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# Well-Posedness of Systems with Weighted Nonlocal Vector Calculus Operators

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## Abstract

The well-posedness of systems with unweighted nonlocal vector calculus operators is leveraged to show well-posedness of analogous systems with weighted nonlocal operators. In particular, a choice in nonlocal kernel is derived which shows equivalence of the weighted and unweighted nonlocal Laplacian operators. This kernel satisfies most desired properties and also is consistent with the fractional Laplacian.

## Background

There has been a rise in the use of nonlocal integral operators in place of their classical differential counterparts due to their potential to allow for more singular solutions. Where classical differential systems require differentiability across the domain, with nonlocal integral operators we can get away with integrability of the defined nonlocal operator. Nonlocal operators come from the idea that information about a point can be gathered by assessing the point's close-by neighbors.

As an example, consider the nonlocal Laplacian operator equipped with the symmetric nonlocal kernel $\boldsymbol{\alpha} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and acting on a scalar function $u : \mathbb{R}^n \to \mathbb{R}$,

$$\mathcal{L}u(\mathbf{x}) = \int_{\mathbb{R}^n} (u(\mathbf{y}) - u(\mathbf{x}))\alpha(\mathbf{x},\mathbf{y})d\mathbf{y}.$$

The choice in $\boldsymbol{\alpha}$ has a major effect on the behavior of the operator. Several results exist based on choices for $\boldsymbol{\alpha}$ which will allow for desired properties such as convergence to the classical differential Laplacian operator.

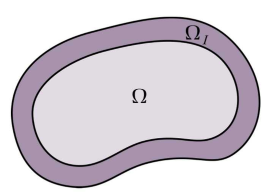
It's not uncommon to have $\boldsymbol{\alpha}$ depend heavily on the distance between $\mathbf{x}$ and $\mathbf{y}$, as we usually want points closer to $\mathbf{x}$ to have a larger impact. Also note, for $0 < s < 1$ with the choice $\boldsymbol{\alpha} = 1/|\mathbf{y}-\mathbf{x}|^{2n+s}$ we have consistency with the fractional Laplacian

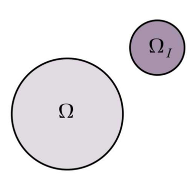
$$(-\Delta)^s u(\mathbf{x}) = c\int_{\mathbb{R}^n} \frac{u(\mathbf{y}) - u(\mathbf{x})}{|\mathbf{y}-\mathbf{x}|^{2n+s}} d\mathbf{y}$$

Let $\Omega \subset \mathbb{R}^n$ be the domain of a function on which we'd like to apply a nonlocal operator. Since the operator depends on the interaction of each point in the domain with any points such that the interaction kernel is nonzero, we have an additional interaction domain

$$\Omega_I = \{\mathbf{y} \in \mathbb{R}^n \setminus \Omega : \boldsymbol{\alpha}(\mathbf{x},\mathbf{y}) \neq 0, \mathbf{x} \in \Omega\}.$$

If we consider $\Omega \subset \mathbb{R}^2$ with the kernel $\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})$ having support on an $\epsilon$ ball about the first input, the domain and interaction domain may look similar to the diagram on the left below




However, the domain and collar need not be so nice. In fact, they may not even be adjacent such as the diagram to the right above.

These systems are then volume-constrained problems instead of boundary value problems; the conditions analogous to boundary conditions are imposed on the interaction domain, which is in the same dimensions as the initial domain.

## Preliminaries

We utilize the definitions of the weighted and unweighted nonlocal operators introduced in [1]. In particular, we set $\boldsymbol{\alpha} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$ to be an antisymmetric vector two-point function which then gives rise to the following operators. For $\boldsymbol{\nu} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$ we have the unweighted nonlocal divergence $\mathcal{D}\boldsymbol{\nu} : \mathbb{R}^n \to \mathbb{R}$ defined

$$\mathcal{D}\boldsymbol{\nu}(\mathbf{x}) := \int_{\mathbb{R}^n} (\boldsymbol{\nu}(\mathbf{x},\mathbf{y}) + \boldsymbol{\nu}(\mathbf{y},\mathbf{x})) \cdot \boldsymbol{\alpha}(\mathbf{x},\mathbf{y})d\mathbf{y} \quad x \in \mathbb{R}^n.$$

Then for $u : \mathbb{R}^n \to \mathbb{R}$ we have the unweighted nonlocal gradient in the form of the adjoint to the unweighted nonlocal divergence $\mathcal{D}^*u : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$ defined

$$\mathcal{D}^*u(\mathbf{x},\mathbf{y}) = -(u(\mathbf{y}) - u(\mathbf{x}))\boldsymbol{\alpha}(\mathbf{x},\mathbf{y}) \quad \mathbf{x}, y \in \mathbb{R}^n.$$

These structures give rise to the weighted nonlocal operators. The major shift between the unweighted and weighted operators is that the weighted operators always act on a function with a single input which is then augmented by a two-point weight function. Throughout, let $\omega : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a non-negative, symmetric scalar function. For $\mathbf{u} : \mathbb{R}^n \to \mathbb{R}^k$, we have the weighted nonlocal divergence $\mathcal{D}_\omega \mathbf{u} : \mathbb{R}^n \to \mathbb{R}$ defined

$$\mathcal{D}_\omega \mathbf{u}(\mathbf{x}) := \mathcal{D}(\omega(\mathbf{x},\mathbf{y})\mathbf{u}(\mathbf{x})) = \int_{\mathbb{R}^n} (\omega(\mathbf{x},\mathbf{y})\mathbf{u}(\mathbf{x}) + \omega(\mathbf{y},\mathbf{x})\mathbf{u}(\mathbf{y})) \cdot \boldsymbol{\alpha}(\mathbf{x},\mathbf{y})dy \quad x \in \mathbb{R}^n.$$

For $u : \mathbb{R}^n \to \mathbb{R}$, we have the weighted nonlocal gradient in the form of the adjoint to the weighted divergence $\mathcal{D}^*_\omega u : \mathbb{R}^n \to \mathbb{R}^k$ defined

$$\mathcal{D}^*_\omega u(\mathbf{x}) := \int_{\mathbb{R}^n} \mathcal{D}^*u(\mathbf{x},\mathbf{y})\omega(\mathbf{x},\mathbf{y})d\mathbf{y} \quad x \in \mathbb{R}^n.$$

There are a myriad of results which show well-posedness of volume-constrained problems which utilize the unweighted versions of the nonlocal operators. Systems with the weighted nonlocal operators are less well studied. However, we can leverage the well posedness of the unweighted case by arguing that both operators have equivalent energy norms. Doing so involves two major facets:

First, the equivalence of the unweighted Laplacian operator

$$\mathcal{D}\mathcal{G}u(\mathbf{x}) = \mathcal{L}u(\mathbf{x}) = \int_{\Omega\cup\Omega_I} (u(\mathbf{x}) - u(\mathbf{y}))\gamma(\mathbf{x},\mathbf{y})d\mathbf{y}.$$

with the weighted Laplacian operator

$$\begin{aligned}\mathcal{D}_\omega\mathcal{G}_\omega u(\mathbf{x}) &= \mathcal{D}_\omega\mathcal{D}^*_\omega u(\mathbf{x}) = \mathcal{D}(\omega(\mathbf{x},\mathbf{y})\mathcal{D}^*_\omega u(\mathbf{x})) \\ &= \int_{\Omega\cup\Omega_I} \Big[\omega(\mathbf{x},\mathbf{y})\mathcal{D}^*_\omega u(\mathbf{x}) + \omega(\mathbf{y},\mathbf{x})\mathcal{D}^*_\omega u(\mathbf{y})\Big] \cdot \boldsymbol{\alpha}(\mathbf{x},\mathbf{y})d\mathbf{y}.\end{aligned}$$

Second, the existence of a weighted nonlocal Green's identity

$$\int_{\Omega\cup\Omega_I} \mathcal{D}_\omega\mathcal{G}_\omega u(\mathbf{x})v(\mathbf{y})d\mathbf{x} = \int_{\Omega\cup\Omega_I} \mathcal{G}_\omega u(\mathbf{x}) \cdot \mathcal{G}_\omega v(\mathbf{x})d\mathbf{x}.$$

## Results

There is a specific choice of kernel $\gamma$ for which the unweighted Laplacian operator $\mathcal{D}\mathcal{G}$ is equivalent to it's weighted counterpart $\mathcal{D}_\omega\mathcal{G}_\omega$. In particular, the equivalence occurs for the choice of kernel

$$\begin{aligned}\gamma(\mathbf{x},\mathbf{y}) = \int_{\Omega\cup\Omega_I} &[\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\omega(\mathbf{x},\mathbf{y}) \cdot \boldsymbol{\alpha}(\mathbf{x},\mathbf{z})\omega(\mathbf{x},\mathbf{z}) \\ &+ \boldsymbol{\alpha}(\mathbf{z},\mathbf{y})\omega(\mathbf{z},\mathbf{y}) \cdot \boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\omega(\mathbf{x},\mathbf{y}) + \boldsymbol{\alpha}(\mathbf{z},\mathbf{y})\omega(\mathbf{z},\mathbf{y}) \cdot \boldsymbol{\alpha}(\mathbf{x},\mathbf{z})\omega(\mathbf{x},\mathbf{z})]d\mathbf{z}\end{aligned}$$

for the unweighted operator.

This kernel satisfies some of the properties we desire in a nonlocal kernel. For instance, it is symmetric. Also, with certain choices of $\boldsymbol{\alpha}$ and $\omega$ the kernel $\gamma$ has been numerically verified as positive. Additionally, for the specific choices

$$\omega(\mathbf{x},\mathbf{y}) = |\mathbf{y}-\mathbf{x}|\phi(|\mathbf{y}-\mathbf{x}|) \;\text{ for }\; \phi(|\mathbf{y}-\mathbf{x}|) = \frac{1}{|\mathbf{y}-\mathbf{x}|^{d+1+s}}$$
$$\boldsymbol{\alpha}(\mathbf{x},\mathbf{y}) = \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}$$

we have that the operator $\mathcal{D}\mathcal{G}$ is consistent with the fractional Laplacian operator.

Letting $E_\omega u$ refer to the energy of $u$ with respect to the weighted nonlocal operators, we have

$$E_\omega u = \int_{\Omega\cup\Omega_I} (\mathcal{D}_\omega u(\mathbf{x}))^2 d\mathbf{x}$$

Utilizing the equivalence kernel above alongside nonlocal Green's identities for both the weighted and unweighted operators, we find that

$$E_\omega u = \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} (u(\mathbf{x}) - u(\mathbf{y}))^2\gamma^2(\mathbf{x},\mathbf{y})d\mathbf{y}d\mathbf{x}.$$

Which garners equivalence to the unweighted nonlocal energy norm

$$|||u||| = \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} (u(\mathbf{x}) - u(\mathbf{y}))^2\boldsymbol{\beta}(\mathbf{x},\mathbf{y})d\mathbf{y}d\mathbf{x}.$$

This equivalence of norms allows us to declare well-posedness of systems with weighted nonlocal operators which are analogous to systems with unweighted nonlocal operators that have already been deemed well-posed.

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