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Entangled Vertex Cover is in P

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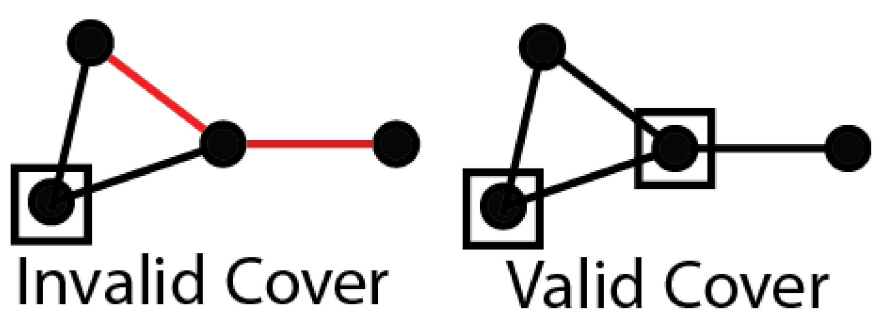
Abstract

Vertex Cover is a well studied classical optimization problem with many applications. Here we define a natural quantum generalization of vertex cover and demonstrate that, counter-intuitively, it is easy to solve when the "constraint" is an entangled quantum state. This result points toward the fact that vertex cover is ill-conditioned in a quantum setting. While vertex cover itself is NP-complete, problems arbitrarily close to it are in P.

Problem Formulation

Definition Given some graph G, we want the smallest set of vertices, A, such that each edge is adjacent to at least one vertex in the set.



Equivalent Formulations:

$$\min \sum_i x_i \qquad \Leftrightarrow \qquad \min \sum_i \mathrm{Tr}\left(\frac{\mathbb{I} - Z_i}{2}\rho\right)$$

$$x_i + x_j \geq 1 \ \forall (i,j) \in E \qquad \mathrm{Tr}[|00\rangle\langle 00|_{ij}\rho] = 0 \ \forall (i,j) \in E$$

$$x_i \in \{0,1\} \ \forall i \in V \qquad \rho \succeq 0, \quad \mathrm{Tr}\rho = 1$$

- Equivalence holds because **1.** Optimal ρ can be assumed pure **2.** Diagonal elements determine cost.
- Motivates following generalization:

Definition Given G and SWAP invariant $|\psi\rangle \in (\mathbb{C}^2)^{\otimes 2}$, quantum vertex cover is:

$$\min \sum_i \mathrm{Tr}[|1\rangle\langle 1|_i \, \rho]$$
$$\mathrm{Tr}[|\psi\rangle\langle\psi|_{ij}\rho] = 0 \ \forall (i,j) \in E$$
$$\rho \succeq 0, \quad \mathrm{Tr}\rho = 1$$

- Results we have cover general 1-local cost observables, for ease of exposition we restrict our attention to diagonal 1-local observables.
- Note we can still assume $\rho = |\phi\rangle\langle\phi|$
- Assume SWAP invariance to capture features of the classical vc. This implies crucial simplification of $|\psi\rangle$:

Lemma If $|\psi\rangle \in (\mathbb{C}^2)^{\otimes 2}$ then $|\psi\rangle = \alpha|00\rangle + \beta|11\rangle$ or one of the Bell states up to local unitaries.

General Idea

- Main idea is to use "transfer matrix approach" pioneered by Bravyi [1] to further restrict the feasible space.
- This result allows us to "propagate" constraints to new constraints, severly restricting feasible space.
- Assumption of SWAP invariance gives nice form for new constraints, allowing complete description of feasible space.
- This characterization is used to explicitly solve QVC.

Propagation for Symmetric States

Quantum States Represented by Tensors

- For n qubit state $|\psi\rangle$ we use a tensor to denote amplitudes in the standard basis: $\psi_x = \langle x|\psi\rangle$.
- Constraint is $\psi_{\alpha_i\alpha_j}\phi_{\alpha_1\alpha_2\ldots\alpha_n} = 0 \ \forall (i,j) \in E$ (Einstein Summation convention)
- We denote singlet tensor using ϵ:

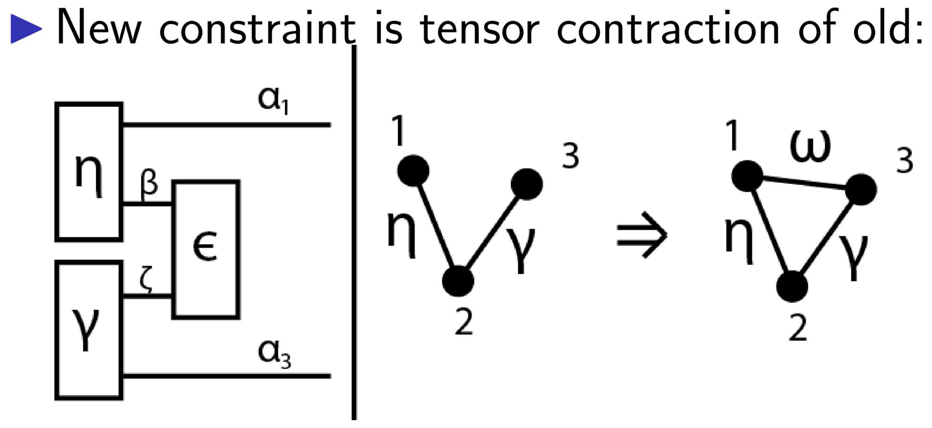
$$\epsilon_{\alpha\beta} = \begin{cases} 1 \text{ if } \alpha = 0 \text{ and } \beta = 1 \\ -1 \text{ if } \alpha = 1 \text{ and } \beta = 0 \\ 0 \text{ o.w.} \end{cases}$$

Theorem [1]

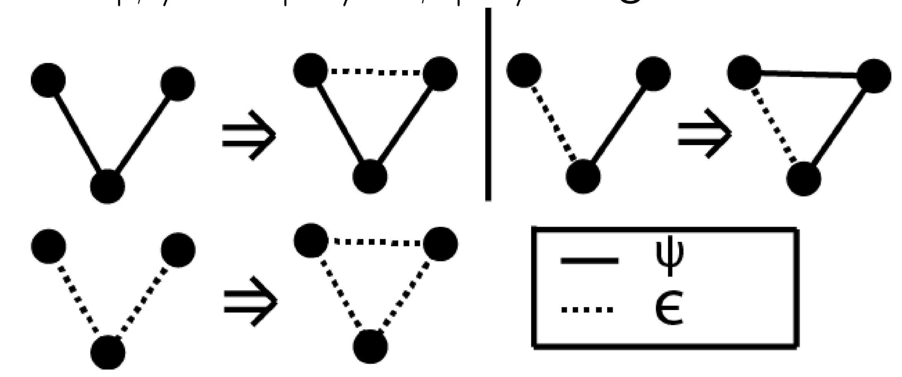
$$\eta_{\alpha_1\alpha_2}\phi_{\alpha_1\alpha_2\alpha_3} = 0 \qquad \Rightarrow \omega_{\alpha_1\alpha_3}\phi_{\alpha_1\alpha_2\alpha_3} = 0$$
$$\gamma_{\alpha_2\alpha_3}\phi_{\alpha_1\alpha_2\alpha_3} = 0$$

with $\omega_{\alpha_1\alpha_3} = \eta_{\alpha_1\beta}\epsilon_{\beta\zeta}\gamma_{\zeta\alpha_3}$.

- New constraint is tensor contraction of old:



- For $|\psi\rangle = \alpha|00\rangle + \beta|11\rangle$ we get:

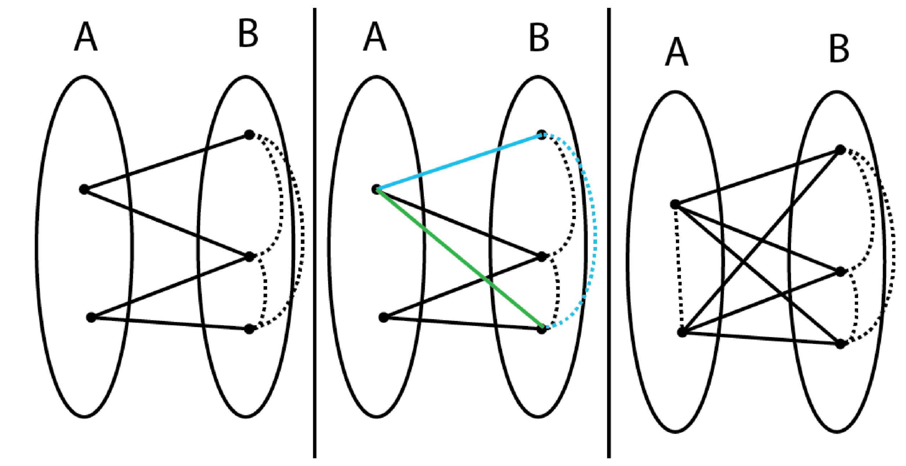


- We can use these propagation relations to see that there are only really two problems here, the bipartite case and the non-bipartite case.

Bipartite Case

Tensor Propagation on Bipartite Graphs

- G is connected bipartite graph with "left set" A and "right set" B.
- Observe that we can connect any pair in B with a singlet (since the graph is connected by assumption).
- These singlets can then be used to connect any point in A with any point in B.



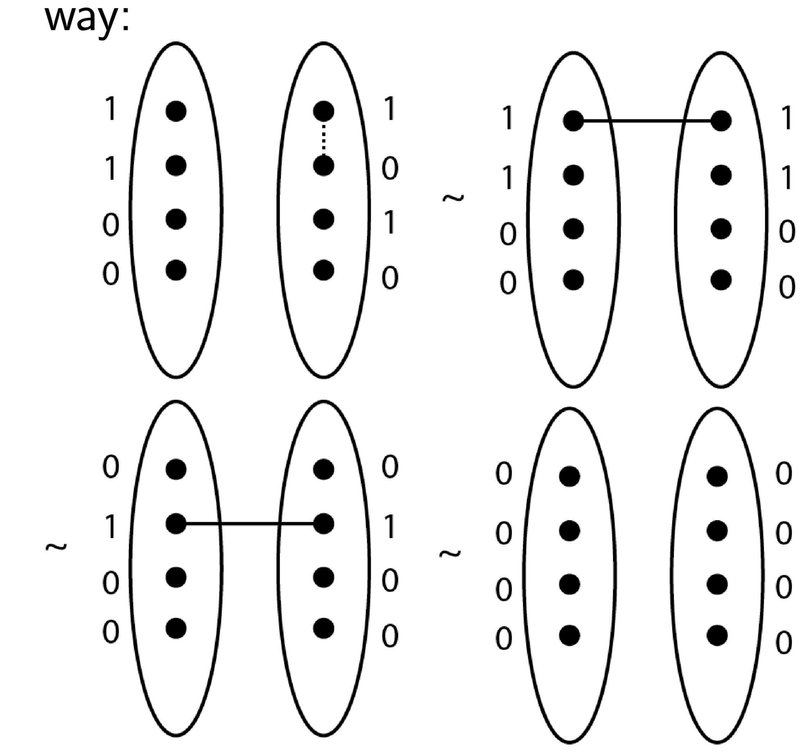
- We can assume a complete bipartite graph of tensors. What does this imply about our state?

Canonical States

- Generally the following relations hold:

$$\epsilon_{\alpha_1\alpha_2}\gamma_{\alpha_1\alpha_2\alpha_3\ldots\alpha_n} = 0 \Leftrightarrow \gamma_{01\alpha_3\ldots\alpha_n} = \gamma_{10\alpha_3\ldots\alpha_n} \tag{1}$$
$$\alpha\gamma_{00\alpha_3\ldots\alpha_n} = -\beta\gamma_{11\alpha_3\ldots\alpha_n} \Leftrightarrow \alpha\gamma_{00\alpha_3\ldots\alpha_n} = -\beta\gamma_{11\alpha_3\ldots\alpha_n}$$

- If two qubits are connected by a singlet, we are can swap ($01 \leftrightarrow 10$) without changing the amplitude.
- If two qubits are connected by ψ, we are free to change $11 \leftrightarrow 00$ while picking up a factor or inverse factor of $-\alpha/\beta$.
- We can relate amplitudes of a feasible tensor in this way:



$$\eta_{1100,1010} = \eta_{1100,1100} = \left(-\frac{\alpha}{\beta}\right)\eta_{0100,0100} = \left(-\frac{\alpha}{\beta}\right)^2 \eta_{0000,0000}$$

- We can use these ideas to construct "canonical" states. These are feasible states that are "closed" under the transformations we have described above. Define: $|p,q\rangle \propto \sum_{\mathbf{x}:|\mathbf{x}|=p,\mathbf{y}:|\mathbf{y}|=q} |\mathbf{x},\mathbf{y}\rangle$.. Then,

Canonical Feasible States

$$|\gamma(a)\rangle \propto \sum_{k=0}^{m-a}\left(-\frac{\alpha}{\beta}\right)^k |k+a,k\rangle$$

$$|\zeta(a)\rangle \propto \sum_{k=0}^{m+a}\left(-\frac{\alpha}{\beta}\right)^k |k,k+a\rangle$$

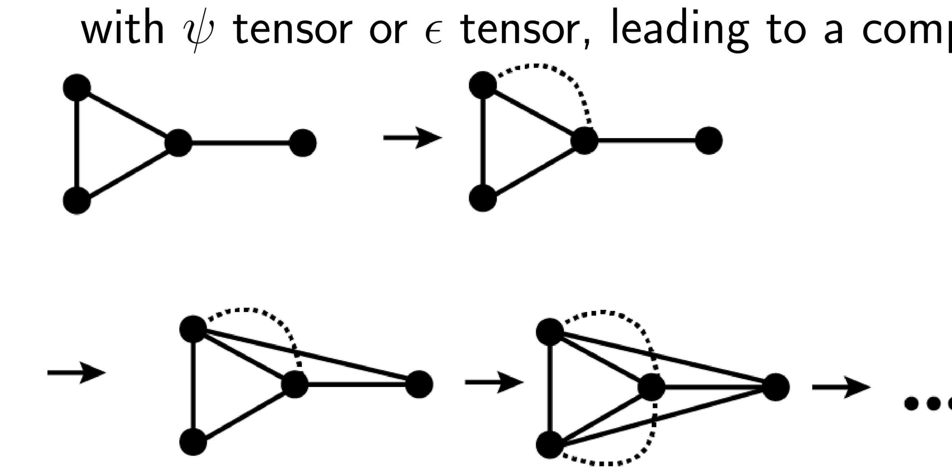
Theorem Let $|\phi\rangle$ be a feasible state to bipartite qvc. Then,

$$|\phi\rangle = \sum_a \alpha_a|\gamma(a)\rangle + \sum_b \beta_b|\zeta(b)\rangle$$

- Observe the cost term does not couple canonical states since it is diagonal in the computational basis.
- Hence $\mathcal{C}(|\phi\rangle) = \sum_a |\alpha_a|^2\mathcal{C}(\gamma(a)) + \sum_b |\beta_b|^2\mathcal{C}(\zeta(b))$
- Optimal state is canonical state with best cost and cost of each canonical state can be easily calculated (For instance $\mathcal{C}(\gamma(0)) = \sum_k (-\alpha/\beta)^{2k} 2k/N^2$ for some normalization N).
- Crucially, only polynomially many canonical states.

Non-bipartite

- This case is even worse because of odd cycle.
- Odd cycle can be used to connect any pair of vertices with ψ tensor or ϵ tensor, leading to a complete graph.



- Yields exactly 2 canonical states, $|even\rangle$ and $|odd\rangle$.
- Costs are still simple to calculate.

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