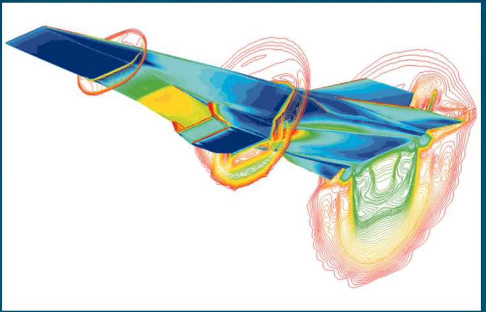
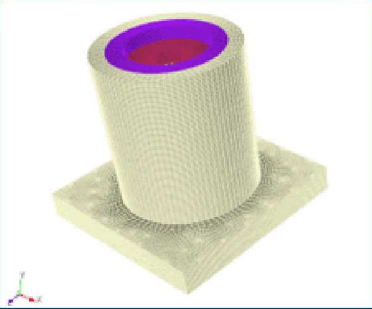
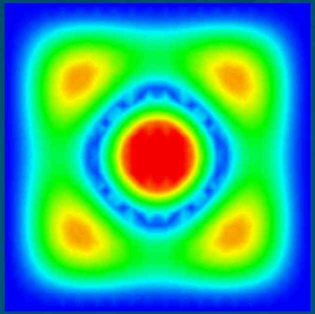
SAND2019-3700PE

Model Order Reduction in Linear and Nonlinear Structural Dynamics







PRESENTED BY

Robert J. Kuether, Sandia National Laboratories

Presented at NMSU on April 5, 2019

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SAND2019-???? PE

Who am I?

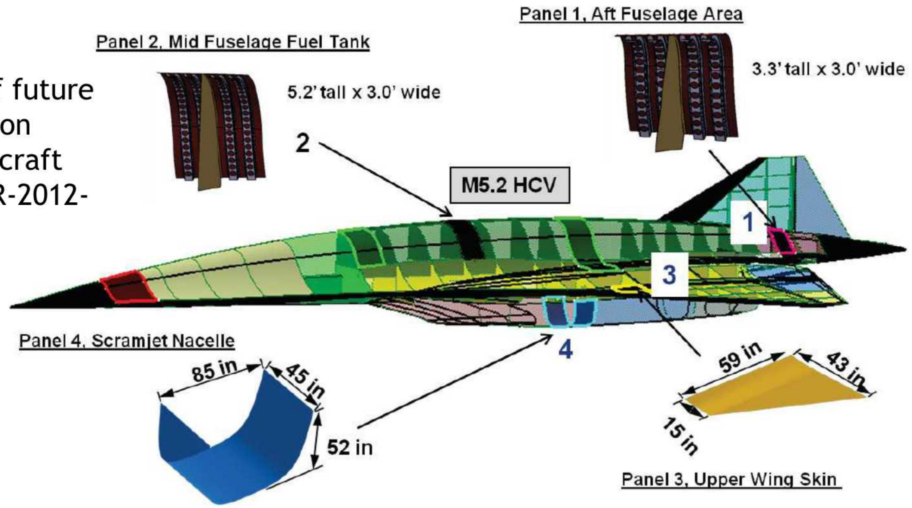
- B.S., M.S. and Ph.D in Engineering Mechanics at University of Wisconsin
  - Focused on computational methods in structural dynamics
  - "Nonlinear Modal Substructuring of Geometrically Nonlinear Finite Element Models"

- Joined Sandia in 2015 as Technical Staff
  - Component Science & Mechanics
  - Research and application work in computational structural dynamics
  - Exploring new nonlinear physics

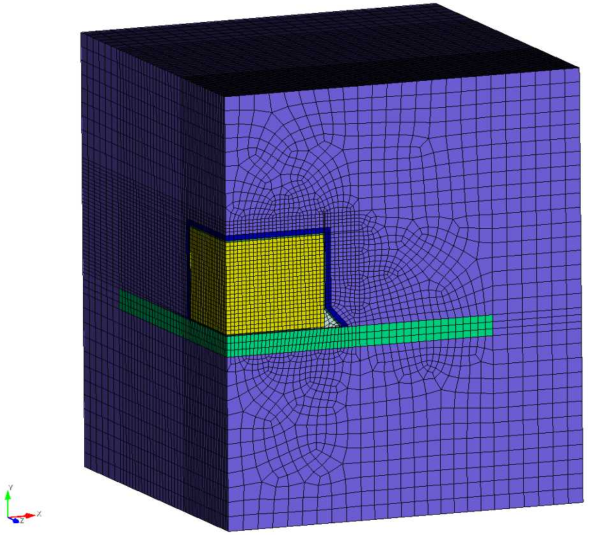
Keywords:

Structural dynamics; reduced order modeling; nonlinear dynamics and vibrations; test-analysis correlation; interface mechanics

Exploratory design of future reusable, long duration cruise high-speed aircraft from AFRL-RQ-WP-TR-2012-0280

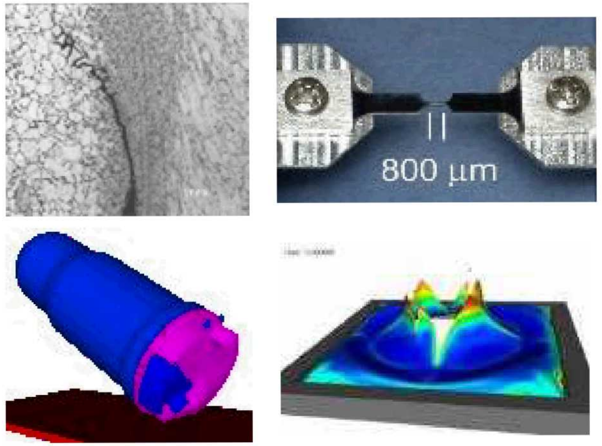


Vibration sensitive electronics potted in foam or polymer to mitigate damaging shock and vibrations

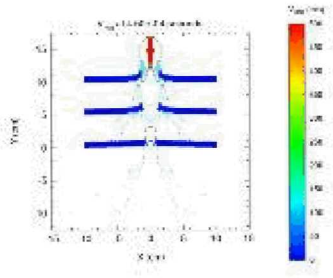


Engineering Sciences Core Technical Areas

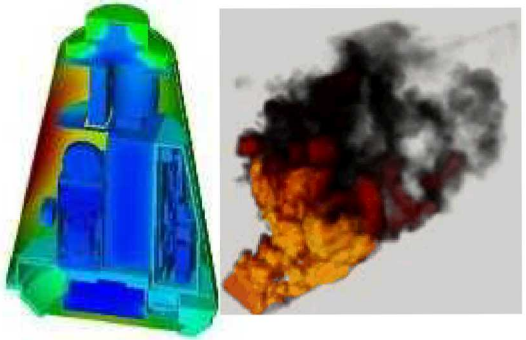
Solid Mechanics



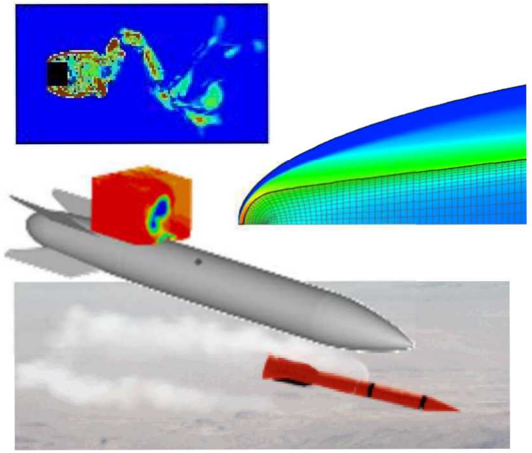
Shock Physics and Energetics



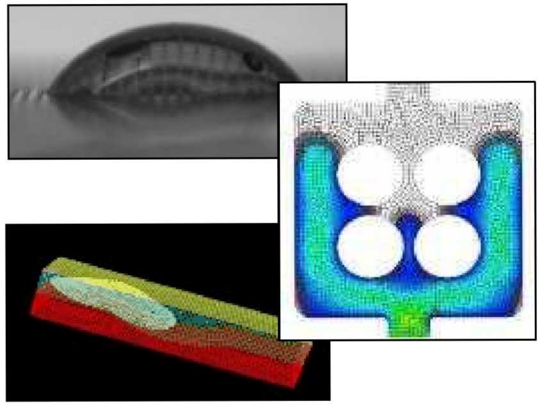
Thermal & Fire Sciences



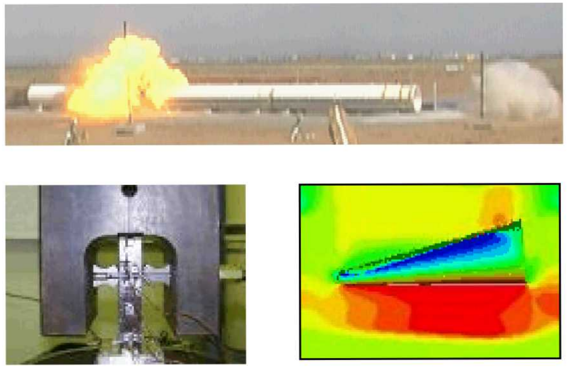
Aerosciences



Fluid Mechanics



Structural Dynamics

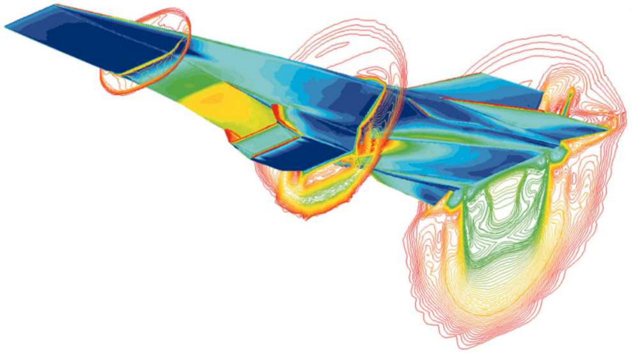


Engineering Sciences Core Technical Areas

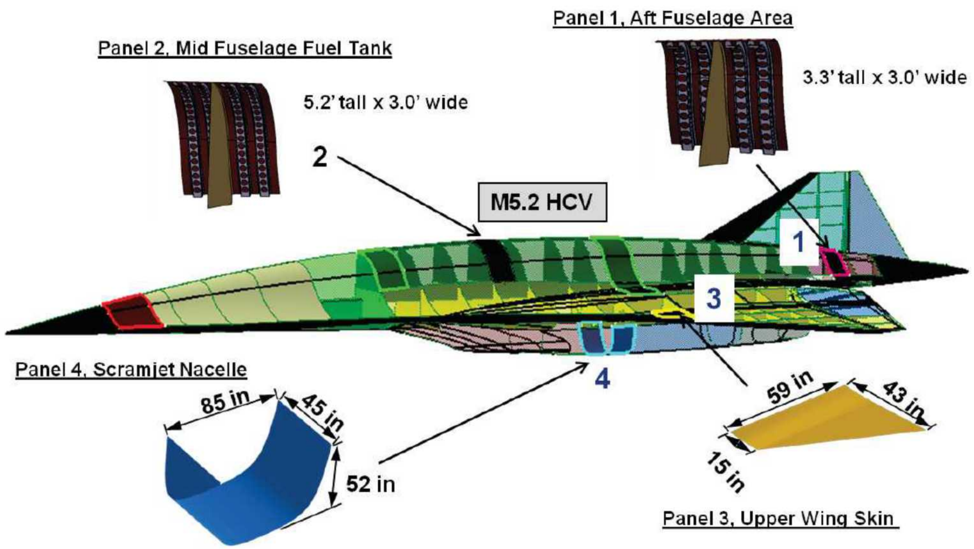
https://www.youtube.com/watch?v=o1qAjLSEv0A

Motivation and Existing Challenges

- Future flight systems seek to push performance envelope
  - High-speed vehicles
  - Spacecraft and satellites
- Harsher Environments / Lighter Designs
  - Less opportunity to overdesign structure
- Design option: operate in nonlinear response regime
  - Need engineering tools to **understand and predict** nonlinear dynamic behavior of large-scale models



NASA X-43A experimental unmanned hypersonic vehicle (left) CFD simulation of Mach 7 flight (right) Mach 7 wind tunnel test full-scale X-43A. Images courtesy of www.dfrc.nasa.gov/Gallery/Photo/X-43A/Large/



Exploratory design of future reusable, long duration cruise hypersonic aircraft from AFRL-RQ-WP-TR-2012-0280 "Predictive Capability for Hypersonic Structural Response and Life Prediction: Phase II – Detailed Design of Hypersonic Cruise Vehicle Hot-Structure"

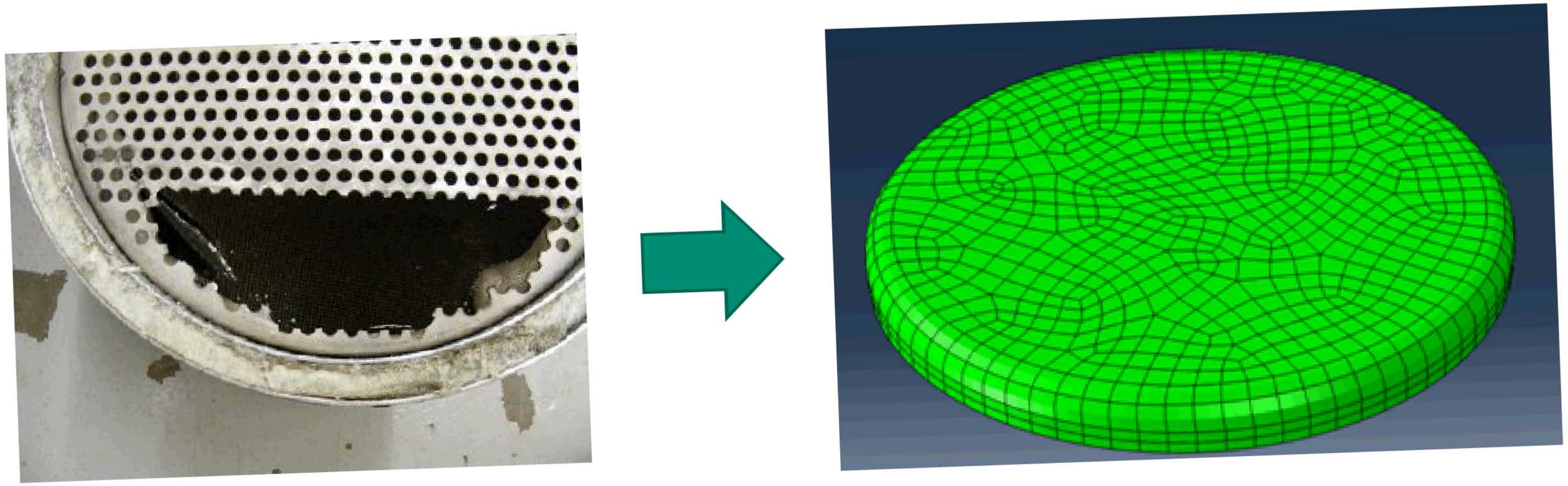
What is a model?

Simplified Models

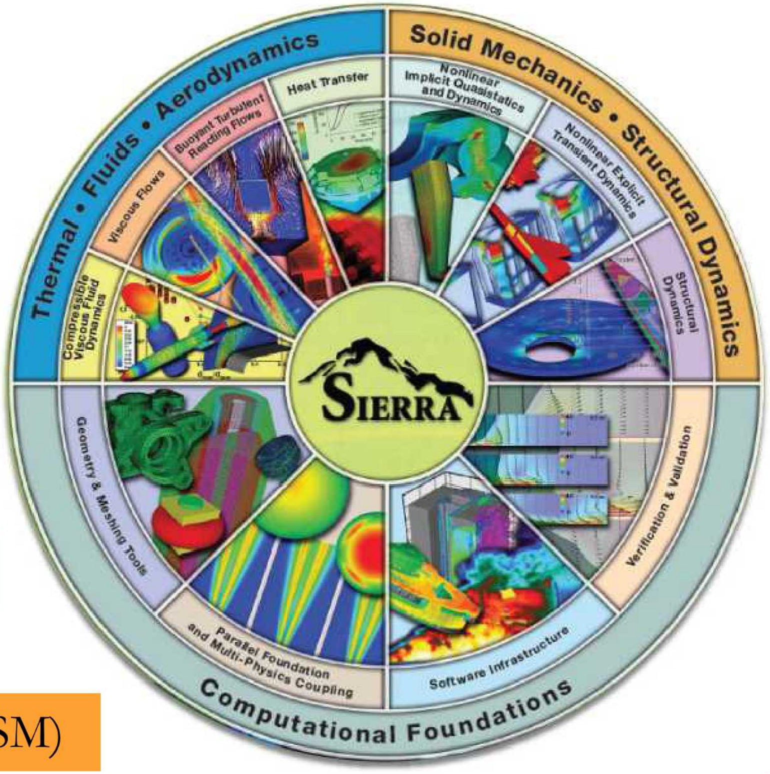
CAD Models

Finite Element Models

Reduced Order Models



Current Approach for Finite Element Models



- Computational simulation capabilities at Sandia
  - High Performance Computing platforms
  - Highly parallelizable, in-house developed finite element software

Current approach to predict structural dynamic response

Sierra Structural Dynamics (Sierra/SD)

- Environmental responses predicted as superposition of linear mode solutions
- Modal analysis assumes linearity (linear elastic, small deflections, no frictional contact, etc..)
- Highly efficient but sacrifices nonlinear physics

Sierra Solid Mechanics (Sierra/SM)

- Captures complex nonlinear phenomena (frictional contact, advance constitutive models, large deflections/rotations, etc..)
- Environmental responses predicted via direct time integration (implicit or explicit)
- Highly representative of physics but lacks efficiency

R&D to Enhance Current Approach

Reduced order models provide a framework that is highly efficient **and** representative of nonlinear physics

Definition of Model Order Reduction

Model order reduction is a technique for reducing the computational complexity and large dimensionality of mathematical models of real-life processes in numerical simulations.

https://en.wikipedia.org/wiki/Model_order_reduction

Modal superposition of undamped MDOF systems

$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{f}(t)$

- Undamped MDOF system

$(\mathbf{K} - \omega_r^2\mathbf{M})\boldsymbol{\phi}_r = \mathbf{0}$

- Real eigenvalue problem

$\boldsymbol{\Phi} = [\boldsymbol{\phi}_1 \quad \boldsymbol{\phi}_2 \quad \cdots \quad \boldsymbol{\phi}_{\widehat{N}}]$

- Collect modes in modal matrix

- Mode superposition introduces the transformation between physical and modal coordinate space

$$\mathbf{x}(t) = \sum_{i=1}^{\widehat{N}} \boldsymbol{\phi}_i q_i(t) = \boldsymbol{\Phi}\mathbf{q}(t)$$

- Substitute into equations-of-motion and pre-multiply by $\boldsymbol{\Phi}^{\mathrm{T}}$

$$\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{M}\boldsymbol{\Phi}\ddot{\mathbf{q}}(t) + \boldsymbol{\Phi}^{\mathrm{T}}\mathbf{K}\boldsymbol{\Phi}\mathbf{q}(t) = \boldsymbol{\Phi}^{\mathrm{T}}\mathbf{f}(t)$$

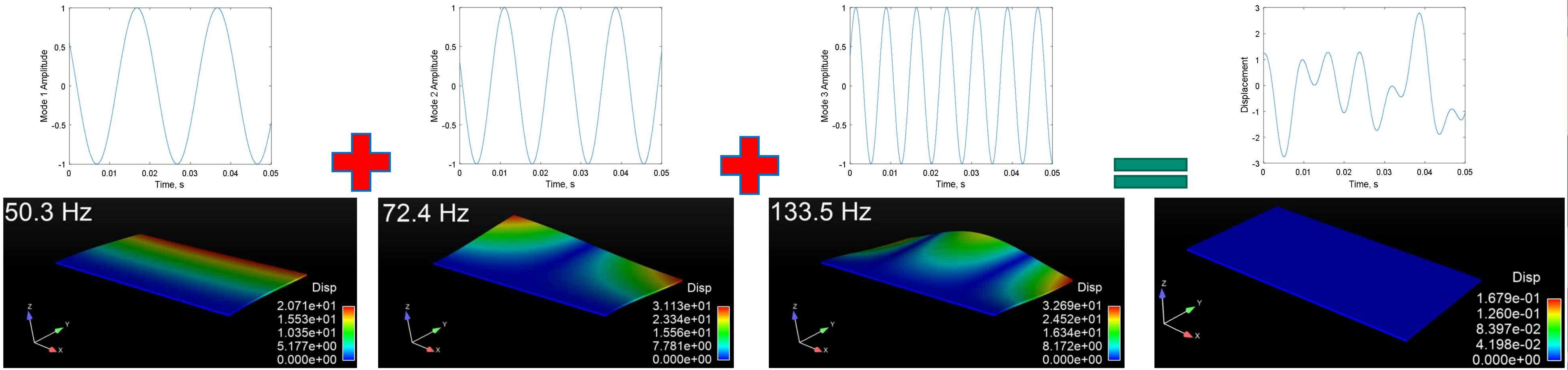
- Orthogonality properties of the real eigenvectors result in decoupled SDOF equations

$$\ddot{q}_r(t) + \omega_r^2 q_r(t) = \boldsymbol{\phi}_r^{\mathrm{T}}\mathbf{f}(t)$$

Modal superposition of undamped MDOF systems

Linear Superposition

$$\mathbf{x}(t) = \sum_{i=1}^{\hat{N}} \boldsymbol{\phi}_i q_i(t)$$

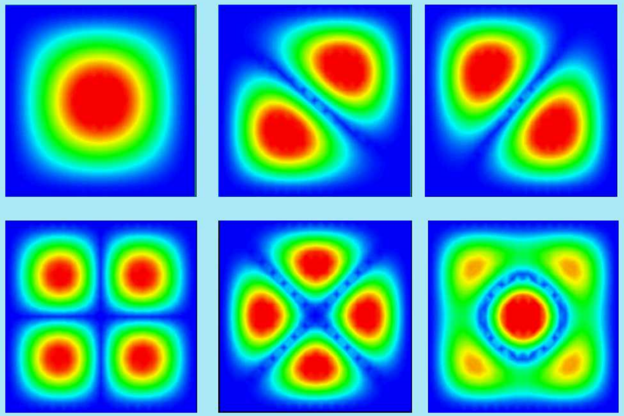


Model order reduction with nonlinearities

FE mesh in physical coordinates

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \mathbf{f}_{int}(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{f}_{ext}(t)$$

Determine appropriate basis, or shape vectors based on the physical equations of motion



Project full equations of motion onto a small set of basis vectors

Related by transformation:

$$\mathbf{x}(\mathrm{t}) = \mathbf{T}\mathbf{q}(t)$$

$$^{*}\mathbf{q}(\mathrm{t}) \ll \mathbf{x}(t)$$

Solve reduced equations

$$\bar{\mathbf{M}}\ddot{\mathbf{q}} + \bar{\mathbf{K}}\mathbf{q} + \mathbf{L}^{\mathrm{T}}\mathbf{f}_{int}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{L}^{\mathrm{T}}\mathbf{f}_{ext}(t)$$

Model order reduction with nonlinearities

Galerkin vs Petrov-Galerkin [1]

Direct or indirect reduction

Proper Orthogonal Decomposition vs Eigenvalue Analysis [2]

Linear subspace vs nonlinear manifold subspace

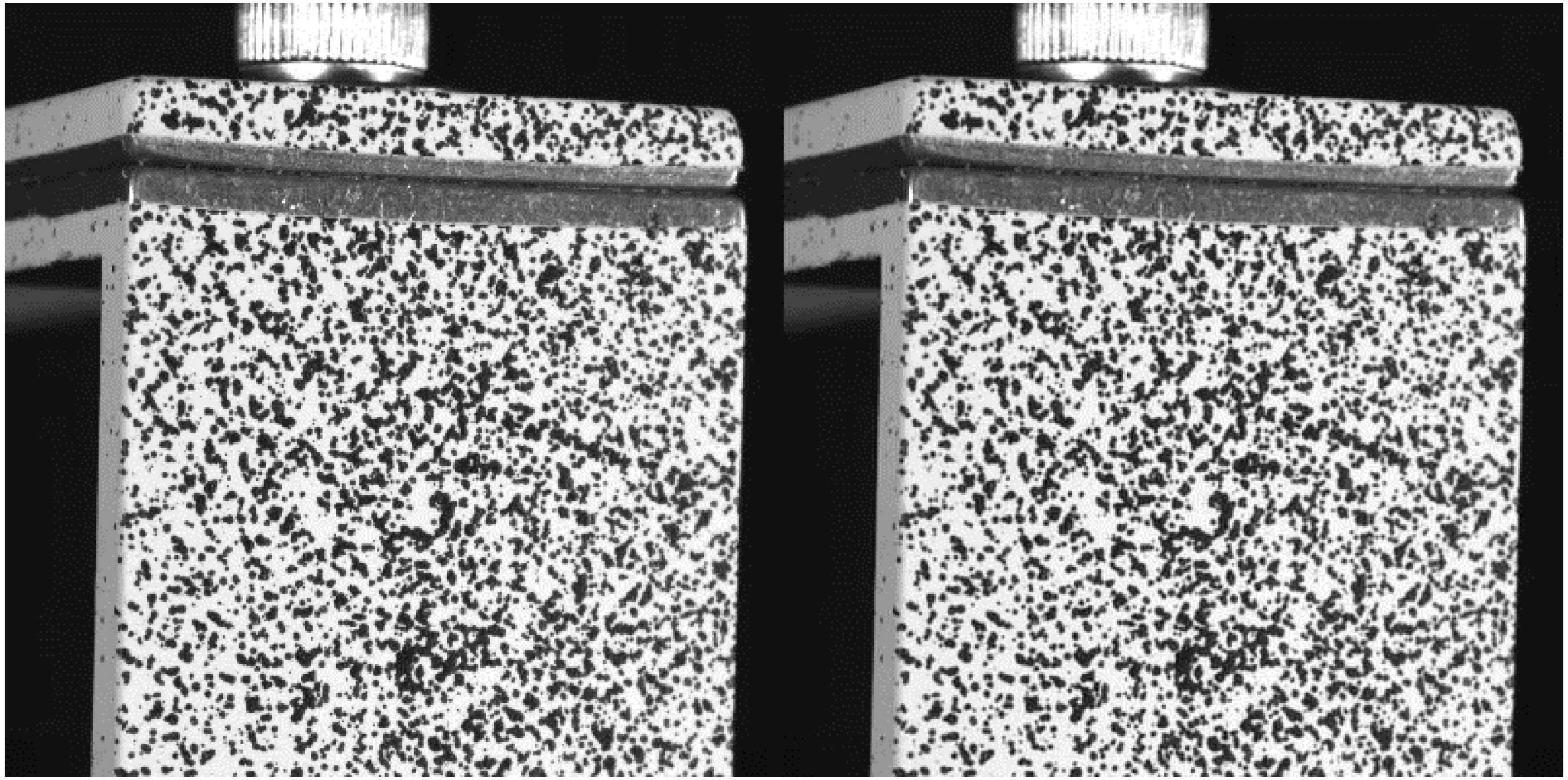
[1] Carlberg, Kevin, Charbel Bou-Mosleh, and Charbel Farhat. "Efficient non-linear model reduction via a least-squares Petrov-Galerkin projection and compressive tensor approximations." *International Journal for Numerical Methods in Engineering* 86.2 (2011): 155-181.
[2] Lülf, Fritz Adrian, Duc-Minh Tran, and Roger Ohayon. "Reduced bases for nonlinear structural dynamic systems: A comparative study." *Journal of Sound and Vibration* 332.15 (2013): 3897-3921.

Overview of NROM topics

Nonlinear frictional contact at mechanical interfaces

- Whole joint modeling
- Interface reduction

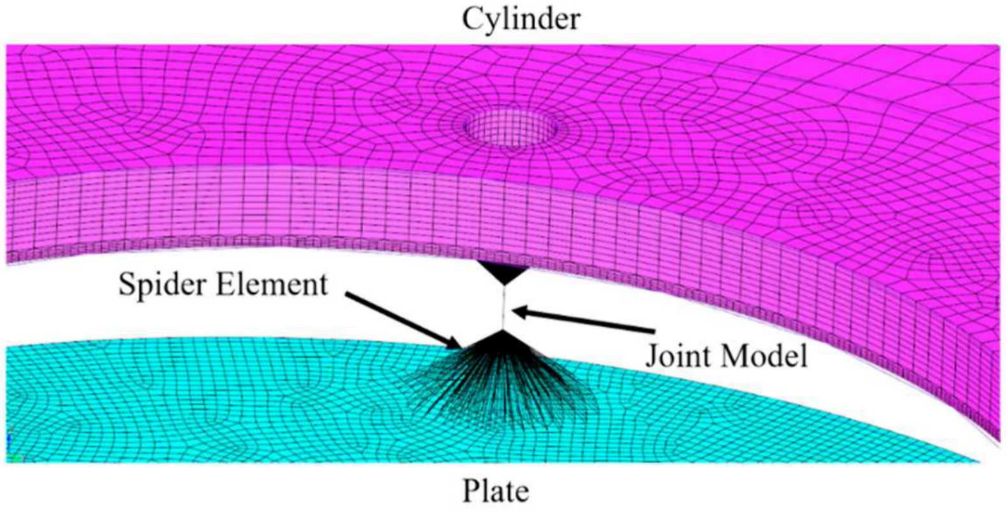
NROMs: Interface Mechanics



NROMs: Interface Mechanics

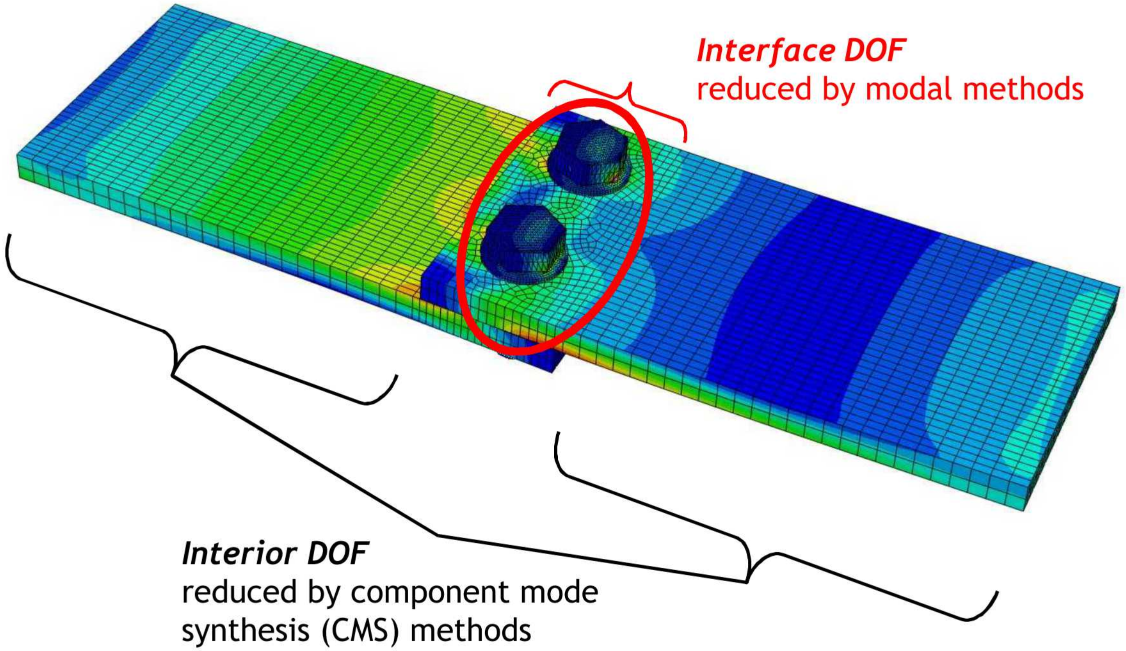
- Two philosophies to develop efficient reduced order models

Whole Joint Modeling
(Rigid Interfaces)



- Goal: Estimate/calibrate the joint parameters in the whole joint reduced order models to match response from high-fidelity models and/or experiments

Interface Reduction
(Flexible Interfaces)



- Goal: keep full kinematics and nonlinear elements, and apply interface reduction

Overview of NROM topics

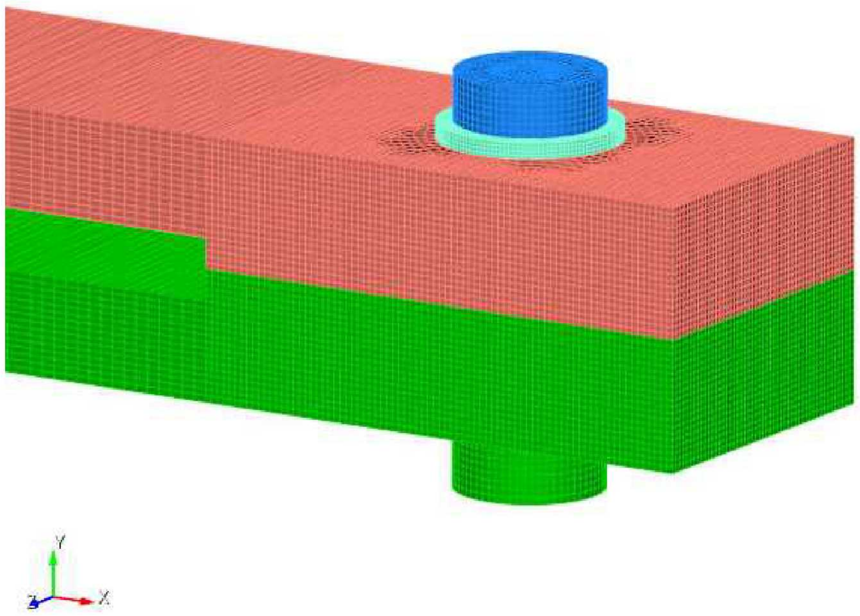
Nonlinear frictional contact at mechanical interfaces

- Whole joint modeling
- Interface reduction

Objectives of whole joint modeling R&D

Contact areas in high fidelity finite element models simplified by “spidering” surface to a single node and modeling joint forces as a 1D constitutive law

Global optimization to calibrate whole joint parameters to match global response



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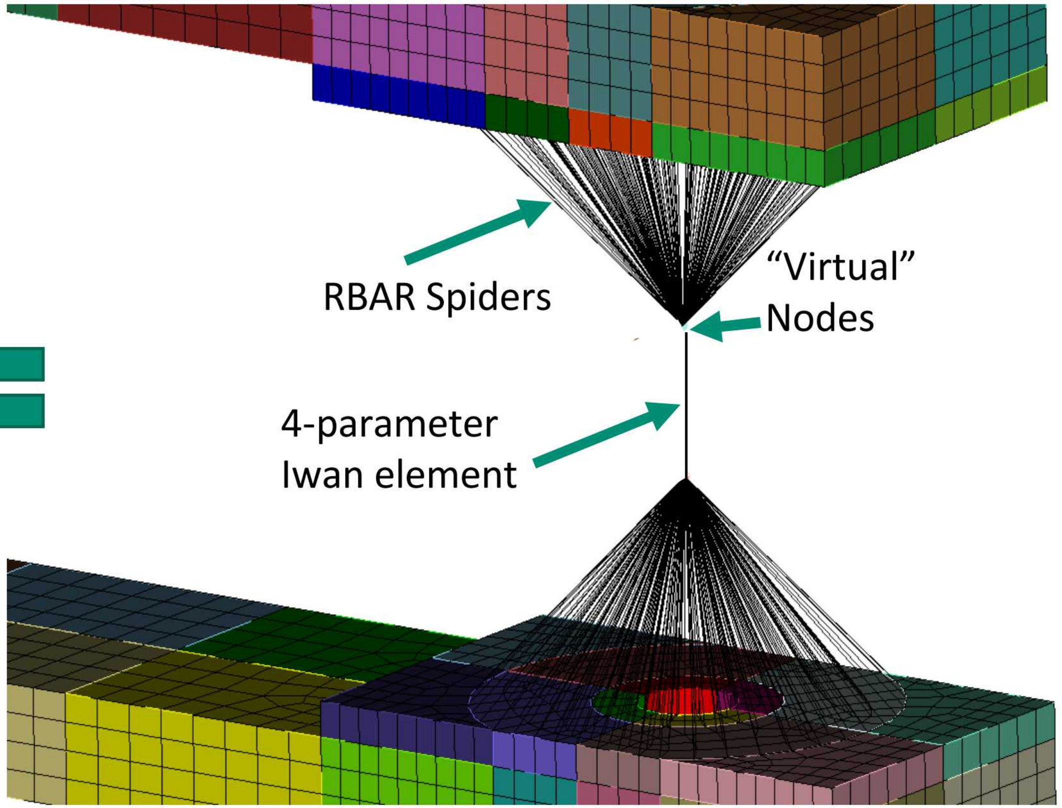

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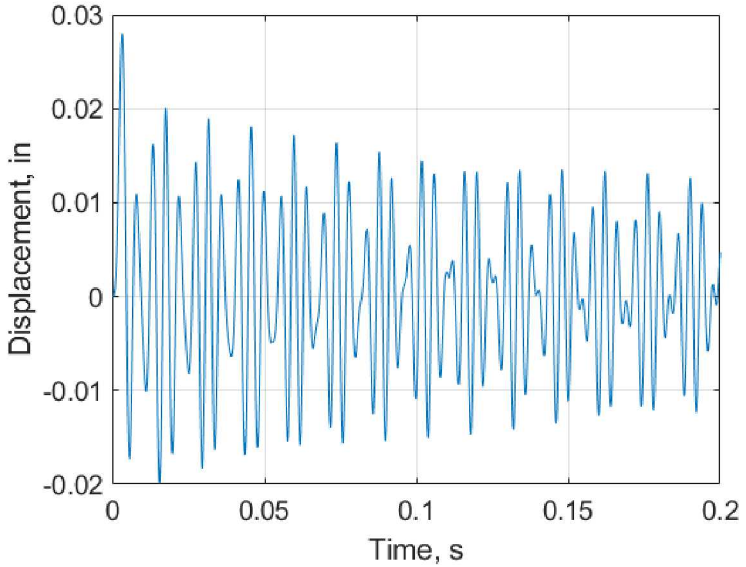
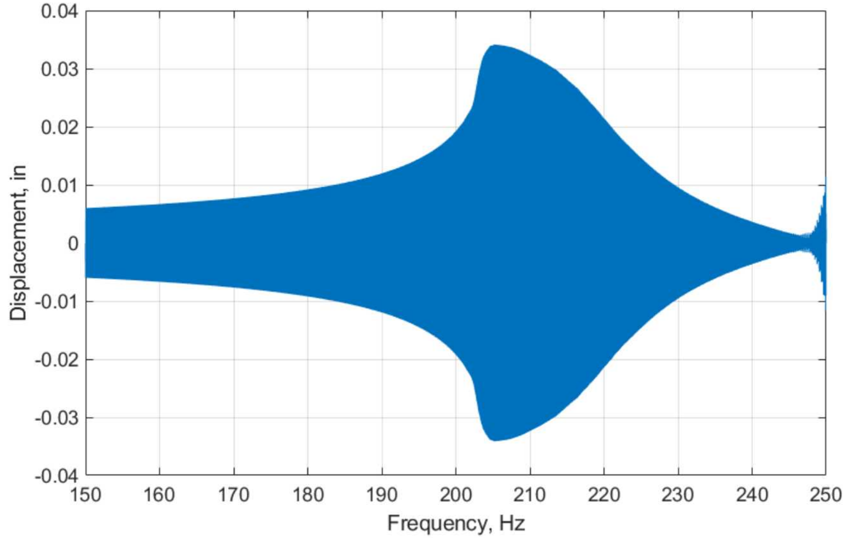
What global response metrics should be preserved?

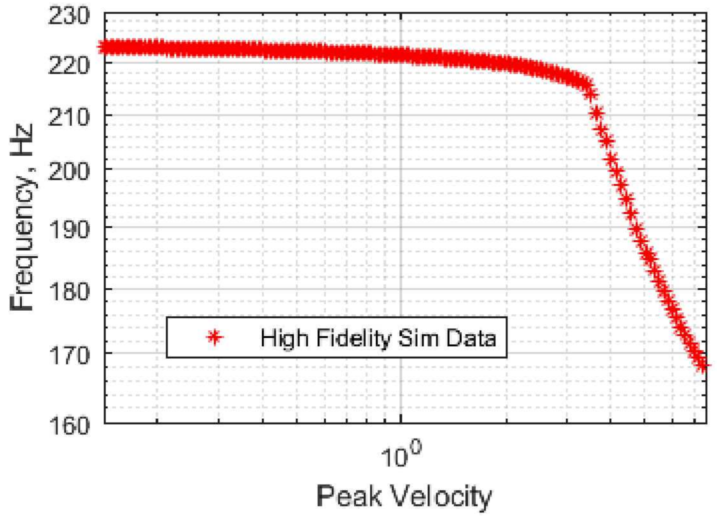
Transient Response?

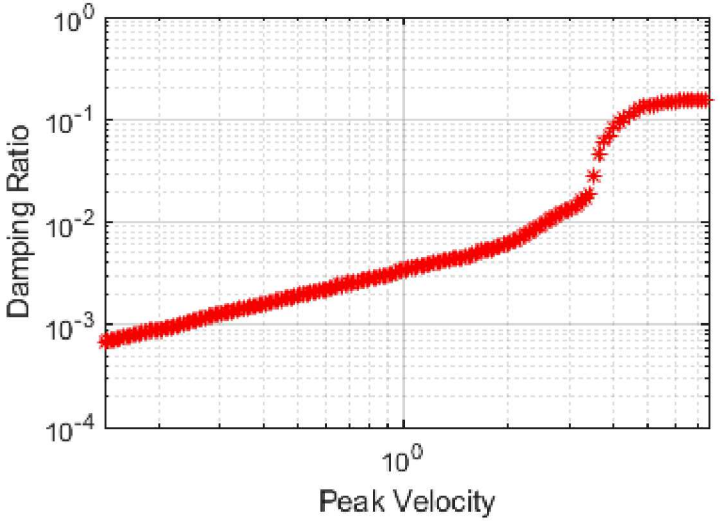


Swept/Stepped Sine Response?

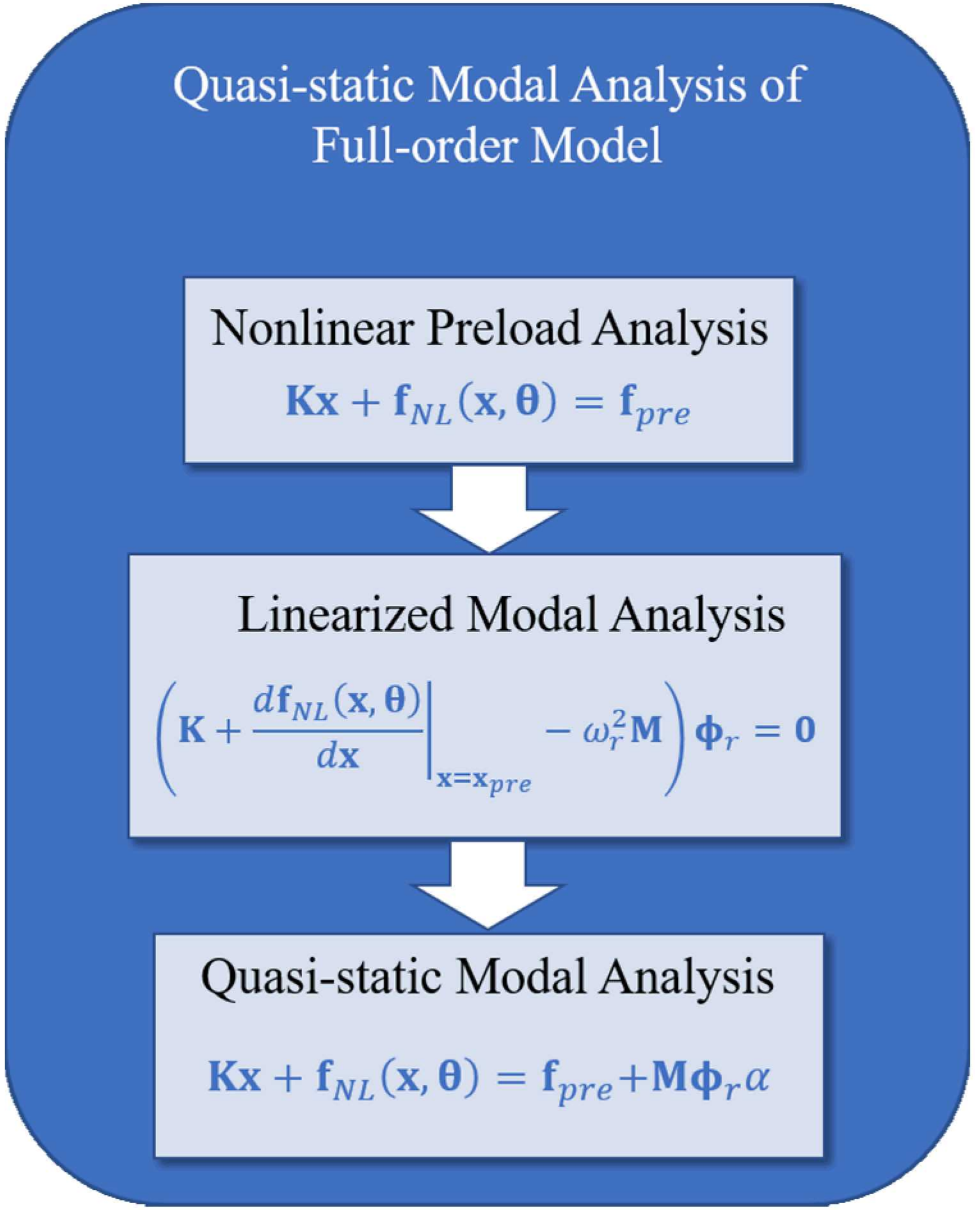


Amplitude dependent natural frequency and damping ratio

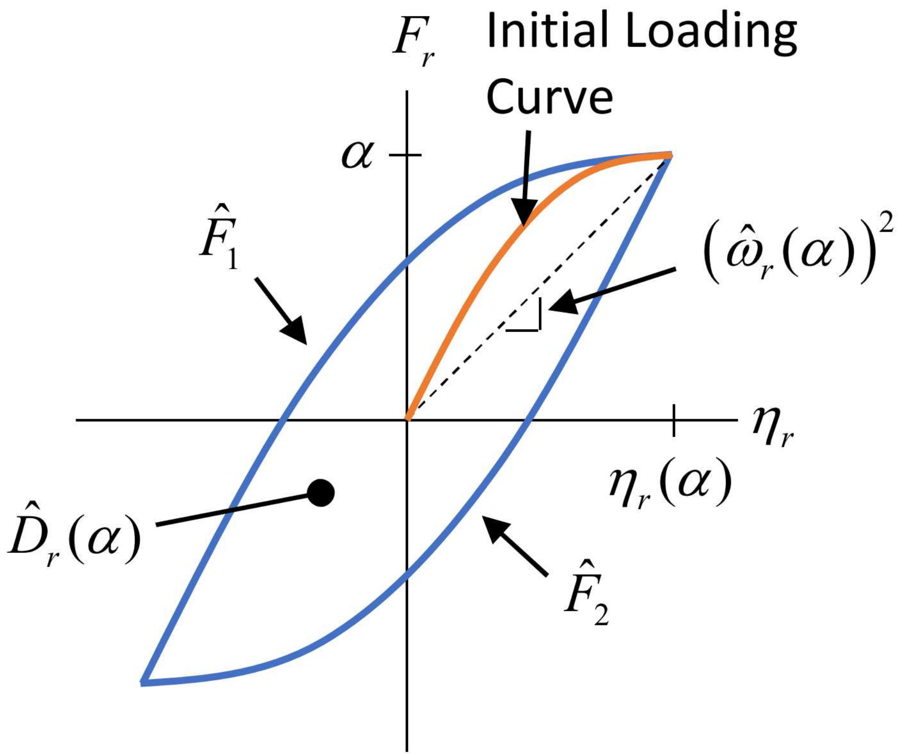




Quasi-static Modal Analysis

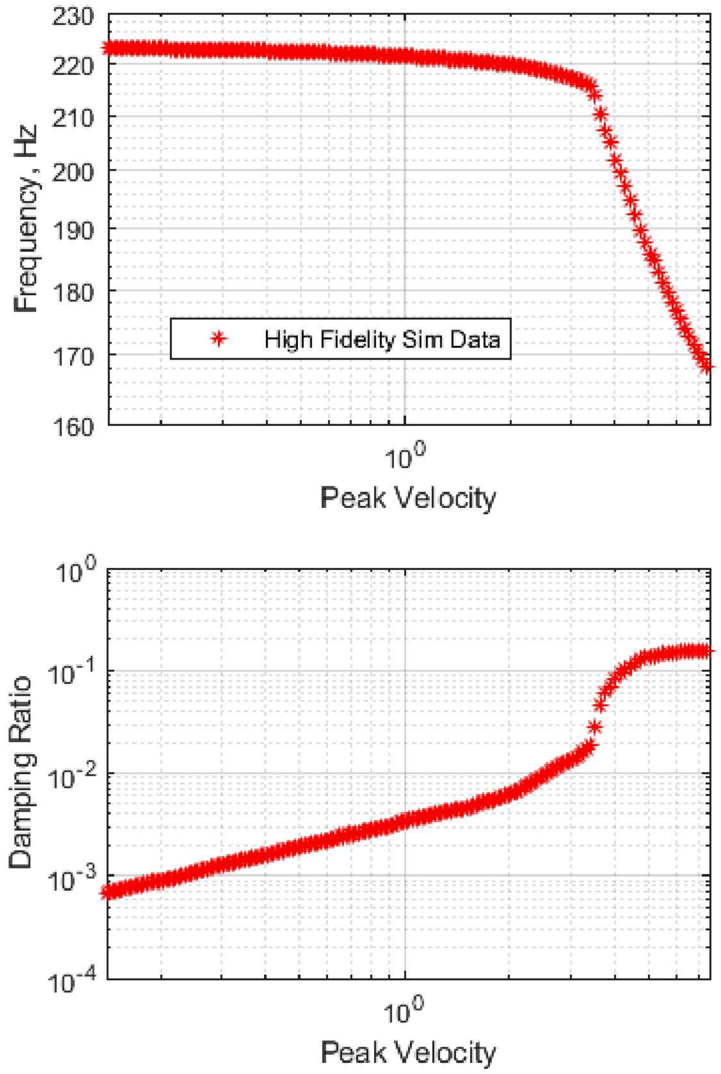


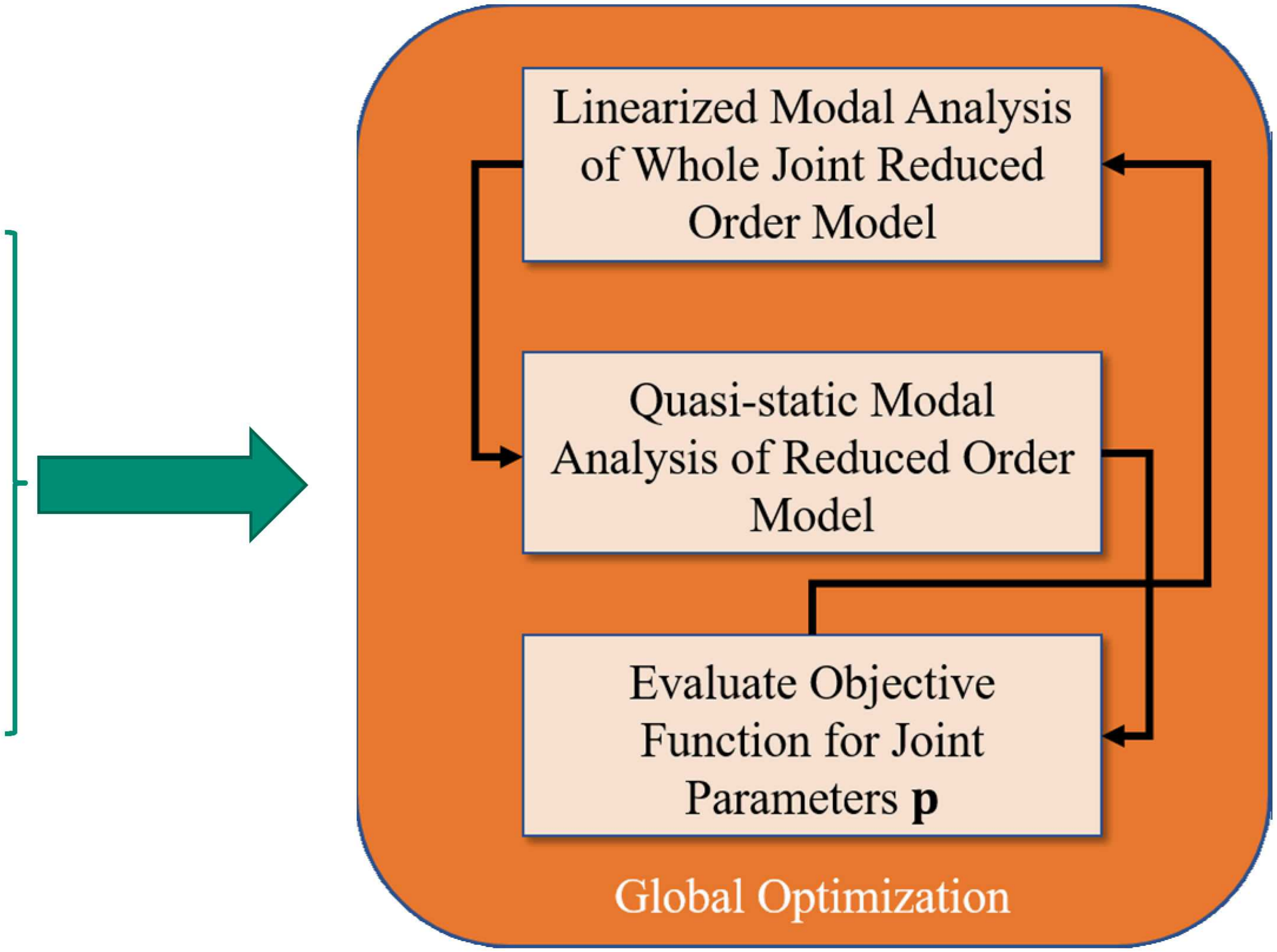
Estimate modal amplitude dependent natural frequencies, $\omega_r(\alpha)$, and damping ratios, $\zeta_r(\alpha)$, of high-fidelity model and reduced models with whole joints [1]



[1] M. S. Allen, R. M. Lacayo, and M. R. W. Brake, "Quasi-static Modal Analysis based on Implicit Condensation for Structures with Nonlinear Joints," presented at the ISMA2016 - International Conference on Noise and Vibration Engineering, Leuven, Belgium, 2016.

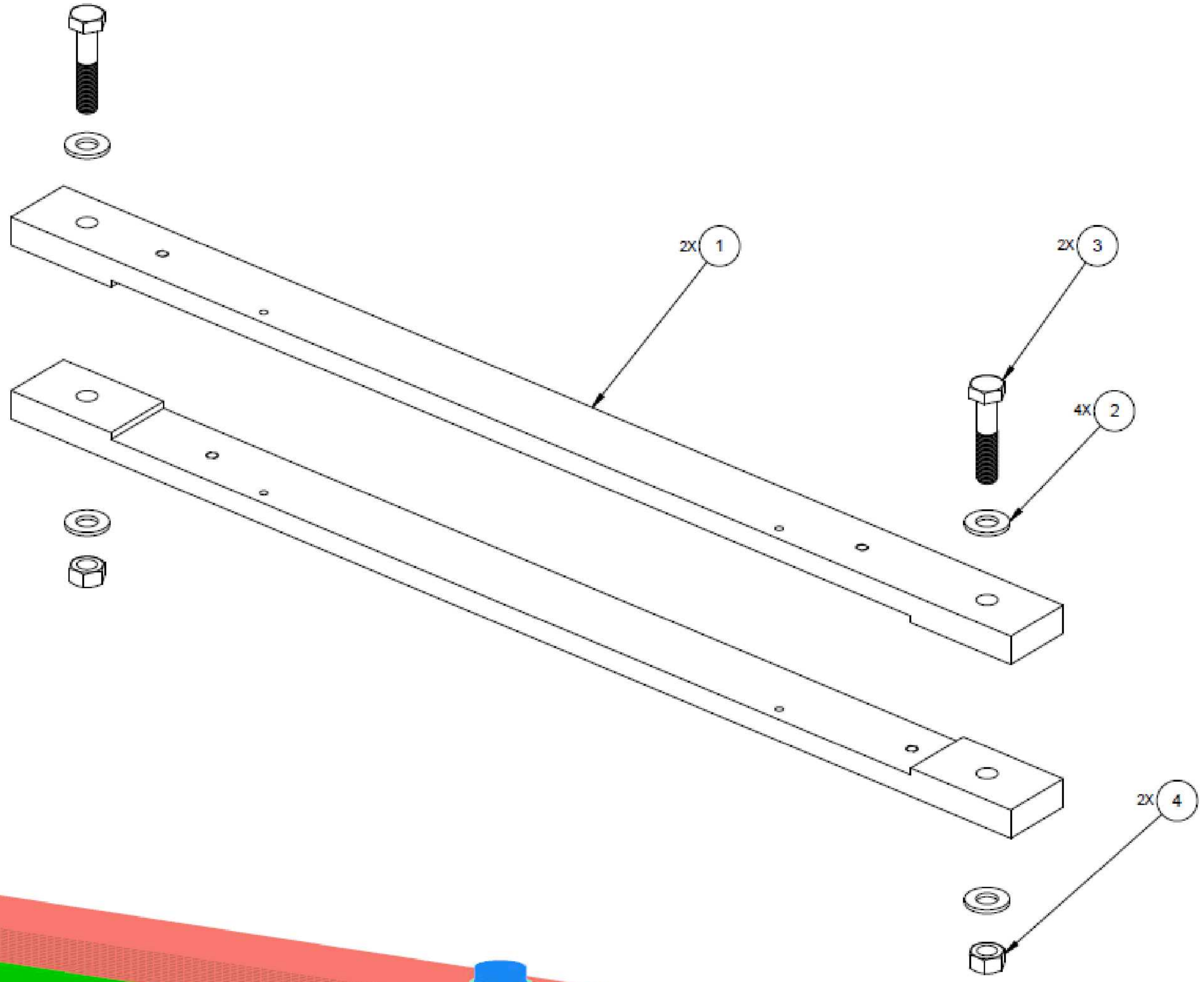
Whole joint calibration via multi-objective optimization [1]

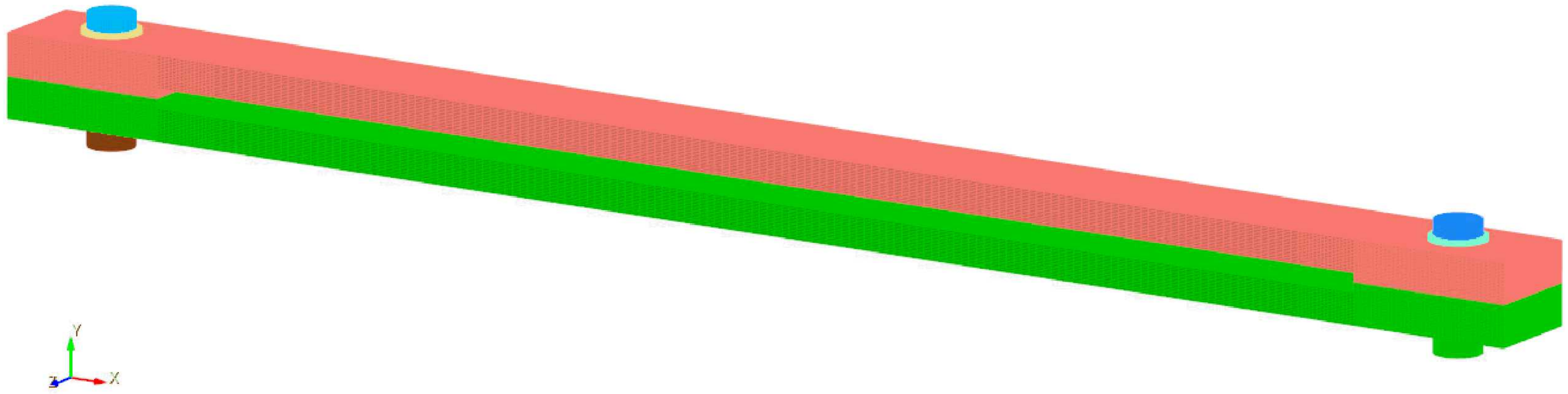




[1] F.-A. Fortin, F.-M. D. Rainville, M.-A. Gardner, M. Parizeau, and C. Gagné, "DEAP: Evolutionary algorithms made easy," *Journal of Machine Learning Research,* vol. 13, pp. 2171-2175, 2012.

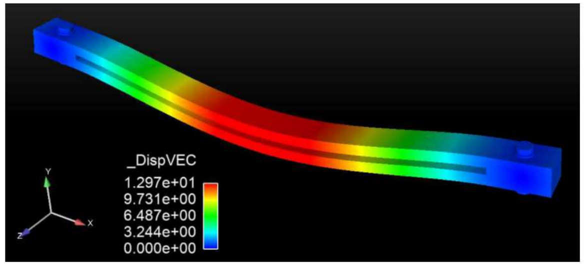
C-Beam Benchmark Example

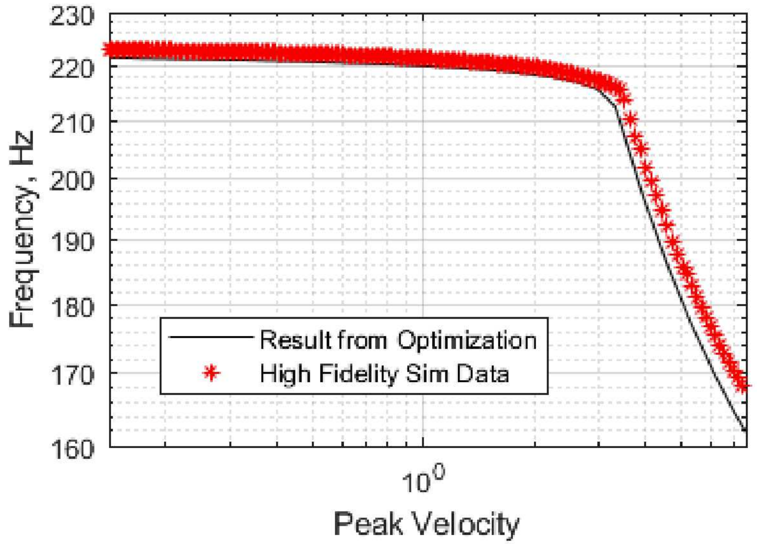


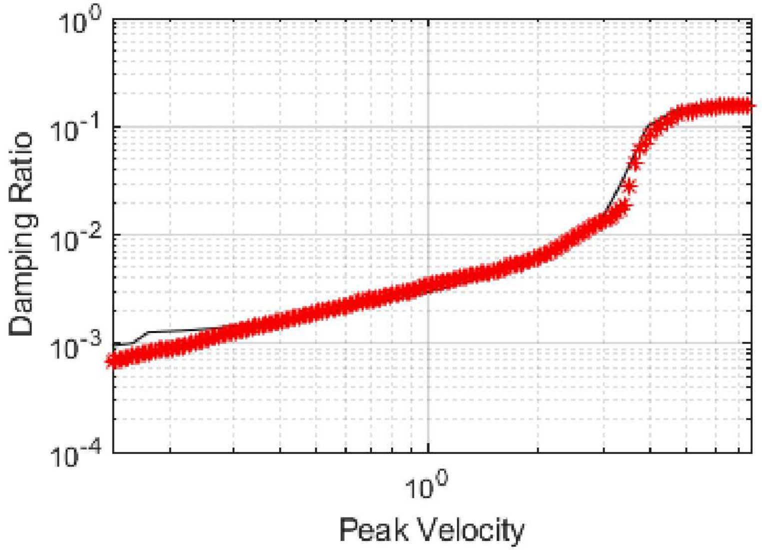


Multi-mode whole joint model calibration

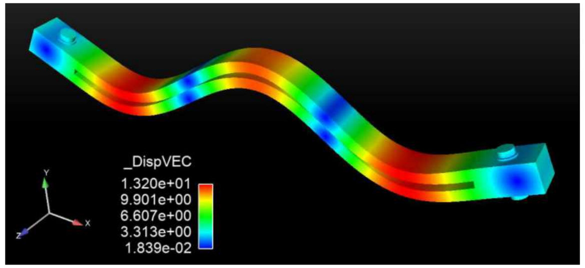
Mode 1 223 Hz

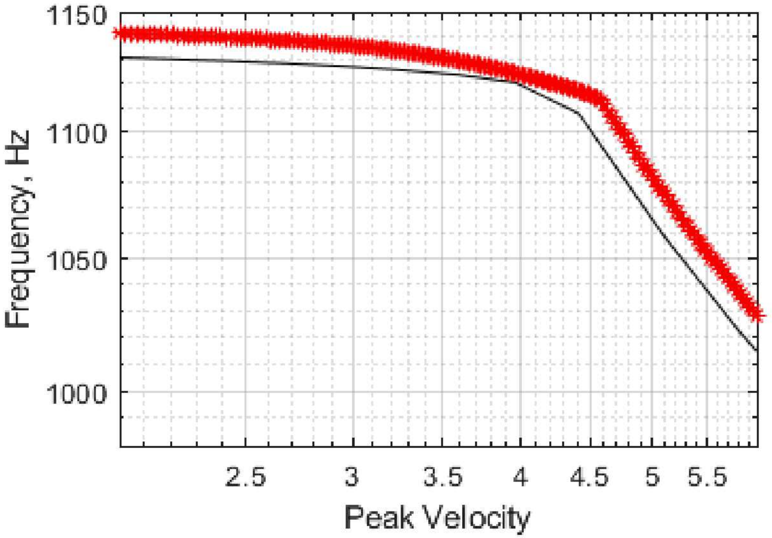


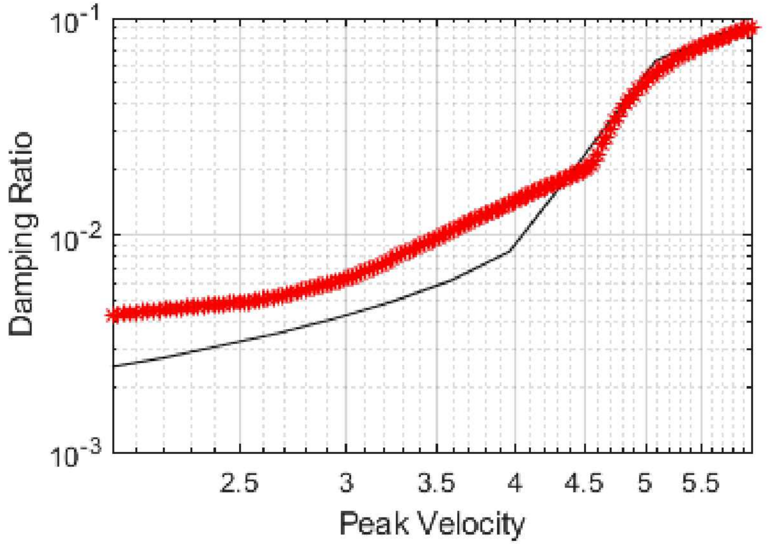




Mode 8 1142 Hz





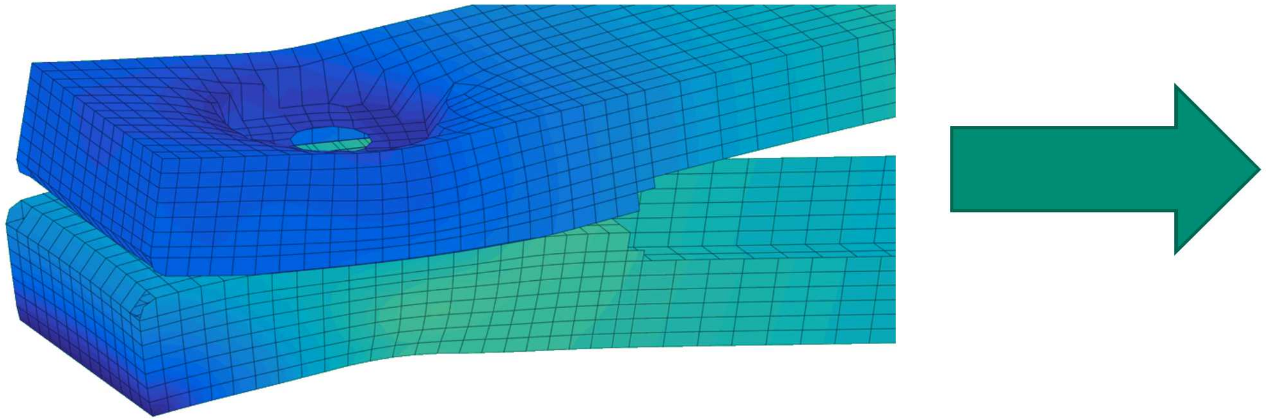


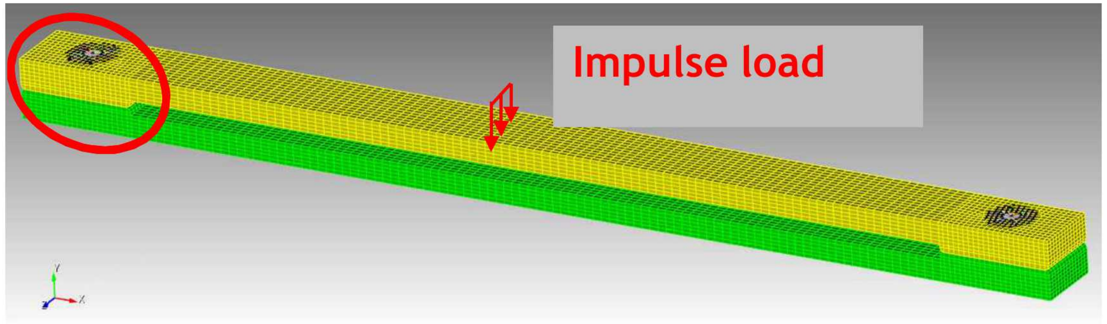
Overview of NROM topics

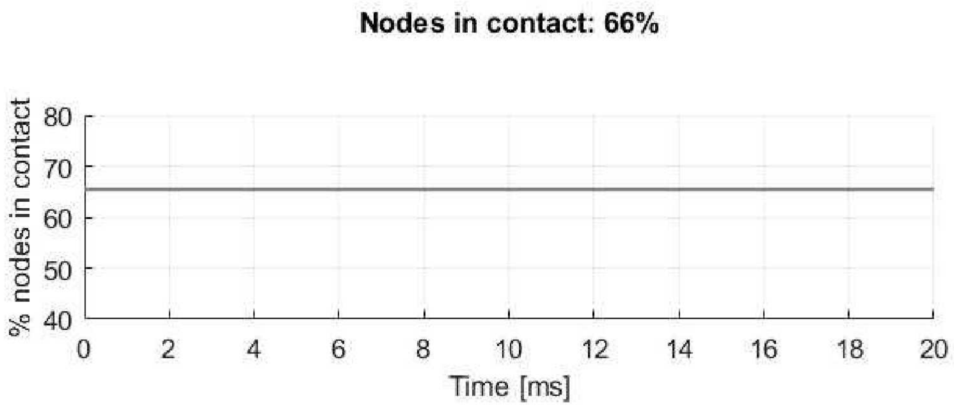
Nonlinear frictional contact at mechanical interfaces

- Whole joint modeling
- Interface reduction

What if the joint is flexible?







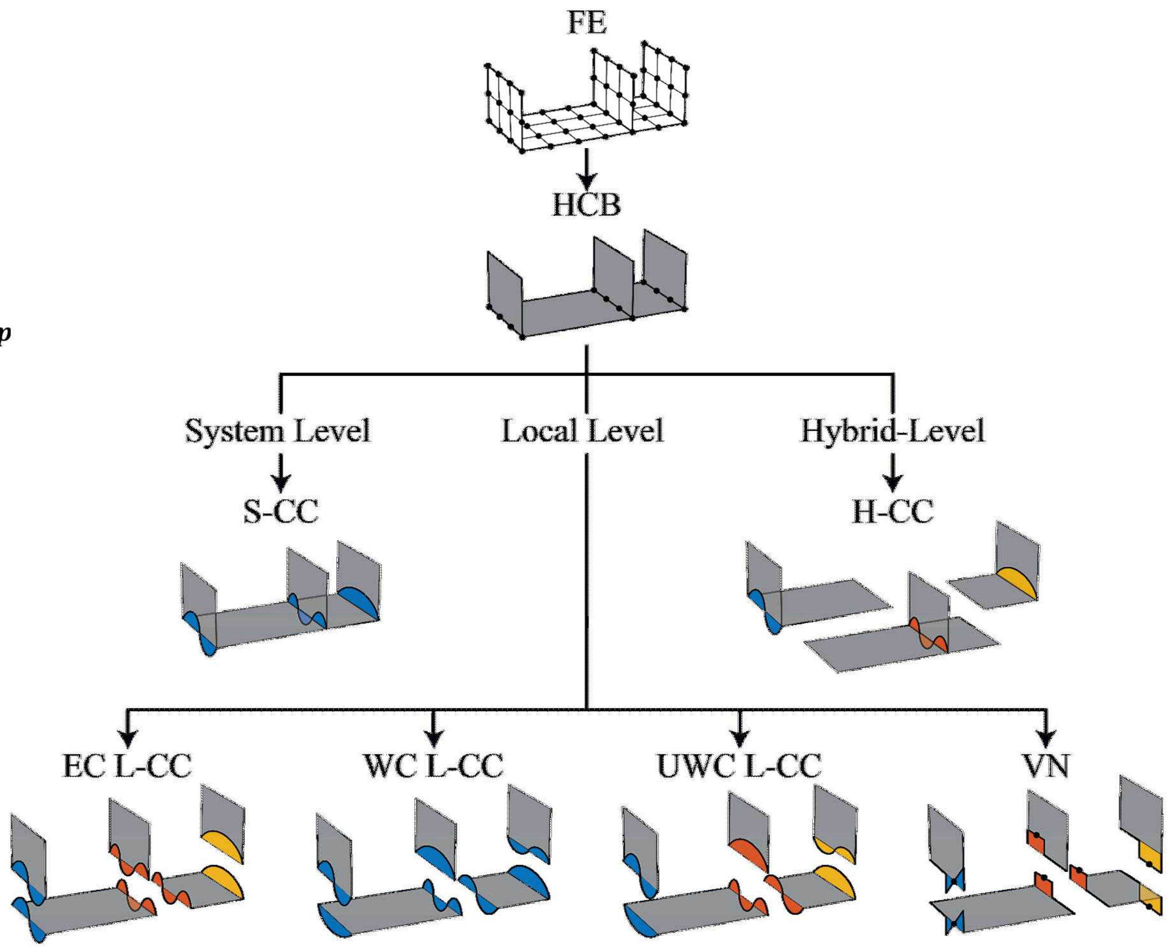
Interface reduction applied to Hurty/Craig-Bampton (HCB) substructures

HCB reduced model dominated by potentially thousands of r-set DOF

$$\begin{bmatrix} \mathbf{I}_{ii} & \mathbf{M}_{ir}^{HCB} & \mathbf{M}_{ip}^{HCB} \\ \mathbf{M}_{ri}^{HCB} & \mathbf{M}_{rr}^{HCB} & \mathbf{M}_{rp}^{HCB} \\ \mathbf{M}_{pi}^{HCB} & \mathbf{M}_{pr}^{HCB} & \mathbf{M}_{pp}^{HCB} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}}_i \\ \ddot{\mathbf{u}}_r \\ \ddot{\mathbf{u}}_p \end{Bmatrix} + \begin{bmatrix} \mathbf{\Lambda}_{ii}^{FI} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{rr}^{HCB} & \mathbf{K}_{rp}^{HCB} \\ \mathbf{0} & \mathbf{K}_{pr}^{HCB} & \mathbf{K}_{pp}^{HCB} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_i \\ \mathbf{u}_r \\ \mathbf{u}_p \end{Bmatrix} + \begin{Bmatrix} \mathbf{0} \\ \mathbf{f}_r(\mathbf{u}_r) \\ \mathbf{0} \end{Bmatrix} = \begin{Bmatrix} \\ \\ \mathbf{f}_p \end{Bmatrix}$$

Research challenge: how can we further reduce these equations?

Explored the extension of interface reduction techniques [1] to problems involving nonlinear contact [2,3]



[1] Krattiger, D. et al. "Interface reduction for Hurty/Craig-Bampton substructured models: Review and improvements," Mechanical Systems and Signal Processing, 114, pp 579-603, 2019.
[2] Kuether RJ, Coffin PB, Brink AR "On Hurty/Craig-Bampton Substructuring With Interface Reduction on Contacting Surfaces," ASME International Design Engineering Technical Conferences and Computers and Information in Engineering Conference, *Volume 8: 29th Conference on Mechanical Vibration and Noise*.
[3] Hughes, P.J. et al. "Interface Reduction on Hurty/Craig-Bampton Substructures with Frictionless Contact," 2018 International Modal Analysis Conference (IMAC) XXXVI, Orlando, FL, 2018.

System-level characteristic constraint modes

Solve the quasi-static version of the HCB model for preloaded equilibrium

$$\begin{bmatrix} \mathbf{\Lambda}_{ii}^{FI} & 0 & 0 \\ 0 & \mathbf{K}_{rr}^{HCB} & \mathbf{K}_{rp}^{HCB} \\ 0 & \mathbf{K}_{pr}^{HCB} & \mathbf{K}_{pp}^{HCB} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_i \\ \mathbf{u}_r \\ \mathbf{u}_p \end{Bmatrix} + \begin{Bmatrix} 0 \\ \mathbf{f}_r(\mathbf{u}_r) \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \mathbf{f}_{\mathbf{pre}} \end{Bmatrix}$$

Apply a secondary reduction about the preloaded equilibrium such that

$$\mathbf{v} = \begin{Bmatrix} \mathbf{q}_i \\ \mathbf{u}_r \\ \mathbf{u}_p \end{Bmatrix} = \mathbf{v}_{pre} + \begin{bmatrix} \mathbf{I} & 0 & 0 \\ 0 & \mathbf{\Phi}^{SCC} & \mathbf{\Psi}^{\mathbf{SCCe}} \\ 0 & 0 & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_i \\ \mathbf{q}_r \\ \mathbf{u}_p \end{Bmatrix} = \mathbf{v}_{pre} + \mathbf{T}^{SCCe}\mathbf{w}$$

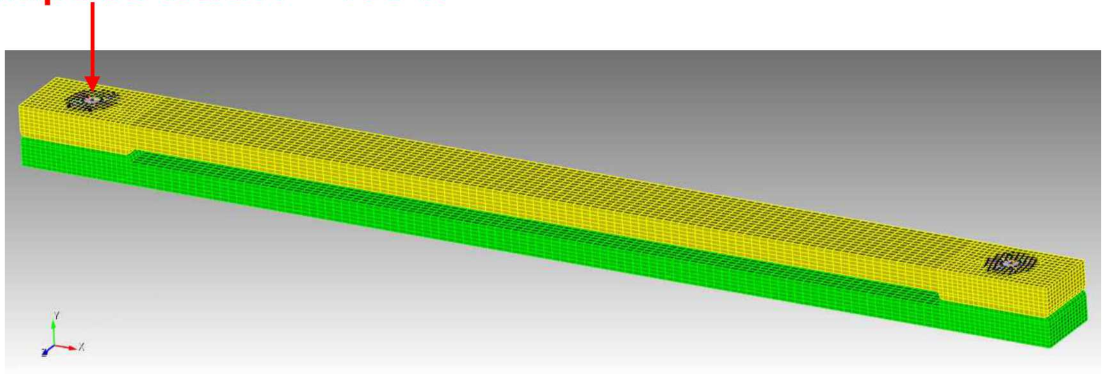
where the tangent S-CC modes and static constraint modes computed about preloaded state

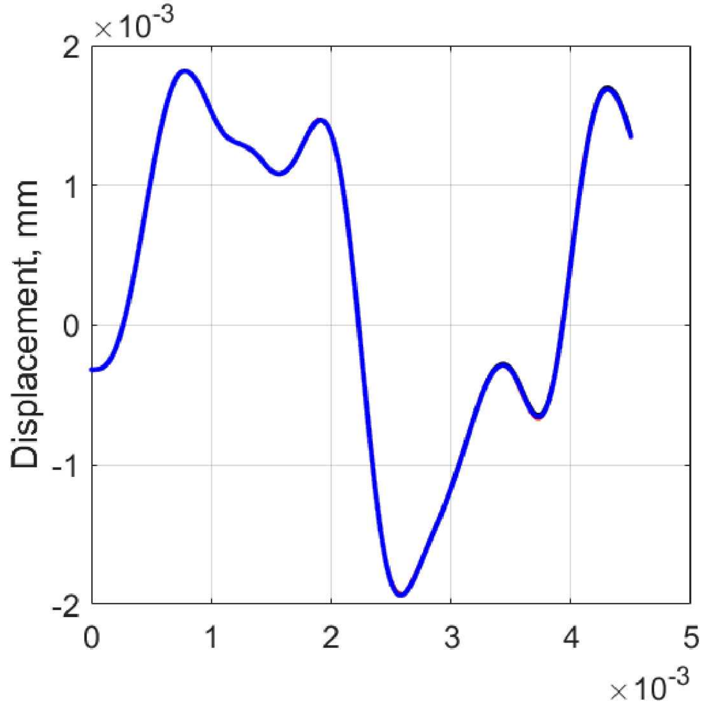
$$\left[\mathbf{K}_{rr}^{HCB} + \left.\frac{\partial \mathbf{f}_r(\mathbf{u}_r)}{\partial \mathbf{u}_r}\right|_{\mathbf{v}_{pre}} - \left(\omega^{SCC}\right)^2 \mathbf{M}_{rr}^{HCB}\right] \mathbf{\phi}_s^{SCC} = 0 \qquad \mathbf{\Psi}^{\mathbf{SCCe}} = -\left(\mathbf{K}_{rr}^{HCB} + \left.\frac{\partial \mathbf{f}_r(\mathbf{u}_r)}{\partial \mathbf{u}_r}\right|_{\mathbf{v}_{pre}}\right)^{-1} \mathbf{K}_{rp}^{HCB}$$

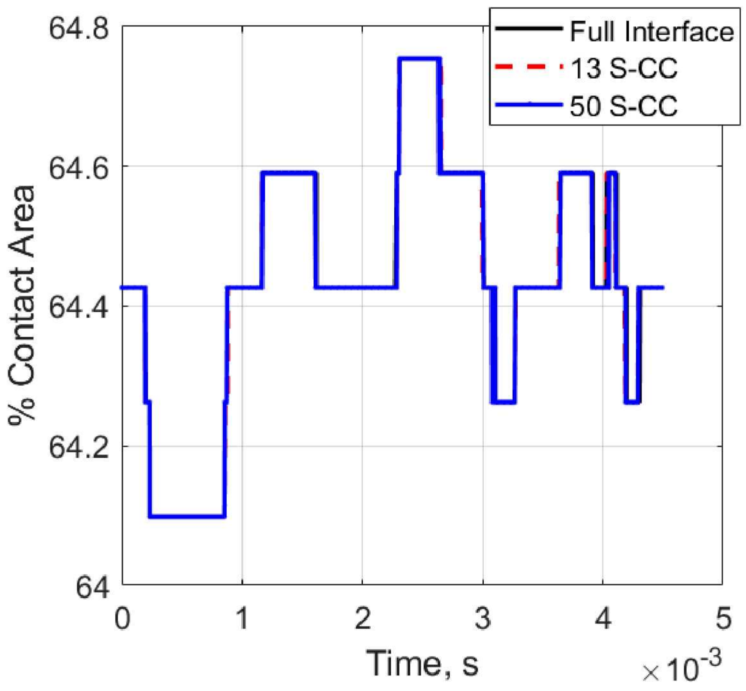
Tangent stiffness contributions about deformed state

Time-domain simulations due to impulse load

Impulse load A = 100 N

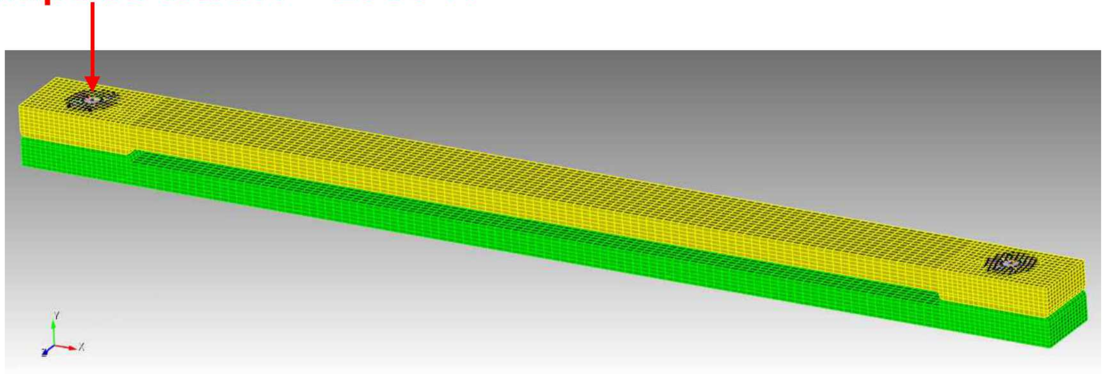


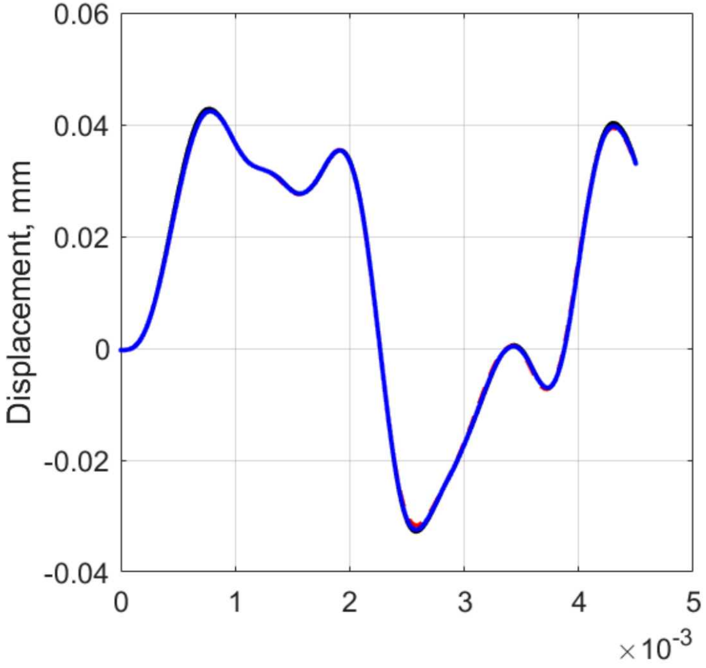


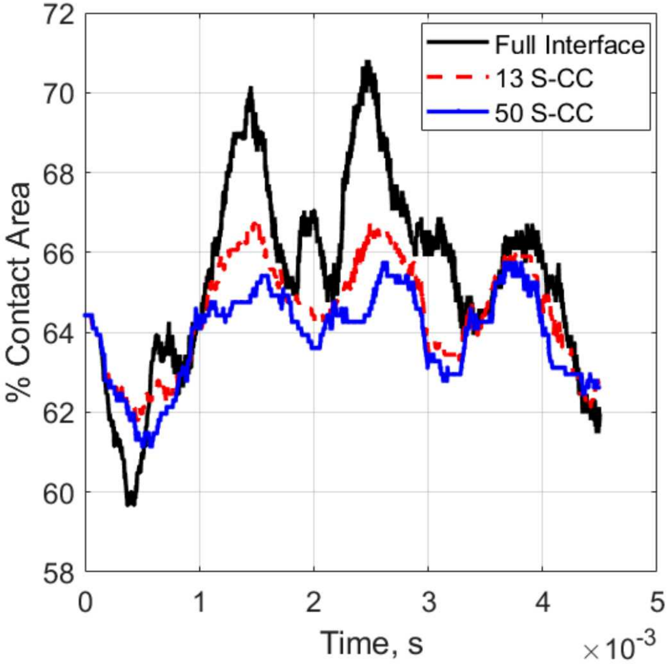


Time-domain simulations due to impulse load

Impulse load A = 2000 N







Enhance basis with trial vector derivatives

Using the S-CC modes from the initial reduction on the interface

$$\mathbf{v} = \begin{Bmatrix} \mathbf{q}_{\mathrm{i}} \\ \mathbf{u}_{\mathrm{r}} \\ \mathbf{u}_{\mathrm{p}} \end{Bmatrix} = \mathbf{v}_{\mathrm{pre}} + \begin{bmatrix} \mathbf{I} & 0 & 0 \\ 0 & \mathbf{\Phi}^{\mathrm{SCC}} & \mathbf{\Psi}^{\mathrm{SCCe}} \\ 0 & 0 & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_{\mathrm{i}} \\ \mathbf{q}_{\mathrm{r}} \\ \mathbf{u}_{\mathrm{p}} \end{Bmatrix} = \mathbf{v}_{\mathrm{pre}} + \mathbf{T}^{\mathrm{SCCe}}\mathbf{w}$$

Take Taylor series expansion around preloaded configuration to get modal derivatives

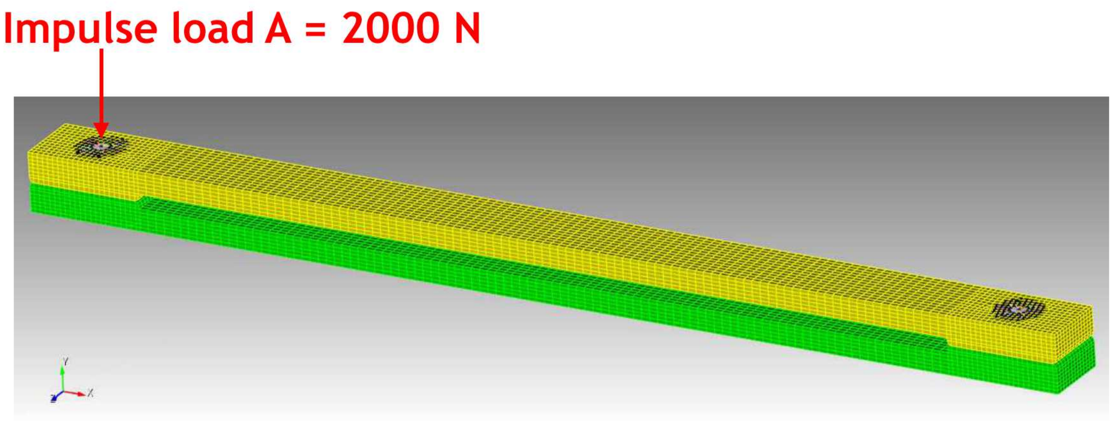
$$\mathbf{T}_{\boldsymbol{i}(\mathbf{w})} = \mathbf{T}_{\boldsymbol{i}}\Big|_{\mathrm{c}} + \underbrace{\sum_{\boldsymbol{j}=1}^{\boldsymbol{n_w}} \frac{\partial \mathbf{T}_{\boldsymbol{i}}}{\partial w_{\boldsymbol{j}}}\Big|_{\mathbf{c}}}\left(w_{\boldsymbol{j}} - \mathrm{w_j(PL)}\right) + \mathrm{H.O.T.}$$

Describe how modes change for a given modal amplitude of response

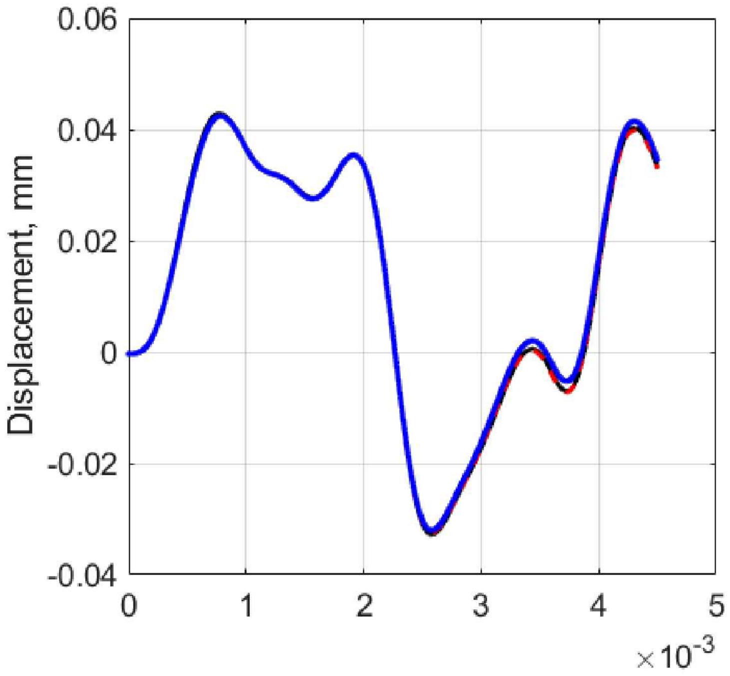
Take Taylor series expansion around preloaded configuration to get modal derivatives

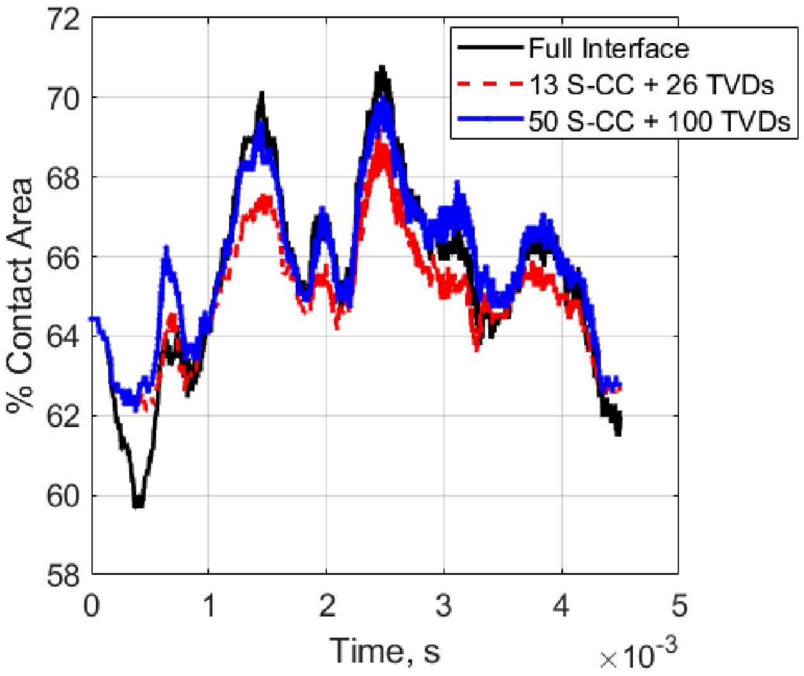
$$\mathbf{T}^{\mathrm{TVD}} = \begin{bmatrix} \mathbf{T}^{\mathrm{SCCe}} & \dfrac{\partial \mathbf{T}^{\mathrm{SCCe}}}{\partial \mathbf{w}} \end{bmatrix}$$

Time-domain simulations due to impulse load



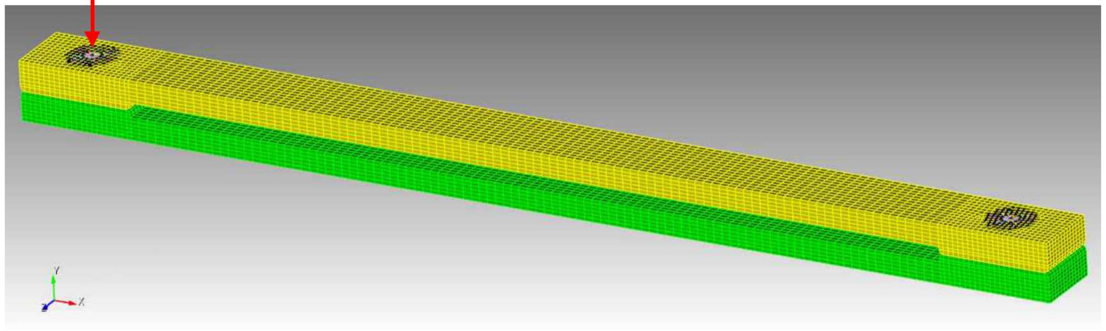
*Full interface ~ 90 minutes
** IR ROMs – 2 minutes





Time-domain simulations due to impulse load

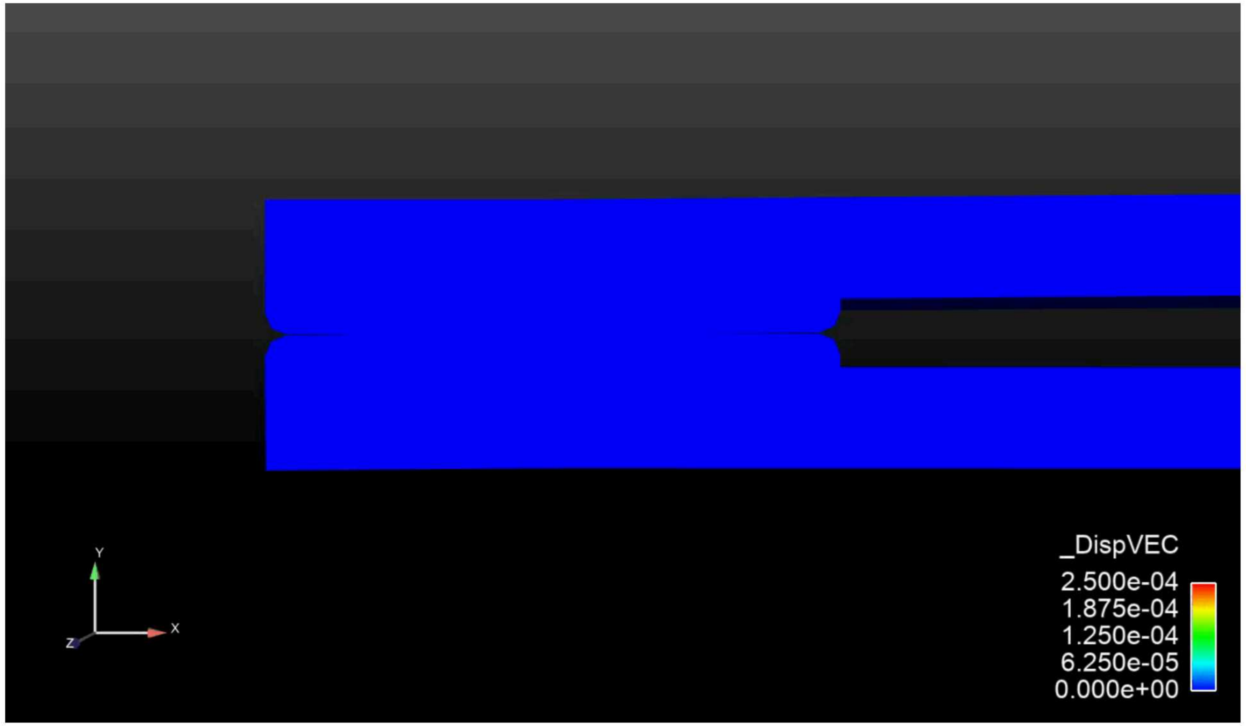
Impulse load A = 2000 N



HCB-SCCe-TVD ROM

272 DOF

2.0 min



Concluding remarks

Covered NLROMs of structures with contact and friction at interfaces

- Whole joint modeling approach and calibration
- Interface reduction to maintain kinematics of joint

Many considerations required for performing model order reduction of nonlinear systems

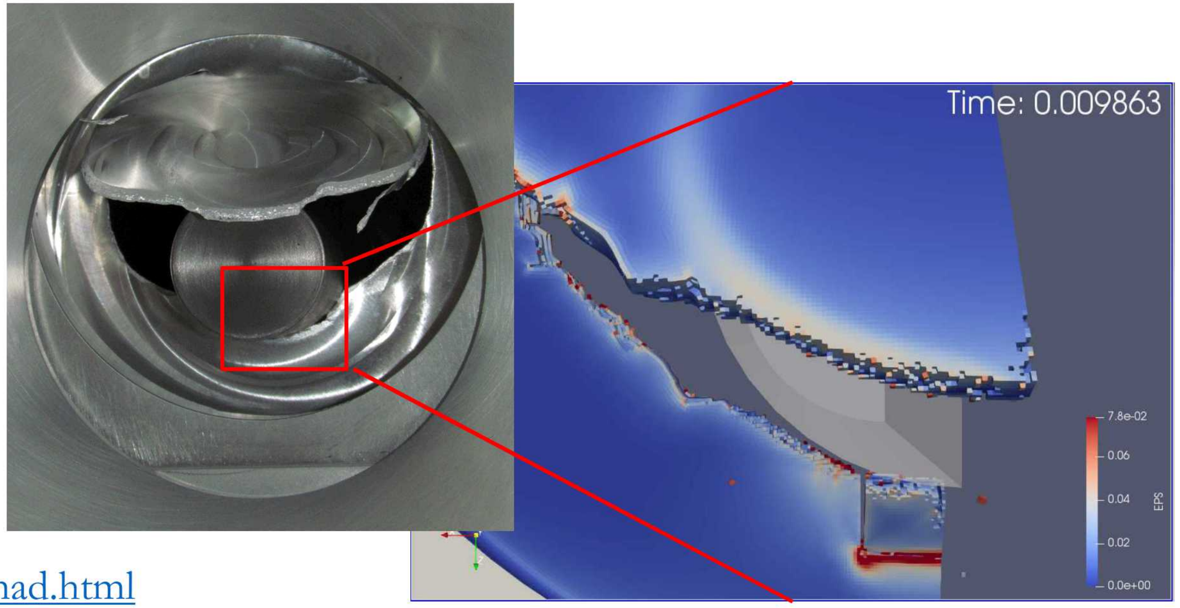
- What type of modes?
- How many modes?
- Introducing stability/convergence issues?

Future challenges/investigations

- Simulation code implementation
- Unifying ROM strategy for all types of nonlinearities (i.e. nonlinear normal modes)?

Research collaboration opportunities for students and professors

- Hosted by Sandia National Laboratories and University of New Mexico
- Collaborative opportunity to work on research in topic areas across nonlinear mechanics and dynamics
- 7 week program held in Albuquerque, New Mexico; open to graduate and highly qualified undergraduate level students



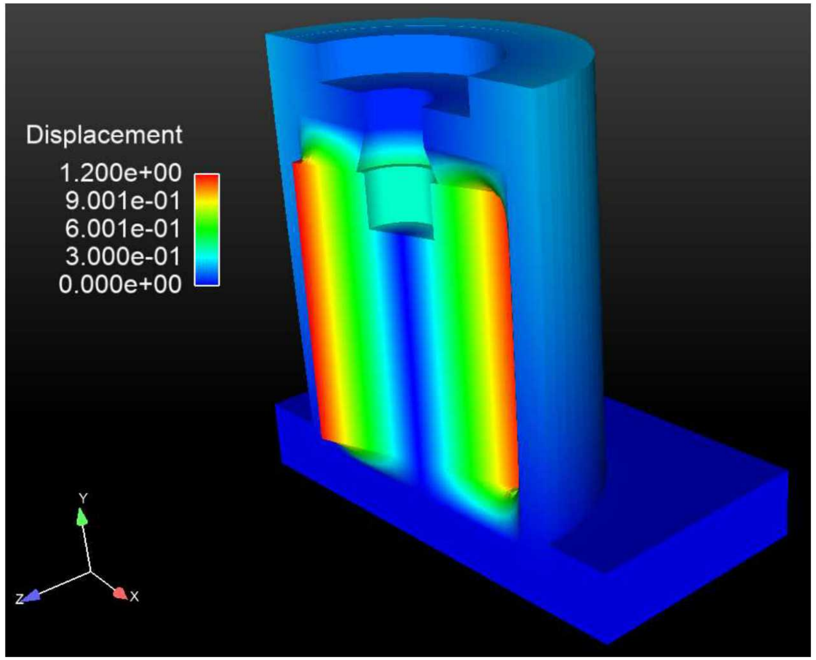
nomad@sandia.gov

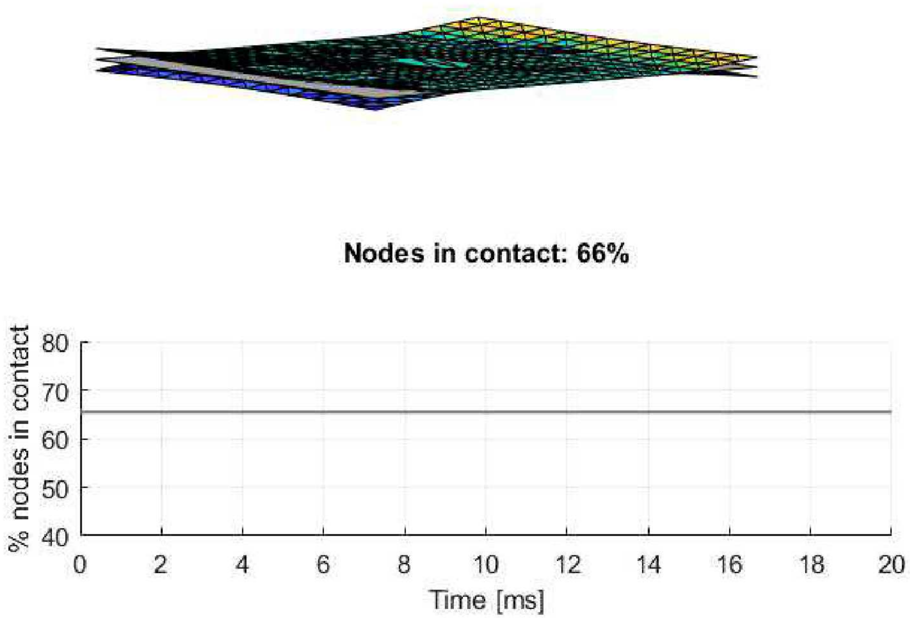
For more information, please visit:
http://www.sandia.gov/careers/students_postdocs/internships/institutes/nomad.html

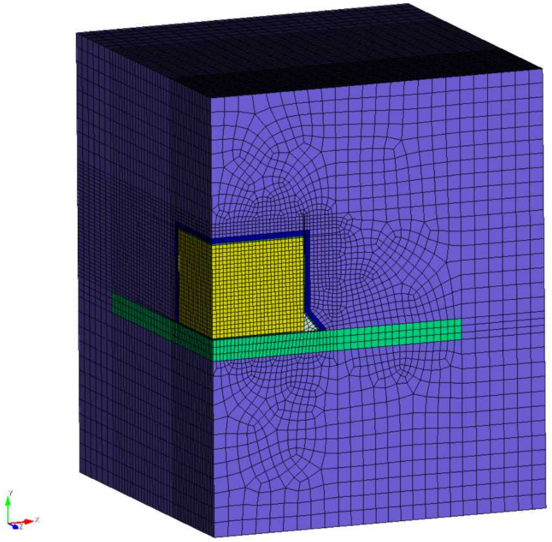
Any questions?

Contact information

- Robert Kuether, rjkueth@sandia.gov







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Overview of NROM topics

Nonlinear frictional contact at mechanical interfaces

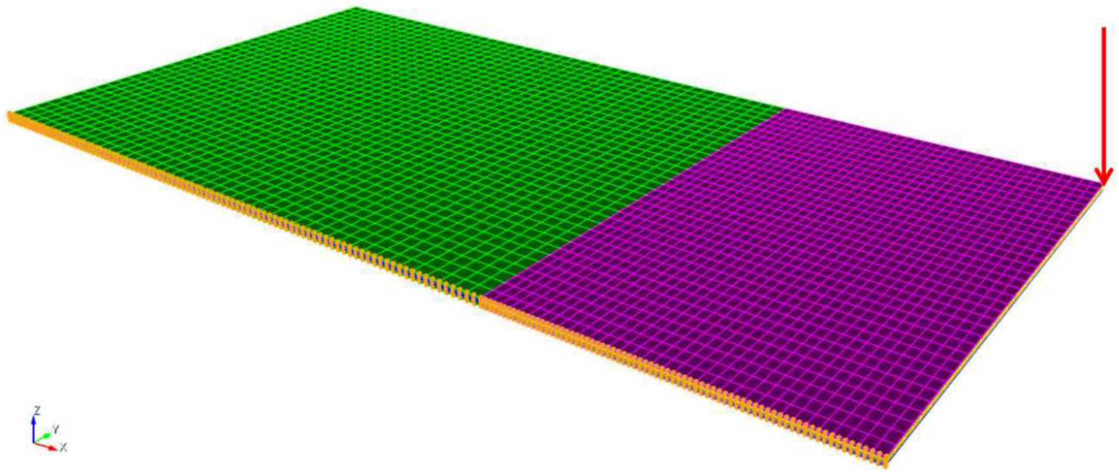
- Whole joint modeling
- Interface reduction

Linear viscoelastic material behavior

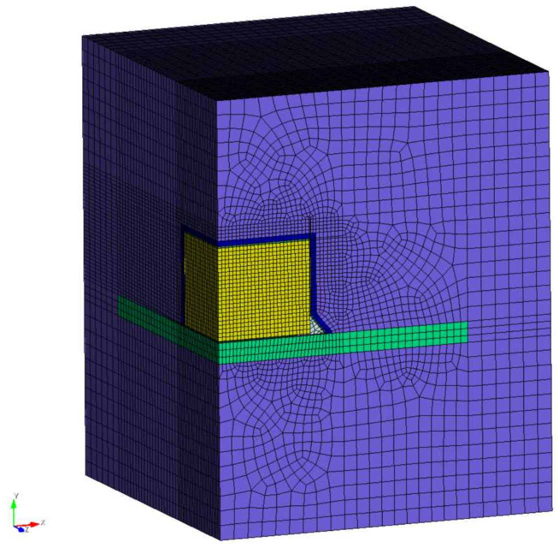
- Composite sandwich plates
- Encapsulate electronics

Model order reduction for linear viscoelastic FEA models

- Finite element models with linear viscoelastic materials require direct time integration
  - ROMs provide solver efficiency while preserving accuracy
  - Limited to lightly damped, linear elastic materials
- Developed a ROM framework for viscoelastic material constitutive laws
  - E.g. low density PMDI foam, cellular silicone, etc..



Sandwich layer damping treatments

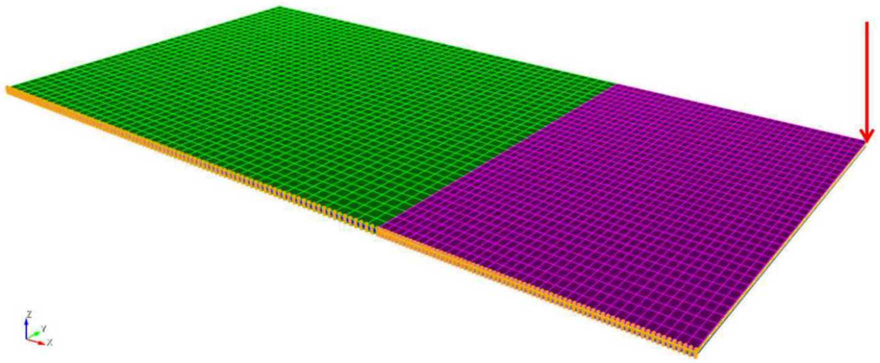


Vibration sensitive electronics potted in foam or polymer

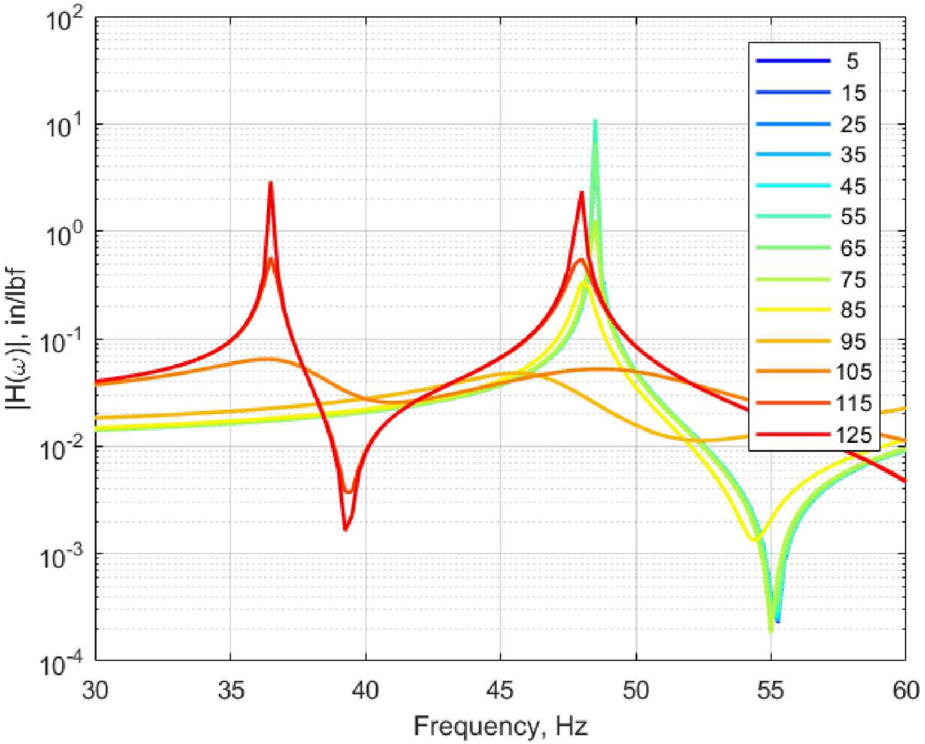
How does viscoelastic material behavior influence the global response?

Model reduction of linear viscoelastic FEMs

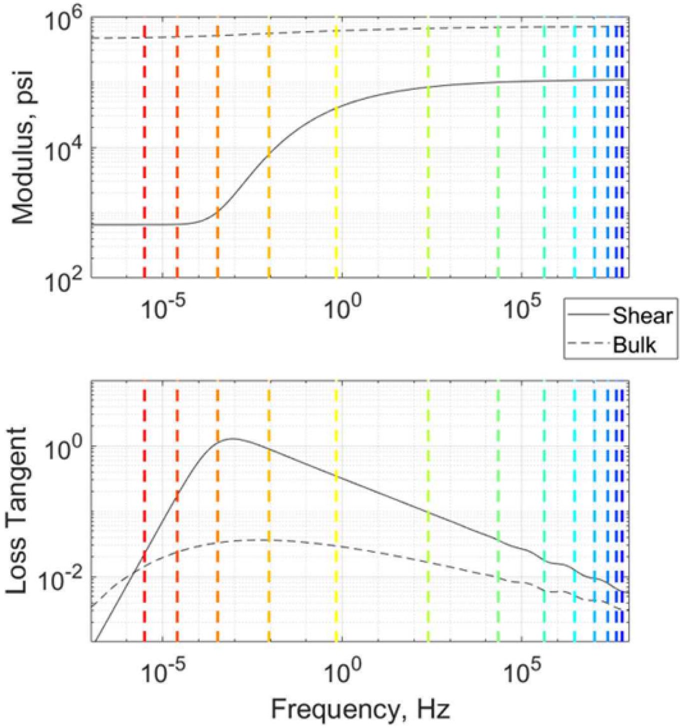
- Frequency-dependent material properties
- Obeys time-temperature superposition
- Results in nonlinear eigenvalue problem



Transfer Function



828 DEA Master Curves



Stiffness and damping change as a function of operating temperature!

Governing equations of motion

Governing equations-of-motion can be reduced using Galerkin approach with a variety of modal bases [1]

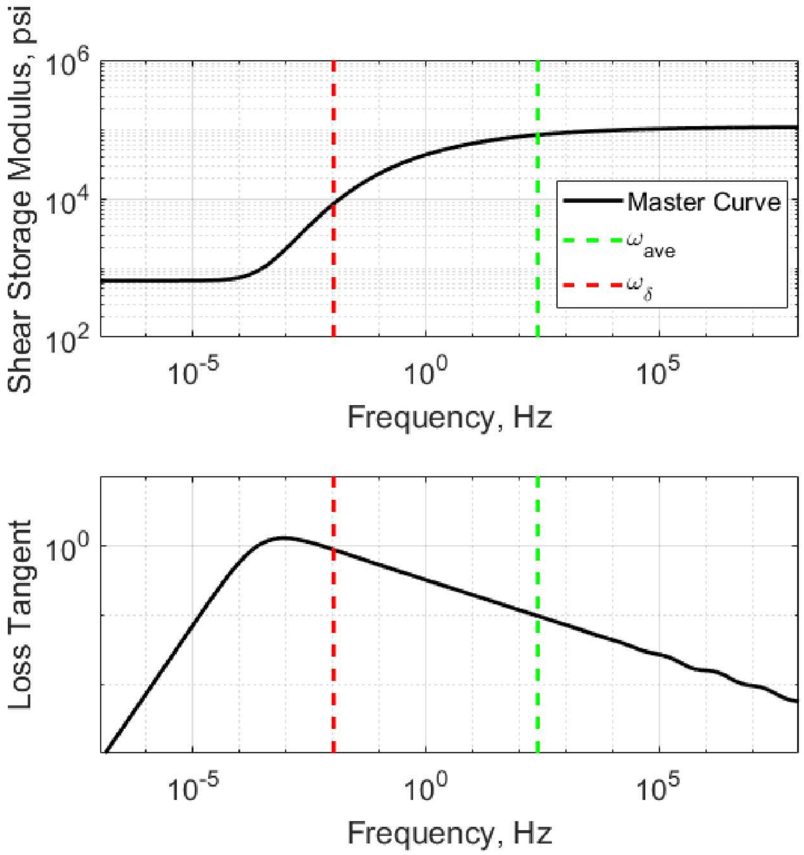
$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}_K \int_0^t \zeta_K(t-\tau)\dot{\mathbf{x}}(\tau)d\tau + \mathbf{K}_G \int_0^t \zeta_G(t-\tau)\dot{\mathbf{x}}(\tau)d\tau + \mathbf{K}_\infty \mathbf{x} = \mathbf{f}(t)$$

Laplace transformation produces a nonlinear eigenvalue problem

$$\left(\lambda_r^2\mathbf{M} + \lambda_r\mathbf{C} + \lambda_r\mathbf{K}_K \sum_{i=1}^{N_K} \frac{\widehat{K}_i}{\lambda_r + 1/\tau_{K,i}} + \lambda_r\mathbf{K}_G \sum_{i=1}^{N_G} \frac{\widehat{G}_i}{\lambda_r + 1/\tau_{G,i}} + \mathbf{K}_\infty\right)\boldsymbol{\phi}_r^* = 0$$

Accurate but expensive – two-tier reduction developed to reduce cost of nonlinear eigensolver [2]

- Step 1: Multi-model Approach [3]
- Step 2: Exact complex modes via iterative Newton solver



[1] L. Rouleau, J.-F. Deü, and A. Legay, "A comparison of model reduction techniques based on modal projection for structures with frequency-dependent damping," *Mechanical Systems and Signal Processing*, vol. 90, pp. 110-125, 2017.

[2] Kuether, R.J., "Two-tier Model Reduction of Viscoelastically Damped Finite Element Models", *Computers & Structures*, (in review).

[3] E. Balmès, "Parametric Families of Reduced Finite Element Models. Theory and Applications," *Mechanical Systems and Signal Processing*, vol. 10, no. 4, pp. 381-394, 1996.

Step 1: Multi-model approach [1]

Starting from the nonlinear eigenvalue problem

$$\left(\lambda_r^2\mathbf{M} + \lambda_r\mathbf{C} + \lambda_r\mathbf{K}_K\sum_{i=1}^{N_K}\frac{\hat{K}_i}{\lambda_r + 1/\tau_{K,i}} + \lambda_r\mathbf{K}_G\sum_{i=1}^{N_G}\frac{\hat{G}_i}{\lambda_r + 1/\tau_{G,i}} + \mathbf{K}_\infty\right)\boldsymbol{\phi}_r^* = 0$$

Linearize Prony series about $\lambda_r = r + i\omega$, perform some mathematical manipulation, and ignore the damping terms to get a real, frequency dependent eigenvalue problem

$$\left(\lambda_r^2\mathbf{M} + \mathbf{K}_K\sum_{i=1}^{N_K}\frac{\hat{K}_i\tau_{K,i}^2\omega^2}{1 + \left(\omega\tau_{K,i}\right)^2} + \mathbf{K}_G\sum_{i=1}^{N_G}\frac{\hat{G}_i\tau_{G,i}^2\omega^2}{1 + \left(\omega\tau_{G,i}\right)^2} + \mathbf{K}_\infty\right)\boldsymbol{\phi}_r = 0$$

Solve for a set real eigenmode bases by sampling at various linearized frequencies, ω

- The extreme limits bound the problem when $\omega \rightarrow 0$ and $\omega \rightarrow \infty$

$$\boldsymbol{\Phi}_{MM} = \left[\boldsymbol{\Phi}_\infty \quad \boldsymbol{\Phi}_g \quad \boldsymbol{\Phi}_\delta \quad \mathbf{R}_\infty \quad \mathbf{R}_g \quad \mathrm{Re}(\mathbf{R}_\delta^*) \quad \mathrm{Im}(\mathbf{R}_\delta^*)\right]$$

[1] E. Balmès, "Parametric Families of Reduced Finite Element Models. Theory and Applications," *Mechanical Systems and Signal Processing*, vol. 10, no. 4, pp. 381-394, 1996.

Step 2: Exact complex modes

Now rewriting the nonlinear eigenvalue problem in the tier 1 reduced space

$$\left(\lambda_r^2\bar{\mathbf{M}} + \lambda_r\bar{\mathbf{C}} + \lambda_r\bar{\mathbf{K}}_K\sum_{i=1}^{N_K}\frac{\hat{K}_i}{\lambda_r + 1/(\tau_{K,i}a_T)} + \lambda_r\bar{\mathbf{K}}_G\sum_{i=1}^{N_G}\frac{\hat{G}_i}{\lambda_r + 1/(\tau_{G,i}a_T)} + \bar{\mathbf{K}}_\infty\right)\bar{\boldsymbol{\Phi}}_r^* = 0$$

Newton's method [1] used to iteratively solve for each $\bar{\boldsymbol{\Phi}}_r^*$ and λ_r

$$\begin{Bmatrix}\bar{\boldsymbol{\Phi}}_{r,k+1}^* \\ \lambda_{r,k+1}\end{Bmatrix} = \begin{Bmatrix}\bar{\boldsymbol{\Phi}}_{r,k}^* \\ \lambda_{r,k}\end{Bmatrix} - \left[\left.\frac{\partial\mathbf{h}}{\partial\bar{\boldsymbol{\Phi}}_r^*}\right|_{\bar{\boldsymbol{\Phi}}_{r,k}^*,\lambda_{r,k}} \quad \left.\frac{\partial\mathbf{h}}{\partial\lambda_r}\right|_{\bar{\boldsymbol{\Phi}}_{r,k}^*,\lambda_{r,k}}\right]^{-1}\mathbf{h}_k\left(\lambda_{r,k},\bar{\boldsymbol{\Phi}}_{r,k}^*\right)$$

where the residual equation is defined as

$$\mathbf{h}_k\left(\lambda_{r,k},\bar{\boldsymbol{\Phi}}_{r,k}^*\right) = \begin{Bmatrix}\left(\lambda_{r,k}^2\bar{\mathbf{M}} + \lambda_{r,k}\bar{\mathbf{C}} + \lambda_{r,k}\bar{\mathbf{K}}_K\sum_{i=1}^{N_K}\frac{\hat{K}_i}{\lambda_{r,k} + 1/(\tau_{K,i}a_T)} + \lambda_{r,k}\bar{\mathbf{K}}_G\sum_{i=1}^{N_G}\frac{\hat{G}_i}{\lambda_{r,k} + 1/(\tau_{G,i}a_T)} + \bar{\mathbf{K}}_\infty\right)\bar{\boldsymbol{\Phi}}_{r,k}^* \\ {\bar{\boldsymbol{\Phi}}_{r,k}^*}^{\mathrm{H}}\bar{\mathbf{M}}\bar{\boldsymbol{\Phi}}_{r,k}^* - 1\end{Bmatrix}$$

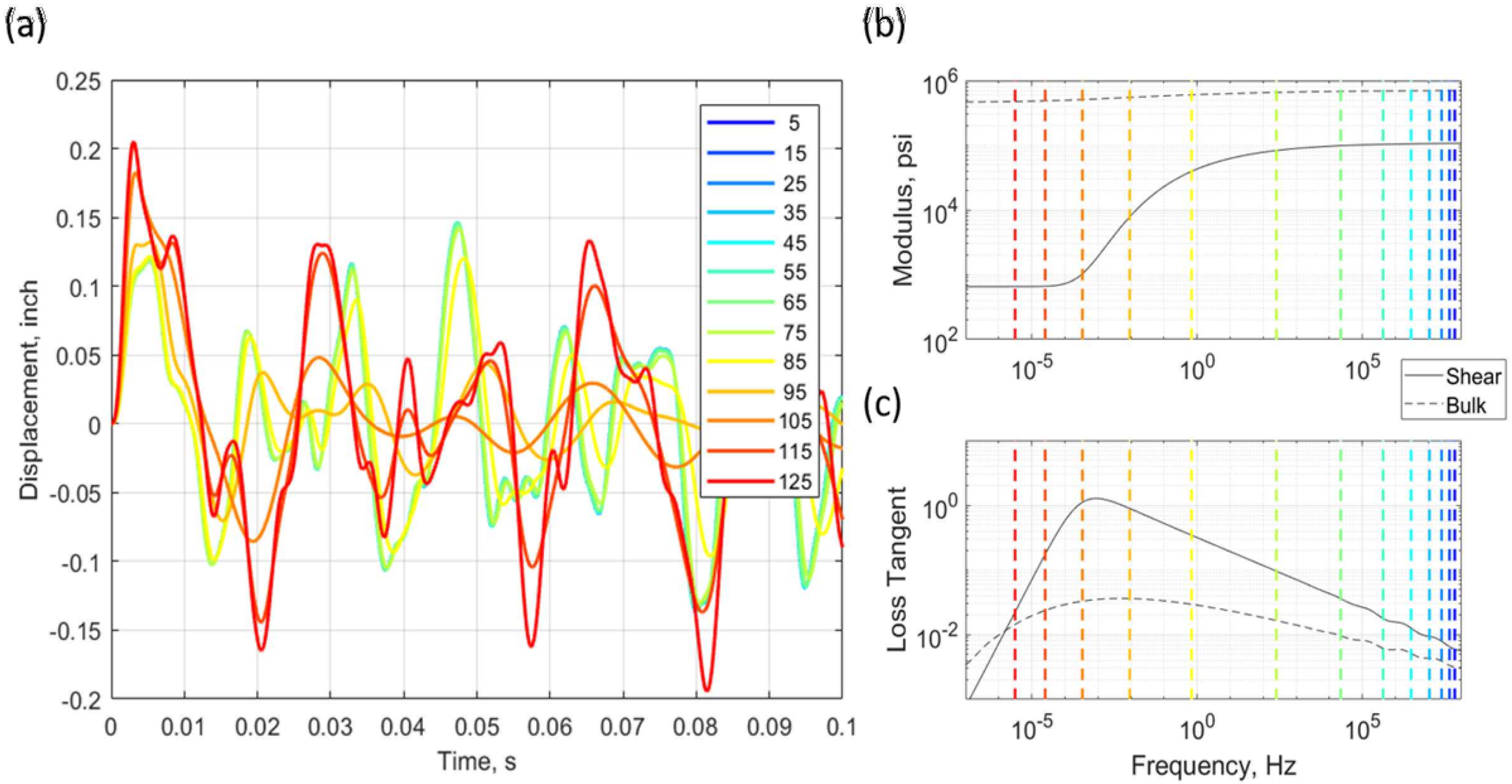
Advantage: solving this equation in a reduced space much more computationally efficient compared to solving it on the full-scale model

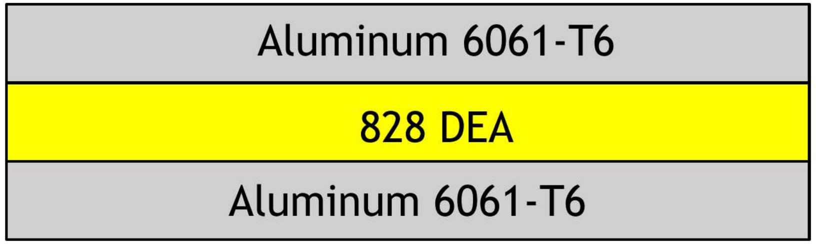
[1] A. Ruhe, "Algorithms for the Nonlinear Eigenvalue Problem," *SIAM Journal on Numerical Analysis*, vol. 10, no. 4, pp. 674-689, 1973.

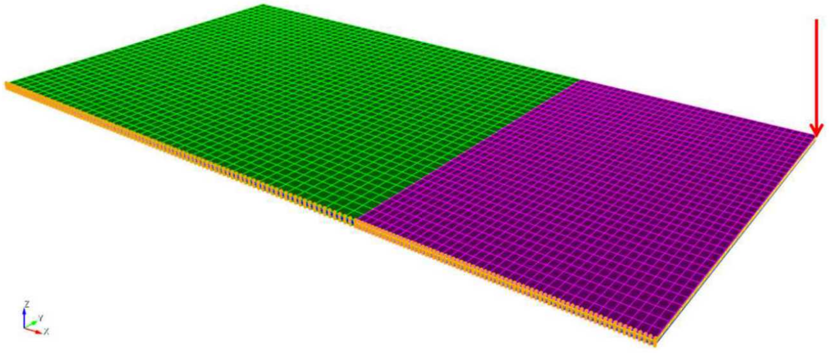
Example demonstration of visco-ROM approach

Sandwich plate structure with haversine impulse applied at the corner

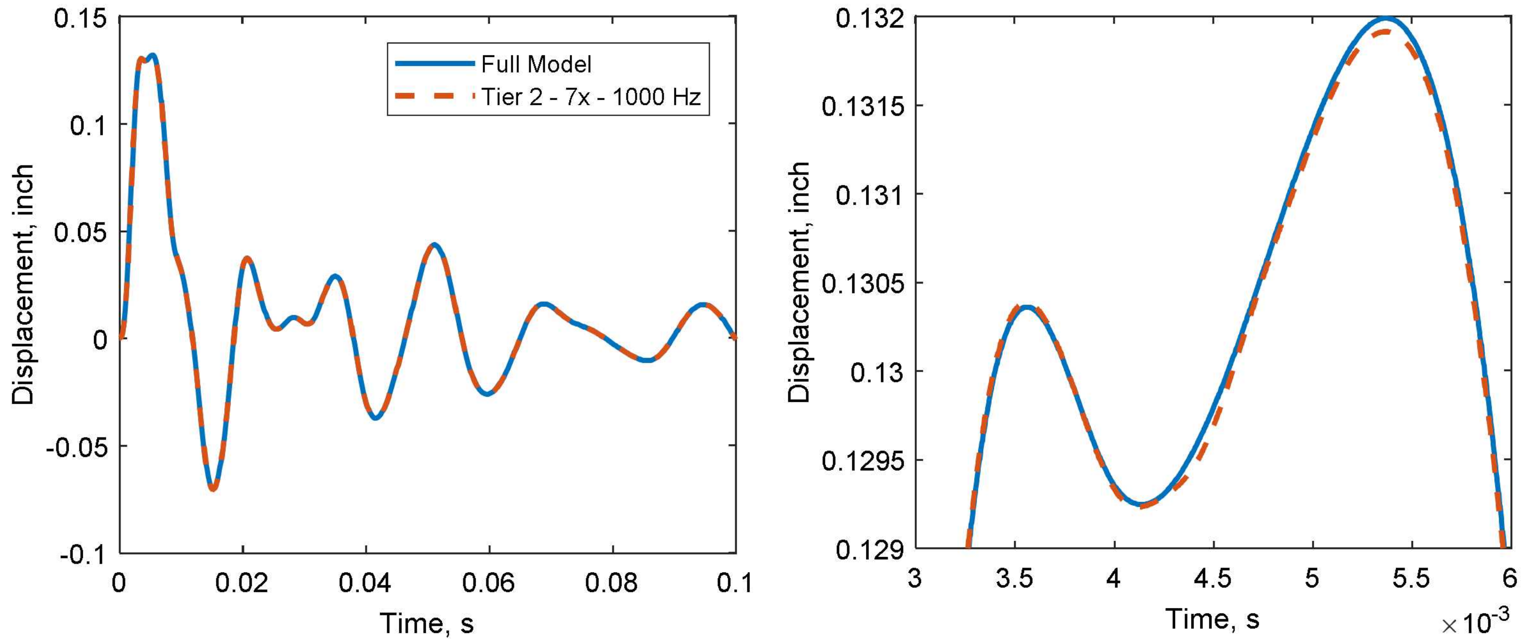
- 130,000 DOF







Example demonstration of visco-ROM approach



Five orders of magnitude in online simulation costs

Global relative error 0.08 %

$$GRE = \frac{\sqrt{\sum_{t\in P}\left(\mathbf{x}(t)-\hat{\mathbf{x}}(t)\right)^{\mathrm{T}}\left(\mathbf{x}(t)-\hat{\mathbf{x}}(t)\right)}}{\sqrt{\sum_{t\in P}\mathbf{x}(t)^{\mathrm{T}}\mathbf{x}(t)}}$$

ROM	Total DOF	Tier-one mode calculation	Tier-two mode calculation	Offline cost (mode calculation)	Online cost (numerical simulation)
Full-order Model	130,305	-	-	0 s	4.51E+06 s
Tier-2 7x-1000 Hz	38	832.4 s	16.4 s	848.8 s	12.8 s

[1] Kuether, R.J., "Two-tier Model Reduction of Viscoelastically Damped Finite Element Models", *Computers & Structures*, (in review).