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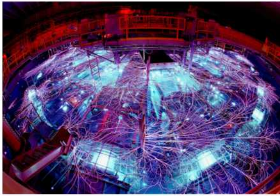

An Anisotropic Plane Stress Model in Linearized Bond-Based Peridynamics

Jeremy Trageser
Collaborator: Pablo Seleson

10/16/19

Outline

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   - Background
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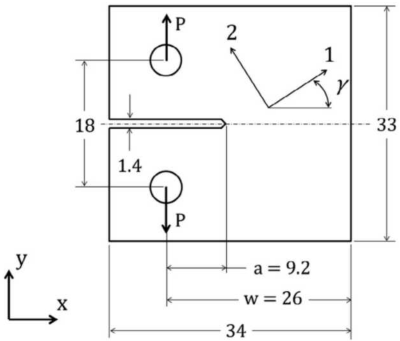
Motivation for a Peridynamic Anisotropic Stress Model

- Many materials are anisotropic so the model can accommodate a larger class of materials.
- Plane stress models are utilized in a number of applications.
- Reduced computational expense.
- Facilitates damage modeling.

Example 1: Compact Tension Test on Cortical Bone

- Cortical bone exhibits transversely isotropic symmetry.
- Cracks tend to propagate along the main axis of the bone.

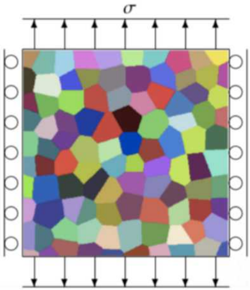


M. Ghajari, L. Iannucci, and P. Curtis. *Computer Methods in Applied Mechanics and Engineering*, 276:431 - 452, 2014.

Example 2: Polycrystalline Structure (Alumina)

- Each grain exhibits transversely isotropic symmetry.
- The plane of isotropy within each grain is randomly oriented.
- Various ratios of the grain fracture energy to the grain boundary fracture energy are considered.

Model



$G_{bdy}/G_{grain} = 1$

$G_{bdy}/G_{grain} = 0.5$

M. Ghajari, L. Iannucci, and P. Curtis. *Computer Methods in Applied Mechanics and Engineering*, 276:431 - 452, 2014.

Why Consider A New Plane Stress Model?

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- Most models begin with a two-dimensional peridynamic model and match peridynamic constants to elasticity constants.

- Most models cannot accommodate all symmetry classes of linear elasticity.

- Most peridynamic plane stress models ignore surface effects.

Classical (Local) Linear Elasticity

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In linear elasticity stresses and strains are related via a generalized Hooke's Law:

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}$$

$\boldsymbol{\sigma}$: stress tensor, ϵ: strain tensor, **C**: elasticity tensor.
C has the following symmetries:

$$\text{Minor Symmetries}: \quad C_{ijkl} = C_{jikl} = C_{ijlk}$$
$$\text{Major Symmetry}: \quad C_{ijkl} = C_{klij}$$

Classical equation of motion:

$$\rho(\mathbf{x})\ddot{u}_i(\mathbf{x},t) = \frac{\partial}{\partial x_j}\sigma_{ij}(\mathbf{x},t) + b_i(\mathbf{x},t) = C_{ijkl}\frac{\partial^2 u_k}{\partial x_j \partial x_l}(\mathbf{x},t) + b_i(\mathbf{x},t)$$

ρ: mass density, **u**: displacement, and **b**: body force density.

Symmetry Groups of **C**

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Definition

An orthogonal transformation **Q** between bases **e** and **e**$'$ is called a symmetry transformation of **C** if

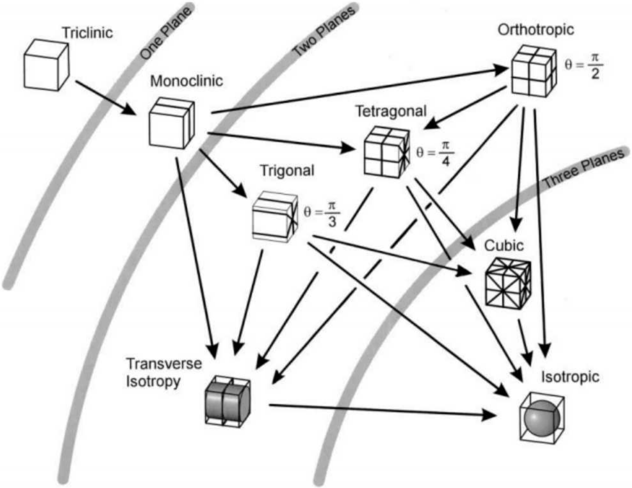
$$C_{ijkl} = Q_{ip}Q_{jq}Q_{kr}Q_{ls}C_{pqrs},$$

i.e. **C** is invariant under the transformation **Q**.

Proposition

The set of symmetry transformations of **C** *forms a group which we call the symmetry group of* **C***.*

The Eight Symmetry Classes of Linear Elasticity



P. Chadwick, M. Vianello, S.C. Cowin, A new proof that the number of linear elastic symmetries is eight, *Journal of Mechanics and Physics of Solids*, 49, 2471-2492, 2001.

Monoclinic Constraints

Suppose a material has monoclinic symmetry. Choose an orientation so that the plane of reflection coincides with the plane $z = 0$. Then the orthogonal transformation **Q** may be represented by the matrix

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Since **Q** is a symmetry transformation of **C**, we know

$$\begin{aligned} C_{ijkl} &= Q_{ip} Q_{jq} Q_{kr} Q_{ls} C_{pqrs} \\ &= (-1)^n \delta_{ip} \delta_{jq} \delta_{kr} \delta_{ls} C_{pqrs} \\ &= (-1)^n C_{ijkl} \end{aligned}$$

where n is the number of threes occurring in $\{i, j, k, l\}$. Consequently,
$C_{1123} = C_{1113} = C_{2223} = C_{2213} = C_{3323} = C_{3313} = C_{2312} = C_{1312} = 0$

Cauchy's Relations

- Term coined by Love in [1].
- Derived from a molecular description of materials assuming central forces between pairs of molecules.
- In three-dimensions it is six relations between the elasticity constants in the elasticity tensor which forces **C** to be completely symmetric:

$$C_{1212} = C_{1122},\ C_{2323} = C_{2233},\ C_{1313} = C_{1133},$$
$$C_{1312} = C_{1123},\ C_{2312} = C_{2213},\ C_{2313} = C_{3312}.$$

- Reduces the number of independent constants in **C** from 21 to 15.
- Determined to be invalid for the majority of materials.

[1] A. E. H. Love. *A Treatise on the Mathematical Theory of Elasticity, Volume I*. Cambridge University Press, 1892.

Planar Approximations of the Classical Linear Elastic Equation of Motion

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There are structural configurations where a three-dimensional object may be simulated by a two-dimensional model.

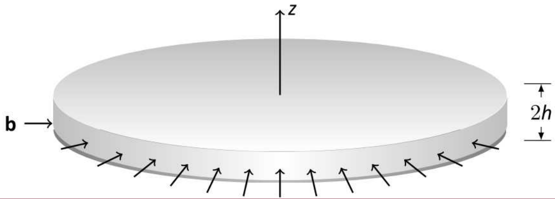
- **Plane strain**: Typically associated with *thick structures*. Due to the thickness of the structure, deformations in the direction of the thickness component are constrained.
- **Plane stress**: Typically associated with *thin structures* such as films. If the film is loaded biaxially, we may suppose there are no normal or shear stresses in the direction perpendicular to the film surfaces.

We consider a derivation of plane stress to motivate analogous derivations in peridynamics. The classical derivation will follow:

T. C. T. Ting. *Anisotropic Elasticity: Theory and Applications*. Oxford University Press, 1996.

Classical Generalized Plane Stress Assumptions

- (Cσ1) The body is a thin plate of thickness $2h$ occupying the region $-h \leqslant z \leqslant h$.
- (Cσ2) The density is constant in the third dimension: $\rho = \rho(x, y)$.
- (Cσ3) The body is subjected to a loading symmetric and parallel relative to the plane $z = 0$.
- (Cσ4) The surfaces of the plate are stress-free, i.e. $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$ for $z = \pm h$.
- (Cσ5) The average stress $\overline{\sigma}_{33}$ is zero throughout the material.
- (Cσ6) The material has at least monoclinic symmetry with a plane of reflection corresponding to the plane $z = 0$.



Generalized Plane Stress Derivation

Step 1: Show the following symmetries for the displacements:

$$u_i(x, y, z) = u_i(x, y, -z), \text{ for } z \in [-h, h], i = 1, 2,$$
$$u_3(x, y, z) = -u_3(x, y, -z), \text{ for } z \in [-h, h].$$

Step 2: Impose monoclinic symmetry on the stress-strain relations and then average in *z* over the thickness of the plate to obtain

$$\overline{\sigma}_{11} = C_{1111}\frac{\partial \overline{u}_1}{\partial x} + C_{1112}\left(\frac{\partial \overline{u}_1}{\partial y} + \frac{\partial \overline{u}_2}{\partial x}\right) + C_{1122}\frac{\partial \overline{u}_2}{\partial y} + C_{1133}[u_3],$$
$$\overline{\sigma}_{22} = C_{1122}\frac{\partial \overline{u}_1}{\partial x} + C_{2212}\left(\frac{\partial \overline{u}_1}{\partial y} + \frac{\partial \overline{u}_2}{\partial x}\right) + C_{2222}\frac{\partial \overline{u}_2}{\partial y} + C_{2233}[u_3],$$
$$\overline{\sigma}_{33} = C_{1133}\frac{\partial \overline{u}_1}{\partial x} + C_{3312}\left(\frac{\partial \overline{u}_1}{\partial y} + \frac{\partial \overline{u}_2}{\partial x}\right) + C_{2233}\frac{\partial \overline{u}_2}{\partial y} + C_{3333}[u_3],$$
$$\overline{\sigma}_{23} = \overline{\sigma}_{13} = 0,$$
$$\overline{\sigma}_{12} = C_{1112}\frac{\partial \overline{u}_1}{\partial x} + C_{1212}\left(\frac{\partial \overline{u}_1}{\partial y} + \frac{\partial \overline{u}_2}{\partial x}\right) + C_{2212}\frac{\partial \overline{u}_2}{\partial y} + C_{3312}[u_3].$$

where

$$[u_3] := \frac{u_3(x, y, h) - u_3(x, y, -h)}{2h} \text{ and } \overline{f} = \frac{1}{2h}\int_{-h}^{h} f dz.$$

Generalized Plane Stress Equation of Motion

Step 3: Utilizing $\overline{\sigma}_{33} = 0$, we may solve for $[u_3]$ in terms of functions of the in-plane displacements:

$$[u_3] = -\left[\frac{C_{1133}}{C_{3333}}\frac{\partial \overline{u}_1}{\partial x_1} + \frac{C_{3312}}{C_{3333}}\left(\frac{\partial \overline{u}_1}{\partial y} + \frac{\partial \overline{u}_2}{\partial x}\right) + \frac{C_{2233}}{C_{3333}}\frac{\partial \overline{u}_2}{\partial y}\right].$$

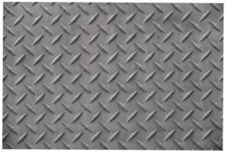
Step 4: Take the average in *z* over the thickness of the plate of the classical equation of motion and then substitute in $\overline{\sigma}_{ij}$:

$$\begin{aligned}
\rho(\mathbf{x})\ddot{\overline{u}}_1 - \overline{b}_1(\mathbf{x}) =& C'_{1111}\frac{\partial^2 \overline{u}_1}{\partial x^2} + C'_{1112}\left(\frac{\partial^2 \overline{u}_1}{\partial x \partial y} + \frac{\partial^2 \overline{u}_2}{\partial x^2}\right) + C'_{1122}\frac{\partial^2 \overline{u}_2}{\partial x \partial y} \\
&+ C'_{1211}\frac{\partial^2 \overline{u}_1}{\partial y \partial x} + C'_{1212}\left(\frac{\partial^2 \overline{u}_1}{\partial y^2} + \frac{\partial^2 \overline{u}_2}{\partial y \partial x}\right) + C'_{1222}\frac{\partial^2 \overline{u}_2}{\partial y^2}, \\
\rho(\mathbf{x})\ddot{\overline{u}}_2 - \overline{b}_2(\mathbf{x}) =& C'_{2111}\frac{\partial^2 \overline{u}_1}{\partial x^2} + C'_{2112}\left(\frac{\partial^2 \overline{u}_1}{\partial x \partial y} + \frac{\partial^2 \overline{u}_2}{\partial x^2}\right) + C'_{2122}\frac{\partial^2 \overline{u}_2}{\partial x \partial y} \\
&+ C'_{2211}\frac{\partial^2 \overline{u}_1}{\partial y \partial x} + C'_{2212}\left(\frac{\partial^2 \overline{u}_1}{\partial y^2} + \frac{\partial^2 \overline{u}_2}{\partial y \partial x}\right) + C'_{2222}\frac{\partial^2 \overline{u}_2}{\partial y^2},
\end{aligned}$$

where $C'_{ijks} := C_{ijks} - \frac{C_{ij33}C_{33ks}}{C_{3333}}$.

Peridynamics

"The objective of peridynamics is to unify the mechanics of discrete particles, continuous media, and continuous media with evolving discontinuities."



Continuous

Discontinuous

- Communicates across length scales.
- Material microstructure aware.
- Models complex damage patterns and applies to highly irregular functions; moreover, the fracture model is implemented before discretization.

S. Silling. *Journal of the Mechanics and Physics of Solids*, 48, 175-209, 2000.
S. Silling, M. Epton, O. Weckner, J. Xu, and E. Askari. *Journal of Elasticity*, 88, 151–184, 2007.

Fundamentals of Bond-Based Peridynamics

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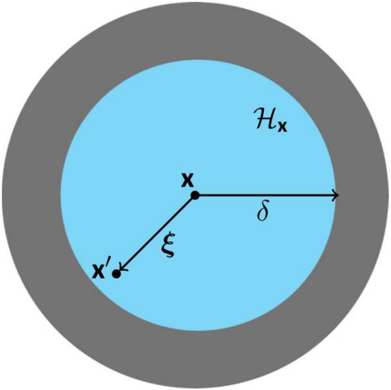
- Points interact with each other over a finite distance δ called the horizon.
- Typically the set of points interacting with $\mathbf{x}$, $\mathcal{H}_{\mathbf{x}}$, is taken to be $B_\delta(\mathbf{x})$.
- In the reference configuration, the vector $\boldsymbol{\xi}$ from $\mathbf{x}$ to $\mathbf{x}'$ within δ is called a bond.
- Each bond has a pairwise force density vector $\mathbf{f}(\mathbf{x}, \mathbf{x}', t)$.
- The equation of motion is an integro-differential equation:

$$\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x}, t) = \int_{\mathcal{H}_{\mathbf{x}}} \mathbf{f}(\mathbf{x}', \mathbf{x}, t) d\mathbf{x}' + \mathbf{b}(\mathbf{x}).$$



Bond-Based Linearized Equation of Motion

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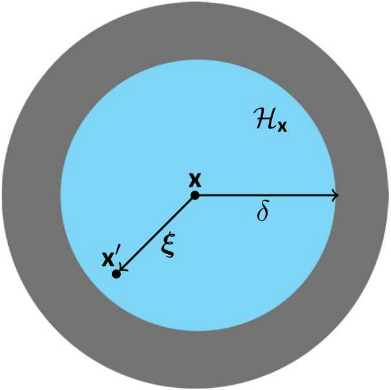
$$\rho(\mathbf{x})\ddot{u}_i(\mathbf{x},t) = \int_{\mathcal{H}_{\mathbf{x}}} \lambda(\boldsymbol{\xi})\xi_i\xi_j(u_j(\mathbf{x}',t) - u_j(\mathbf{x},t))d\mathbf{x}' + b_i(\mathbf{x},t)$$

Here ρ is the mass density, $\mathbf{u}$ is the displacement field, λ is a micromodulus function, and $\boldsymbol{\xi} := \mathbf{x}' - \mathbf{x}$ is the reference bond.



- The micromodulus measures the interaction between material points.
- How to determine λ?

Micromodulus

One way of informing the micromodulus λ is to relate it to commonly measured quantities in classical linear elasticity such as the elasticity tensor $\mathbf{C}$.

Use the change of variable $\mathbf{x}' = \boldsymbol{\xi} + \mathbf{x}$, apply a Taylor expansion about $\mathbf{x}$, and then eliminate antisymmetric terms in the integral:

$$\rho(\mathbf{x})\ddot{u}_i(\mathbf{x}, t) \approx \frac{1}{2}\frac{\partial^2 u_k}{\partial x_j \partial x_l}(\mathbf{x}, t)\left(\int_{B_\delta(\mathbf{0})} \lambda(\boldsymbol{\xi})\xi_i\xi_j\xi_k\xi_l d\boldsymbol{\xi}\right) + b_i(\mathbf{x}, t).$$

Equating with the classical elasticity equation results in

$$C_{ijkl} = \frac{1}{2}\int_{B_\delta(\mathbf{0})} \lambda(\boldsymbol{\xi})\xi_i\xi_j\xi_k\xi_l dV_{\boldsymbol{\xi}}. \tag{3}$$

Cauchy's relations $C_{ijkl} = C_{ikjl}$ are imposed in bond-based peridynamics.

Symmetry Groups of λ

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Definition

An orthogonal transformation **Q** is a symmetry transformation of $\lambda(\boldsymbol{\xi})$ if

$$\lambda(\mathbf{Q}\boldsymbol{\xi}) = \lambda(\boldsymbol{\xi}), \ \forall \boldsymbol{\xi} \in \mathbb{R}^d.$$

Proposition

Suppose λ and **C** *are related through* (3). *If* **Q** *is a symmetry transformation of λ, then it is also a symmetry transformation of* **C***.*

The converse implication is not necessarily true for an arbitrarily chosen λ.

Anisotropic Linearized Elastic Peridynamic Model

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Objectives:

1. Model is informed by the classical elasticity tensor.

2. Imposing material symmetries on the elasticity tensor is equivalent to imposing symmetries on the micromodulus.

3. Model can describe elastic materials in any of the eight symmetry classes of three-dimensional linear elasticity.

4. The peridynamic model has the same degrees of freedom as the classical model (Cauchy's relations imposed).

Three-Dimensional Linear Peridynamic Model

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We propose a micromodulus of the form

$$\lambda(\boldsymbol{\xi}) = \frac{\omega(\|\boldsymbol{\xi}\|)}{\|\boldsymbol{\xi}\|^2} \frac{(\boldsymbol{\xi} \otimes \boldsymbol{\xi})\boldsymbol{\Lambda}(\boldsymbol{\xi} \otimes \boldsymbol{\xi})}{\|\boldsymbol{\xi}\|^4} = \frac{\omega(\|\boldsymbol{\xi}\|)}{\|\boldsymbol{\xi}\|^2} \frac{\xi_i \xi_j \xi_k \xi_l \Lambda_{ijkl}}{\|\boldsymbol{\xi}\|^4}$$

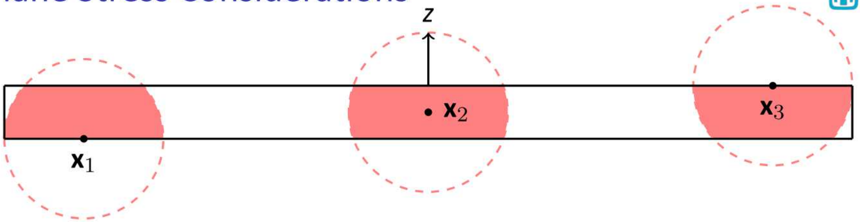
where $\boldsymbol{\Lambda}$ is a completely symmetric fourth-order tensor. Using the relationship between λ and **C** we find

$$\begin{bmatrix} \Lambda_{1111} \\ \Lambda_{2222} \\ \Lambda_{3333} \\ \Lambda_{1122} \\ \Lambda_{1133} \\ \Lambda_{2233} \end{bmatrix} = \frac{2}{m} \begin{bmatrix} 15 & \frac{15}{8} & \frac{15}{8} & -\frac{45}{2} & -\frac{45}{2} & \frac{15}{4} \\ \frac{15}{8} & 15 & \frac{15}{8} & -\frac{45}{2} & \frac{15}{4} & -\frac{45}{2} \\ \frac{15}{8} & \frac{15}{8} & 15 & \frac{15}{4} & -\frac{45}{2} & -\frac{45}{2} \\ -\frac{15}{4} & -\frac{15}{4} & \frac{5}{8} & \frac{255}{8} & -\frac{25}{8} & -\frac{25}{8} \\ -\frac{15}{4} & \frac{5}{8} & -\frac{15}{4} & -\frac{25}{8} & \frac{255}{8} & -\frac{25}{8} \\ \frac{5}{8} & -\frac{15}{4} & -\frac{15}{4} & -\frac{25}{8} & -\frac{25}{8} & \frac{255}{8} \end{bmatrix} \begin{bmatrix} C_{1111} \\ C_{2222} \\ C_{3333} \\ C_{1122} \\ C_{1133} \\ C_{2233} \end{bmatrix}$$

$$\begin{bmatrix} \Lambda_{1112} \\ \Lambda_{2212} \\ \Lambda_{3312} \end{bmatrix} = \frac{2}{m} \begin{bmatrix} \frac{105}{4} & -\frac{105}{8} & -\frac{105}{8} \\ -\frac{105}{8} & \frac{105}{4} & -\frac{105}{8} \\ -\frac{35}{8} & -\frac{35}{8} & 35 \end{bmatrix} \begin{bmatrix} C_{1112} \\ C_{2212} \\ C_{3312} \end{bmatrix}$$

$$\vdots$$

Plane Stress Considerations



Challenges:

- Surface effects are intrinsic to the problem.
- Attempting to relate **C** to λ becomes problematic.

$$\rho(\mathbf{x})\ddot{u}_i(\mathbf{x},t) \approx \frac{\partial u_k}{\partial x_j}(\mathbf{x},t)\left(\int_{\mathcal{H}_\mathbf{x}} \lambda(\boldsymbol{\xi})\xi_i\xi_j\xi_k d\mathbf{x}'\right) + \frac{\partial^2 u_k}{\partial x_j \partial x_l}(\mathbf{x},t)\left(\int_{\mathcal{H}_\mathbf{x}} \lambda(\boldsymbol{\xi})\xi_i\xi_j\xi_k\xi_l d\mathbf{x}'\right) + b_i(\mathbf{x},t).$$

Possible Remedy:

- Let $\lambda := \lambda(\mathbf{x}, \mathbf{x}')$.
- Averaged over the thickness, surface effects can be negated.

Peridynamic Generalized Plane Stress Assumptions

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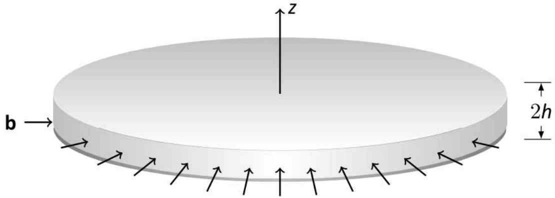
(Pσ1) The body is a thin plate of thickness $2h \leq \delta$ occupying the region $-h \leqslant z \leqslant h$.

(Pσ2) The body is subjected to a loading $\mathbf{b}(\mathbf{x})$ symmetric and parallel to the plane $z = 0$.

(Pσ3) The micromodulus function $\lambda(\mathbf{x}', \mathbf{x})$ is null when $\mathbf{x}'$ and $\mathbf{x}$ are not material points of the plate.

(Pσ4) $\overline{\tau_3(\mathbf{x}, \mathbf{e}_3)} = 0$, where $\overline{\tau_3(\mathbf{x}, \mathbf{e}_3)}$ is the third component of the peridynamic stress averaged in the third spatial component.

(Pσ5) The density is constant in the third dimension: $\rho = \rho(x, y)$.



Peridynamic Generalized Plane Stress Assumptions

(Pσ6) The micromodulus function has the form:

$$\lambda(\mathbf{x}', \mathbf{x}) := \lambda(\xi_1, \xi_2, z', z).$$

(Pσ7) The micromodulus function has at least monoclinic symmetry with a plane of reflection corresponding to $z = 0$:

$$\lambda(\xi_1, \xi_2, z', z) = \lambda(\xi_1, \xi_2, z, z') = \lambda(\xi_1, \xi_2, -z', -z).$$

(Pσ9) The displacement $\mathbf{u}(\mathbf{x})$ is smooth in z and its third component $u_3(\mathbf{x})$ is smooth in $\mathbf{x}$.

Derivation of Peridynamic Plane Stress (4 Steps)

Step 1: Show the following symmetries for the displacements:

$$u_i(x,y,z) = u_i(x,y,-z), \text{ for } z \in [-t,t], i = 1,2,$$
$$u_3(x,y,z) = -u_3(x,y,-z), \text{ for } z \in [-t,t].$$

Step 2: Average over the thickness of the plate and apply (Pσ3):

$$\begin{aligned}\rho\ddot{\bar{u}}_i(\mathbf{x}) =& \frac{1}{2h}\int_{-h}^{h}\int_{-h}^{h}\int_{B_r^{2D}(x,y)} \lambda(\mathbf{x}',\mathbf{x})\xi_i\xi_1(u_1(\mathbf{x}') - u_1(\mathbf{x}))d\mathbf{x}'dz \\ &+ \frac{1}{2h}\int_{-h}^{h}\int_{-h}^{h}\int_{B_r^{2D}(x,y)} \lambda(\mathbf{x}',\mathbf{x})\xi_i\xi_2(u_2(\mathbf{x}') - u_2(\mathbf{x}))d\mathbf{x}'dz \\ &+ \frac{1}{2h}\int_{-h}^{h}\int_{-h}^{h}\int_{B_r^{2D}(x,y)} \lambda(\mathbf{x}',\mathbf{x})\xi_i\xi_3(u_3(\mathbf{x}') - u_3(\mathbf{x}))d\mathbf{x}'dz + \bar{b}_i(\mathbf{x}),\end{aligned}$$

where $r = \sqrt{\delta^2 - (z'-z)^2}$.

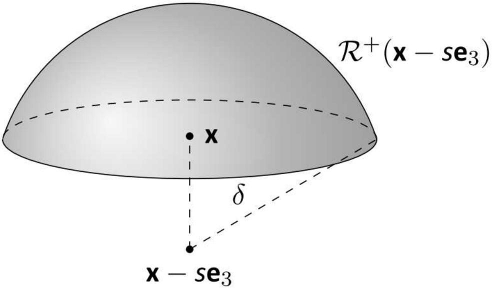
Peridynamic Stress Tensor

The areal force density $\boldsymbol{\tau}(\mathbf{x}, \mathbf{e}_3)$ at a point $\mathbf{x}$ in the directions of $\mathbf{e}_3$:

$$\tau_i(\mathbf{x}, \mathbf{e}_3) := \int_0^\delta \int_{R^+(\mathbf{x}-s\mathbf{e}_3)} \lambda(\hat{\boldsymbol{\xi}})\hat{\xi}_i\hat{\xi}_j(u_j(\mathbf{x}', t) - u_j(\hat{\mathbf{x}}, t))d\mathbf{x}'ds.$$

where $\hat{\mathbf{x}} = \mathbf{x} - s\mathbf{e}_3$, $\hat{\boldsymbol{\xi}} = \mathbf{x}' - \hat{\mathbf{x}}$, and

$$R^+(\mathbf{x} - s\mathbf{e}_3) = \left\{\mathbf{x}' \in B_\delta(\mathbf{x} - s\mathbf{e}_3) : x_3' > x_3\right\}.$$



S. Silling. *Journal of the Mechanics and Physics of Solids*, 48, 175-209, 2000.

Utilizing Peridynamic Stress Tensor

Step 3: Use $\overline{\tau_3(\mathbf{x}, \mathbf{e}_3)}$ to replace u_3 in the in-plane equations of motion with expressions in u_1 and u_2.

Lemma

Under the Assumptions of peridynamic generalized plane stress, we have the following approximations:

$$\frac{\partial u_3}{\partial z}(x,y,0) \approx \frac{-\int_{-h}^{h}\int_{-h}^{h}\int_{B_r^{2D}(x,y)} \lambda(\mathbf{x}',\mathbf{x})\xi_3^2\xi_j(u_j(\mathbf{x}')-u_j(\mathbf{x}))d\mathbf{x}'dz}{\int_{-h}^{h}\int_{-h}^{h}\int_{B_r^{2D}(\mathbf{0})} \lambda(\mathbf{x}',\mathbf{x})\xi_3^4 d\xi_1 d\xi_2 dz' dz}$$

for $j = 1, 2$ and for $i = 1, 2$ we additionally have:

$$\int_{-h}^{h}\int_{-h}^{h}\int_{B_r^{2D}(x,y)} \lambda(\mathbf{x}',\mathbf{x})\xi_i\xi_3(u_3(\mathbf{x}')-u_3(\mathbf{x}))d\mathbf{x}'dz$$
$$\approx \frac{1}{2}\int_{-h}^{h}\int_{-h}^{h}\int_{B_r^{2D}(x,y)} \lambda(\mathbf{x}',\mathbf{x})\xi_i\xi_3^2\frac{\partial u_3}{\partial z}(x',y',0)d\mathbf{x}'dz.$$

Peridynamic Generalized Plane Stress

Step 4: Utilize approximations to remove integrals in z, z'.

$$\begin{aligned}
\rho\ddot{\bar{u}}_1(\mathbf{x}) \approx & \frac{1}{2h}\int_{B_\delta^{2D}(\mathbf{x})} \lambda_0(\mathbf{x}',\mathbf{x})\xi_1\left[\xi_1(\overline{u_1}(\mathbf{x}')-\overline{u_1}(\mathbf{x}))+\xi_2(\overline{u_2}(\mathbf{x}')-\overline{u_2}(\mathbf{x}))\right]d\mathbf{x}' \\
& -\frac{1}{4tB}\int_{B_\delta^{2D}(\mathbf{x})}\int_{B_\delta^{2D}(\mathbf{x}')} \lambda_2(\mathbf{x}',\mathbf{x})\lambda_2(\mathbf{x}'',\mathbf{x}')\xi_1\left(\zeta_1(\overline{u_1}(\mathbf{x}'')-\overline{u_1}(\mathbf{x}'))\right)d\mathbf{x}''d\mathbf{x}' \\
& -\frac{1}{4tB}\int_{B_\delta^{2D}(\mathbf{x})}\int_{B_\delta^{2D}(\mathbf{x}')} \lambda_2(\mathbf{x}',\mathbf{x})\lambda_2(\mathbf{x}'',\mathbf{x}')\xi_1\left(\zeta_2(\overline{u_2}(\mathbf{x}'')-\overline{u_2}(\mathbf{x}'))\right)d\mathbf{x}''d\mathbf{x}' + \bar{b}_1(\mathbf{x}). \\
\rho\ddot{\bar{u}}_2(\mathbf{x}) \approx & \frac{1}{2h}\int_{B_\delta^{2D}(\mathbf{x})} \lambda_0(\mathbf{x}',\mathbf{x})\xi_2\left[\xi_1(\overline{u_1}(\mathbf{x}')-\overline{u_1}(\mathbf{x}))+\xi_2(\overline{u_2}(\mathbf{x}')-\overline{u_2}(\mathbf{x}))\right]d\mathbf{x}' \\
& -\frac{1}{4tB}\int_{B_\delta^{2D}(\mathbf{x})}\int_{B_\delta^{2D}(\mathbf{x}')} \lambda_2(\mathbf{x}',\mathbf{x})\lambda_2(\mathbf{x}'',\mathbf{x}')\xi_2\left(\zeta_1(\overline{u_1}(\mathbf{x}'')-\overline{u_1}(\mathbf{x}'))\right)d\mathbf{x}''d\mathbf{x}' \\
& -\frac{1}{4tB}\int_{B_\delta^{2D}(\mathbf{x})}\int_{B_\delta^{2D}(\mathbf{x}')} \lambda_2(\mathbf{x}',\mathbf{x})\lambda_2(\mathbf{x}'',\mathbf{x}')\xi_2\left(\zeta_2(\overline{u_2}(\mathbf{x}'')-\overline{u_2}(\mathbf{x}'))\right)d\mathbf{x}''d\mathbf{x}' + \bar{b}_2(\mathbf{x}).
\end{aligned}$$

where $\mathbf{x} \in \mathbb{R}^2$,

$$B := \int_{-h}^{h}\int_{-h}^{h}\int_{B_r^{2D}(\mathbf{0})} \lambda(\mathbf{x}'',\mathbf{x}')\zeta_3^4 d\zeta_1 d\zeta_2 dz'' dz' \text{ and } \lambda_i(x'y',x,y) := \int_{-h}^{h}\int_{-h}^{h} \lambda(\mathbf{x}',\mathbf{x})\xi_3^i dz dz'.$$

Peridynamic Generalized Plane Stress Reformulation

We can reformulate the peridynamic generalized plane stress model as a two-dimensional state-based peridynamic model:

$$\rho(\mathbf{x})\ddot{\overline{\mathbf{u}}}(\mathbf{x},t) = \int_{\mathcal{H}} \left\{ \underline{\mathbf{T}}[\mathbf{x},t]\langle\boldsymbol{\xi}\rangle - \underline{\mathbf{T}}[\mathbf{x}+\boldsymbol{\xi},t]\langle-\boldsymbol{\xi}\rangle \right\} d\boldsymbol{\xi} + \overline{\mathbf{b}}(\mathbf{x},t),$$

where

$$\underline{\mathbf{T}}[\mathbf{x},t]\langle\boldsymbol{\xi}\rangle = \frac{1}{4h}\left[\lambda_0(\boldsymbol{\xi})\boldsymbol{\xi}\otimes\boldsymbol{\xi}(\overline{\mathbf{u}}(\mathbf{x}+\boldsymbol{\xi},t) - \overline{\mathbf{u}}(\mathbf{x},t)) - \lambda_2(\boldsymbol{\xi})A(\mathbf{x},t)\boldsymbol{\xi}\right]$$

and

$$A(\mathbf{x},t) = \frac{1}{\int_{\mathcal{H}} \lambda_4(\boldsymbol{\zeta})d\boldsymbol{\zeta}} \int_{\mathcal{H}} \lambda_2(\boldsymbol{\zeta})\boldsymbol{\zeta}\cdot(\overline{\mathbf{u}}(\mathbf{x}+\boldsymbol{\zeta},t) - \overline{\mathbf{u}}(\mathbf{x},t))d\boldsymbol{\zeta}.$$

Connection to the Classical Plane Stress Model

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Proposition

Suppose the micromodulus function λ satisfies

$$2hC_{ijkl} = \frac{1}{2}\int_{-h}^{h}\int_{-h}^{h}\int_{B_\delta^{2D}(x,y)} \lambda(\mathbf{x},\mathbf{x}')\xi_i\xi_j\xi_k\xi_l d\mathbf{x}' dz.$$

Then under a smooth deformation, the peridynamic plane stress equation of motion agrees with the classical equation of motion up to second-order terms.

Plane Stress Simulations

	Isotropic (Pyroceram 9608)	Cubic ($MgAl_2O_4$)	Orthotropic (Te_2W)
Classical			
Peridynamics			

Conclusions and Future Work

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Conclusions:

- Developed a three-dimensional anisotropic peridynamic model for every symmetry class in linear elasticity.
- Derived peridynamic analogue of plane stress from a three-dimensional anisotropic peridynamic model.

Future Work:

- Derive anisotropic state-based models.
- Remove restrictions on the peridynamic plane stress model.
- Bond-breaking criteria for anisotropic and planar models.
- Validation of the peridynamic plane strain and stress models.
- Material stability.

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