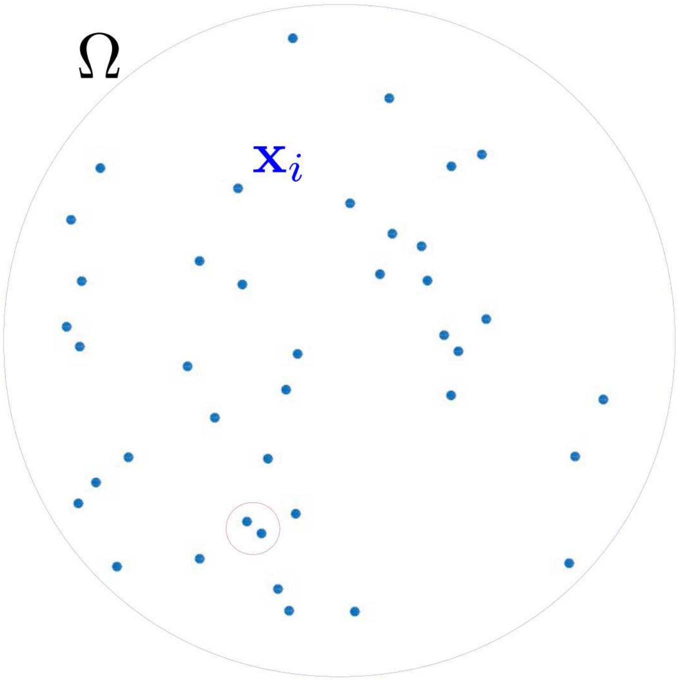
point cloud:

$$X^h = \{\mathbf{x}_i\}_{i=1}^{N_h} \subset \Omega$$

filling distance:

$$h := \sup_{\mathbf{x}\in\Omega} \inf_{\mathbf{x}_i \in X^h} |\mathbf{x} - \mathbf{x}_i|$$

Moving Least Squares



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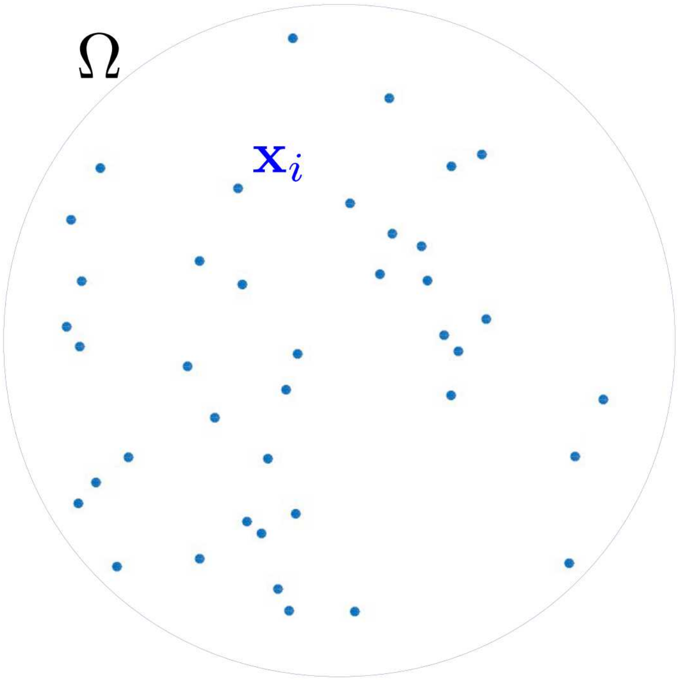
$$h_s := \min_{\mathbf{x}_i \neq \mathbf{x}_j \in X^h} \frac{1}{2} |\mathbf{x}_i - \mathbf{x}_j|$$

Moving Least Squares



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$$h_s := \min_{\mathbf{x}_i \neq \mathbf{x}_j \in X^h} \frac{1}{2} |\mathbf{x}_i - \mathbf{x}_j|$$

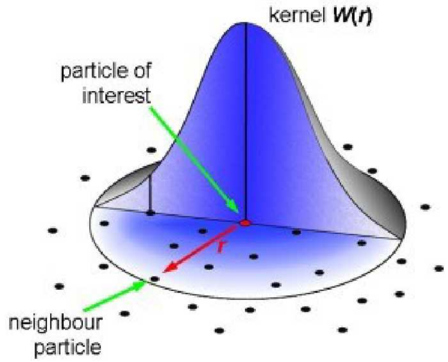
the point cloud is quasi-uniform if

$$h \leq c_{\mathrm{qu}} h_s$$

Moving Least Squares

Least Square formulation:

$$\boxed{u^h(\bar{\mathbf{x}}) := p_{\bar{\mathbf{x}},u}(\bar{\mathbf{x}}),} \quad p_{\bar{\mathbf{x}},u} = \arg\min_{p \in P} \sum_{i=1}^{N_h} \left(u(\mathbf{x}_i) - p(\mathbf{x}_i)\right)^2 W(\bar{\mathbf{x}}, \mathbf{x}_i^h)$$



Moving Least Squares

Least Square formulation:

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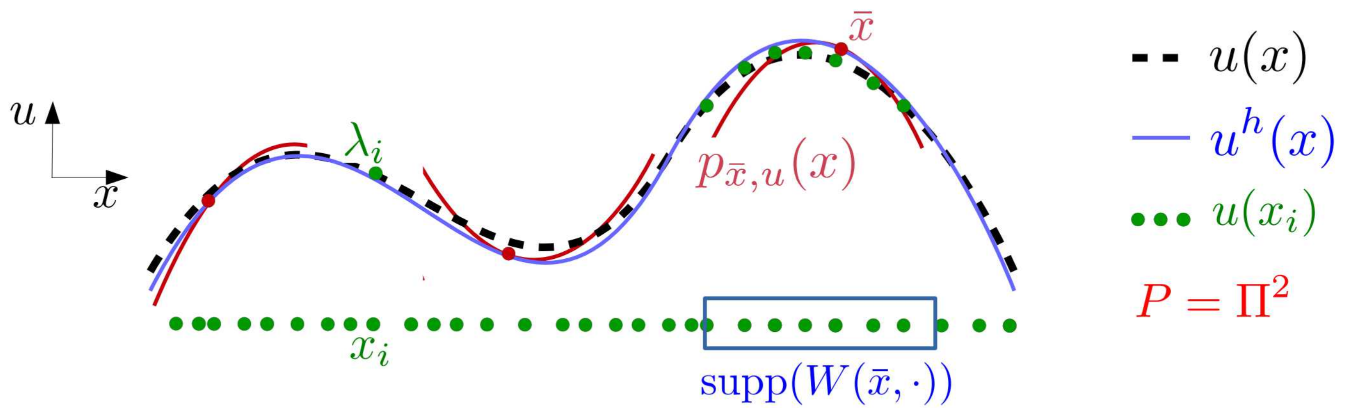
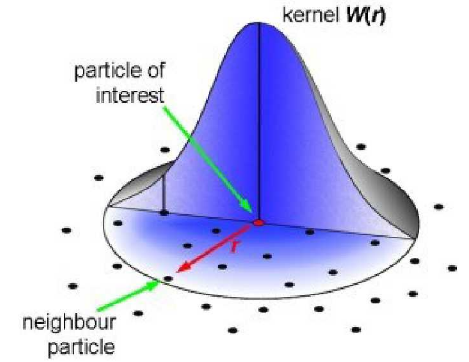
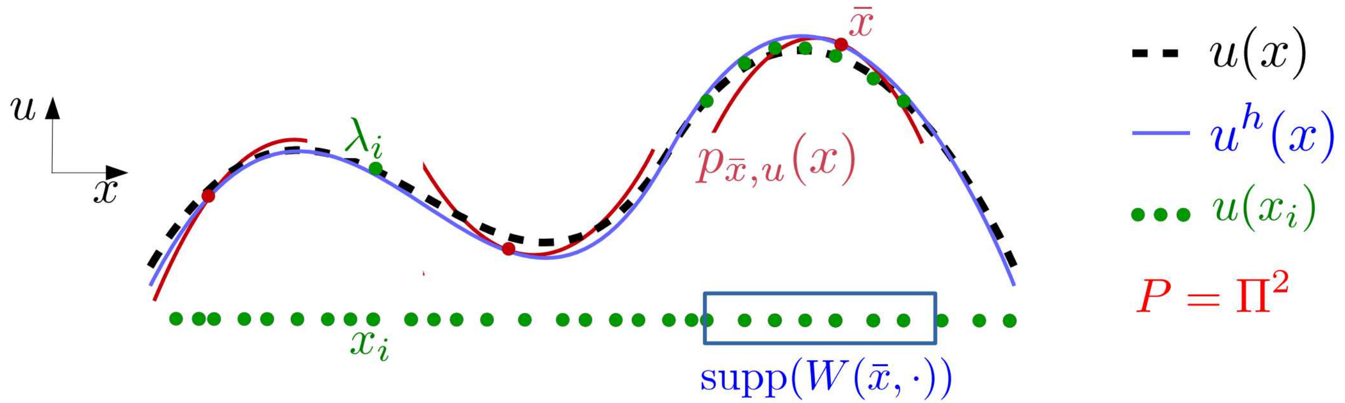
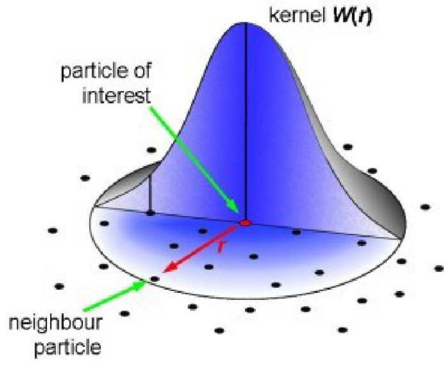




Least Square formulation:

$$\boxed{u^h(\bar{\mathbf{x}}) := p_{\bar{\mathbf{x}},u}(\bar{\mathbf{x}}),} \quad p_{\bar{\mathbf{x}},u} = \arg\min_{p \in P} \sum_{i=1}^{N_h} (u(\mathbf{x}_i) - p(\mathbf{x}_i))^2 \, W(|\bar{\mathbf{x}} - \mathbf{x}_i|)$$





$$u^h(\mathbf{x}) = \sum_{i=1}^{N_h} \sum_{j=1}^{M} p_j(\mathbf{x}) \, \Theta_{ji}(\mathbf{x}) \, u(\mathbf{x}_i)$$

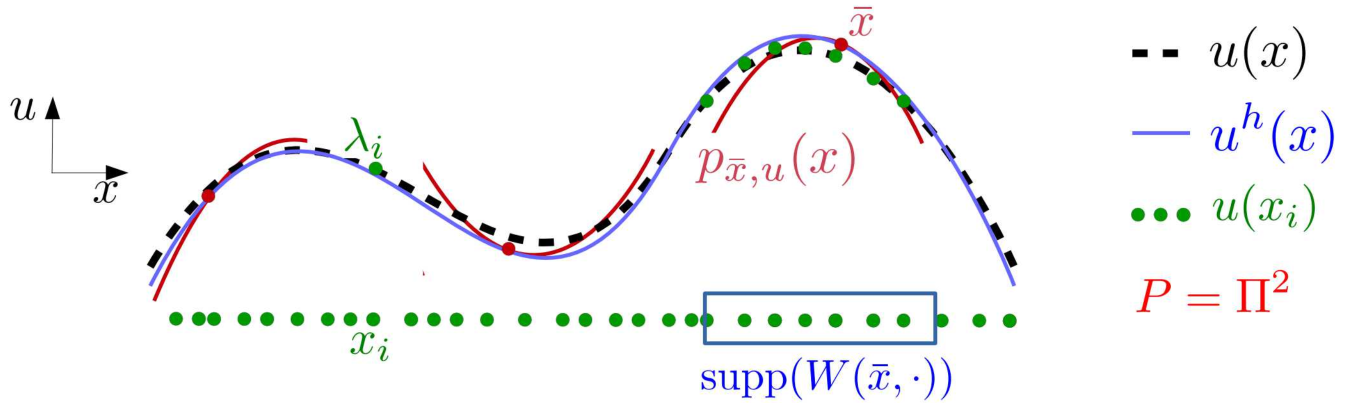
where $\{p_j\}_{j=1}^{M}$ is a basis for P, $\quad \Theta := (B^T W^{\mathbf{x}} B)^{-1} B^T W^{\mathbf{x}}$,

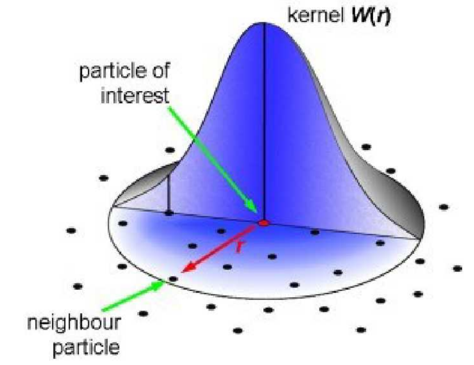
with $W^{\mathbf{x}}_{ik} := \delta_{ik} W(|\mathbf{x} - \mathbf{x}_i|), \quad B_{ij} := p_j(\mathbf{x}_i)$

Moving Least Squares

Least Square formulation:

$$\boxed{u^h(\bar{\mathbf{x}}) := p_{\bar{\mathbf{x}},u}(\bar{\mathbf{x}}),} \quad p_{\bar{\mathbf{x}},u} = \arg\min_{p\in P} \sum_{i=1}^{N_h} \left(u(\mathbf{x}_i) - p(\mathbf{x}_i)\right)^2 W(|\bar{\mathbf{x}} - \mathbf{x}_i|)$$



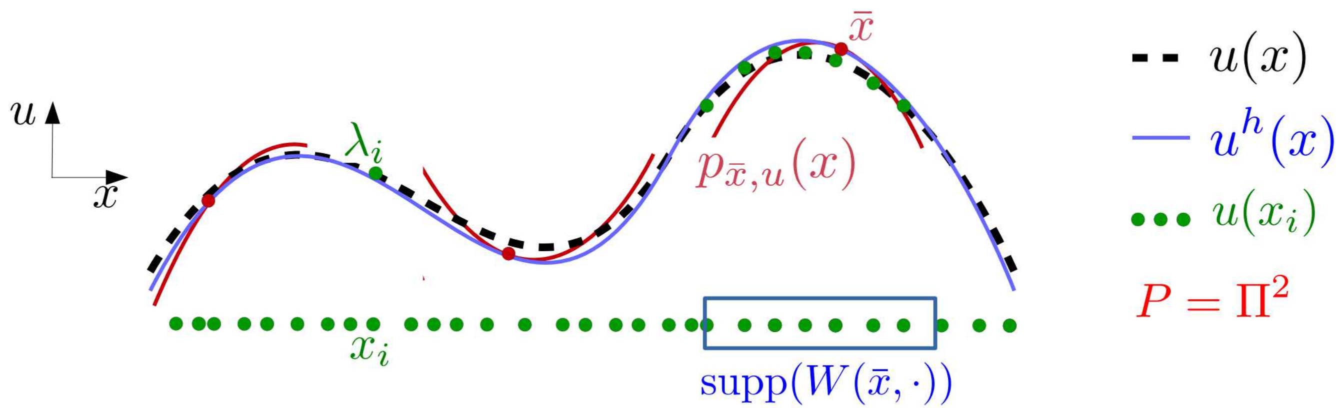


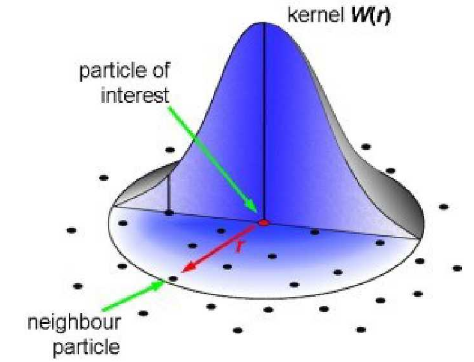
$$u^h(\mathbf{x}) = \sum_{i=1}^{N_h} \sum_{j=1}^{M} p_j(\mathbf{x})\,\Theta_{ji}(\mathbf{x})\,u(\mathbf{x}_i) = \boxed{\sum_{i=1}^{N_h} a_i(\mathbf{x})\,u(\mathbf{x}_i)} = \boxed{\sum_{j=1}^{M} p_j(\mathbf{x})\,b_j(\mathbf{x})}$$

Moving Least Squares

Least Square formulation:

$$\boxed{u^h(\bar{\mathbf{x}}) := p_{\bar{\mathbf{x}},u}(\bar{\mathbf{x}}),} \quad p_{\bar{\mathbf{x}},u} = \arg\min_{p\in P} \sum_{i=1}^{N_h} \left(u(\mathbf{x}_i) - p(\mathbf{x}_i)\right)^2 W(|\bar{\mathbf{x}} - \mathbf{x}_i|)$$





$$u^h(\mathbf{x}) = \sum_{i=1}^{N_h}\sum_{j=1}^{M} p_j(\mathbf{x})\,\Theta_{ji}(\mathbf{x})\,u(\mathbf{x}_i) = \boxed{\sum_{i=1}^{N_h} a_i(\mathbf{x})\,u(\mathbf{x}_i)} = \boxed{\sum_{j=1}^{M} p_j(\mathbf{x})\,b_j(\mathbf{x})}$$

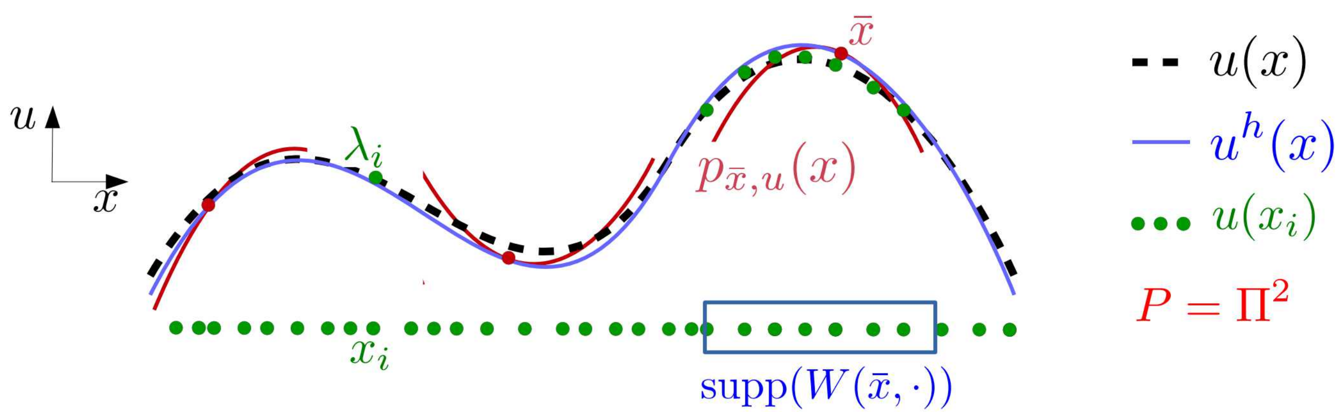
$$\frac{\partial}{\partial \mathbf{x}} u^h(\mathbf{x}) = \sum_{j=1}^{M} \frac{\partial}{\partial \mathbf{x}} p_j\Big|_{\mathbf{x}} b_j(\mathbf{x}) + \cancel{\sum_{j=1}^{M} p_j(\mathbf{x}) \frac{\partial}{\partial \mathbf{x}} b_j\Big|_{\mathbf{x}}}$$

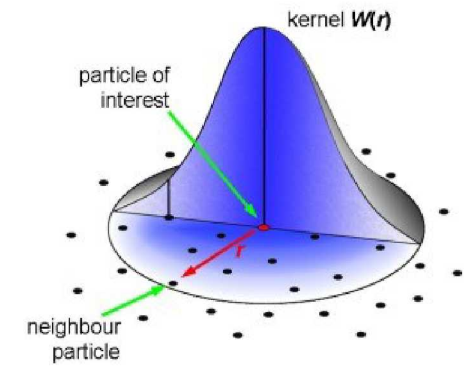
diffuse derivative

Moving Least Squares

Least Square formulation:

$$\boxed{u^h(\bar{\mathbf{x}}) := p_{\bar{\mathbf{x}},u}(\bar{\mathbf{x}}),} \qquad p_{\bar{\mathbf{x}},u} = \arg\min_{p \in P} \sum_{i=1}^{N_h} \left(u(\mathbf{x}_i) - p(\mathbf{x}_i)\right)^2 W(|\bar{\mathbf{x}} - \mathbf{x}_i|)$$





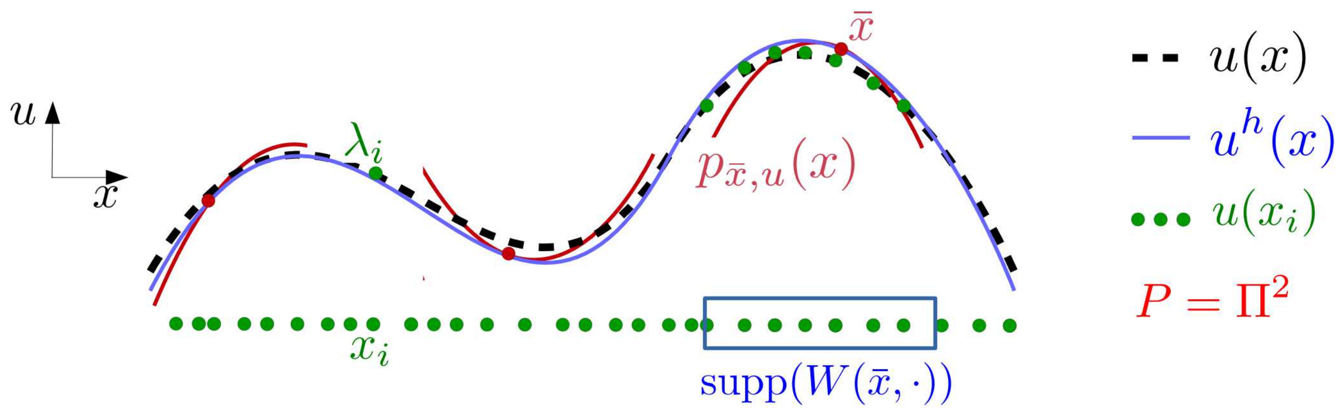
$$u^h(\mathbf{x}) = \sum_{i=1}^{N_h} \sum_{j=1}^{M} p_j(\mathbf{x})\, \Theta_{ji}(\mathbf{x})\, u(\mathbf{x}_i) = \boxed{\sum_{i=1}^{N_h} a_i(\mathbf{x})\, u(\mathbf{x}_i)} = \boxed{\sum_{j=1}^{M} p_j(\mathbf{x})\, b_j(\mathbf{x})}$$

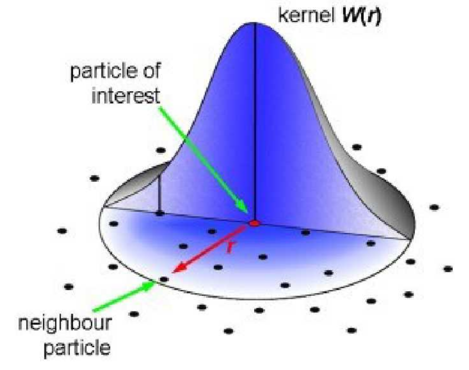
$$\tau(u^h) = \sum_{j=1}^{M} \tau(p_j)\, b_j(\mathbf{x}) \longleftarrow \qquad \tau \text{ is a generic linear functional}$$

Moving Least Squares

Least Square formulation:

$$\boxed{u^h(\bar{\mathbf{x}}) := p_{\bar{\mathbf{x}},u}(\bar{\mathbf{x}}),} \quad p_{\bar{\mathbf{x}},u} = \arg\min_{p \in P} \sum_{i=1}^{N_h} \left(u(\mathbf{x}_i) - p(\mathbf{x}_i)\right)^2 W(|\bar{\mathbf{x}} - \mathbf{x}_i|)$$





Constrained Optimization formulation:

$$\boxed{u^h(\bar{\mathbf{x}}) := \sum_{i=1}^{N_h} a^i_{\bar{\mathbf{x}}} u(\mathbf{x}_i),} \quad \{a^i_{\bar{\mathbf{x}}}\} = \arg\min_{a^i} \sum_{i=1}^{N_h} \frac{|a^i|^2}{W(|\bar{\mathbf{x}} - \mathbf{x}_i|)},$$

$$\text{s. t. } p(\bar{\mathbf{x}}) = \sum_{i=1}^{N_h} a^i p(\bar{\mathbf{x}}_i), \ \forall p \in P$$

Reproducibility:
$\left(p^h(\mathbf{x}) = p(\mathbf{x})\right)$

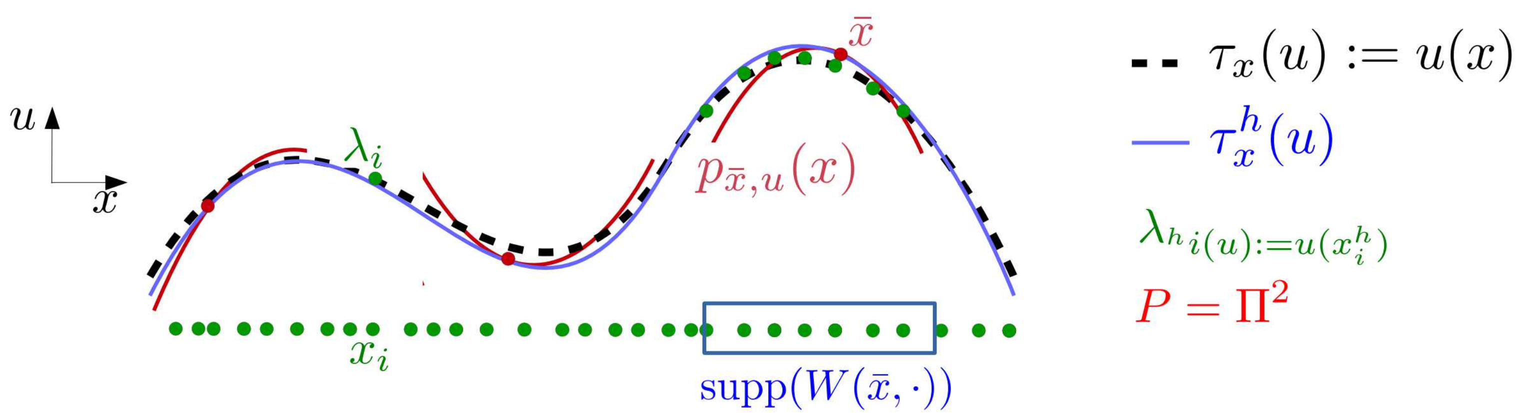
Generalized Moving Least Squares

Recipes (2 equivalent formulations):

1. Least Square formulation:

$$\boxed{\tau^h_{\bar{\mathbf{x}}}(u) := \tau_{\bar{\mathbf{x}}}(p_{\bar{\mathbf{x}},u}),} \quad p_{\bar{\mathbf{x}},u} = \arg\min_{p\in P} \sum_{i\in I_{\bar{\mathbf{x}}}} \left(\lambda^h_i(u) - \lambda^h_i(p)\right)^2 W(\bar{\mathbf{x}}, \mathbf{x}^h_i)$$

$$\mathrm{I}_{\bar{\mathbf{x}}} := \{i : W(\bar{\mathbf{x}}, \mathbf{x}^h_i) > 0\}$$



2. Constrained Optimization formulation:

$$\boxed{\tau^h_{\bar{\mathbf{x}}}(u) := \sum_{i\in I_{\bar{\mathbf{x}}}} a^i_{\tau_{\bar{\mathbf{x}}}} \lambda^h_i(u),} \quad \{a^i_{\tau_{\mathbf{x}}}\} = \arg\min_{a^i} \sum_{i\in I_{\bar{\mathbf{x}}}} \frac{|a^i|^2}{W(\bar{\mathbf{x}}, \mathbf{x}^h_i)},$$

$$\text{s. t. } \tau_{\bar{\mathbf{x}}}(p) = \sum_{i\in I_{\bar{\mathbf{x}}}} a^i \lambda^h_i(p), \ \forall p \in P \quad \left(\tau^h_{\bar{\mathbf{x}}}(p) = \tau_{\bar{\mathbf{x}}}(p)\right)$$

Generalized Moving Least Squares
(approximation theory)

Goal: collect and consolidate approximation results* for GMLS

In particular we are interested in these generalizations:

Approximation space: we want to consider vector polynomials and subspaces of complete polynomial spaces (e.g. div-free, curl-free spaces)

Sampling/target functionals: we want to consider differential forms and in particular volume integrals, fluxes over (virtual) faces and line integral over (virtual) edges

$$\lambda_i^e(\mathbf{u}) := \frac{1}{|e_i|}\int_{e_i} \mathbf{u}\cdot\mathbf{t}_i \qquad \lambda_i^f(\mathbf{u}) = \frac{1}{|f_i|}\int_{f_i} \mathbf{u}\cdot\mathbf{n}_i \qquad \lambda_i^v(u) := \frac{1}{|V_i|}\int_{V_i} u(\boldsymbol{y})\,d\boldsymbol{y}$$

Applications: ***PDE discretizations and data transfer***.

In the following we present

generalization of definition of "local polynomial reproduction"

existence and approximation results for different sampling/target functionals

* *Mirzaei, Narcowich, Rieger, Schaback, Ward, Wendland*

Generalized Moving Least Squares

Ingredients:

V, V^* - *a function space (e.g. continuous functions) and its dual*

$P = span\{p_i\}_{i=1}^{Q} \subset V$ - a finite dimensional *approximation space*, e.g. polynomials

$\Lambda^h := \{\lambda_1^h, \ldots, \lambda_{N_h}^h\} \subset \Lambda \subset V^*$ - a finite set of *sampling functionals* (e.g. point evaluations)
FE analog: *degrees of freedom*

$\tau \in \mathcal{T} \subset V^*$ - a *target functional* (or a family of target functionals)

$W(\tau, \lambda_i) : (\mathcal{T} \cup \Lambda) \times (\mathcal{T} \cup \Lambda) \to \mathbb{R}$ - a *window function* correlating functionals (e.g. a radial kernel) determines the **smoothness** of reconstruction

Example, MLS case:

Point cloud $X^h = \{\mathbf{x}_i^h\}_{i=1}^{N_h} \subset \Omega$, with filling distance $h = \sup_{\mathbf{x}^h \in \Omega} \min_{i=1,\ldots,N_h} |\mathbf{x} - \mathbf{x}_i|$.

$$u \in V = C^{k+1}(\Omega), \quad P = \Pi^k(\Omega), \quad \lambda_i^h(u) = u(\mathbf{x}_i^h), \quad \tau_{\mathbf{x}} = u(\mathbf{x}), \quad W = W(|\mathbf{x} - \mathbf{x}_i^h|)$$

Properties of Generalized Moving Least Squares

Main questions: **When is the reconstruction possible? How good is the reconstruction?**

- **Unisolvency**:
  In order for $\tau^h_{\mathbf{x}}$ to exist, $\Lambda_{\mathbf{x}} := \{\lambda_i\}_{i \in I_{\mathbf{x}}}$ must be P-unisolvent, i.e. for $p \in P$

$$\forall \lambda_i \in \Lambda_{\mathbf{x}},\ \lambda_i(p) = 0 \iff p = 0.$$

- **Approximation property**:
  $|\tau^h_{\mathbf{x}}(u) - \tau_{\mathbf{x}}(u)| \sim O(h^s),$ for some s.

➢ Sufficient results for unisolvancy and approx. error have been provided by Wendland* for (non generalized) MLS.

➢ Non local smapling operators**

➢ Results has been generalized by Mirzaei***, for point derivatives as target functionals:
$|D^{\alpha,h}_{\mathbf{x}}(u) - D^{\alpha}_{\mathbf{x}}(u)| \sim O(h^{(k+1-|\alpha|)}),$ where k is the degree of the polynomials.

and for weak derivatives in Sobolev spaces***.

* H. Wendland, Scattered data approximation, Vol. 17. Cambridge university press, 2004
** Christian Rieger, PhD thesis, personal communication.
*** D. Mirzaei et Al., IMA J. Num. Analysis, 2011, D. Mirzaei, Comp. And App. Math. 2016

Result (**local polynomial reproduction**) [MLS]

If, for each $\mathbf{x} \in \Omega \subset \mathbb{R}^d$, $\Lambda^h \subset \Lambda \subset V^*$, exist $C_{\text{loc}} > 0$ and $C_a > 0$, s.t.

$$\begin{cases} r^h_{\mathbf{x}}(p) := \sum_i a^i_{\mathbf{x}}\, p(\mathbf{x}_i) = p(\mathbf{x}),\ \forall p \in \Pi^m & \text{- consistency} \\ \sum_i |a^i_{\mathbf{x}}| \leq C_a & \text{- uniform boundness} \\ |\mathbf{x} - \mathbf{x}_i| > C_{\text{loc}} h \ \Rightarrow\ a^i_{\mathbf{x}} = 0 & \text{- local support} \end{cases}$$

$$\Longrightarrow\ |r^h_{\mathbf{x}}(u) - u(\mathbf{x})| \leq C_{u,m} h^{m+1}$$

Result (**generalized local reproduction**) [GMLS]

If, for each $\tau \in \mathcal{T} \subset V^*$, $\Lambda^h \subset \Lambda \subset V^*$ there exist $C^h_{\text{loc}} \to 0$, $C^h_{u,\mathcal{T}} \to 0$, $C^h_a C^h_{u,\Lambda} \to 0$, s.t.

$$\begin{cases} \tau^h(p) := \sum_i a^i_\tau \lambda_i(p) = \tau(p),\ \forall p \in P & \text{- consistency} \\ \sum_i |a^i_\tau| \leq C^h_a & \text{- uniform boundness} \\ d(\tau, \lambda_i) > C^h_{\text{loc}} \ \Rightarrow\ a^i_\tau = 0, & \text{- local support} \\ \forall f \in V,\ \exists p \in P \text{ such that:} & \\ \quad |\tau(u-p)| \leq C^h_{u,\mathcal{T}} \text{ and} & \text{- approximation} \\ \quad |\lambda_i(u-p)| \leq C^h_{u,\Lambda},\ \forall \lambda_i : d(\tau, \lambda_i) \leq C^h_{\text{loc}}. & \end{cases}$$

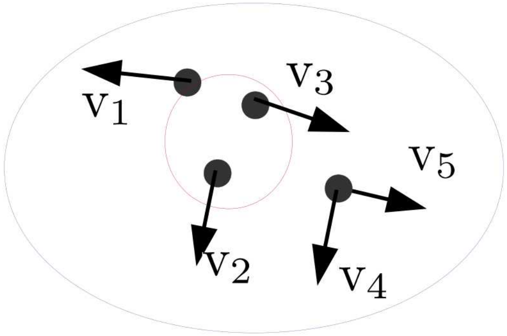
$$\Longrightarrow\ |\tau^h(u) - \tau(u)| \leq C^h_{u,\mathcal{T}} + C^h_a C^h_{u,\Lambda}$$

Reconstructing **vector** functions
(from projections along given directions)

Associate to each particle i a position $\mathbf{x}_i$ and a unit vector $\mathbf{v}_i \in \mathbb{R}^d$.

Sampling functional: $\boxed{\lambda_i(\mathbf{u}) := \mathbf{u}(\mathbf{x}_i) \cdot \mathbf{v}_i}$

Filling distance h_ω is the radius of the smallest ball that centered at any point of Ω contains d particles whose versors $\mathbf{v}_1, \ldots, \mathbf{v}_d$ contain a basis for $\mathbb{R}^d$ with associated determinant bigger than ω ($|\,\mathbf{det}[\mathbf{v_1}, \ldots, \mathbf{v_d}]| \geq \boldsymbol{w}$).



$$\boxed{\begin{array}{l} \text{For } h_\omega \leq C(\theta, R, m, w),\ \forall p \in [\Pi^m]^d, \\ \exists\, \lambda_{i_p} \in \Lambda_h,\ \text{such that } |\lambda_{i_p}(p)| \geq \rho_\omega \|p\|_{\Omega,\infty} \end{array}}$$

Other sampling functionals we considered:

$$\lambda_i^e(\mathbf{u}) := \frac{1}{|e_i|} \int_{e_i} \mathbf{u} \cdot \mathbf{t}_i \qquad \lambda_i^f(\mathbf{u}) = \frac{1}{|f_i|} \int_{f_i} \mathbf{u} \cdot \mathbf{n}_i \qquad \lambda_i^v(u) := \frac{1}{|V_i|} \int_{V_i} u(\boldsymbol{y})\, d\boldsymbol{y}$$

GMLS approximation results:

Basic technique:

$$
\begin{aligned}
|\tau_{\mathbf{x}}(u) - \tau_{\mathbf{x}}^h(u)| &\leq |\tau_{\mathbf{x}}(u) - \tau_{\mathbf{x}}(p)| + |\tau_{\mathbf{x}}(p) - \tau_{\mathbf{x}}^h(u)|, \quad (\forall p \in P) \\
&\leq |\tau_{\mathbf{x}}(u) - \tau_{\mathbf{x}}(p)| + |\tau_{\mathbf{x}}^h(p - u)|, \quad \longleftarrow \text{ reconstruction property} \\
&\leq |\tau_{\mathbf{x}}(u - p)| + \left| \sum_{i=1}^{N_p} \lambda_i(u - p)\, a_{\tau_{\mathbf{x}}}^i \right| \quad \longleftarrow \text{ GMLS definition} \\
&\leq |\tau_{\mathbf{x}}(u - p)| + \max_{i \in I_{\mathbf{x}}} |\lambda_i(u - p)| \sum_{i \in I_{\mathbf{x}}} |a_{\tau_{\mathbf{x}}}^i|.
\end{aligned}
$$

$$\boxed{\sum_{i \in I_{\mathbf{x}}} |a_{\tau_{\mathbf{x}}}^i| \leq C_W \|\tau_{\mathbf{x}}\|_{P^*} \|\Lambda_{\mathbf{x}}^{-1}\|}$$

Holds for any target functional and approximation space:

$$\boxed{|\tau_{\mathbf{x}}(u) - \tau_{\mathbf{x}}^h(u)| \leq |\tau_{\mathbf{x}}(u - p)| + C_W \|\tau_{\mathbf{x}}\|_{P^*} \|\Lambda_{\mathbf{x}}^{-1}\| \max_{i \in I_{\mathbf{x}}} |\lambda_i(u - p)|, \quad p \in P}$$

GMLS approximation results: **derivatives** as target functionals

$$\lambda_i(u) := u(\mathbf{x}_i), \quad \tau_{\mathbf{x}}(u) = D^\alpha(u)\big|_{\mathbf{x}}, \quad V = C^{k+1}(\Omega), \quad P = \Pi^k(\Omega)$$

Take $p = p^k_{\mathbf{x},u}$, the Taylor polynomial of degree k of u at $\mathbf{x}$.

$$|D^\alpha_{\mathbf{x}}(u) - D^{\alpha,h}_{\mathbf{x}}(u)| \le \underbrace{|D^\alpha_{\mathbf{x}}(u - p^k_{\mathbf{x},u})|}_{\substack{\text{Taylor approx. } (|\alpha| \le k)\\ \downarrow\\ 0}} + C_W \underbrace{\|D^\alpha_{\mathbf{x}}\|_{P^*} \|\Lambda^{-1}_{\mathbf{x}}\|}_{\substack{\text{Mirzaei**}\\ \downarrow\\ \mathcal{O}\left(h^{-|\alpha|}\right)}} \underbrace{\max_{i \in I_{\mathbf{x}}} |\lambda_i(u - p^k_{\mathbf{x},u})|}_{\substack{\text{Taylor approx.}\\ \downarrow\\ \mathcal{O}\left(h^{k+1}\right)}}$$

$$\boxed{|D^\alpha_{\mathbf{x}}(u) - D^{\alpha,h}_{\mathbf{x}}(u)| \sim \mathcal{O}\left(h^{k+1-|\alpha|}\right)}$$

** D. Mirzaei et Al., IMA J. Num. Analysis, 2011.

GMLS approximation results: **div free** spaces

$$V = \{\mathbf{u} \in [C^{k+1}(\Omega)]^d, \mathrm{div}(\mathbf{u}) = 0\}, \quad P = \{\mathbf{p} \in [\Pi^k(\Omega)]^d, \mathrm{div}(\mathbf{p}) = 0\}$$

$$\lambda_i^j(\mathbf{u}) := \mathbf{u}(\mathbf{x}_i) \cdot \mathbf{e}_j, \quad \tau_{\mathbf{x}}^j(\mathbf{u}) = \mathbf{u}(\mathbf{x}) \cdot \mathbf{e}_j, \quad \boldsymbol{\tau}_{\mathbf{x}} = (\tau_{\mathbf{x}}^1, \ldots, \tau_{\mathbf{x}}^d)$$

The Taylor polynomial $\mathbf{p}_{\mathbf{x},u}^k$ of $\mathbf{u} \in V$, belongs to P (it's divergence free).

$$|\tau_{\mathbf{x}}^j(\mathbf{u}) - \tau_{\mathbf{x}}^{j,h}(\mathbf{u})| \leq |\tau_{\mathbf{x}}^j(\mathbf{u} - \mathbf{p}_{\mathbf{x},u}^k)| + C_W \|\tau_{\mathbf{x}}^j\|_{P^*} \|\Lambda_{\mathbf{x}}^{-1}\| \max_{i \in I_{\mathbf{x}}} |\lambda_i^j(\mathbf{u} - \mathbf{p}_{\mathbf{x},\mathbf{u}}^k)|$$

Taylor approx.

$(|\alpha| \leq k)$

Taylor approx.

0

$\mathcal{O}\left(h^{k+1}\right)$

$$\boxed{|\boldsymbol{\tau}_{\mathbf{x}}(\mathbf{u}) - \boldsymbol{\tau}_{\mathbf{x}}^h(\mathbf{u})| \sim \mathcal{O}(h^{k+1})}$$

Note that, in general, $\mathrm{div}(\tau_{\mathbf{x}}^h(\mathbf{u})) \neq 0$.

GMLS approximation results:
integral target functionals (e.g. used for nonlocal problems)

$$\boxed{\lambda_i^e(\mathbf{u}) := \frac{1}{|e_i|}\int_{e_i} \mathbf{u}\cdot\mathbf{t}_i,} \qquad V = C^{k+1}(\Omega), \qquad P = \left[\Pi^k(\Omega)\right]^d$$

$$\boxed{\tau_{\mathbf{x}}(u) = \operatorname{div}(\mathbf{u})|_{\mathbf{x}}} \qquad\qquad |\lambda_i^e(\mathbf{u})| \le \|\mathbf{u}\|_{L^\infty(e_i)}$$

Take $p = p^k_{\mathbf{x},u}$, the Taylor polynomial of degree k of u at $\mathbf{x}$.

$$|\tau_{\mathbf{x}}(\mathbf{u}) - \tau^h_{\mathbf{x}}(\mathbf{u})| \le |\tau_{\mathbf{x}}(\mathbf{u} - p^k_{\mathbf{x},u})| + C_W \|\tau_{\mathbf{x}}\|_{P^*} \|\Lambda_{\mathbf{x}}^{-1}\| \max_{i\in I_{\mathbf{x}}} |\lambda_i(\mathbf{u} - p^k_{\mathbf{x},u})|$$

Taylor approx.
$(k \ge 1)$

$\Downarrow$

Mirzaei**

$$|\tau_{\mathbf{x}}(\mathbf{u}) - \tau^h_{\mathbf{x}}(\mathbf{u})| \le \qquad 0 \qquad + \qquad Ch^{-1}\, h^{k+1} \|\mathbf{u}\|_{\infty,\Omega}, \qquad \text{if } |e_i| \sim h$$

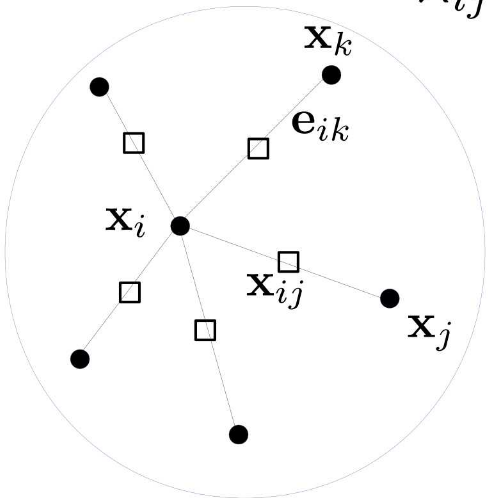
$$\Downarrow$$

$$\boxed{|\tau_{\mathbf{x}}(\mathbf{u}) - \tau^h_{\mathbf{x}}(\mathbf{u})| \le Ch^k \|\mathbf{u}\|_{\infty,\Omega}}$$

** D. Mirzaei et Al., IMA J. Num. Analysis, 2011.

Application to PDE: staggered scheme for Darcy

$$\begin{cases} \operatorname{div}(\mathbf{u}) = \mathbf{0} \\ \mathbf{u} = -K\nabla\phi \end{cases}$$



$$\lambda_{ij}(\mathbf{u}) := \int_{\mathbf{e}_{ij}} \mathbf{u}\cdot\mathbf{t}_{ij} \approx K(\mathbf{x}_{ij})(\phi_j - \phi_i)$$

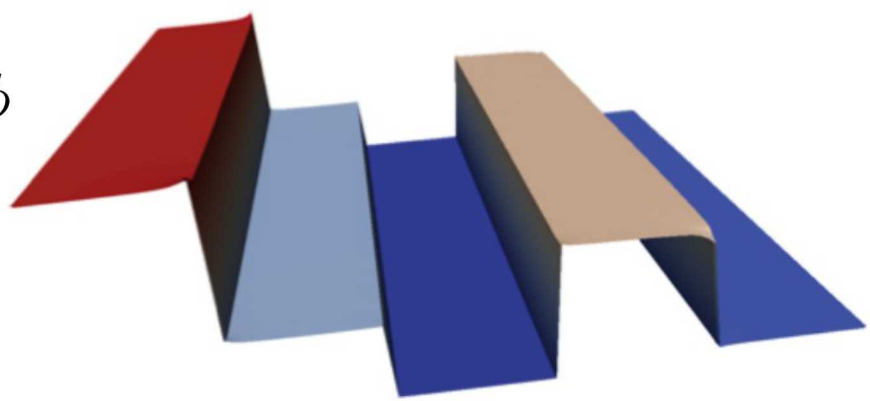
$$\tau_{\mathbf{x}_i}(\mathbf{u}) = \operatorname{div}(\mathbf{u})|_{\mathbf{x}_i}$$

$$P = [\Pi_m]^d$$

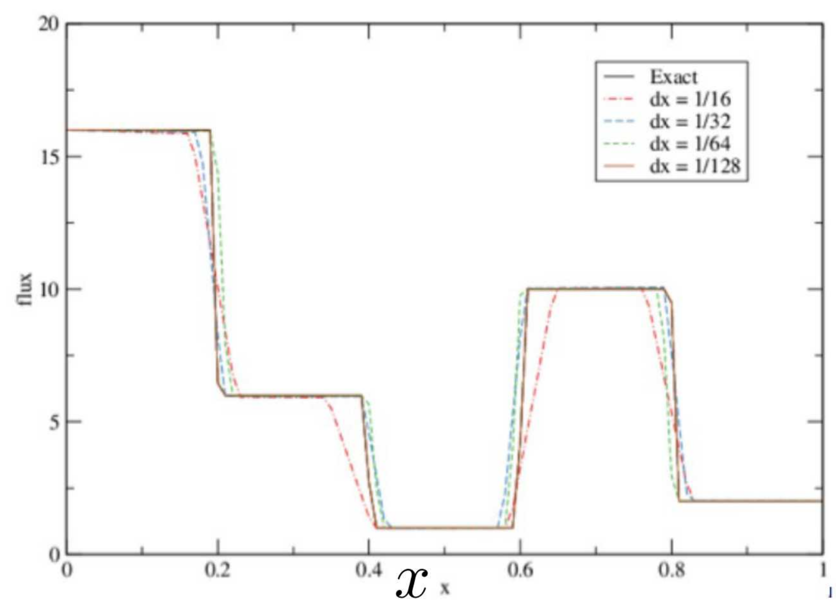
$$W(\tau_{\mathbf{x}}, \lambda_{ij}) = W(|\mathbf{x} - \mathbf{x}_{ij}|)$$

Solution of PDE w/ jumps in permeability

$\nabla\phi$



$\nabla\phi$



x

Galerkin Discretization

$$a(u,v) = f(v), \quad \forall v \in H^1(\Omega)$$

$$\sum_k a_k(u,v) = \sum_k f_k(v), \quad \forall v \in H^1(\Omega)$$

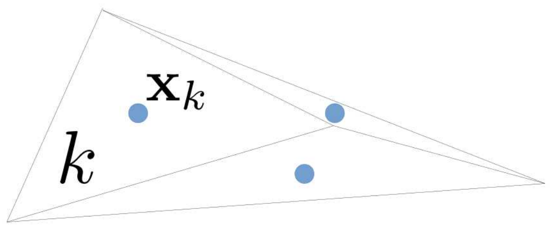
Sum over mesh elements

$$V^h = \mathrm{span}\{v_i \in V,\ v_i(\mathbf{x}_j) = \delta_{ij}\}_{i=1}^{N_h}$$

Finite dimensional space

$$\sum_{j,k} a_k(v_j, v_i) u_j^h = \sum_k f_k(v_i)$$

Finite dimensional formulation



Galerkin Discretization

$$\sum_{j,k} a_k(v_j, v_i) u_j^h = \sum_k f_k(v_i)$$

Finite dimensional formulation

Recall that:

$$\tau_{\bar{\mathbf{x}}}^h(u) := \tau_{\bar{\mathbf{x}}}(p_{\bar{\mathbf{x}},u}), \quad p_{\bar{\mathbf{x}},u} = \arg\min_{p \in P} \sum_{i \in I_{\bar{\mathbf{x}}}} \left(\lambda_i^h(u) - \lambda_i^h(p)\right)^2 W(\bar{\mathbf{x}}, \mathbf{x}_i^h)$$

$$\sum_{j,k} a_k(v_j, p_{\mathbf{x}_k, v_i}) u_j^h = \sum_k f_k(p_{\mathbf{x}_k, v_i})$$

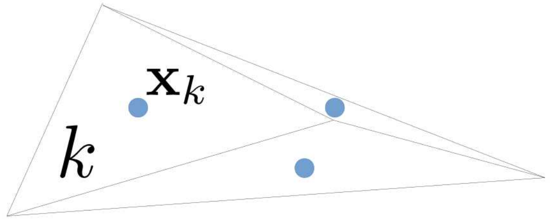
Linearity of bilinear form

$$\sum_{j,k} a_k(p_{\mathbf{x}_k, v_j}, p_{\mathbf{x}_k, v_i}) u_j^h = \sum_k f_k(p_{\mathbf{x}_k, v_i})$$

Linearity of bilinear form

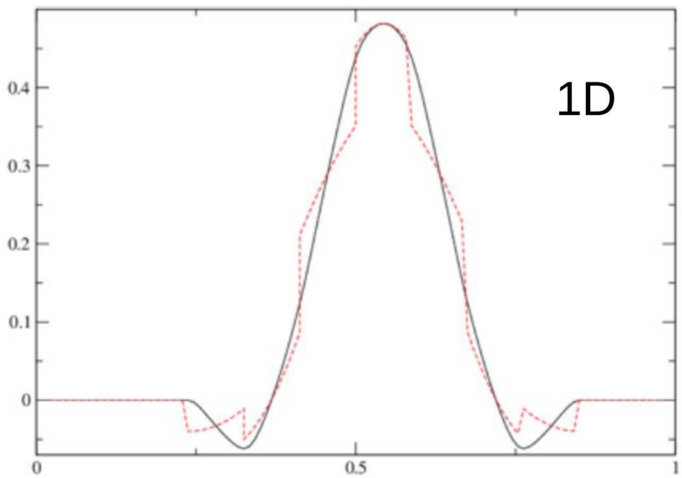
$$\sum_{j} a(\phi^j, \phi^i) u_j^h = f(\phi^i)$$

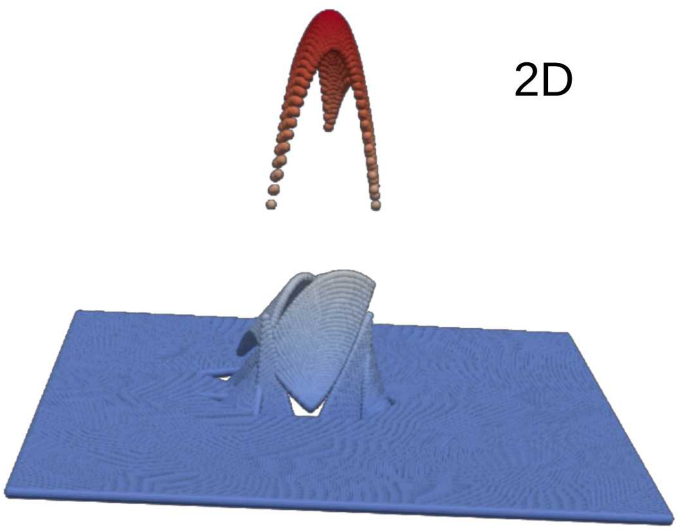
where $\phi^i(\mathbf{x}) := \sum_k p_{\mathbf{x}_k, v_i} \chi_{c_k}(\mathbf{x})$



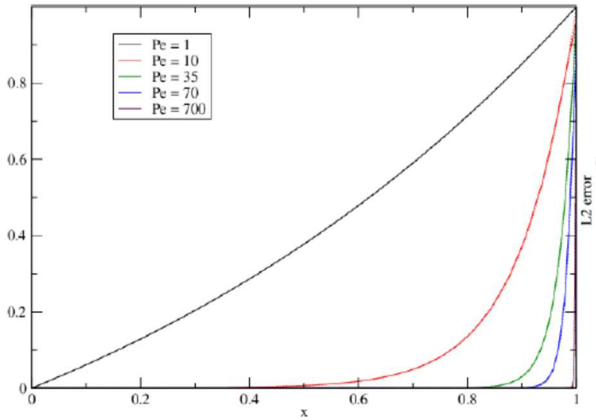
Galerkin Discretization (DG)

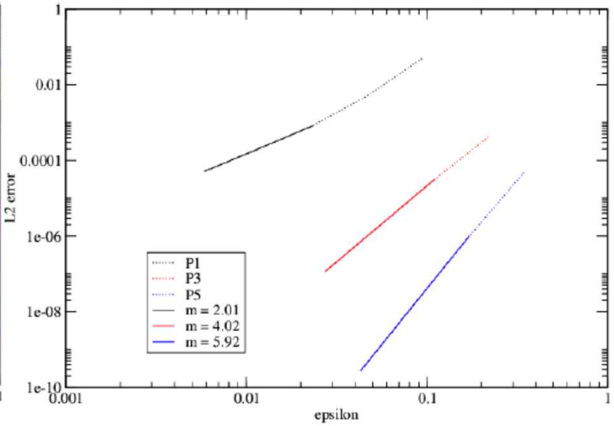
Basis functions





Example (1D)





$$-\varepsilon \Delta u + \mathbf{b} \cdot \nabla u = f$$

Conclusion

- GMLS are fun (?)
- Lot of flexibility
- Lot to do
  - Prove well posedness of GMLS DG formulation
  - 2D/3D implementation
  - ...

Thank you!