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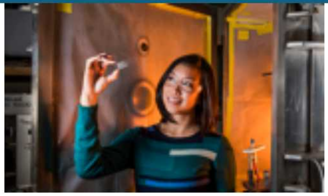
Almost optimal classical approximation algorithms for a quantum generalization of Max Cut



PRESENTED BY

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DOE/ASCR Quantum Algorithms Teams Program

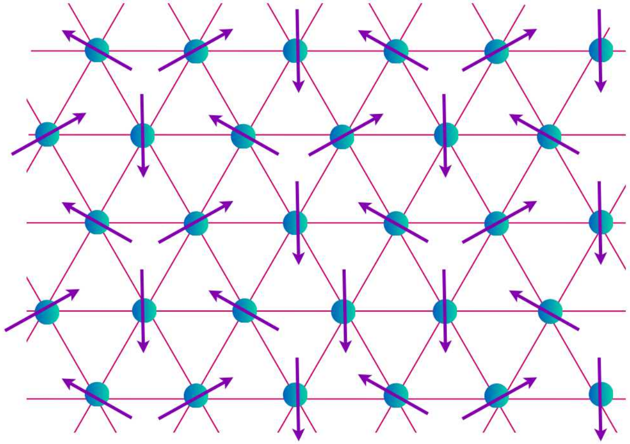
Anti-ferromagnetic Max Heisenberg Model

Max Cut Hamiltonian:
$\sum(I - Z_i Z_j)$

Max Quantum Heisenberg generalization:
$\sum(I - X_i X_j - Y_i Y_j - Z_i Z_j)$

Motivation

The Heisenberg model is fundamental for describing quantum magnetism, superconductivity, and charge density waves. Beyond 1 dimension, the properties of the anti-ferromagnetic Heisenberg model are notoriously difficult to analyze.



Anti-ferromagnetic Heisenberg model: roughly neighboring quantum particles aim to align in opposite directions. This kind of Hamiltonian appears, for example, as an effective Hamiltonian for so-called Mott insulators.
[Image: Sachdev, http://arxiv.org/abs/1203.4565]

Problem

Find max-energy state of $\sum(I - X_i X_j - Y_i Y_j - Z_i Z_j)$

($\equiv$ Find min-energy state of $\sum(X_i X_j + Y_i Y_j + Z_i Z_j)$,
but different from approximation point of view)

Nearly optimal product-state approximation algorithm for this problem:

0.498-approx via a product state, where 1/2 is best possible for product states

(also 0.649-approx for XY model, where 2/3 is best possible for product states, as well as results for some generalizations of the Max Heisenberg Model)

Related Work

Theme: P approximations for QMA-hard local Hamiltonian problems

- PTAS for bounded-degree planar *k-local Hamiltonian* (*k*-LH, sum of *k*-local terms) [Bansal, Bravyi, Terhal '09]
- PTAS for dense *k*-LH with positive terms [Gharibian, Kempe '12]
- PTAS for planar, dense, or low threshold-rank *k*-LH [Brandão, Harrow '13]

QMA-hard 2-LH problem class	NP-hard specialization	P approximation for NP-hard specialization	Product-state approximation for QMA-hard 2-LH problem
Max traceless 2-LH: $\sum_{ij} H_{ij}$, H_{ij} traceless	Max -$\sum_{ij} z_i z_j$, $z_i \in \{-1,1\}$	O(1/log n) [Charikar, Wirth '04]	O(1/log n) [Bravyi,Gosset,Koenig,Temme '18]
Max positive 2-LH: $\sum_{ij} H_{ij}$, $H_{ij} \succcurlyeq 0$	Max 2-CSP	0.874 [Lewin, Livnat, Zwick '02]	0.25 [Random assignment] 0.282 [Hallgren, Lee '19] **0.328 [Hallgren, Lee, P '19]** 0.5 best possible
Max Heisenberg: $\sum_{ij} I - X_iX_j - Y_iY_j - Z_iZ_j$	Max Cut: Max $\sum_{ij} I - z_i z_j$, $z_i \in \{-1,1\}$	0.878 [Goemans, Williamson '95]	(special case of above) **0.498 [Gharibian, P '19]** 0.5 best possible

Max Cut vs Quantum Max Cut

Classical Max Cut

2-variable constraint: $x_i \oplus x_j$

$$\begin{array}{r} \\ x_i, x_j = 0,0 \\ x_i, x_j = 0,1 \\ x_i, x_j = 1,0 \\ x_i, x_j = 1,1 \end{array} \begin{array}{c} \begin{matrix} 0,0 & 0,1 & 1,0 & 1,1 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Diagonal matrix

$1/2(I - Z_i Z_j)$

Maximum eigenvector:
$(0, 1, 0, 0) \rightarrow |01\rangle$,
with energy 1

quantum generalization

Quantum Max Cut

Max Heisenberg model

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Non-diagonal matrix

$1/4(I - X_i X_j - Y_i Y_j - Z_i Z_j)$

Maximum eigenvector:

$$\left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right) \rightarrow \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle,$$

with energy 1

Maximum product state:
$|01\rangle$
with energy 1/2

Max Cut Semidefinite Programming Relaxation

Max $\sum_{ij \in E}(1 - m_{ij})/2$

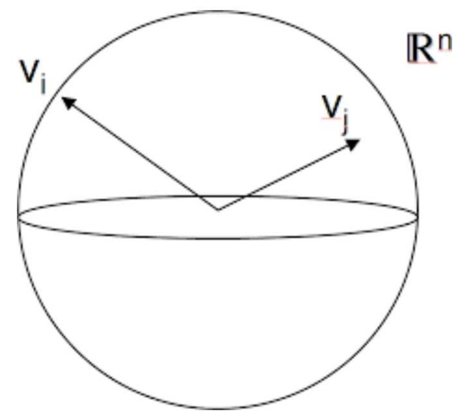
$$\begin{bmatrix} 1 & m_{12} & m_{13} & \cdots \\ m_{12} & 1 & m_{23} & \\ m_{13} & m_{23} & 1 & \\ \vdots & & & \ddots \end{bmatrix} \succcurlyeq 0$$

$\equiv$

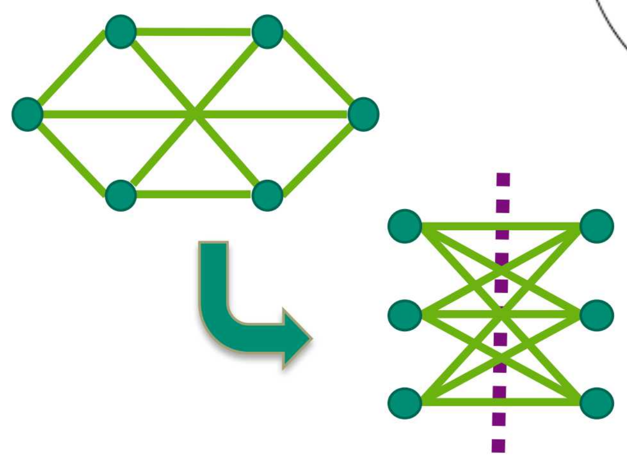
Max $\sum_{ij \in E}(1 - v_i \cdot v_j)/2$

$\|v_i\| = 1$, for all $i \in V$

$(v_i \in \mathbb{R}^n)$



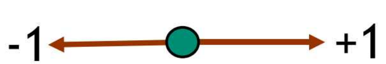
Equivalent perspective: unit vectors v_i, with $m_{ij} = v_i \cdot v_j$



Max Cut

Find cut with max # of crossing edges in graph $G = (V, E)$

Exact solution when $v_i \in \mathbb{R}^1$:



Quantum Moment Matrices are Positive

State on n qubits

$\langle\psi| \in \mathbb{C}^{2^n}$

$$V = \begin{bmatrix} \langle x_1| = \langle\psi|X_1 \\ \langle y_1| = \langle\psi|Y_1 \\ \langle z_1| = \langle\psi|Z_1 \\ \vdots \\ \langle x_n| = \langle\psi|X_n \\ \langle y_n| = \langle\psi|Y_n \\ \langle z_n| = \langle\psi|Z_n \end{bmatrix}, \quad M_{ij} = \begin{bmatrix} \langle\psi|X_iX_j|\psi\rangle & \langle x_i|y_j\rangle & \langle x_i|z_j\rangle \\ \langle y_i|x_j\rangle & \langle y_i|y_j\rangle & \langle y_i|z_j\rangle \\ \langle z_i|x_j\rangle & \langle z_i|y_j\rangle & \langle z_i|z_j\rangle \end{bmatrix}$$

$$\begin{array}{c|ccccccccc} & X_1 & Y_1 & Z_1 & X_2 & Y_2 & Z_2 & X_3 & Y_3 & Z_3 \\ \hline \begin{matrix} X_1 \\ Y_1 \\ Z_1 \\ X_2 \\ Y_2 \\ Z_2 \\ X_3 \\ Y_3 \\ Z_3 \end{matrix} & \multicolumn{9}{c}{\begin{bmatrix} M_{11} & M_{12} & M_{13} & \cdots \\ M_{12}^\dagger & M_{22} & M_{23} & \\ M_{13}^\dagger & M_{23}^\dagger & M_{33} & \\ \vdots & & & \ddots \end{bmatrix}} \end{array} = VV^\dagger \succcurlyeq 0 \Longrightarrow Re\left(VV^\dagger\right) \succcurlyeq 0$$

Entries of this $3n \times 3n$ *moment matrix* are expectation values of all 2-local Pauli terms

Quantum Max Cut SDP Relaxation

$$
\begin{array}{c|ccccccccc c}
 & X_1 & Y_1 & Z_1 & X_2 & Y_2 & Z_2 & X_3 & Y_3 & Z_3 & \\
\hline
X_1 & 1 & 0 & 0 & & & & & & & \cdots \\
Y_1 & 0 & 1 & 0 & & M_{12} & & & M_{13} & & \\
Z_1 & 0 & 0 & 1 & & & & & & & \\
X_2 & & & & 1 & 0 & 0 & & & & \\
Y_2 & & M_{12}^T & & 0 & 1 & 0 & & M_{23} & & \\
Z_2 & & & & 0 & 0 & 1 & & & & \\
X_3 & & & & & & & 1 & 0 & 0 & \\
Y_3 & & M_{13}^T & & & M_{23}^T & & 0 & 1 & 0 & \\
Z_3 & & & & & & & 0 & 0 & 1 & \\
 & \vdots & & & & & & & & & \ddots
\end{array} \succcurlyeq 0
$$

$$
M_{ij} = \begin{bmatrix} x_i \cdot x_j & x_i \cdot y_j & x_i \cdot z_j \\ y_i \cdot x_j & y_i \cdot y_j & y_i \cdot z_j \\ z_i \cdot x_j & z_i \cdot y_j & z_i \cdot z_j \end{bmatrix}
$$

Real part of moment matrix

Max Cut vector relaxation

$$\text{Max } \sum_{ij \in E} (1 - v_i \cdot v_j)$$

$\|v_i\| = 1$, for all $i \in V$

$(v_i \in \mathbb{R}^n)$

Quantum Max Cut vector relaxation

$$\text{Max } \sum_{ij \in E} (1 - x_i \cdot x_j - y_i \cdot y_j - z_i \cdot z_j)$$

$\|x_i\|, \|y_i\|, \|z_i\| = 1$, for all $i \in V$

$x_i \cdot y_i = x_i \cdot z_i = y_i \cdot z_i = 0$, for all $i \in V$

$(v_i \in \mathbb{R}^{3n})$

Approximating Quantum Max Cut

We use hyperplane rounding generalization by Briët, de Oliveira Filho, and Vallentin [arXiv 1011.1754] to round the vectors x_i, y_i, z_i to scalars $\alpha_i, \beta_i, \gamma_i$ to obtain:

$$\boldsymbol{\rho = \frac{1}{2^n}\prod_i (I + \alpha_i X_i + \beta_i Y_i + \gamma_i Z_i)\,,\ \alpha_i^2 + \beta_i^2 + \gamma_i^2 = 1}$$

Classical rounding ($\mathbb{R}^n \longrightarrow \mathbb{R}^1$)

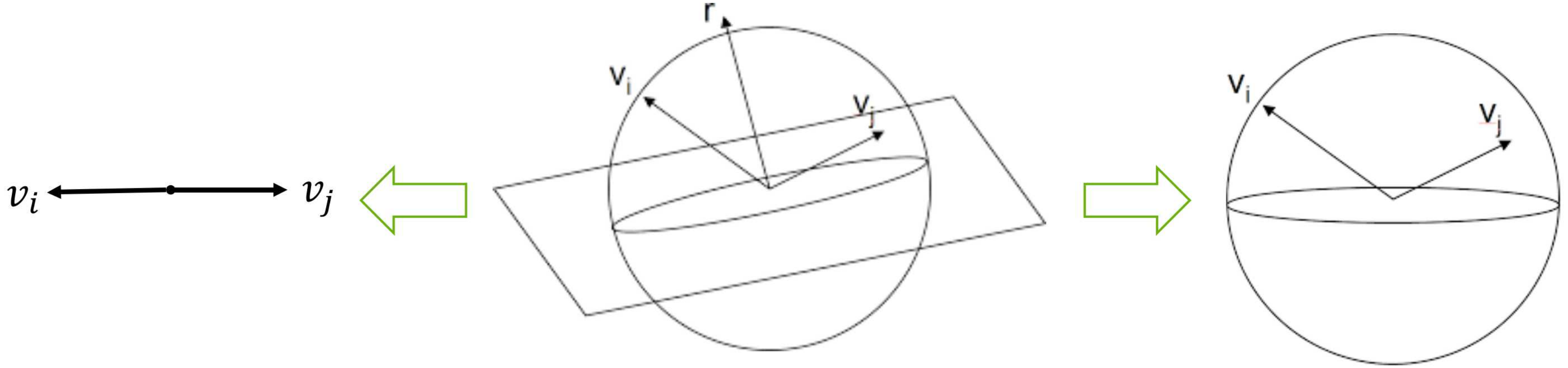
$$v_i \in \mathbb{R}^n \longrightarrow \alpha_i = \frac{r^T v_i}{|r^T v_i|}$$

$$r \sim N(0,1)^n$$

Product-state rounding ($\mathbb{R}^{9n} \longrightarrow \mathbb{R}^3$)

$$v_i = (x_i, y_i, z_i) \in \mathbb{R}^{9n} \longrightarrow (\alpha_i, \beta_i, \gamma_i) = \frac{R^T v_i}{\| R^T\ v_i \|}$$

$$R^T \sim N(0,1)^{3\times 9n}$$



Open questions

- Is a tight ratio of 1/2 possible within P? Is it NP-hard? (such a product state always exists by [Gharibian, Kempe '12])
- SDP gap? Unique games hardness?
- New analysis techniques for more general quantum CSPs (e.g. Quantum k-SAT)?
- BQP approximation algorithms (i.e., quantum approximation algorithms)? Can we find problems with provably better quantum approximation ratios?

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QOALAS

Quantum Optimization and Learning and Simulation

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Quantum approaches to discrete optimization, Theoretical computer science, Quantum and classical simulation of many-body quantum systems, QCVV, Quantum dynamics and Control

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Quantum computing, Quantum algorithms, Quantum complexity theory, Machine learning, Quantum many-body physics, Quantum gravity, Quantum machine learning, Optimization

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Quantum computing, Condensed matter theory

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Quantum computing, High-energy physics, Quantum error correction and fault tolerance

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Quantum algorithms and complexity theory

QOALAS

Quantum Optimization and Learning and Simulation

Goal: New quantum techniques and algorithms from the interplay of quantum simulation, optimization, and machine learning

Optimization

(Approximate) extremal energy states of physically-inspired Hamiltonians

Variational approaches and QAOA

Adiabatic quantum evolution

Convex and gradient-based optimization

Convex/semidefinite relaxations

New ML-inspired optimization problems

Quantum Simulation

Machine Learning

Sampling from max-entropy distributions

Hamiltonian simulation

Thanks!

Full paper: https://arxiv.org/abs/1909.08846

We're hiring: https://qoalas.sandia.gov