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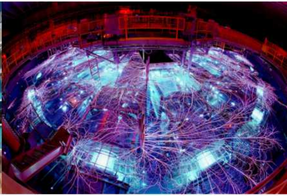

Bulk-Interface-Flux-Recovery (Bulk-IFR) Method for Coupling the Atmosphere-Ocean Interface

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Main Idea

- Does solving the monolithic problem give improvements over explicit treatment of a Robin condition?
- Well informed coupling scheme: Utilize Schur Complement of monolithic problem to produce boundary conditions. Similar to variational flux recovery

1. Define interface-flux variable $\boldsymbol{\lambda}$ with its own DOF space (could be either model's or a 3rd set). No remapping necessary!
2. At each time step, isolate interface DOF's using Schur complement
3. Isolate and solve for $\boldsymbol{\lambda}$ with second Schur complement
4. Back substitute $\boldsymbol{\lambda}$ into interface systems to find fluxes on the interface for each model (exact with respect to the data at the last time step).
5. Fluxes act as boundary conditions for interior evolution
6. Repeat

Atmosphere–Ocean Coupling

- Abstract formulation of coupled model (Lemarié, et al. 2015)
- Γ is Atmosphere–Ocean interface
- Linear operator $\mathcal{L}$, flux function $\mathcal{F}$, BC operator $\mathcal{B}$, flux coupling $\mathbf{F}_{ao}$

$$
\begin{aligned}
\frac{\mathrm{d}U_a}{\mathrm{d}t} + \mathcal{L}_a\mathcal{F}_a(U_a) &= f_a && \text{in } \Omega_a \times [0, T] \\
\mathcal{B}_a U_a &= b_a && \text{on } \partial\Omega_a \setminus \Gamma \times [0, T] \\
\mathcal{F}_a(U_a) &= \alpha(U_a - U_o) && \text{on } \Gamma \times [0, T] \\
\frac{\mathrm{d}U_o}{\mathrm{d}t} + \mathcal{L}_o\mathcal{F}_a(U_a) &= f_o && \text{in } \Omega_o \times [0, T] \\
\mathcal{B}_o U_o &= b_o && \text{on } \partial\Omega_o \setminus \Gamma \times [0, T] \\
\mathcal{F}_o(U_o) &= \alpha(U_o - U_a) && \text{on } \Gamma \times [0, T]
\end{aligned}
$$

- Bulk condition on interface with constant α

$$
\mathcal{F}_a(U_a) = \mathcal{F}_o(U_o) = \alpha(U_a - U_o) \leftrightarrow k_a \boldsymbol{\nabla} U_a \cdot \mathbf{n} = -k_a \boldsymbol{\nabla} U_o \cdot \mathbf{n} = \alpha(U_a - U_o)
$$

- More Generally: Strategy for $\alpha(x, t)$ clear, $\alpha(U_a, U_o)$ not clear.

Weak Formulation

- Assume Dirichlet boundary conditions are satisfied on $\partial\Omega_a \setminus \Gamma$ and $\partial\Omega_o \setminus \Gamma$ (i.e non Γ boundaries)
- Introduce boundary flux variable $\boldsymbol{\lambda} = \mathcal{F}_a(U_a) = \mathcal{F}_o(U_o) = \alpha(U_a - U_o)$
- Test functions $\psi_a \in V_a$, $\psi_o \in V_o$, $\mu \in V_\Gamma$,

$$\begin{aligned}
\left(\psi_a, \frac{\mathrm{d}U_a}{\mathrm{d}t}\right)_{\Omega_a} + (\psi_a, \mathcal{L}_a\mathcal{F}_a(U_a))_{\Omega_a} &= (\psi_a, f_a)_{\Omega_a} && \text{in } \Omega_a \times [0, T] \\
\left(\psi_o, \frac{\mathrm{d}U_o}{\mathrm{d}t}\right)_{\Omega_o} + (\psi_o, \mathcal{L}_o\mathcal{F}_o(U_o))_{\Omega_o} &= (\psi_o, f_o)_{\Omega_o} && \text{in } \Omega_o \times [0, T] \\
(\mu, \alpha(U_a - U_o) - \lambda)_\Gamma &= 0 && \text{on } \Gamma \times [0, T]
\end{aligned}$$

- Integration-by-parts

$$\begin{aligned}
\left(\psi_a, \frac{\mathrm{d}U_a}{\mathrm{d}t}\right)_{\Omega_a} - (\widetilde{\mathcal{L}_a}\psi_a, \widetilde{\mathcal{F}_a}(U_a))_{\Omega_a} + (\psi_a, \lambda)_\Gamma &= (\psi_a, f_a)_{\Omega_a} && \text{in } \Omega_a \times [0, T] \\
\left(\psi_o, \frac{\mathrm{d}U_o}{\mathrm{d}t}\right)_{\Omega_o} - (\widetilde{\mathcal{L}_o}\psi_o, \widetilde{\mathcal{F}_o}(U_o))_{\Omega_o} - (\psi_o, \lambda)_\Gamma &= (\psi_o, f_o)_{\Omega_o} && \text{in } \Omega_o \times [0, T] \\
(\mu, \alpha(U_a - U_o) - \lambda)_\Gamma &= 0 && \text{on } \Gamma \times [0, T]
\end{aligned}$$

Discrete Problem

- Expand U_a, U_o, and λ with trial functions

$$\mathbf{M}_a \frac{\mathrm{d}\mathbf{U}_a}{\mathrm{d}t} - \mathbf{K}_a\mathbf{U}_a + \mathbf{G}_a^\top \boldsymbol{\lambda} = \mathbf{f}_a$$
$$\mathbf{M}_o \frac{\mathrm{d}\mathbf{U}_o}{\mathrm{d}t} - \mathbf{K}_o\mathbf{U}_o - \mathbf{G}_o^\top \boldsymbol{\lambda} = \mathbf{f}_o$$
$$\alpha\mathbf{G}_a\mathbf{U}_a - \alpha\mathbf{G}_o\mathbf{U}_o - \widetilde{\mathbf{M}}\boldsymbol{\lambda} = 0$$

- Explicit-Explicit time-stepping

$$\mathbf{M}_a\mathbf{U}_a^n + \Delta t\mathbf{G}_a^\top \boldsymbol{\lambda} = \mathbf{g}_a(\mathbf{U}_a^{n-1})$$
$$\mathbf{M}_o\mathbf{U}_o^n - \Delta t\mathbf{G}_o^\top \boldsymbol{\lambda} = \mathbf{g}_o(\mathbf{U}_o^{n-1})$$
$$\alpha\mathbf{G}_a\mathbf{U}_a^n - \alpha\mathbf{G}_o\mathbf{U}_o^n - \widetilde{\mathbf{M}}\boldsymbol{\lambda} = 0$$

$$\begin{pmatrix} \mathbf{M}_a & 0 & \Delta t\mathbf{G}_a^\top \\ 0 & \mathbf{M}_o & \Delta t\mathbf{G}_o^\top \\ \alpha\mathbf{G}_a & \alpha\mathbf{G}_o & \widetilde{\mathbf{M}} \end{pmatrix} \begin{pmatrix} \mathbf{U}_a^n \\ \mathbf{U}_o^n \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{g}_a^{n-1} \\ \mathbf{g}_o^{n-1} \\ 0 \end{pmatrix}$$

- RHS is

$$\mathbf{g}_i^{n-1} = \Delta t\mathbf{f}_i + \Delta t\mathbf{K}_i\mathbf{U}_i^{n-1} + \mathbf{M}\mathbf{U}_i^{n-1} \quad i = a, o$$

Interface Problem: Explicit-Explicit Coupling

- Separate DOFs on interior and interface

$$\begin{pmatrix} \mathbf{M}_{a,\Gamma} & 0 & \Delta t\mathbf{G}_a^T & \mathbf{M}_{a,\Gamma\Omega} & 0 \\ 0 & \mathbf{M}_{o,\Gamma} & -\Delta t\mathbf{G}_o^T & 0 & \mathbf{M}_{o,\Gamma\Omega} \\ \alpha\mathbf{G}_a & -\alpha\mathbf{G}_o & \widetilde{\mathbf{M}}_\Gamma & 0 & 0 \\ \mathbf{M}_{a,\Omega\Gamma} & 0 & 0 & \mathbf{M}_{a,\Omega} & 0 \\ 0 & \mathbf{M}_{o,\Omega\Gamma} & 0 & 0 & \mathbf{M}_{o,\Omega} \end{pmatrix} \begin{pmatrix} \mathbf{U}^n_{a,\Gamma} \\ \mathbf{U}^n_{o,\Gamma} \\ \boldsymbol{\lambda} \\ \mathbf{U}^n_{a,\Omega} \\ \mathbf{U}^n_{o,\Omega} \end{pmatrix} = \begin{pmatrix} \mathbf{g}_{a_\Gamma}^{n-1} \\ \mathbf{g}_{o_\Gamma}^{n-1} \\ 0 \\ \mathbf{g}_{a_\Omega}^{n-1} \\ \mathbf{g}_{o_\Omega}^{n-1} \end{pmatrix}$$

- Interface System

$$\begin{pmatrix} \mathbf{A}_{a,\Gamma} & 0 & \Delta t\mathbf{G}_a^T \\ 0 & \mathbf{A}_{o,\Gamma} & -\Delta t\mathbf{G}_o^T \\ \alpha\mathbf{G}_a & -\alpha\mathbf{G}_o & \widetilde{\mathbf{M}}_\Gamma \end{pmatrix} \begin{pmatrix} \mathbf{U}^n_{a,\Gamma} \\ \mathbf{U}^n_{o,\Gamma} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \widehat{\mathbf{g}}^{n-1}_{a,\Gamma} \\ \widehat{\mathbf{g}}^{n-1}_{o,\Gamma} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{i,\Gamma} = \mathbf{M}_{i,\Gamma} - \mathbf{M}_{i,\Gamma\Omega}\mathbf{M}^{-1}_{i,\Omega}\mathbf{M}_{i,\Omega\Gamma} \quad \widehat{\mathbf{g}}^{n-1}_{i,\Gamma} = \mathbf{g}^{n-1}_{i,\Gamma} - \mathbf{M}_{i,\Gamma\Omega}\mathbf{M}^{-1}_{i,\Omega}\mathbf{g}^{n-1}_{i,\Omega} \quad i = a, o$$

- Schur Complement: solve for interface flux

$$\boldsymbol{\lambda} = \mathbf{S}^{-1}\left(\alpha\Delta t\mathbf{G}_a^T\mathbf{A}^{-1}_{a,\Gamma}\mathbf{g}^{n-1}_{a,\Gamma} - \alpha\Delta t\mathbf{G}_o^T\mathbf{A}^{-1}_{o,\Gamma}\mathbf{g}^{n-1}_{o,\Gamma}\right)$$

$$\mathbf{S} = \left(\alpha\Delta t\mathbf{G}_a^T\mathbf{A}^{-1}_{a,\Gamma}\mathbf{G}_a + \alpha\Delta t\mathbf{G}_o^T\mathbf{A}^{-1}_{o,\Gamma}\mathbf{G}_o - \widetilde{\mathbf{M}}_\Gamma\right)$$

Explicit-Implicit Coupling

- Explicit-Implicit time-stepping

$$
\begin{aligned}
\mathbf{M}_a\mathbf{U}_a^n + \Delta t\mathbf{G}_a^\top\boldsymbol{\lambda} &= \mathbf{g}_a(\mathbf{U}_a^{n-1}) \\
\mathbf{M}_o\mathbf{U}_o^n - \Delta t\mathbf{K}_o\mathbf{U}_o^n - \Delta t\mathbf{G}_o^\top\boldsymbol{\lambda} &= \widetilde{\mathbf{g}_o}(\mathbf{U}_o^{n-1}) \\
\alpha\mathbf{G}_a\mathbf{U}_a^n - \alpha\mathbf{G}_o\mathbf{U}_o^n - \widetilde{\mathbf{M}}\boldsymbol{\lambda} &= 0
\end{aligned}
\qquad
\begin{pmatrix} \mathbf{M}_a & 0 & \Delta t\mathbf{G}_a^\top \\ 0 & \mathbf{W}_o & \Delta t\mathbf{G}_o^\top \\ \alpha\mathbf{G}_a & \alpha\mathbf{G}_o & \widetilde{\mathbf{M}} \end{pmatrix}
\begin{pmatrix} \mathbf{U}_a^n \\ \mathbf{U}_o^n \\ \boldsymbol{\lambda} \end{pmatrix}
=
\begin{pmatrix} \mathbf{g}_a^{n-1} \\ \widetilde{\mathbf{g}_o}^{n-1} \\ 0 \end{pmatrix}
$$

- RHS is

$$
\widetilde{\mathbf{g}}(\mathbf{U}^{n-1}) = \Delta t\mathbf{f} + \mathbf{M}\mathbf{U}^{n-1}\,, \quad \mathbf{W}_o = \mathbf{M}_o - \Delta t\mathbf{K}_o
$$

- Separate DOFs $\rightarrow$ Schur Complement $\rightarrow$ Interface System

$$
\begin{pmatrix} \mathbf{A}_{a,\Gamma} & 0 & \Delta t\mathbf{G}_a^T \\ 0 & \widetilde{\mathbf{A}}_{o,\Gamma} & -\Delta t\mathbf{G}_o^T \\ \alpha\mathbf{G}_a & -\alpha\mathbf{G}_o & \widetilde{\mathbf{M}}_\Gamma \end{pmatrix}
\begin{pmatrix} \mathbf{U}_a^{n-1}{}_\Gamma \\ \mathbf{U}_o^{n-1}{}_\Gamma \\ \boldsymbol{\lambda} \end{pmatrix}
=
\begin{pmatrix} \widehat{\mathbf{g}}_{a,\Gamma}^{n-1} \\ \widehat{\mathbf{g}}_{o,\Gamma}^{n-1} \\ 0 \end{pmatrix}
$$

$$
\mathbf{A}_{a,\Gamma} = \mathbf{M}_{a,\Gamma} - \mathbf{M}_{a,\Gamma\Omega}\mathbf{M}_{a,\Omega}^{-1}\mathbf{M}_{a,\Omega\Gamma}\,, \quad \widetilde{\mathbf{A}}_{o,\Gamma} = \mathbf{W}_{o,\Gamma} - \mathbf{W}_{o,\Gamma\Omega}\mathbf{W}_{o,\Omega}^{-1}\mathbf{W}_{o,\Omega\Gamma}
$$

$$
\widehat{\mathbf{g}}_{a,\Gamma}^{n-1} = \mathbf{g}_{a}{}_\Gamma^{n-1} - \mathbf{M}_{a,\Gamma\Omega}\mathbf{M}_{a,\Omega}^{-1}\mathbf{g}_{a}{}_\Omega^{n-1}
$$

$$
\widehat{\mathbf{g}}_{o,\Gamma}^{n-1} = \widetilde{\mathbf{g}_o}{}_\Gamma^{n-1} - \mathbf{W}_{o,\Gamma\Omega}\mathbf{W}_{o,\Omega}^{-1}\widetilde{\mathbf{g}_o}{}_\Omega^{n-1}
$$

Advection-Diffusion Equations

- Advection in x direction, diffusion in y direction
- Denote a by 1 and o by 2

$$\frac{\partial \varphi_i}{\partial t} - \frac{\partial}{\partial y} k_i \frac{\partial \varphi_i}{\partial y} + u_i \frac{\partial \varphi_i}{\partial x} = f_i \quad \text{in } \Omega_i \times [0, T] \quad i = 1, 2$$

- Initial condition

$$\varphi_i(x, y, 0) = \varphi_{i,0}(x, y) \quad \text{in } \Omega_i \quad i = 1, 2$$

- Non-interface boundary conditions

$$\varphi_i = g_i \quad \text{on } \partial\Omega_i \quad i = 1, 2\,.$$

- Bulk condition

$$k_1 \frac{\partial \varphi_1}{\partial y} = k_2 \frac{\partial \varphi_2}{\partial y} = \alpha(\varphi_1 - \varphi_2) \quad \text{on } \Gamma\,.$$

Test Case Setup

Figure: Atmosphere-Ocean domain with interface Γ.

- Domain size $(x, y) \in [0, x_{\max}] \times [-y_{\max}, y_{\max}]$, with $x_{\max} = 10$ km and $y_{\max} = 500$ m. Γ is line $y(x) = 0$.
- Method of manufactured solutions to give realistic test case
- Explicit – Explicit coupling

Manufactured solution

- Manufactured solution: Oscillation about some reference temperature

$$\varphi_1 = -\frac{A}{2N\pi k_1}\cos(N\pi(x-ut))(y-a)^2 + T$$

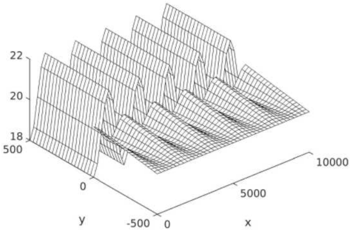
$$\varphi_2 = \frac{B}{N\pi k_2}\cos(N\pi(x-ut))(y+y_{\max}) + T \ ,$$

- Forcing: $f_1 = \frac{A}{N\pi}\cos(N\pi(x-u_1 t))$ and $f_2 = 0$
- Bulk conditions on $y(x) = 0$ demands

$$\frac{aA}{N\pi} = \frac{B}{N\pi} = \alpha\left(-a^2\frac{A}{2N\pi k_1} - \frac{B}{N\pi k_2}b\right) .$$

- Implies that

$$aA = B \ , \quad \alpha = -\frac{Aa}{\left(\frac{a^2 A}{2k_1} + \frac{B}{k_2}b\right)}$$



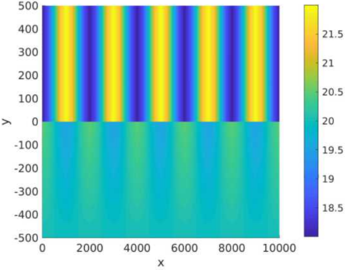


Figure: Initial Condition

Parameters

- Maximum amplitude around T must be bounded.

$$\max|\varphi_2 - T| = (\frac{By_{\max}}{N\pi k_2}) = M_2 \Leftrightarrow B = \frac{M_2 N\pi k_2}{y_{\max}}$$

$$\max|\varphi_1 - T| = \frac{Aa^2}{2N\pi k_1} = M_1 \Leftrightarrow a = \frac{2M_1 N\pi k_1}{B}$$

Parameter	Value
T	293.15 K = 20 °C
N	$\frac{1}{1000}$
u	2 m s^{-1}
M_1	2.0 °C
M_2	0.5 °C

Table: Model parameters for the simplified atmosphere-ocean test case

- k_1 and k_2 have yet to be specified.

Numerical Results

- Time Horizon: 2 minutes, time step: 0.05 sec, SUPG stabilization, consistent mass matrices, linear Lagrange elements, forward Euler.

- Advection Dominated: $k_1 = 2 \times 10^{-3}$ and $k_1 = 1 \times 10^{-5}$

h_x	h_y	Rel. $\|\|err\|\|_{L_2}$	Rel. $\|err\|_{H_1}$
333	33.3	1.661e-01	4.101e-01
166	16.6	3.670e-02	2.019e-01
83.3	8.33	8.349e-03	1.011e-01
Rate	-	2.136	0.997

- Medium Diffusion: $k_1 = 2$ and $k_1 = 1 \times 10^{-1}$

h_x	h_y	Rel. $\|\|err\|\|_{L_2}$	Rel. $\|err\|_{H_1}$
333	33.3	1.627e-01	3.937e-01
166	16.6	3.581e-02	2.005e-01
83.3	8.33	8.152e-03	1.011e-01
Rate	-	2.135	0.987

Numerical Results

- Large Diffusion: $k_1 = 200$ and $k_2 = 1$

h_x	h_y	Rel. $\|\|err\|\|_{L_2}$	Rel. $\|err\|_{H_1}$
333	33.3	1.441e-01	3.876e-01
166	16.6	3.240e-02	2.003e-01
83.3	8.33	7.740e-03	1.011e-01
Rate	-	2.065	0.986

- Convergence rates are reasonable

Non Matching Grids

- Ocean grid is approximately 1.5 times finer
- Advection Dominated: $k_1 = 2 \times 10^{-3}$ and $k_1 = 1 \times 10^{-5}$
- Atmosphere is mortar side

$h_{1,x}$	$h_{1,y}$	$h_{2,x}$	$h_{2,y}$	Rel. $\|\|err\|\|_{L_2}$	Rel. $\|err\|_{H_1}$
333	33.3	222	22.2	7.169e-02	2.737e-01
166	16.6	111	11.1	1.611e-02	1.365e-01
83.3	8.33	55.5	5.55	3.676e-03	6.837e-02
Rate	–	–	–	2.131	0.997

- Similar errors when ocean is mortar side

Comparison to Direct Flux Transfer

- Direct Method: Let $\boldsymbol{\lambda} = \alpha(\varphi_1^{n-1} - \varphi_2^{n-1})$
- Large Diffusion: $k_1 = 200$ and $k_2 = 1$
- Time Horizon: 1 hour, Time-step: 0.05

Method	$h_{\mathbf{1},x}$	$h_{\mathbf{1},y}$	Rel. $\|\|err\|\|_{L_2}$	Rel. $\|err\|_{H_1}$
Direct	166	16.6	1.673e-02	8.192e-02
IFR	166	16.6	1.525e-02	7.962e-02

Conclusions and Future Work

- Interface flux can be implicit solved for any combination of explicit or implicit time integration. Heterogeneous time-integration
- Rigorous stability analysis required
- Extend to Primitive equations and FD-FV models.
- Extend to IMEX methods, multi-rate methods, and exploit horizontally-explicit vertically-implicit type methods.
- Heterogeneous Asynchronous Time Integration methods (HATI)
- Typically, one must compromise between conservation or unconditional stability
- We want Conservative Heterogeneous Asynchronous Time Integration that is Unconditionally Stable

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