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A Comparison of Computational Issues in the Evaluation of Hankel Transforms for Geophysical Electromagnetics

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SUMMARY

Hankel transforms are prevalent in a vast number of geophysical problems. They arise in analytic solutions which reduce to radial symmetry (e.g. dipole antenna problems, loop antennas, etc.). In electromagnetic geophysical applications, particular interest lies in the computational cost and accuracy of evaluating them and improving current methods of evaluating Hankel transforms to result in faster, more accurate solutions. The current preferred method for fast Hankel transform evaluation is the digital filter. However, this approach suffers from various drawbacks, not the least of which is the complete absence of an error estimate. However, as the digital filter method is comprised of only weights and abscissae used to calculate various Hankel transforms, it is incredibly fast. In contrast, adaptive quadrature methods remain the 'gold standard' for accuracy as they can be evaluated within a given error tolerance. This accuracy often results in more computationally expensive solutions which make repeated Hankel transforms cumbersome to efficiently compute. Recent analysis by Key (2012) found that in some circumstances a well-designed quadrature-with-extrapolation (QWE) algorithm can be computationally competitive with digital filters. Here, we further examine the Key (2012) approach by focusing on error propagation in both the quadrature and extrapolation steps of the Wynn epsilon algorithm used to accelerate convergence in the Key (2012) approach. To complement this work, we also investigate a novel convolution method based on logarithmic change of variables that can exploit the exceptionally fast FFT algorithm. We find that the accuracy of the convolution algorithm is dependent upon poorly-determined constants necessary for the log transform and also necessitates an excessive number of FFT samples.

1. Background

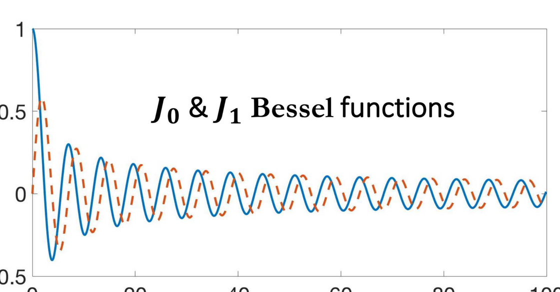
Hankel transforms appear in the analytic solutions of many electromagnetic problems. When confronted with a PDE where we can assume radial symmetry, our typical analytic strategy is to Fourier transform away two of the invariant directions. This double Fourier transform, while successful in reducing the problem to a much simpler ODE F(u,v), still requires a double Fourier transform to return to the original solution f(x,y), which is quite computationally expensive, even with a Fast Fourier Transform (FFT) algorithm.

$$f(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\, e^{2\pi i(ux+vy)} du\, dv$$

Instead of applying a double Fourier Transform, we can instead apply a 1D Hankel Transform, resulting in a more computationally efficient result f(r):

$$f(r) = \int_0^{\infty} \lambda F(\lambda)\, J_n(\lambda r)\, d\lambda \qquad \lambda = \sqrt{u^2+v^2} \quad r = \sqrt{x^2+y^2}$$

There are two common methods generally used to numerically evaluate Hankel transforms: adaptive quadrature and digital filters. Adaptive quadrature methods such as the one introduced by Chave (1983) evaluate the Hankel transform by breaking up domain into a series of subdomains from 0 to infinity, performing adaptive quadrature on each of the subdomains, and using an extrapolation method, such as the Padé series, to predict series sum. An extrapolation method is generally used as the oscillatory kernel (a Bessel function, $J_n(\lambda r)$, in this case) decays slowly, which would require a large amount of sums for convergence and these partial sums nearly cancel out, resulting in a large amount of error. The adaptive quadrature methods can be potentially slow but has the advantage of having error estimates available.



Digital filters, on the other hand, are often commonly used in geophysical forward codes because of their fast computational speed. Digital filters are designed and implemented by converting the Hankel transform into a convolution through a change of variables, solving the convolution integrals through either performing deconvolution in the spectral domain (most common) or through the Wiener-Hopf method. The resultant weights and abscissae are then used to calculate other Hankel transforms. While fast, the digital filter method is problem specific and may not apply to every application. Additionally, there are no error estimations available for the digital filter method, making it difficult to estimate accuracy.

2. Hankel transform methods

Within this study, we restrict our evaluations to two commonly used digital filters (Guptasarma & Singh, 1997; Kong 2007), two adaptive quadrature methods (Chave, 1983; Key, 2012), and examine another method which uses a log-transform to coach the problem into a convolution which can exploit the FFT algorithm (Bisseling & Kosloff, 1985):

$$f(r) = \int_0^{\infty} \lambda F(\lambda)\, J_n(\lambda r)\, d\lambda$$

$$\Downarrow$$

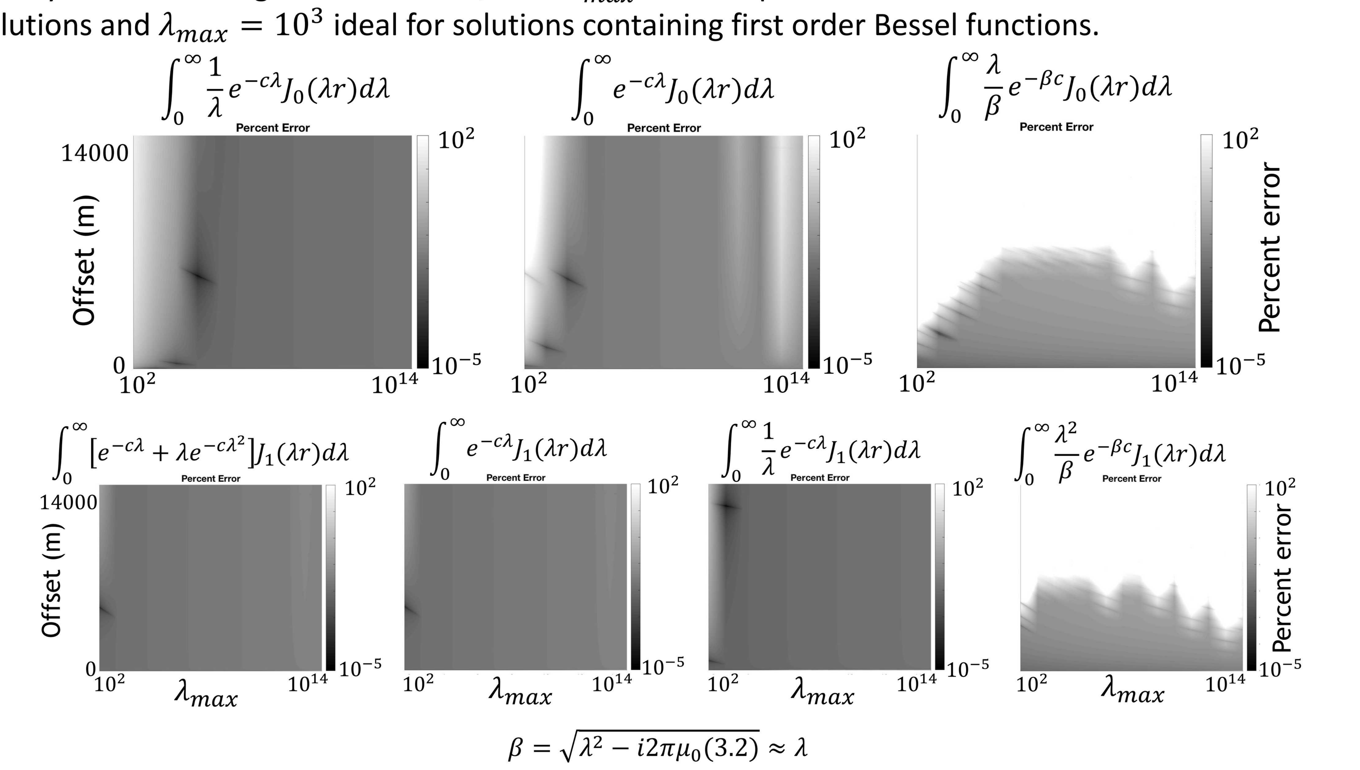
$$f\left(r_{min}e^{j\delta}\right) = {\lambda_{max}}^2\delta \sum_{j=0}^{N-1} e^{-2m\delta} F\left(\lambda_{max}e^{-m\delta}\right) J_n\left(\lambda_{max} r_{min} e^{(j-m)\delta}\right)$$

$$r = r_{min}e^{j\delta},\ \lambda = \lambda_{max}e^{-m\delta} \text{ where } m = j = 0,..,N-1$$

We examine the ability of the log-transform (LT) method to compete with the above listed methods in both computationally efficiency and accuracy. We additionally develop a homegrown version of the Key (2012) QWE method in FORTRAN and evaluate the error propagation associated with various parts of the QWE algorithm which is divided into three separate sections for our analysis: generation of the zeros and Bessel weight vector, adaptive quadrature evaluation, and epsilon algorithm extrapolation.

3. Log - transform method

The LT method depends on three unknown constants to result in a viable solution: δ (step size), λ_{max} (maximum value in the Hankel domain), and N (sample length). We can reduce the parameters we need to explicitly state If we know the range which we are solving for (i.e. r_{max} and r_{min}), we can start at $\lambda_{min} = \frac{\pi}{r_{max}}$ (the upper bound of λ_{min}) and half λ_{min} and λ_{max}, using $\delta = \frac{\log\left(\frac{\lambda_{max}}{\lambda_{min}}\right)}{\mathrm{N}-1}$ for the step size until we reach criteria: $\lambda_{max}\, r_{max}\delta < \pi$. This now leaves us with determining the ideal choice for two parameters: λ_{max} and N. The optimal choice for λ_{max} was generally found to be fairly stable, and can be any value in a range of 10^3 to 10^6, with $\lambda_{max} = 10^6$ optimal for solutions with zeroth-order Bessel solutions and $\lambda_{max} = 10^3$ ideal for solutions containing first order Bessel functions.



$$\beta = \sqrt{\lambda^2 - i2\pi\mu_0(3.2)} \approx \lambda$$

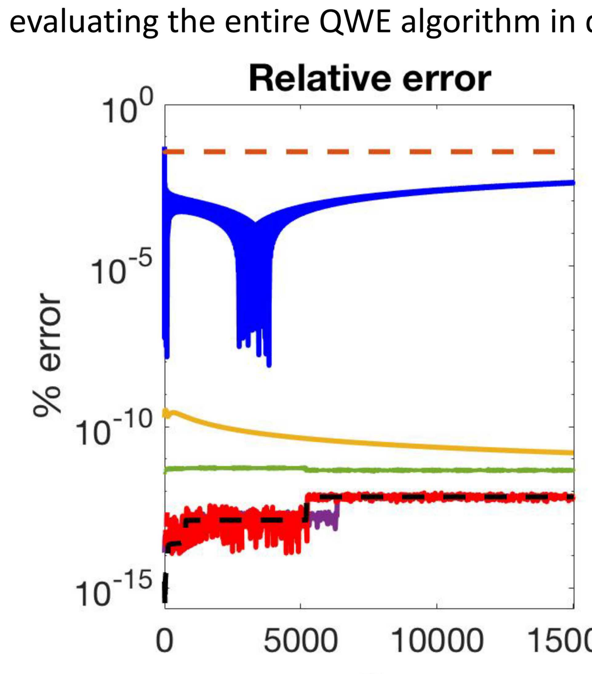
The **above figures** show the results for several Hankel transforms at c=10 and N=2^{19}. The darker grey indicates a better fit. Upon examining this method, it quickly became apparent that it was only viable for large sample lengths (N=2^{19} or greater), and not as accurate as initially hoped.

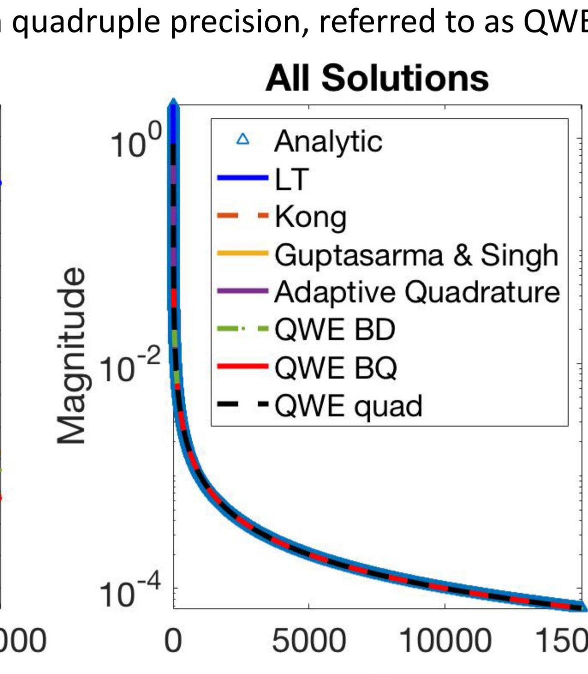
4. QWE and comparison

We compare three different versions of our homegrown QWE solution, developed following Key (2012), against the LT solution, the two previously mentioned digital filters, and the Chave (1983) adaptive quadrature solution (AQ). We select a test function to focus on:

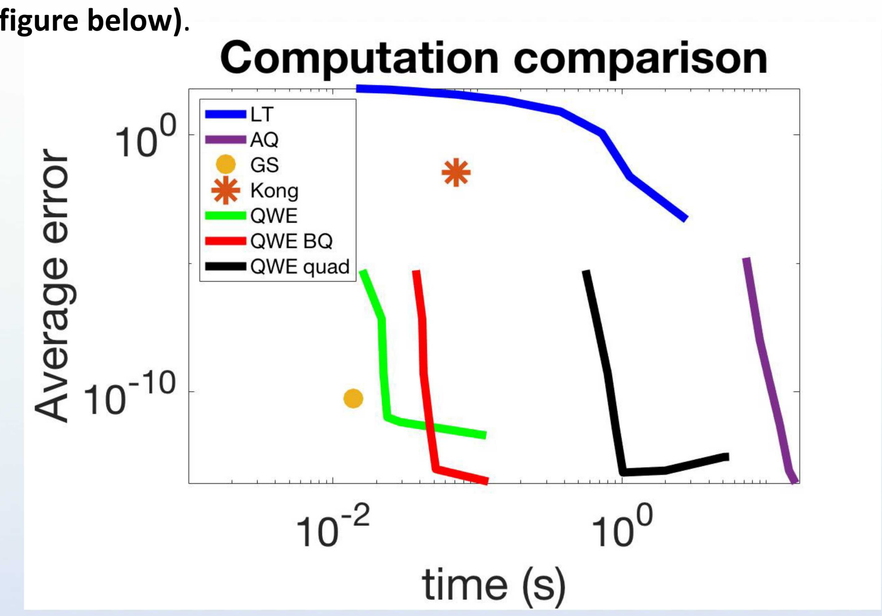
$$\int_0^{\infty} \frac{1}{\lambda} e^{-c\lambda} J_0(\lambda r) d\lambda = \frac{1}{\sqrt{(c^2+r^2)}}$$

We first evaluate overall relative error for the above methods as a function of r (**figures below**). To obtain an estimate of where error propagates the most in the QWE formulation, we evaluated the QWE solution in double precision using double precision Bessel functions obtained from Zhang & Jin (1996), referred to henceforth as QWE (BD), evaluating the QWE code in double precision with quadruple precision Bessel functions (via the ARB C library), henceforth referred to as QWE BQ, and evaluating the entire QWE algorithm in quadruple precision, referred to as QWE quad.





We then compare the average percent error as a function of time for increasing N value for the LT method, increasing quadrature tolerance for the QWE solutions, and increasing absolute tolerance for the AQ solution (**figure below**).



5. QWE tolerances

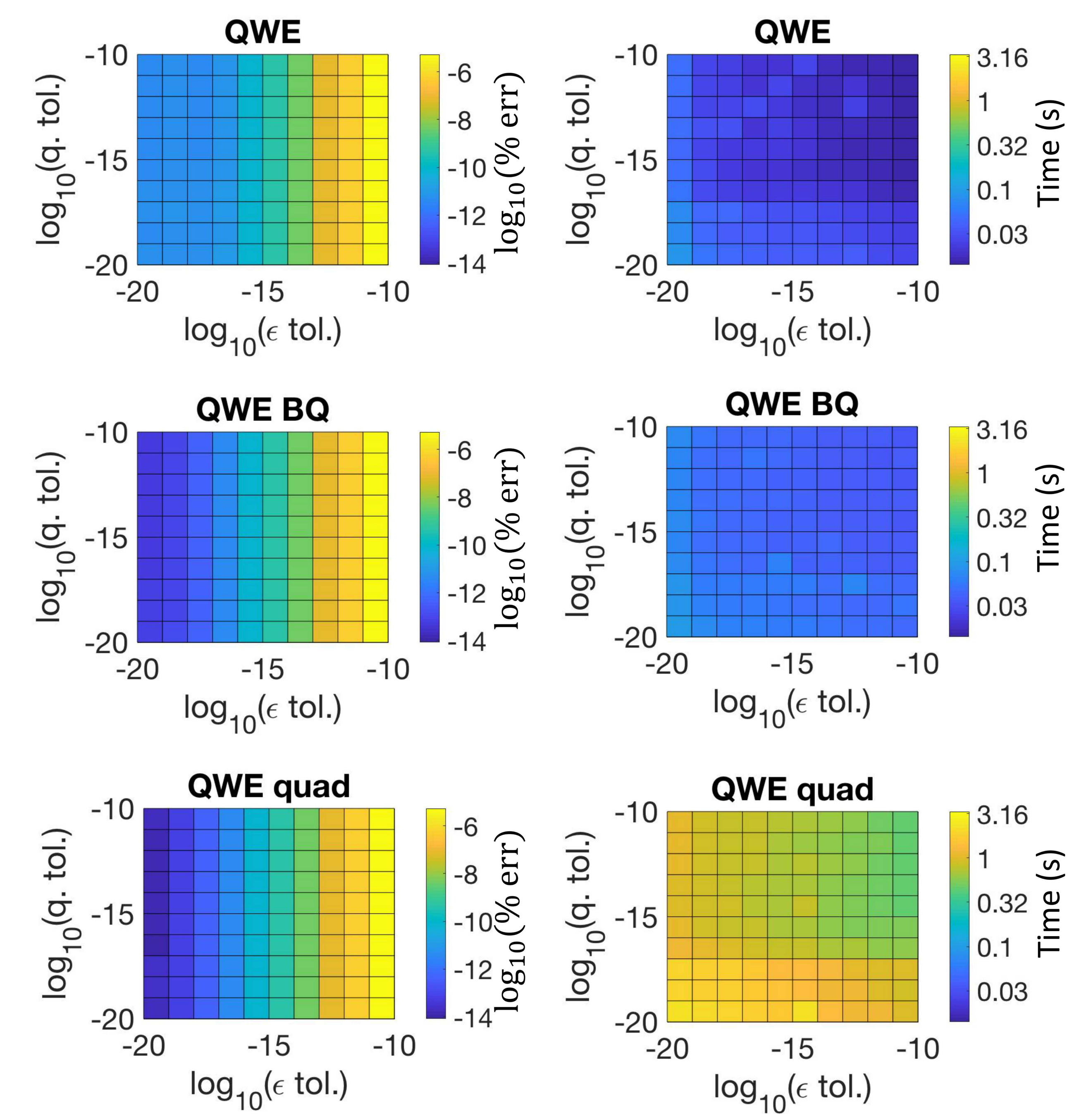
In addition to comparing the overall precision between the QWE codes, we also compared the overall effect of given tolerances in two different sections. There are two potential areas in the QWE code where tolerance might affect the accuracy of the final result: the tolerance on the initial quadrature result before it is passed to the epsilon algorithm (termed quadrature tolerance), and the final tolerance that determines the termination criteria for the QWE iterative sequence. In Key (2012) this is formulated using the difference (δS_n^*) in the current (S_n^*) and previous (S_{n-1}^*) extrapolated sum, which needs to be smaller than an expression containing an absolute (β) and relative tolerance (α):

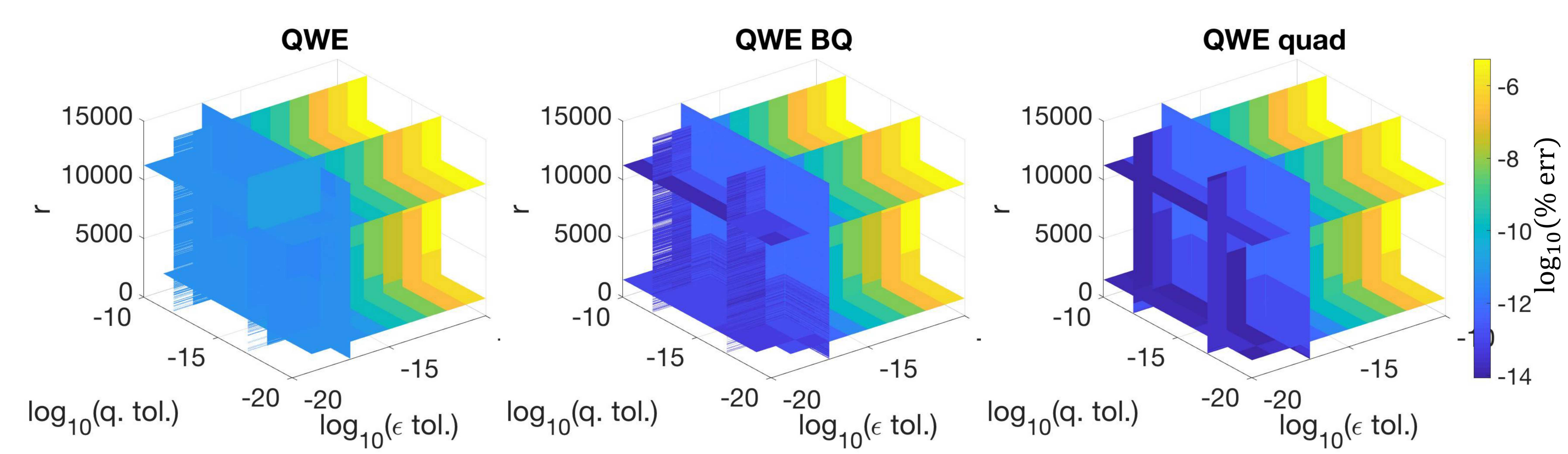
$$\delta S_n^* = |S_n^* - S_{n-1}^*|$$
$$\delta S_n^* \le \alpha|S_n^*| + \beta$$

To simplify our analysis, we reduce this down to a single tolerance which we term the ϵ tolerance, reducing the above criteria to:

$$\delta S_n^* \le \epsilon$$

We then evaluate each QWE version (QWE, QWE BQ, and QWE quad) for a range of quadrature tolerances (10^{-10} to 10^{-20}) and ϵ tolerances (10^{-10} to 10^{-20}) to identify which part of the QWE code has the most impact on the overall accuracy of each QWE code. For each quadrature and ϵ tolerance combination we evaluated the percent error **(bottom figures)** as well as the overall average percent error and the time it took to evaluate each combination (**figures to right**). For all QWE versions, the ϵ tolerances (10^{-10} to 10^{-20}) had the largest influence on the total accuracy of the solution while the quadrature tolerance generally only affected the computational efficiency, with a smaller quadrature tolerance resulting in a generally faster and just as accurate solution.





6. Conclusions

LT method

- The Logarithmic transform method, while potentially promising, had a large number of drawbacks that resulted in the method being relatively unusable for evaluating Hankel transforms.
- The LT method requires three poorly defined constants to return a viable solution, resulting in a degree of ambiguity for other potential results that were not tested in this evaluation (it is difficult to predict if the prescribed λ_{max} for zeroth and first order Bessel functions would result in similar accuracies for different functions than those tested).
- Additionally, the LT method requires a range of values with the r_{min} value required to be near zero, further reducing its usefulness for generic solutions.
- Lastly, the LT method requires extremely large sample sizes (N=2^{19} or greater) to result in a still rather low accuracy which is overall not as computationally efficient as the much more accurate double precision QWE results.

QWE error analysis

- We examined the overall precision of the QWE code by evaluating the QWE code with different precisions and examining how the quadrature and epsilon algorithm tolerance affected the resulting solution accuracy.
- Examining the relative error, the QWE BD solution is relatively close in error to the AQ, QWE BQ , and QWE quad solutions, however it is still not as accurate as the rest of the adaptive quadrature solutions (with all QWE solutions at quadrature tol. = 10^{-12}, ϵ tol. = 10^{-17} and the AQ solution at abs. tol.= 10^{-17}, rel. tol.= 10^{-17}) .
- With only the addition of quadruple precision Bessel functions to the double precision QWE, we immediately see an improvement in accuracy, resulting in a (less stable at lower ϵ tolerances) QWE BQ solution that is around as accurate as both the AQ and QWE quad solutions.
- As the QWE BQ solution has a much lower relative error at the same tolerance than the QWE BD solution, we can interpret that to mean that the majority of the precision is tied up in the accuracy of the Bessel function zeros and overall precision, especially as the QWE quad solution remains at the same relative accuracy with only the benefit of an increase in stability for lower ϵ tolerances.
- Additionally, the quadrature tolerance had minimal effect on the overall accuracy of the final QWE solutions, but a large impact on the time each solution required, indicating that it is preferable to run the QWE code with a relatively small ϵ tolerance and a large quadrature tolerance.
- Finally, the overall time and average accuracy for each method indicates that the two double precision QWE solutions are as fast or faster than a digital filter with much higher accuracy. As the QWE BQ solution is nearly as fast as the QWE BD solution and as accurate as the AQ and QWE quad solution, the ideal choice would be to use the QWE BQ solution with a relatively small ϵ tolerance and a large quadrature tolerance for the optimal solution for both accuracy and computational efficiency.

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