The Stabilizer Rank of the T-Gate Magic State

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Stabilizer states:
- given a state |Ψ>, the stabilizer group is the set of tensor products of Pauli operators that stabilizes |Ψ>
e.g. I|0> = |0> and Z|0> = |0>.

- stabilizer states $\{|\Psi_i>\}$ on n d-dimensional qudits are the set of states stabilized by d n commuting stabilizer group elements
Clifford gateset can be fully generated by the CNOT gate, Hadamard and the phase gate

- gateset is not universal — it takes stabilizer states to themselves

- projection onto Paulis also takes stabilizer states to themselves
Pauli measurement + Clifford gateset + (convex combos of) stabilizer states = Clifford subtheory

Gottesman-Knill theorem: there is a polynomial-time classical algorithm to simulate the Clifford subtheory.

- in fact, the optimal qubit (strong simulation) algorithm scales as $O(n^2)$ for n qubits and scales as O(n) for n odd-dimensional qudits.

- stabilizer states $\{|\phi_i>\}_i$ form an overcomplete basis.
  - therefore, any state |Ψ> can be expressed as $|\Psi> = \Sigma^{\chi}_i c_i |\phi_i>$.

- the stabilizer rank $\chi(\Psi)$ of a pure state |Ψ> is the minimal number χ of states required in a stabilizer state decomposition of |Ψ>.

Trivial tensor bound property: Let $\chi_n(\Psi)$ be the stabilizer rank of $|\Psi>^n$. Since the tensor product of two stabilizer states is a stabilizer state, it follows that $\chi_{n+m}(\Psi) \leqslant \chi_n(\Psi)\chi_m(\Psi)$.

T-Gate Magic State:

$|T> = 2^{-1/2}(|0> + e^{\pi i/4} |1>)$.

The T-gate magic state extends the Clifford subtheory to universality.

It has been postulated that $\chi(T)$ grows slowest with increasing number of qubits.
- for t = 1 qubit T gate magic state, $\chi_1 = 2$.
- for t = 2 qubit T gate magic states, $\chi_2 = 2$.
- for t = 3 qubit T gate magic states, $\chi_3 = 3$.
- for t = 6 qubit T gate magic states, $\chi_6 \leqslant 7$.

It is has been conjectured that this bound is tight.

- $\chi_1{}^t = 2^t$.
- $\chi_2{}^{t/2} = 2^{0.5t}$ where t is even.
- $\chi_3{}^{t/3} = 3^{t/3} \approx 2^{0.53t}$ where t is a multiple of 3.
- $\chi_6{}^{t/6} = 7^{t/3} \approx 2^{0.47t}$ where t is a multiple of 6.

The last bound provides the most favorable asymptotic scaling, and so the outcome of this procedure is often reported as a scaling of $O(2^{0.468t})$for the qubit T gate magic state.

- stabilizer state decompositions are a pretty good way to measure the cost of strong simulation (i.e. classical computation of the probability outcome of a measurement)

- the inner product of two stabilizer states $<\phi_i|\phi_j>$ is governed by Gaussian elimination and therefore scales as $O(n^3)$.

-in a Pauli-based computation, given a projector
$\Pi = \Pi^t_{i=1}(I+\sigma_i P_a)/2$ where P_a is a Pauli operator, it follows that $\Sigma_{ij}{}^{\chi n}=c_i{}^* c_j <\phi_i|\Pi|\phi_j>$

- since the number of terms is $\chi_n{}^2$, we want to use the lowest χ_n that we can to simulate this classically

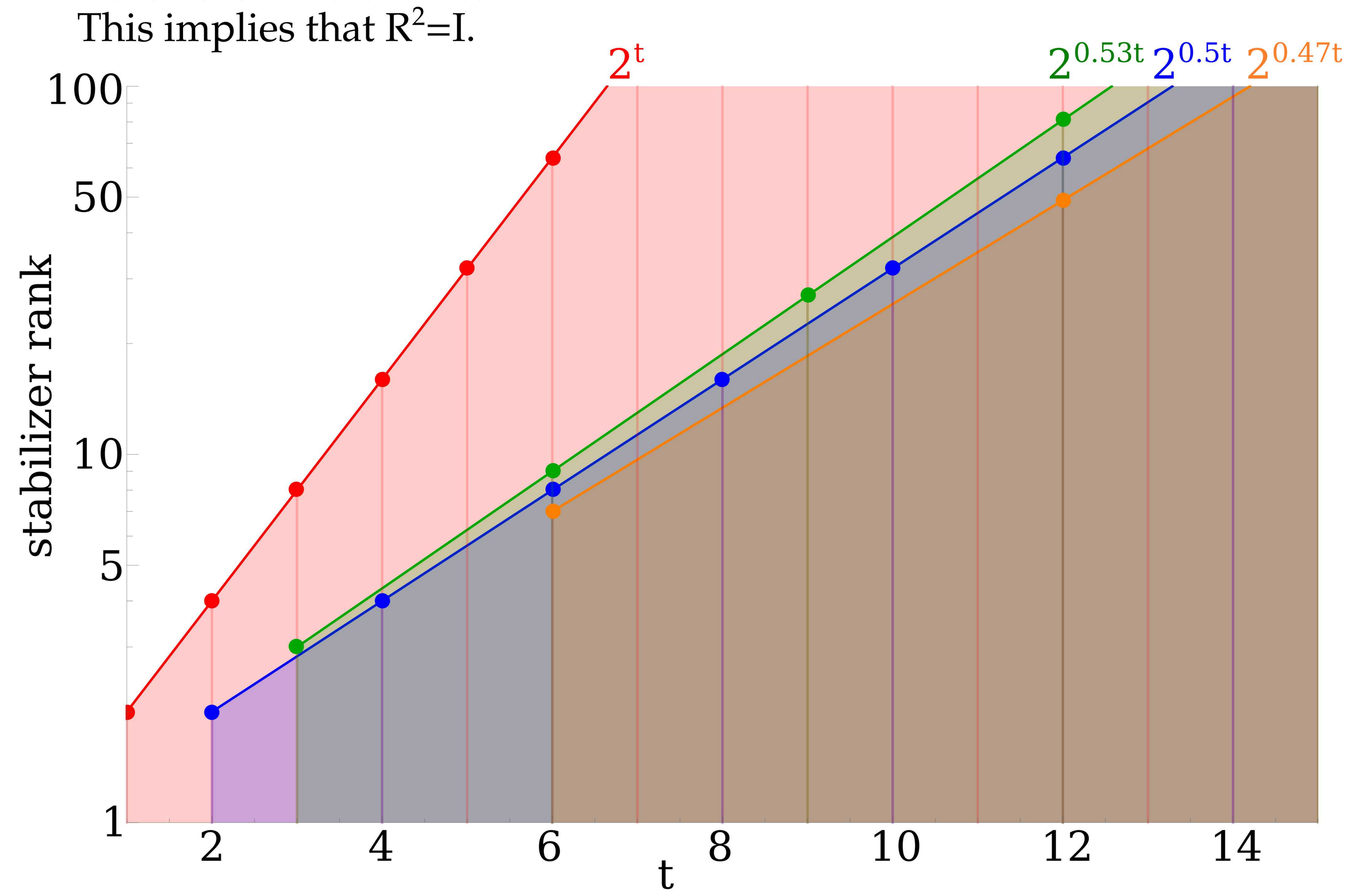
Wigner Function:
In odd d arithmetic, $-1/2 \equiv (d+1)/2$.
Therefore,
$T(\xi_p,\xi_q) = \omega^{-(d+1)/2} Z^{\xi p} X^{\xi q}$
and
$T^{-1}(\xi_p,\xi_q) = T(-\xi_p,-\xi q) = T^{\dagger}(\xi_p,\xi_q)$.

T are Hilbert-Schmidt orthogonal
and so can be used as a complete operator basis for any operator A:
$A = d^{-1}\sum_{\xi p,\xi q \in Z/dZ} Tr(T(-\xi_p,-\xi_q)A)T(\xi_p,\xi_q)$
$\equiv d^{-1}\sum_{\xi p,\xi q \in Z/dZ} A_{\xi}(\xi_p,\xi_q)T(\xi_p,\xi_q)$.

We also note that the T satisfy the translation group structure with an additional phase:
$T(\xi_2)T(\xi_1) = T(\xi_1+\xi_2)\omega^{(-\xi 1p \xi 2q+\xi 1q \xi 2p)(d+1)/2}$.

We denote the symplectic Fourier transform of T:
$R(xp,xq) = d^{-1}\sum_{\xi p,\xi q \in Z/dZ} \omega^{\xi p \xi q - \xi q \xi p} T(\xi_p,\xi_q)$.

It is easy to show that R satisfies the following properties of a reflection operator, with added phase:
$R(x)T(\xi) = R(x-\xi/2)\omega^{xp\xi q - xq\xi p}$,
$T(\xi)R(x) = R(x+\xi/2)\omega^{-xp\xi q + xq\xi p}$,
$R(x_1)R(x_2) = T(2(x_2-x_1))\omega^{x1px2q-x1qx2p}$.
This implies that $R^2=I$.



Therefore, R are Hilbert-Schmidt orthogonal, Hermitian, self-inverse and unitary operators.
This means that they can serve as an operator basis for any operator A with cofficients (denoted $A_x(x_p,x_q)$ below) which will be real-valued:
$A = d^{-1}\sum_{xp,xq} Tr(R(x_p,x_q)A)R(x_p,x_q) \equiv A_x(x_p,x_q)R(x_p,x_q)$.

As a result, it can then be shown that $A_x(x_p,x_q)$ satisfies the following properties:
$\sum_{xp,xq \in Z/dZ} A_x(x_p,x_q) = 1$,
$\sum_{xp \in Z/dZ} A_x(x_p,x_q) = \langle x_q|A|x_q\rangle$,
$\sum_{xq \in Z/dZ} A_x(x_p,x_q) = \langle x_p|A|x_p\rangle$.

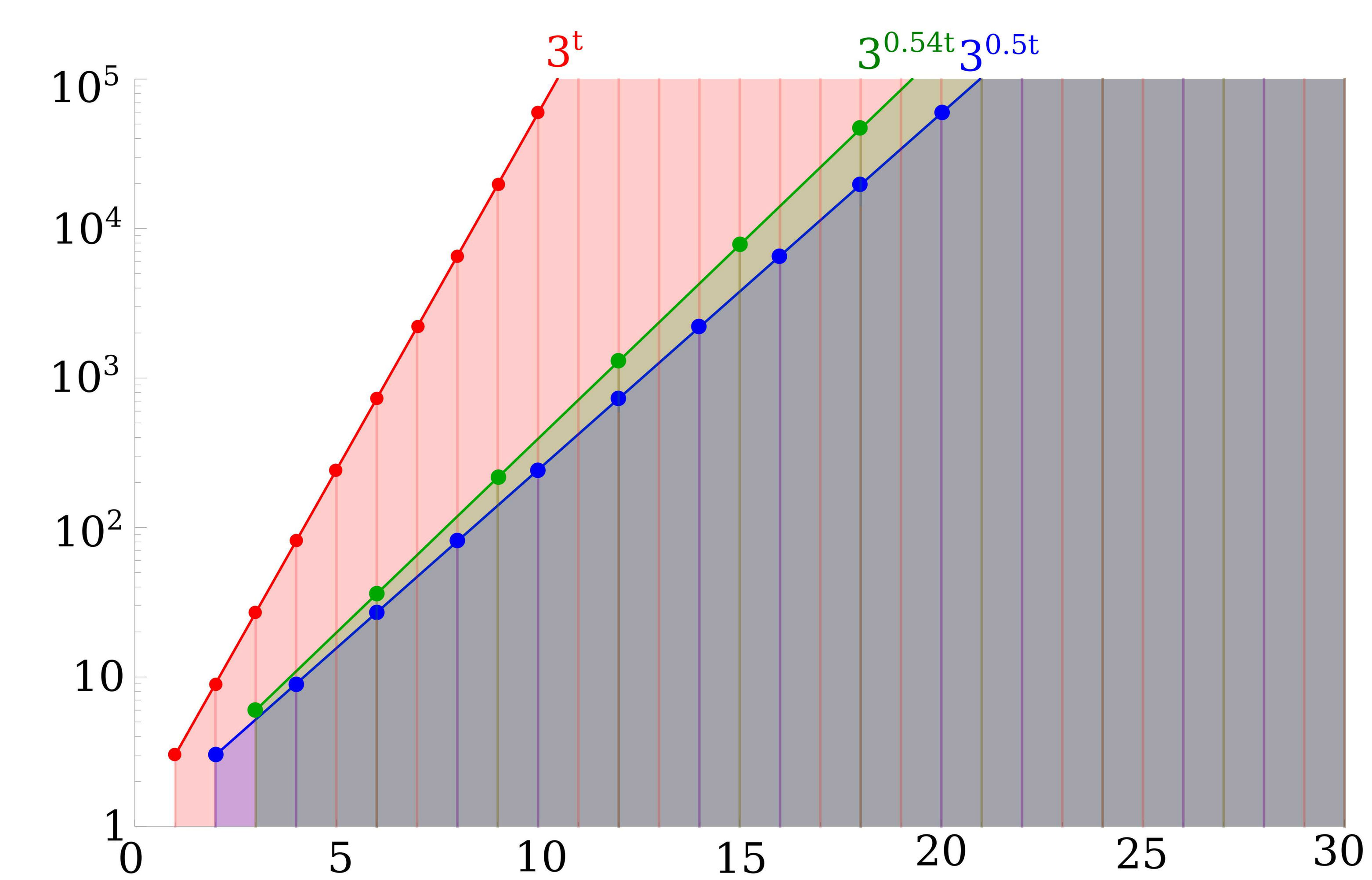
This representation of finite odd-dimensional quantum states is especially simple for the Clifford subtheory. For stabilizer states ρ, Gross proved $\rho_x(x_p,x_q) \in R_+ \cup \{0\}$.
For Clifford gates O, Almeida proved $O_x(x_p,x_q) = \exp(2\pi i S(x_p,x_q)/d)$,
where $S(x_p,x_q)$ is a quadratic function with integer coefficients.

Furthermore, stabilizer states can be expressed in terms of quadratic Gauss sums.

It also seems that for the T-gate magic state Wigner function, the number of quadratic Gauss sums necessary to express the function seems to follow the T-gate magic state's stabilizer rank:

$$\rho_T(x) = 9^{-1/2}\sum_{x2q \in Z/3Z} \exp 2\pi i/9(-x_{2q} + 2x_q) - x_{2q} + 2\times 3(x_{2q} - x_q)x_p$$

$$\rho^2_T(x) = \exp(2\pi i/9\, (7X^3_{2q} + 6X^2_{2q}(x_{q2} + \sigma_{2q}) + 3X_{2q}(2x_{p2} + 2x^2_q + 2x_{q2}\sigma_{2q} + \sigma^2_{2q}) + 6(x_{q1} + x_{q2})\sigma^2_{2q} + 3(2x_{p1} + x_{p2} + 2x^2_q + x^2_q)\sigma_{2q} + 3x_{p1}x_{q1} + 8x^3_q + 3x_{p2}x_{q2} + 8x^3_{q2}))$$



Can this correspondence be used to push the search for the T-gate magic state stabilizer rank beyond six qudits?

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