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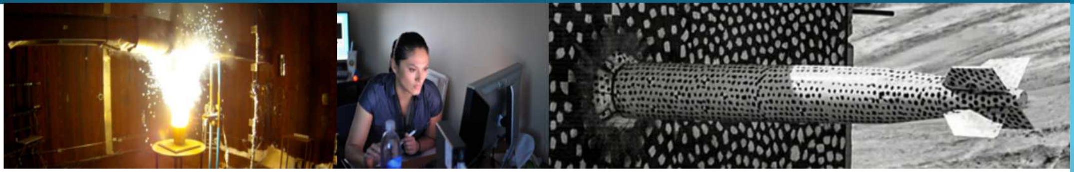
The Pollution Effect and Novel Methods to Reduce Its Influence in the Mid-Frequency Range



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2-6 December 2019

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Assumptions

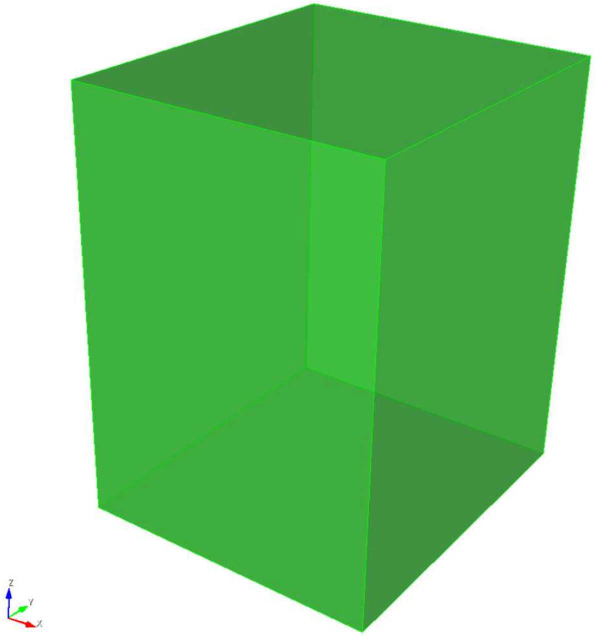
- Consider only deterministic problems
  - Assume fluid properties and damping are known
- Demonstrate pollution using acoustics
  - Results and methods applicable to elastodynamics
- Considering only finite element solution in frequency domain
  - Helmholtz equation

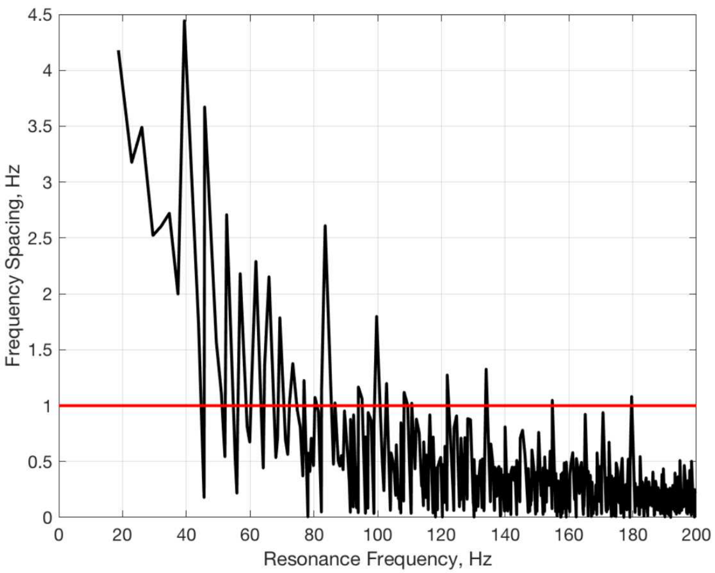
Outline

- Illustration of mid-frequency and pollution error
- History of pollution error
- Description of pollution error
- Methods to mitigate pollution error
- Conclusions

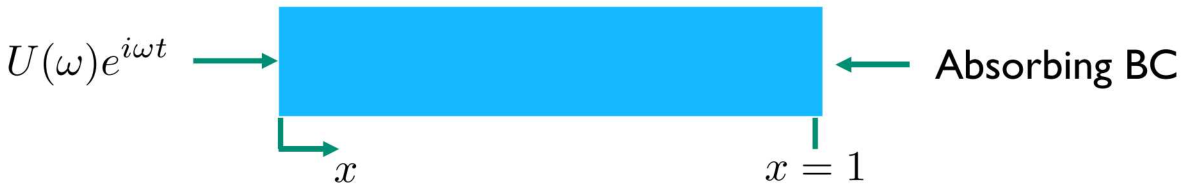
Mid-Frequency Illustration

- Sandia Reverberation Chamber
- Dimensions:
  - $L_x = 6.58$ m (21.6 ft)
  - $L_y = 7.50$ m (24.6 ft)
  - $L_z = 9.17$ m (30.1 ft)
- Volume:
  - $V = 453\ m^3$ (15994 ft^3)
- Length along diagonal:
  - $L_d = 13.6$ m (44.5 ft)
- Schroeder frequency:
  - $f_s \sim 300$ Hz
- Mid-frequency onset:
  - $f_{mid} \sim 100$ Hz





Classical Galerkin Finite Element Method – 1D



$$\frac{\partial^2 p(x)}{\partial x^2} + k^2 p(x) = 0$$ Helmholtz equation

Wavenumber: $k = \dfrac{\omega}{c}$

$$\left.\frac{\partial p}{\partial x}\right|_{x=0} = -i\omega\rho_o U$$
$$\left.\frac{\partial p}{\partial x}\right|_{x=1} = -ikp(1)$$
Boundary conditions

- Construct weak formulation:

$$\int_0^1 \Big[-k^2 p(x) w(x) + \frac{\partial p}{\partial x}\frac{\partial w}{\partial x} \Big] dx + ikp(1)w(1) = i\omega\rho_o U(\omega) w(0)$$

Test (weighting) function: $w(x)$

$$p^h(x) = \sum_{i=1}^{N} \phi_i(x) p_i$$
$$w^h(x) = \sum_{i=1}^{N} \phi_i(x) c_i$$

Galerkin formulation

- Linear algebraic finite element system:

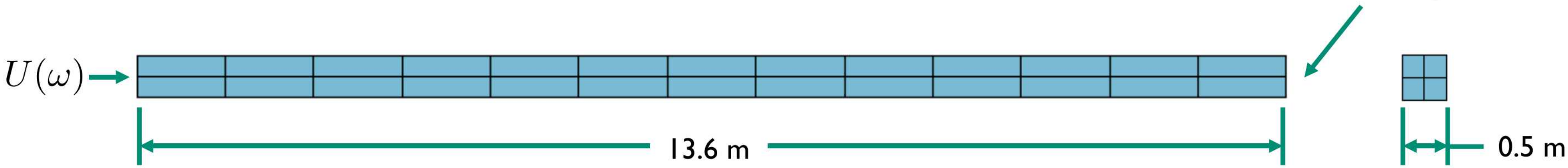
$$(-k^2\mathbf{M} + \mathbf{K})\mathbf{p} = \mathbf{u}$$

pressure at nodes

Semi-Infinite 1D Duct Example – 20 Hz

- Duct length 13.6 m
  - Equal to diagonal of Sandia reverb chamber
- Excitation frequency: **20 Hz**
- **16 Hex8 elements per wavelength**

Absorbing BC

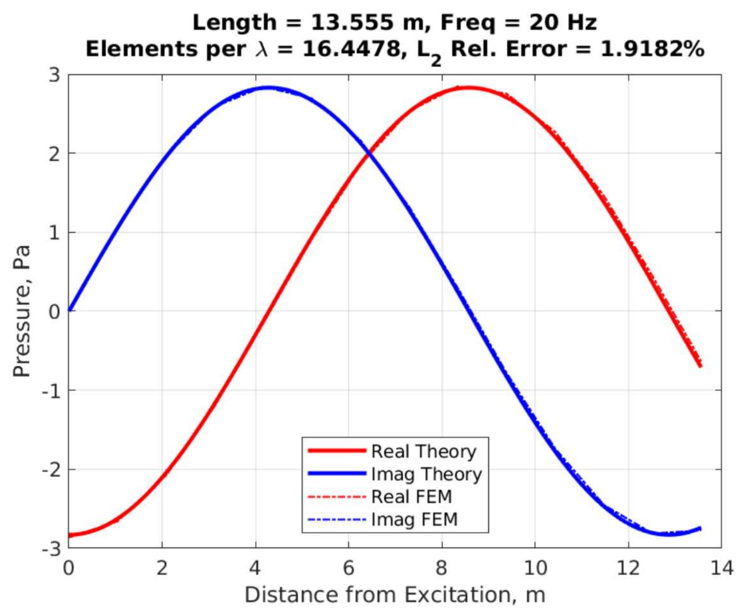


- Analytical solution: $p(z) = \rho_o c U(\omega) e^{-ikz}$

L_2 relative error = 1.9 %

$$\tilde{e}_2 = \frac{||p - p_h||_2}{||p||_2}$$

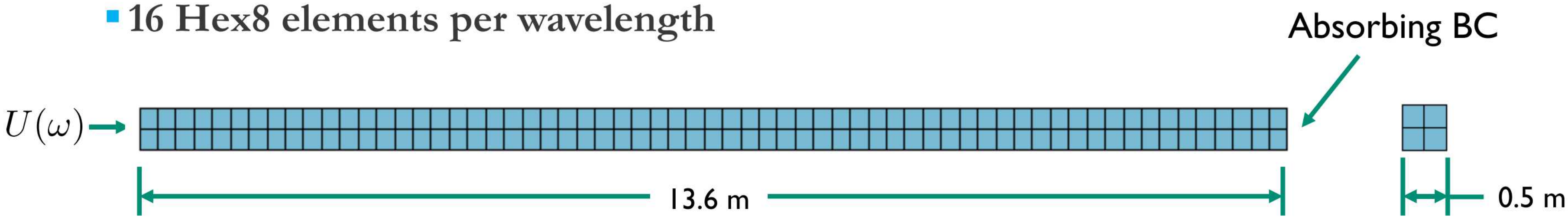
$$||u||_2 = \left[\int_0^L |u(z)|^2 dz \right]^{1/2}$$



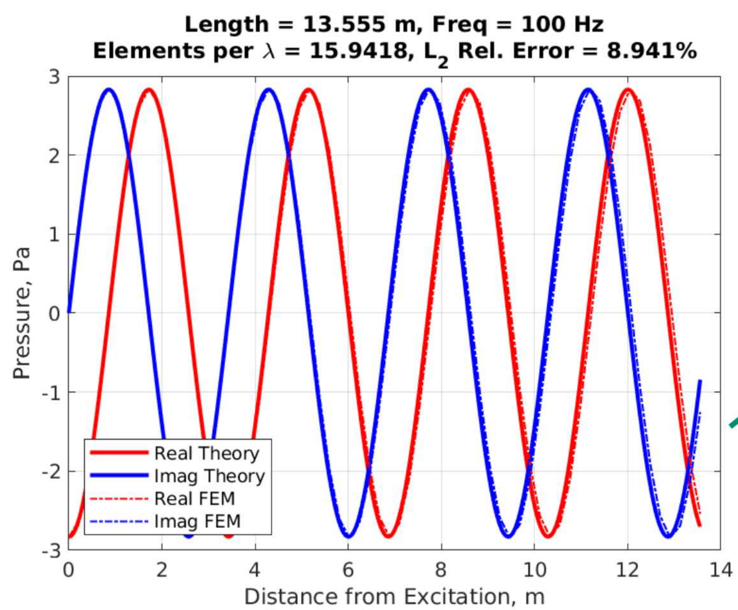
Percent error at end = 2.9 %

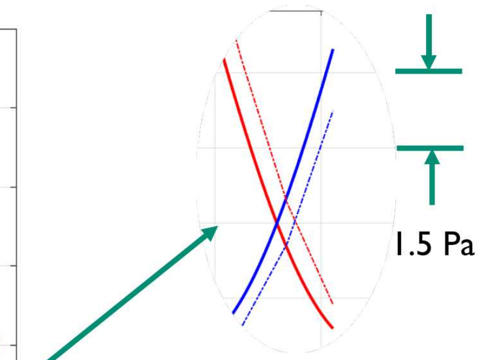
Semi-Infinite 1D Duct Example – 100 Hz

- Excitation frequency: **100 Hz**
- **16 Hex8 elements per wavelength**



L_2 relative error = 8.9 %



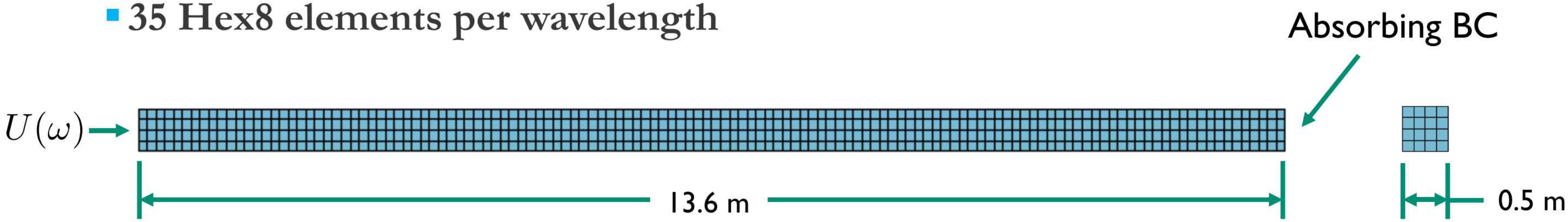


Percent error at end = 15.5 %

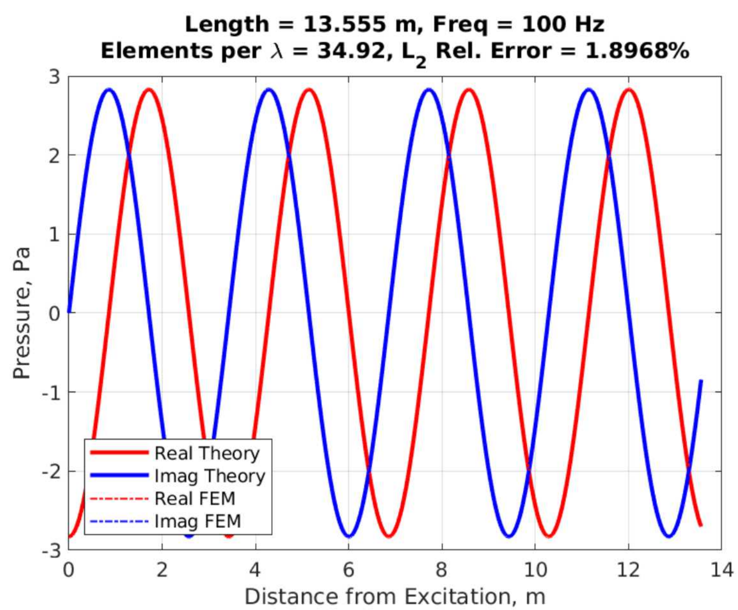
Same number of elements per wavelength as 20 Hz case, 4.7X the L_2 error!

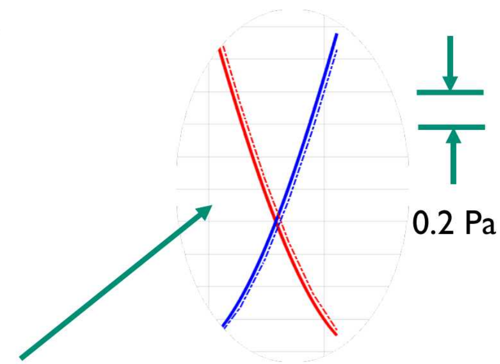
Semi-Infinite 1D Duct Example – 100 Hz

- Excitation frequency: **100 Hz**
- **35 Hex8 elements per wavelength**



L_2 relative error = 1.9 %



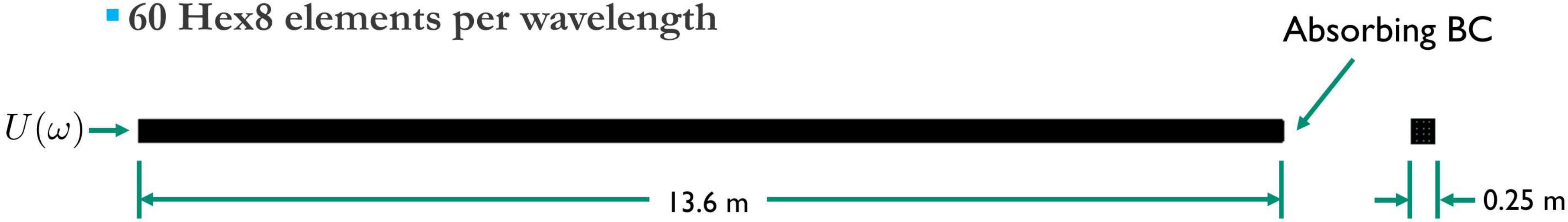


Percent error at end = 3.3 %

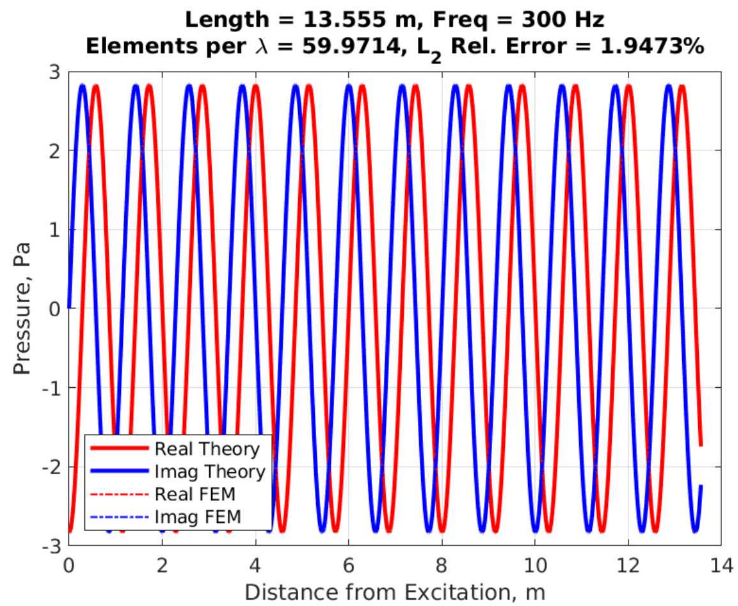
Require 35 elements per wavelength at 100 Hz for same error as 16 elements per wavelength at 20 Hz

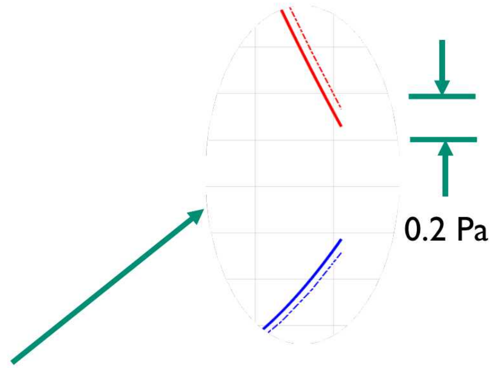
Semi-Infinite 1D Duct Example – 300 Hz

- Excitation frequency: **300 Hz** (onset of diffuse field)
- **60 Hex8 elements per wavelength**



L_2 relative error = 1.9 %



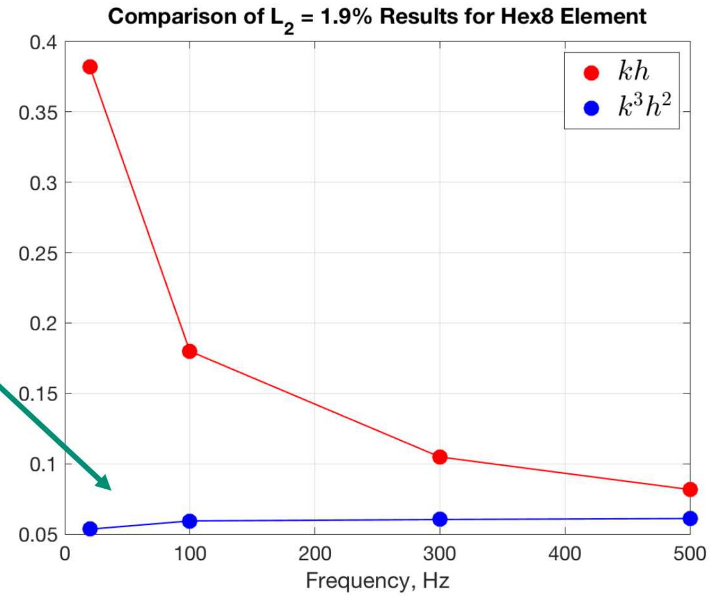


Percent error at end = 3.4 %

Require 60 elements per wavelength at 300 Hz for same error as 16 elements per wavelength at 20 Hz

Semi-Infinite 1D Duct Example – Results

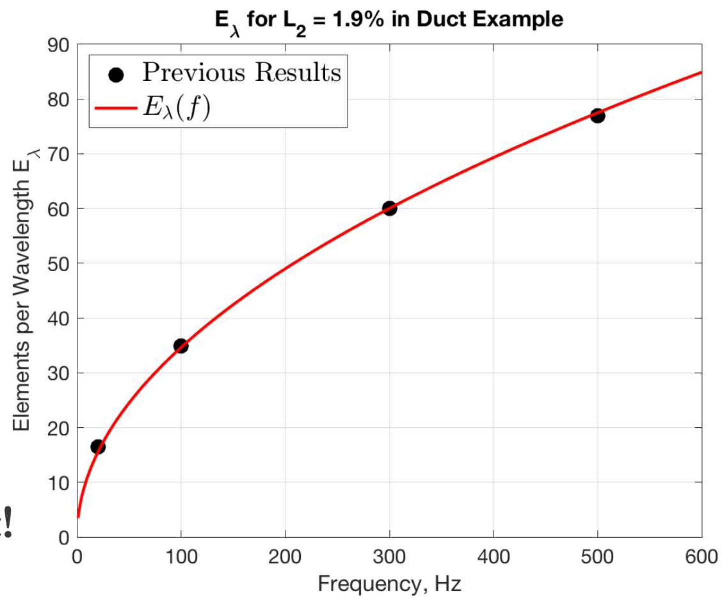
- Examine results from previous slides for 1.9% relative error solutions
- Find L_2 error remains constant if $k^3h^2 = C_1^2$
  - Constant dependent on duct length



- Expression for elements-per-wavelength needed for 1.9% error:

$$k = 2\pi/\lambda$$
$$h = \lambda/E_\lambda$$
$$\longrightarrow \quad k^3h^2 = \frac{8\pi^3}{\lambda E_\lambda^2}$$

$$E_\lambda(f) = C_1\sqrt{\frac{8\pi^3 f}{c}}$$



- **Standard rule-of-thumb kh = const insufficient!**

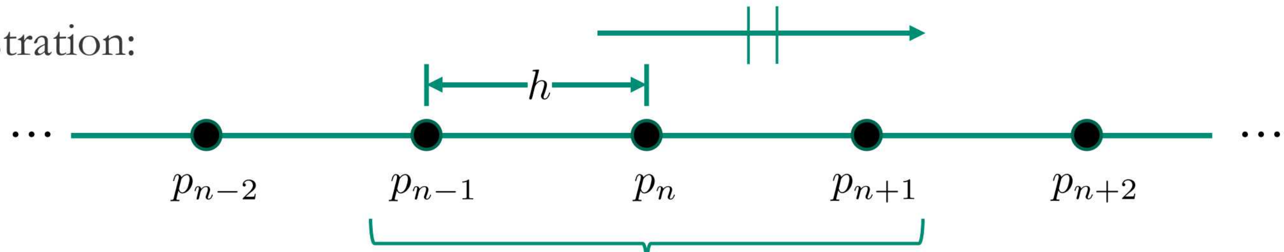
Brief History

- First paper illustrating these results was Bayliss, et al.[1] in 1985
  - Performed an analysis similar to that just presented
- Harari and Hughes[2] 1991
  - Examined dispersion
  - Proposed 10 elements per wavelength 'rule-of-thumb' for first-order elements
- Ihlenburg and Babuška[3] 1995
  - Derived expression for relative error for h-refinement with first-order (p=1) elements
- Ihlenburg and Babuška[4] 1997
  - Derived general expression for relative error for h- and p-refinement.
- Babuška and Sauter[5] 1997
  - Impossible to eliminate pollution effect in two or more spatial dimensions in Galerkin FE
- Ihlenburg[6] 1998
  - Authors book providing overview of accurate numerical analysis for Helmholtz equation

[1] A. Bayliss, C. I. Goldstein, and E. Turkel. *Journal of Computational Physics*, 59(3):396-404, 1985.
[2] I. Harari and T. J. R. Hughes. *Computer Methods in Applied Mechanics and Engineering*, 87(1):59-96, 1991.
[3] F. Ihlenburg and I. Babuška. *Computers and Mathematics with Applications*, 30(9):9-37, 1995.
[4] F. Ihlenburg and I. Babuška. *SIAM Journal on Numerical Analysis*, 34(1):315-358, 1997.
[5] I. M. Babuška and S. A. Sauter. *SIAM Journal on Numerical Analysis*, 34(6):2393-2423, 1997.
[6] F. Ihlenburg, Finite Element Analysis of Acoustic Scattering, Springer, 1998.

Dispersion Error

- Numerical solutions of the Helmholtz equation are dispersive
  - Both finite element and finite difference
- 1D Illustration:



FE row equation for node *n*

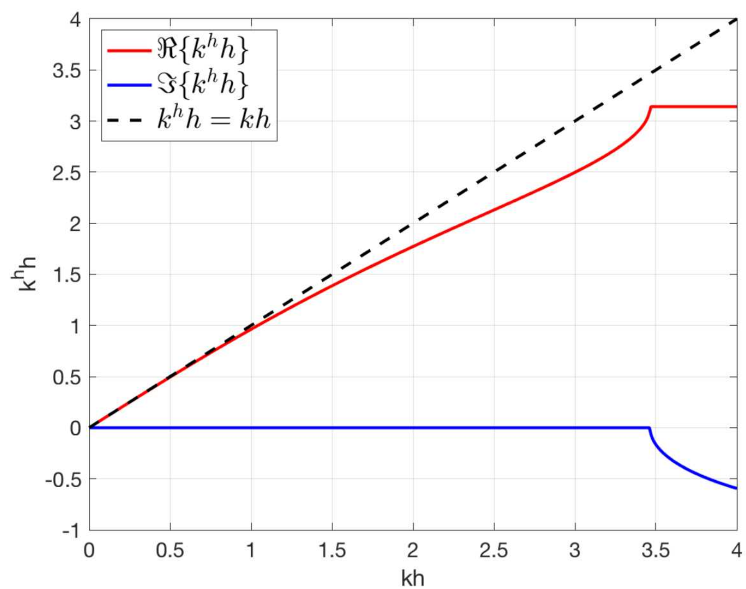
$$-\frac{k^2h^2}{6}(p_{n-1} + 4p_n + p_{n+1}) - p_{n-1} + 2p_n - p_{n+1} = 0$$

Mass matrix **Stiffness matrix**

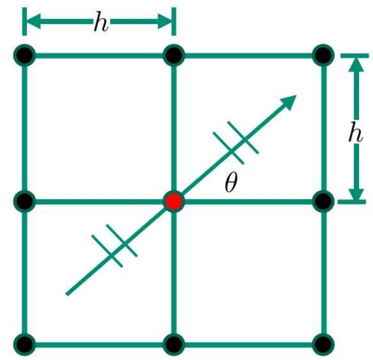
- Exact solution for plane wave: $p_{exact} = e^{-ikx}$
- Substitute into row equation: $p_j = e^{-ik^h hj}$
  - Expression for discrete wavenumber k^h :

$$k^h h = \cos^{-1}\left[\frac{1-(kh)^2/3}{1+(kh)^2/6}\right]$$

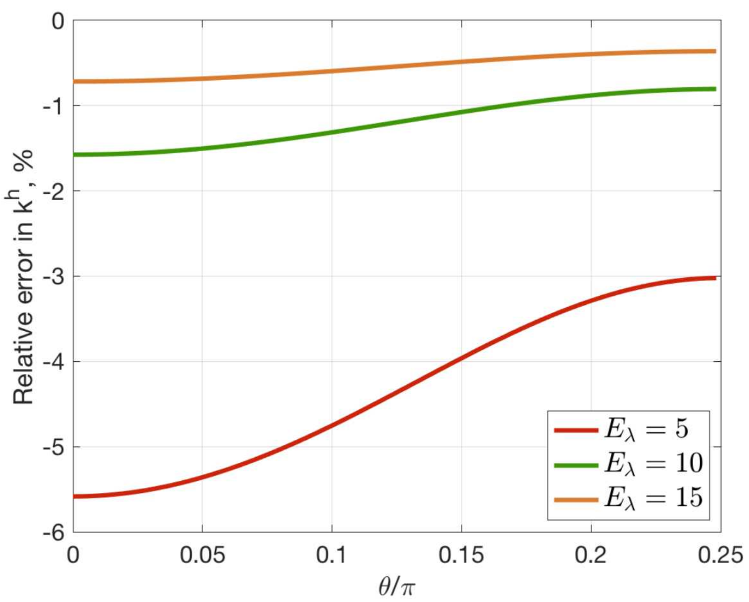
- The dispersion is a phase lead: $c^h \geq c$
- Evanescent FE prediction unless $kh < \sqrt{12}$

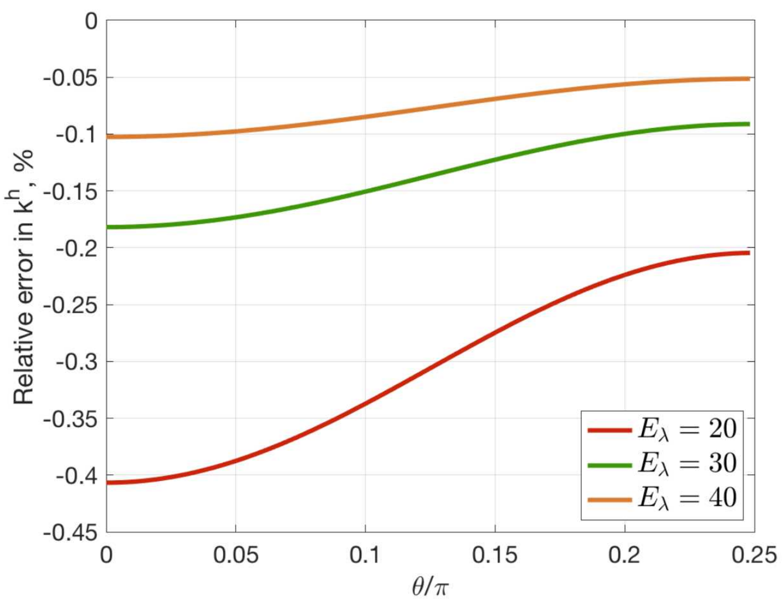


Dispersion Error – 2D Illustration



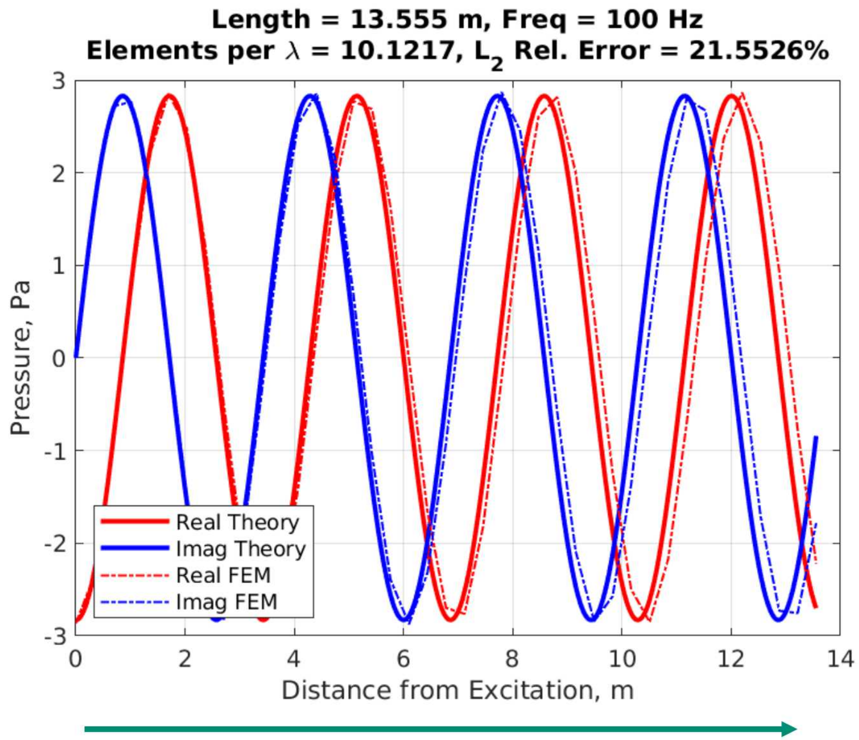
- Bilinear quadrilateral elements
- Exact plane wave solution: $p_{exact}(x, y, \theta) = e^{-ik[x\cos(\theta)+y\sin(\theta)]}$
- As before substitute into FE row equation for center node
  - Obtain expression for discrete wavenumber: $k^h h = f(kh, \theta)$
- Largest dispersion when plane wave is aligned with mesh $(\theta = 0)$
- Dispersion always present for $E_\lambda < \infty$





Pollution Effect

- Dispersion analysis shows error depends on mesh refinement kh
  - The error is local to the elements, due to interpolation by shape functions
- However, accumulation of dispersion over the domain 'pollutes' the prediction
  - The longer the domain relative to a wavelength, the greater the pollution
- Return to prediction at 100 Hz with 10 elements per wavelength:



Accumulating interpolation error

Relative Error Equation and the 'Pollution Effect'

- Ihlenburg and Babuška derived general expression for relative error of Galerkin finite element solution of Helmholtz equation:

$$\frac{|u-u_h|_1}{|u|_1} \leq C_1\left(\frac{kh}{p}\right)^p + C_2 Lk\left(\frac{kh}{p}\right)^{2p}, \quad kh < 1$$

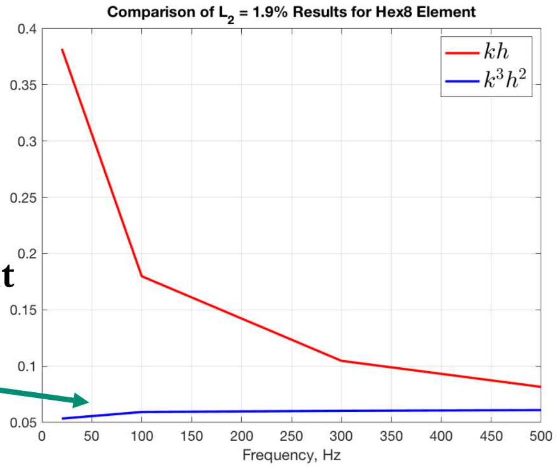
Element size: h
Element order: p
Domain length: L

Interpolation Error (dispersion)

'Pollution Effect'

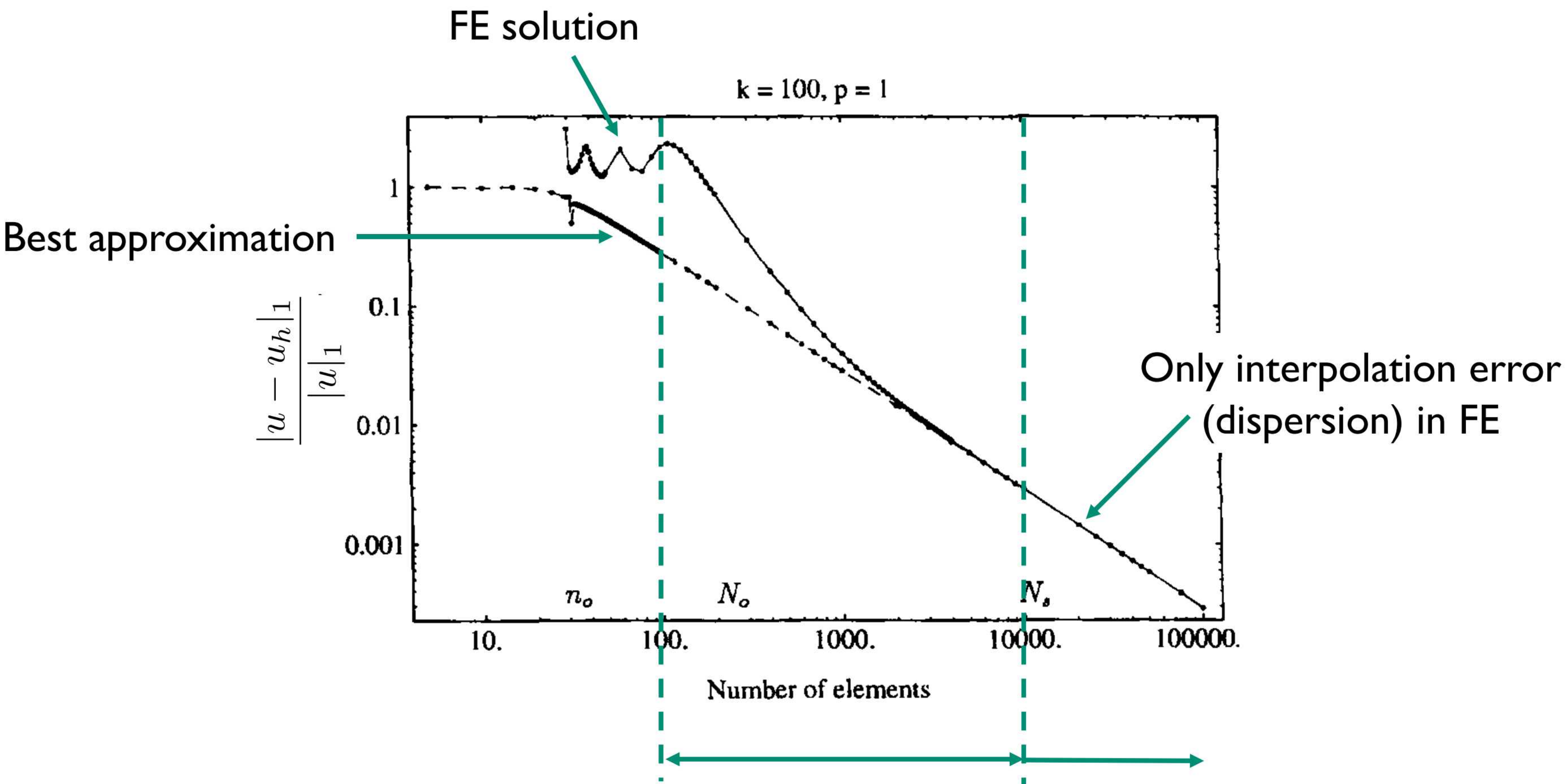
The dominate issue at mid-frequencies

- Applies to oscillatory solutions
- Derived using the Green's function, Galerkin error analysis, FE stability and approximation statements
  - Expressed in terms of the H^1-seminorm
- Note kL product in the pollution term
- **For constant relative error: second term must remain constant**
  - Discovered previously in the 1D duct results



F. Ihlenburg and I. Babuška. *SIAM Journal on Numerical Analysis*, 34(1):315-358, 1997.
F. Ihlenburg, Finite Element Analysis of Acoustic Scattering. Springer, 1998.

Convergence using Linear Elements



$$\frac{|u-u_h|_1}{|u|_1} \leq C_1 kh + C_2 L k^3 h^2$$

$$\frac{|u-u_h|_1}{|u|_1} \leq C_1 kh$$

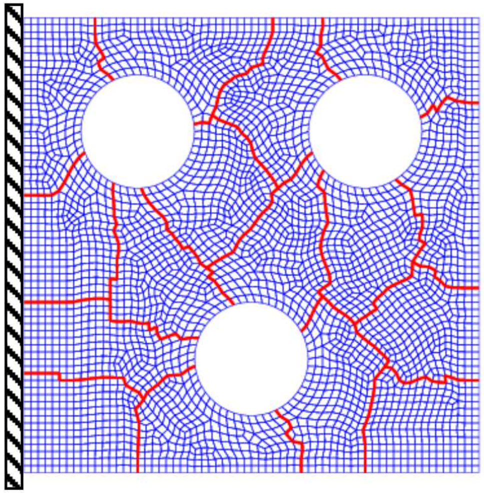
Figure: F. Ihlenburg and I. Babuška. *International Journal For Numerical Methods In Engineering*, 38(22):3745-3774, 1995.

Methods to Reduce Pollution

- Brute force method
- Higher-order shape functions
- Isogeometric analysis
- Enriched/Wave-based finite element methods
- Discontinuous Petrov-Galerkin

Brute Force - Domain Decomposition and Massively Parallel

- Domain Decomposition
  - First performed by Schwarz in the 1870s
  - Decompose model into smaller subdomains, each often assigned to one processor
  - Two-level methods have "local" subdomain and "global" coarse solves
  - Solve using preconditioned conjugate gradients or GMRES
- Massively Parallel
  - Distribution of processors (nodes), each with own memory, linked together by a specialized network communication system



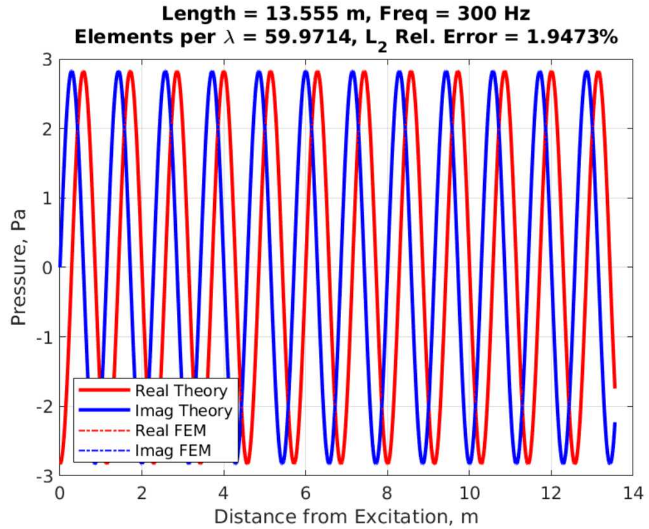
C. Farhat and F.-X. Roux. *International Journal for Numerical Methods in Engineering*, 32:1205-1227, 1991.
C. R. Dohrmann, and O. B. Widlund. *International Journal for Numerical Methods in Engineering*, 82(2):157-183, 2010.

Higher-order Shape Functions

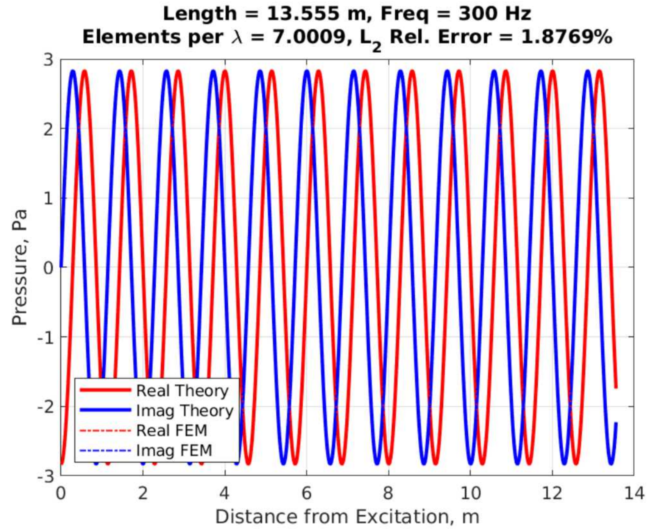
- Express relative error in terms of elements-per-wavelength E_λ

$$\frac{|u-u_h|_1}{|u|_1} \leq \tilde{C}_1 \frac{1}{E_\lambda^p} + \tilde{C}_2 \frac{Lf}{c\,E_\lambda^{2p}}, \quad kh < 1$$

- As order of shape functions p increases, pollution effect decreases by E_λ^{-2p}
- Comparison of Hex8 (p=1) and Hex20 (p=2) at 300 Hz:
  - Same relative error with ¼ the degrees of freedom



Hex8: 60 elements/wavelength
L_2 relative error = 1.9 %



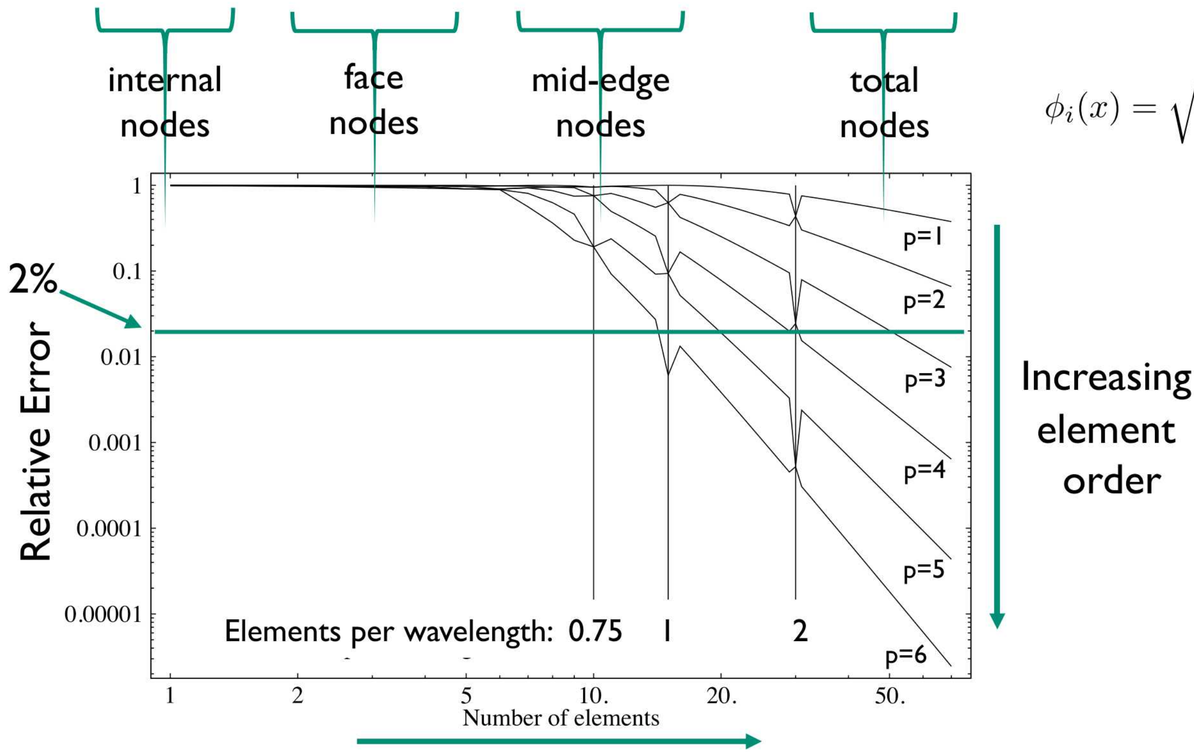
Hex20: 7 elements/wavelength
L_2 relative error = 1.9 %

L. L. Thompson and P. M. Pinsky. *Computational Mechanics*, 13(4):255-275, 1994.
F. Ihlenburg and I. Babuška. *SIAM Journal on Numerical Analysis*, 34(1):315-358, 1997.

Hierarchical Polynomial Bases

- High wavenumber and large domains ($kL >> 1$) require $p > 2$ for accurate predictions and computationally feasible FE system
- Hierarchical Legendre polynomials
  - Add mid-edge, face and internal nodes to standard Hex8:

$$(p-1)^3 + 6(p-1)^2 + 12(p-1) + 8 = (p+1)^3, \quad p \geq 2$$



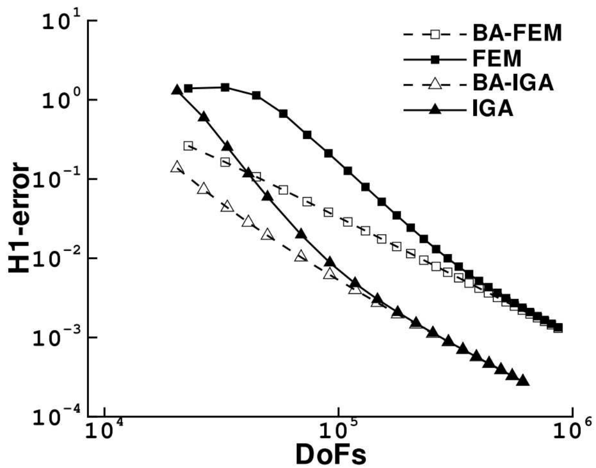
$$\phi_i(x) = \sqrt{\frac{2i-1}{2}} \int_{-1}^{\xi} P_{i-1}(\tau)d\tau$$

Legendre polynomial

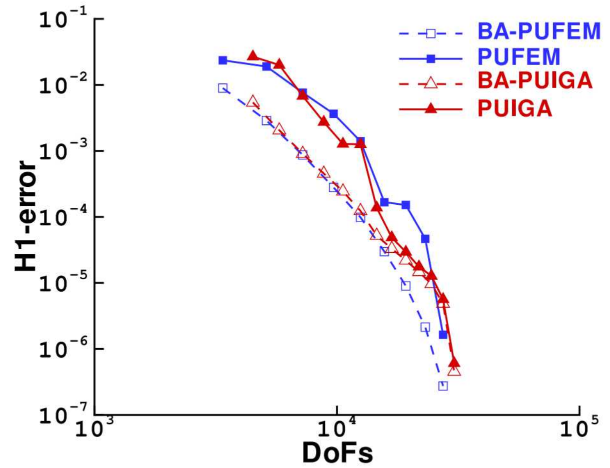
I. Babuška, B. A. Szabo and I. N. Katz. *SIAM Journal on Numerical Analysis*, 19(3):515-545, 1981.
Figure: F. Ihlenburg and I. Babuška. *SIAM Journal on Numerical Analysis*, 34(1):315-358, 1997.

Enriched Isogeometric Analysis

- Proposed by Hughes et al[1]
- Uses non-uniform B-splines (NURBS)
  - Defines geometry in CAD models
  - Used as shaped functions for the method
- Improved accuracy relative to classical polynomial shape functions
- An enriched method has been proposed with greater accuracy relative to higher-order piecewise polynomials[2]



$k = 100\pi,\ p = 3$

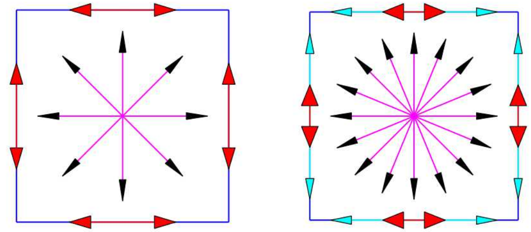


$k = 100\pi,\ p = 3,$ 20 plane waves

[1] T. J. R. Hughes, J. A. Cottrell, and Y. Bazilevs. *Comp. Methods in Appl. Mech. and Eng.*, 194(39-41):4135-4195, 2005.
[2] M. Dinachandra and S. Raju. *Computer Methods in Applied Mechanics and Engineering*, 335:380-402, 2018.
Figures: G. C. Diwan and M. S. Mohamed. *Computer Methods in Applied Mechanics and Engineering*, 350:701-718, 2019.

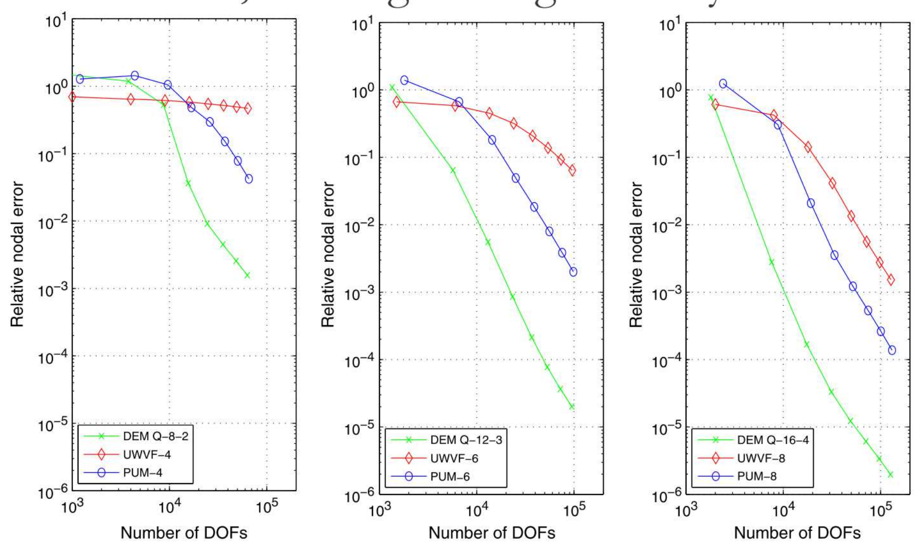
Enriched / Wave-based Finite Elements

- Incorporate piecewise plane waves into the standard basis functions
- Specific examples:
  - Discontinuous enrichment method[1]
  - Ultra weak variational formulation[2]
  - Partition of unity method[3]



Quad 8-2 Quad 16-4

- DEM and UWF discontinuous, weakly enforce continuity across adjacent elements
- Due to plane wave enrichment, convergence significantly faster than Galerkin FE



[1] R. Tezaur and C. Farhat. *International Journal for Numerical Methods in Engineering*, 66(5):796-815, 2006.

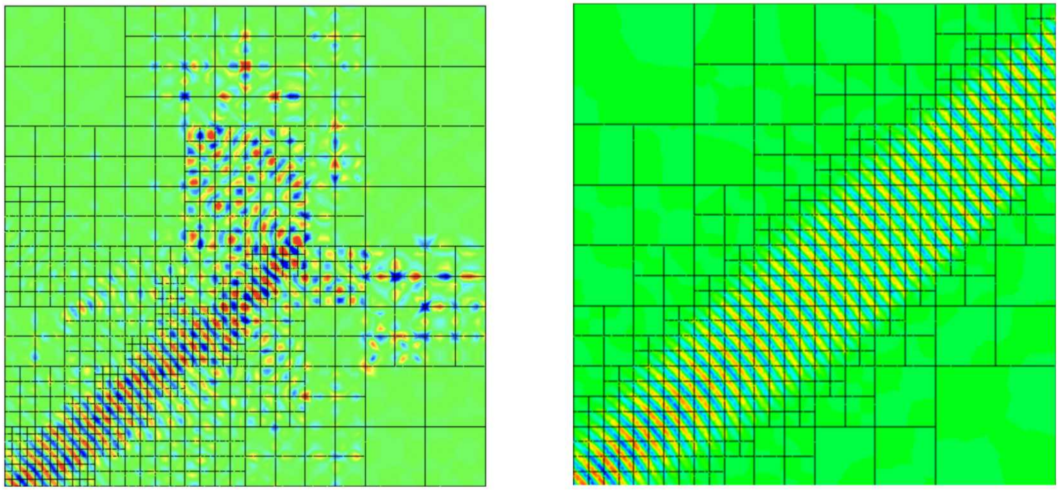
[2] T. Huttunen, J. P. Kaipio, and P. Monk. *SIAM Journal on Numerical Analysis*, 213(1):166-185, 2008.

[3] A. El Kacimi and O. Laghrouche. *International Journal for Numerical Methods in Engineering*, 77(12):1646-1669, 2009.

Figures: D. Wang, R. Tezaur, J. Toivanen, and C. Farhat. *Inter. Journal for Num. Methods in Engineering*, 89(4):403-417, 2012.

Discontinuous Petrov-Galerkin

- Developed by Demkowicz at UT Austin
- Trial functions are continuous, test functions are discontinuous
  - Computes optimal test functions on the fly
- Ultraweak formulation
  - Positive definite matrices
- Pollution is diffusive rather than dispersive
  - Pollution free in 1D
- Unconditionally stable
  - Allows for hp-adaptivity



Galerkin FEM DPG

L. Demkowicz and J. Gopalakrishnan. *ICES REPORT 15-20, The University of Texas at Austin*, 2015.

J. Zitelli and I. Muga and L. Demkowicz and J. Gopalakrishnan and D. Pardo and V. M. Calo. *Journal of Computational Physics*, 230(7): 2406-2432, 2011.

Figures: S. Petrides. *PhD Dissertation, The University of Texas at Austin*, 2019.

Conclusions

- Pollution effect is the dominate accuracy issue in the mid-frequency range
  - Occurs in one, two and three spatial dimensions
  - Can not eliminate in 2D and 3D for Galerkin FE

- Pollution is the accumulation of interpolation error over the domain
  - Increases with increasing kL

- Much of the understanding was developed in the mid-1990s
  - Began with the famous paper by Bayliss, Goldstein and Turkel

- Many methods to alleviate pollution
  - All methods seek to converge as best approximation with coarsest discretization
  - Main methods are p-refinement and plane wave enrichment
  - DPG provides unconditional stability for hp-adapativity

- Related note: pollution also occurs in the Boundary Element Method[1]
  - Appears as diffusion

[1] S. Marburg. *Journal of Theoretical and Computational Acoustics*, 26(2):1850018, 2018.