

Eigenvalue Uncoupling of Electrokinetic Flows

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There is a need to solve coupled physics problems efficiently.

We present a simple approach to uncoupling electrokinetic flow using a matrix technique (Lo et al., 2009). Electrokinetics is movement of ions and water in a porous medium under fluid pressure (p) and electrostatic potential (ψ) gradients:

$$\begin{bmatrix} C^* & 0 \\ 0 & nc \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} \psi \\ p \end{bmatrix} = \begin{bmatrix} \sigma_0 & L_{12} \\ L_{21} & \frac{k_0}{\mu} \end{bmatrix} \nabla^2 \begin{bmatrix} \psi \\ p \end{bmatrix}$$

Consolidating coefficients leads to:

$$\frac{\partial}{\partial t} \begin{bmatrix} \psi \\ p \end{bmatrix} = \begin{bmatrix} \alpha_E & \alpha_E K_S \\ \alpha_H K_E & \alpha_H \end{bmatrix} \nabla^2 \begin{bmatrix} \psi \\ p \end{bmatrix}$$

Non-dimensionalizing the equations reduces free parameters from 4 to 2, the diffusivity ratio ($\alpha_D = \alpha_E/\alpha_H$) and the electrokinetic coupling strength ($K_D = K_E K_S$):

$$A \frac{\partial}{\partial t_D} \begin{bmatrix} \psi_D \\ p_D \end{bmatrix} = \begin{bmatrix} \alpha_D & \alpha_D \\ K_D & 1 \end{bmatrix} \nabla_D^2 \begin{bmatrix} \psi_D \\ p_D \end{bmatrix} \quad \mathbf{d}_D$$

Using the eigenvalue decomposition from linear algebra:

$$\frac{\partial \mathbf{d}_D}{\partial t} = A \nabla_D^2 \mathbf{d}_D \quad \frac{\partial}{\partial t_D} \mathbf{S}^{-1} \mathbf{d}_D = (\mathbf{S}^{-1} \mathbf{S}) \Lambda \nabla_D^2 \mathbf{S}^{-1} \mathbf{d}_D$$

$$A = \mathbf{S} \Lambda \mathbf{S}^{-1} \quad \mathbf{d}_D = \mathbf{S} \boldsymbol{\delta}$$

We get two uncoupled governing equations (diagonal Λ matrix), but in terms of a vector of new potentials ($\boldsymbol{\delta} = \mathbf{S}^{-1} \mathbf{d}_D$). Eigenvector matrices (\mathbf{S}) and eigenvalue matrix (Λ) are below.

Uncoupling Procedure:

- 1) Pose problem in terms of δ_1, δ_2
- 2) Solve 2 uncoupled diffusion problems
- 3) Compute p, ψ via matrix multiply ($\mathbf{d}_D = \mathbf{S} \boldsymbol{\delta}$)

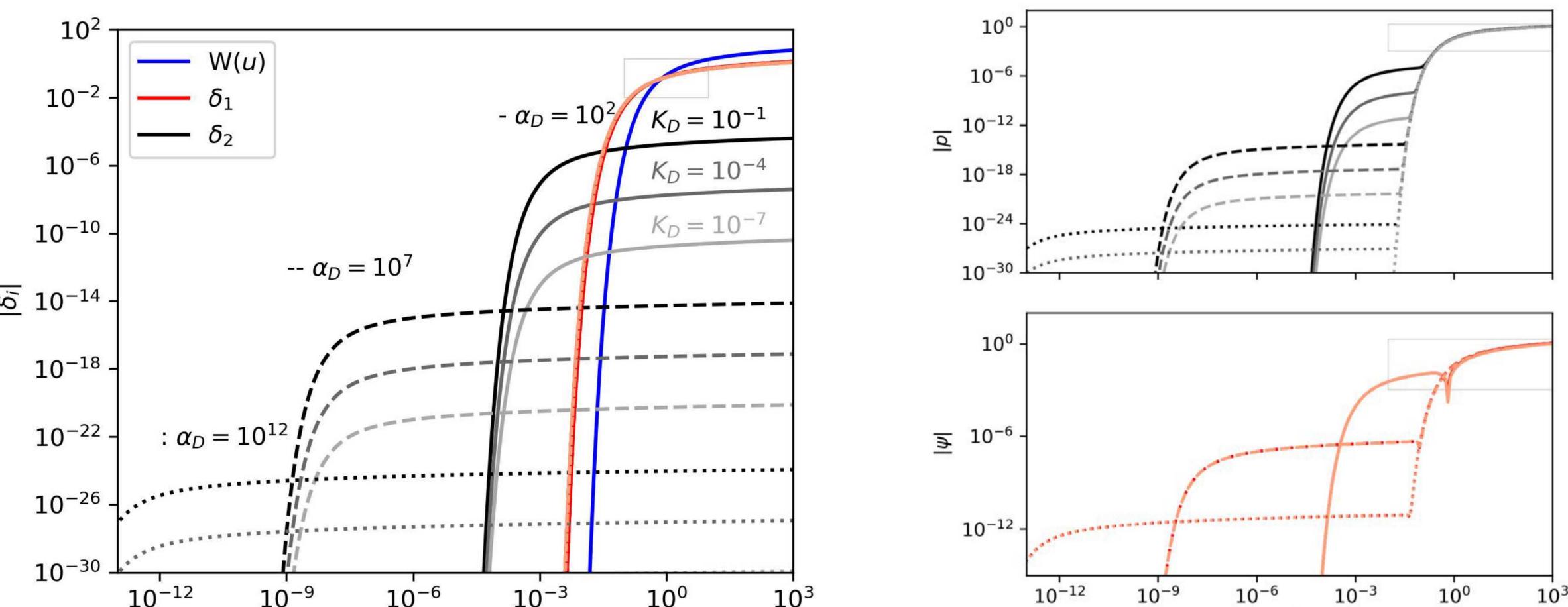
$$\mathbf{S} = \begin{bmatrix} \frac{2\alpha_D}{1-\alpha_D-\Delta} & -\frac{1}{2K_D}(1-\alpha_D-\Delta) \\ 1 & 1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} \frac{2\alpha_D(1-K_D)}{1+\alpha_D+\Delta} & 0 \\ 0 & \frac{1}{2}(1+\alpha_D+\Delta) \end{bmatrix}$$

$$\mathbf{S}^{-1} = \frac{1}{\Delta} \begin{bmatrix} -K_D & -\frac{1}{2}(1-\alpha_D-\Delta) \\ K_D & \frac{-2K_D\alpha_D}{1-\alpha_D-\Delta} \end{bmatrix}$$

$$\Delta = \sqrt{1+\alpha_D(4K_D-2+\alpha_D)}$$
Lo, Sposito & Majer, 2009. Analytical decoupling of poroelasticity equations for acoustic-wave propagation and attenuation in a porous medium containing two immiscible fluids. *Journal of Engineering Mathematics*, 64:219-235.

Example 1. Flow to Pumping Well (Theis)

1D radial flow to a confined well surrounded by insulators for electrokinetic problem. The solution to the decoupled intermediate equations (δ_1, δ_2) comes from the Theis (1935) well function (exponential integral).Initial type curve (blue), δ_1 solutions (red), and δ_2 solutions (black) for a range of α_D and K_D (left). The solution for (p, ψ) is recombined (right) using $\mathbf{d}_D = \mathbf{S} \boldsymbol{\delta}$.

Using only type curve methods, we solve coupled streaming potential and electroosmosis.

$$\text{x-axis: } \ln \left[\frac{t}{r^2} \right] = \ln \left[\frac{1}{4\Lambda_{ii}} \right] + \ln \left[\frac{1}{u} \right]$$

$$\text{y-axis: } \ln [\delta_i] = \ln \left[\frac{Q_i}{4\pi\Lambda_{ii}} \right] + \ln [E_1(u)]$$

Variable	Description	Dimensions
α_H, α_E	hydraulic/electrical diffusivity	m^2/sec
t	time	sec
k_0	formation permeability	m^2
c	formation compressibility	$1/\text{Pa}$
n	formation porosity	—
C^*	electric capacitance of formation	$\text{C}/(\text{m}^3 \cdot \text{V})$
σ_0	electric conductivity	S/m
p	change in fluid pressure	Pa
ψ	change in electrostatic potential	V
μ	viscosity of water	$\text{Pa} \cdot \text{sec}$
L_{12}, L_{21}	electrokinetic coupling coefficient	$\text{A}/(\text{Pa} \cdot \text{m}) = \text{m}^2/(\text{V} \cdot \text{sec})$
K_S	streaming potential	V/Pa
K_E	electroosmosis pressure	Pa/V

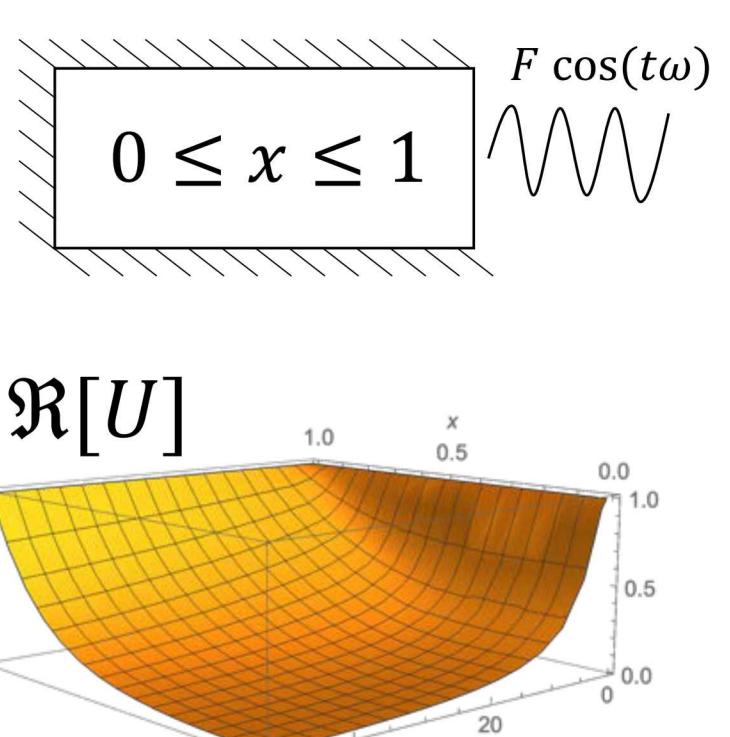
Theis, 1935. The relation between lowering of the piezometric surface and the rate and duration of discharge of a well using ground-water storage. *Transactions, American Geophysical Union*, 16(2):519-524.

Example 2. Driven Electrokinetic Core

1D flow in a box driven by periodic Dirichlet BC on one end, no-flow / insulating on other end (Pengra et al., 1999). Steady-state solution is:

$$\delta_i(x_D, t_D) = \delta_{i\text{SS}}(x_D, t_D) + \delta_{i\text{TR}}(x_D, t_D),$$

$$\delta_{i\text{SS}}(x_D, t_D) = \Re [U(x_D) e^{j\omega_D t_D}]$$

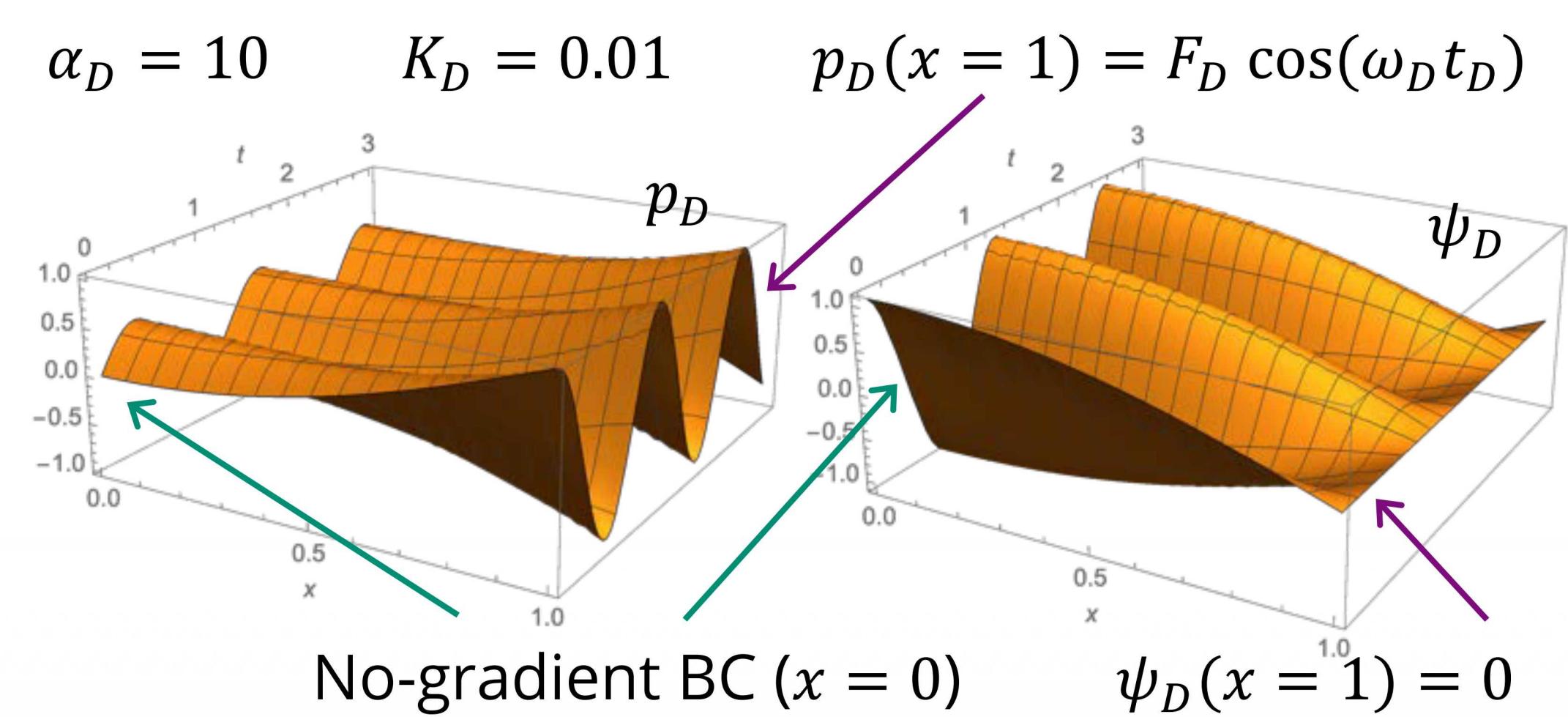
Solve ODE for complex amplitude (U), given BC

$$\frac{d^2 U}{dx_D^2} - \frac{j\omega_D}{\Lambda_{ii}} U = 0 \quad U(x_D) = c_1 e^{\zeta x_D} + c_2 e^{-\zeta x_D} \quad \zeta = \sqrt{j\omega_D/\Lambda_{ii}}$$

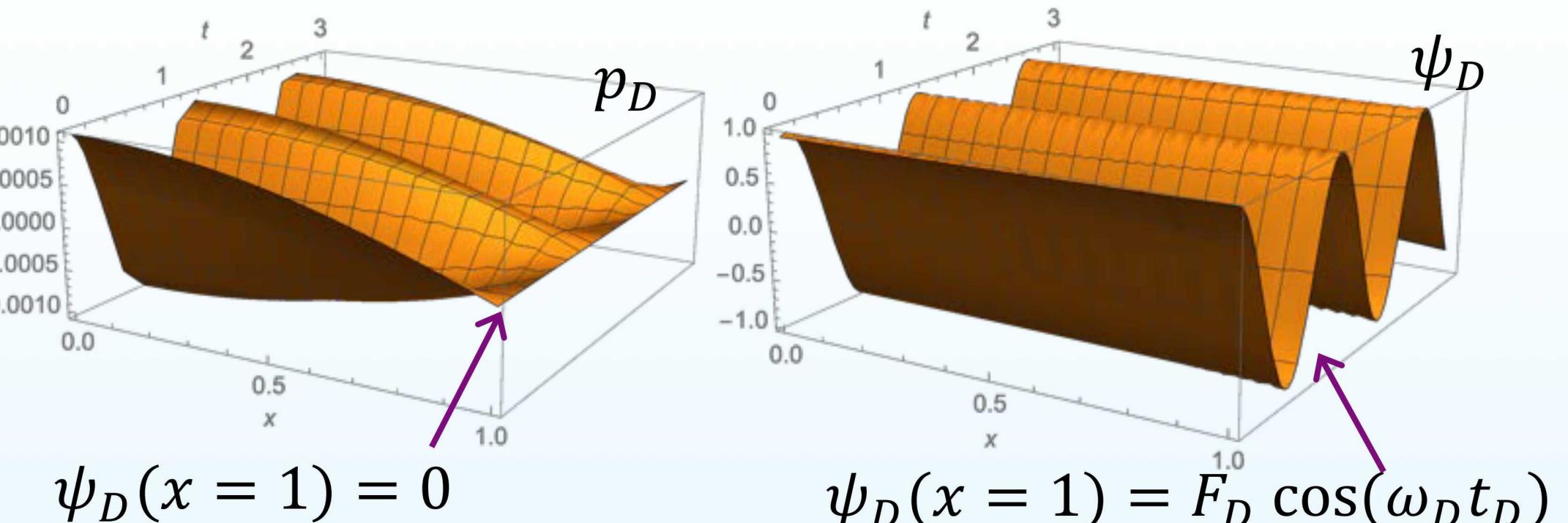
$$\delta_{i\text{SS}}(x_D, t_D) = \Re \left\{ \frac{jK_D F_D \zeta}{\omega_D \Delta (e^\zeta - 1)} \left[e^{i\omega_D t_D + \zeta(1+x_D)} + e^{i\omega_D t_D - \zeta x_D} \right] \right\}$$

Solution is applied to electrokinetics via decoupling

Applied Pressure BC (Streaming Potential)



Applied Voltage BC (Electroosmosis)

Pengra, Li & Wong, 1999. Determination of rock properties by low-frequency AC electrokinetics. *Journal of Geophysical Research*, 104(B12):29485-29508.