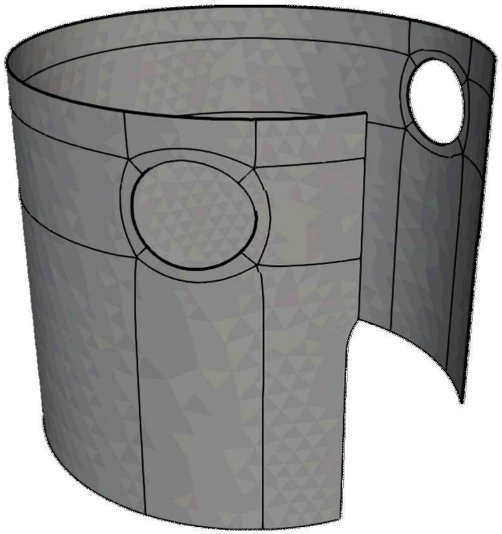
SAND2019-12336C

COARSE QUAD LAYOUTS THROUGH ROBUST …IFICATION OF CROSS …SEPARATRIX …TIONS



…ertel

October 16th 2019

28th International Meshing Roundtable

U.S. DEPARTMENT OF ENERGY NNSA National Nuclear Security Administration CASL Consortium for Advanced Simulation of LWRs Sandia National Laboratories

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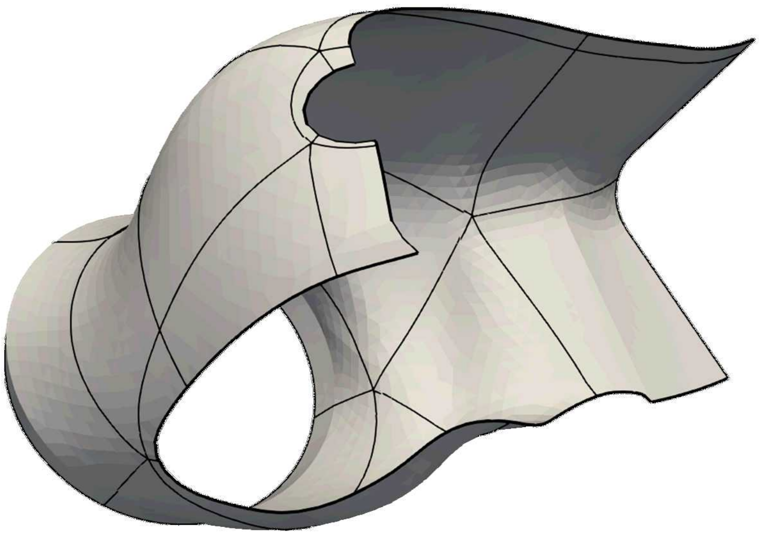
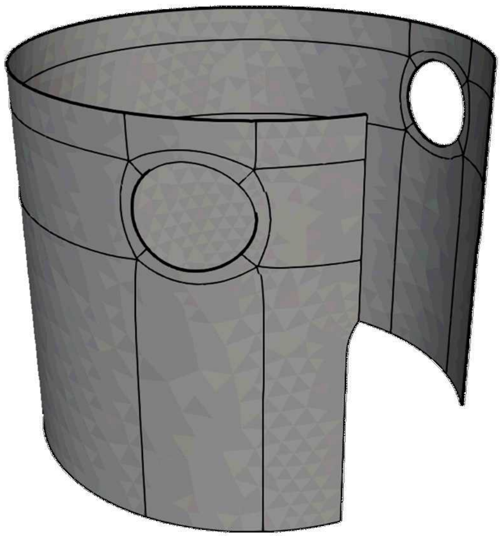
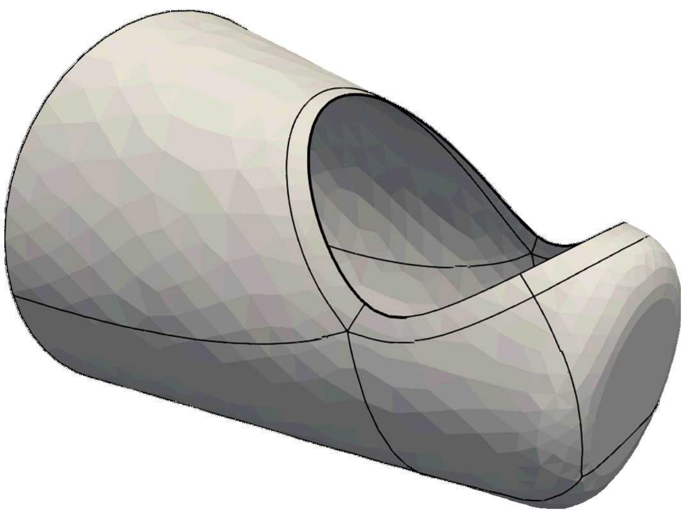
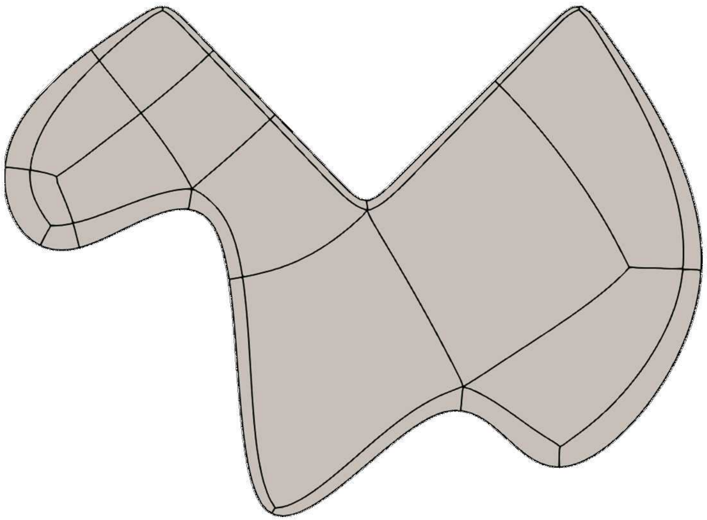
Goal: Coarse boundary aligned quad partitions









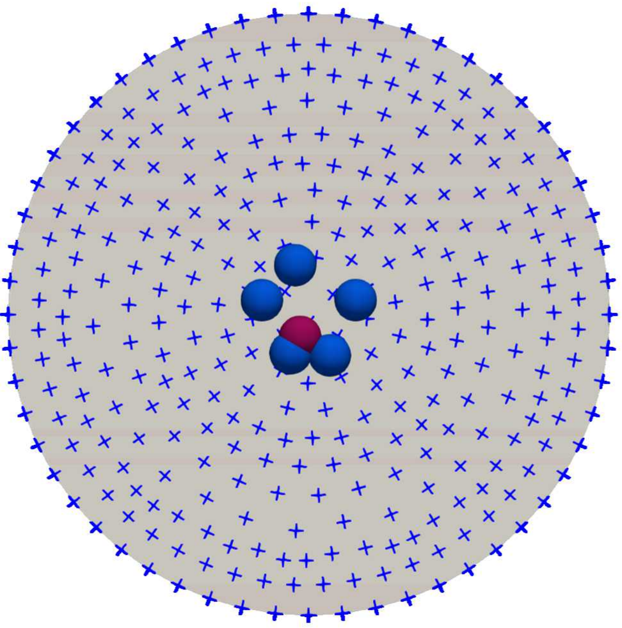
Contributions:

1. Extend MBO method for cross field design to curved surfaces
2. Prove that near singularities, streamlines of a cross field are hyperbolic under a conformal map.
3. Partition simplification algorithm to generate coarse quad layouts

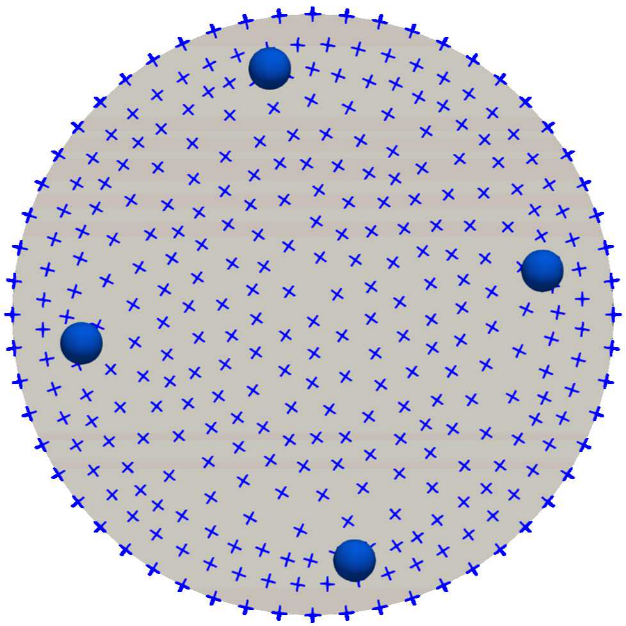
Merriman-Bence-Osher (MBO) Method for Cross Field Design

Why MBO?

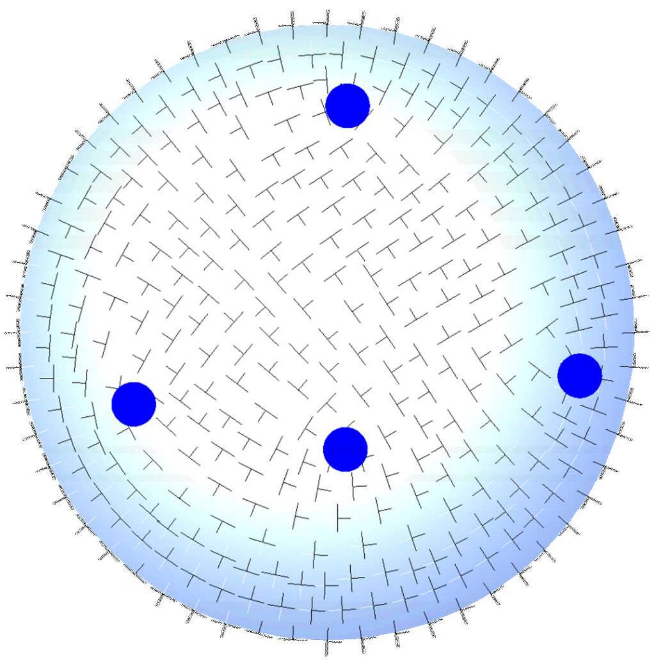
- Easy to implement
- Fast
- Good singularity placement



Knoppel et al. 2013



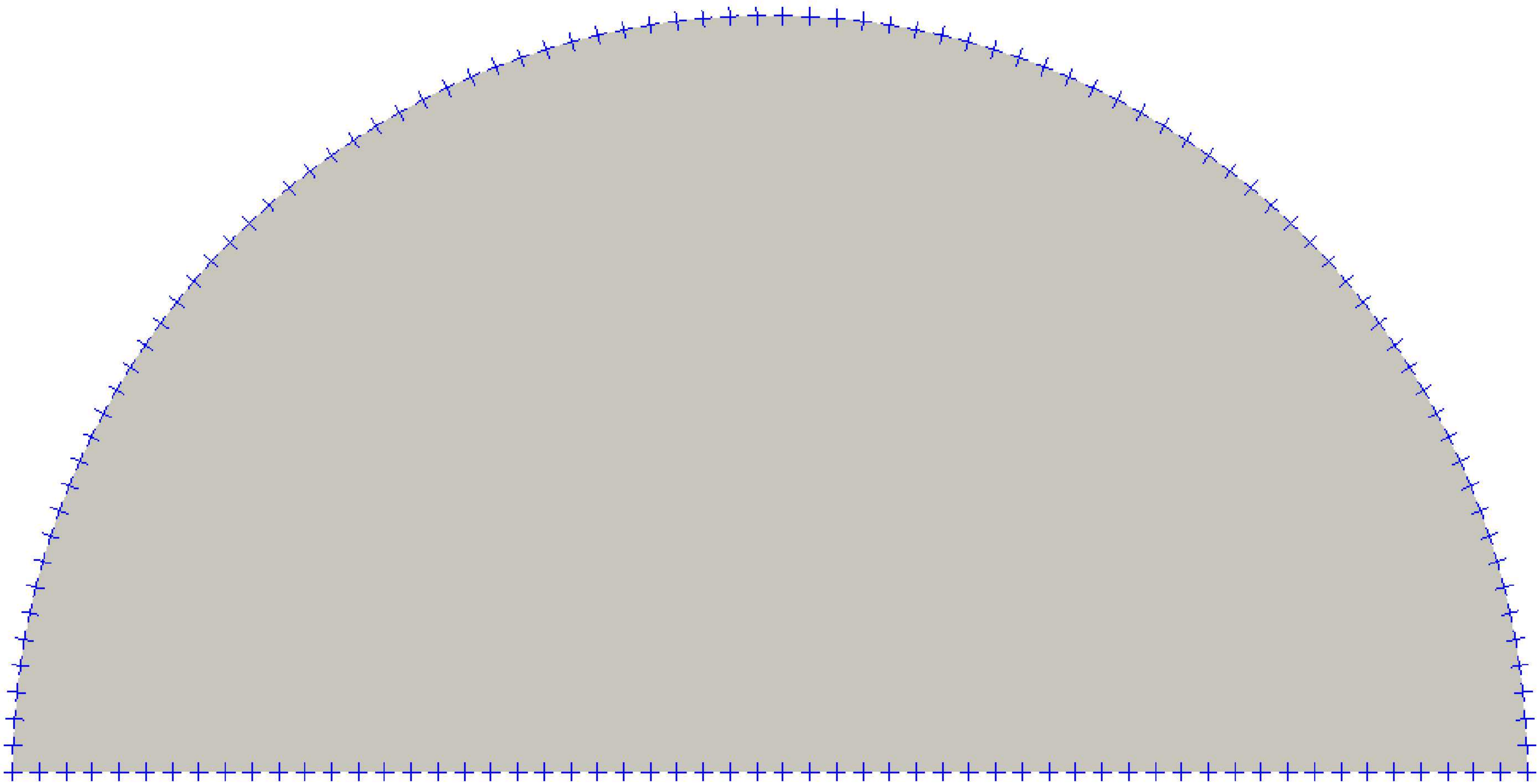
MBO



Jakob et al. 2014

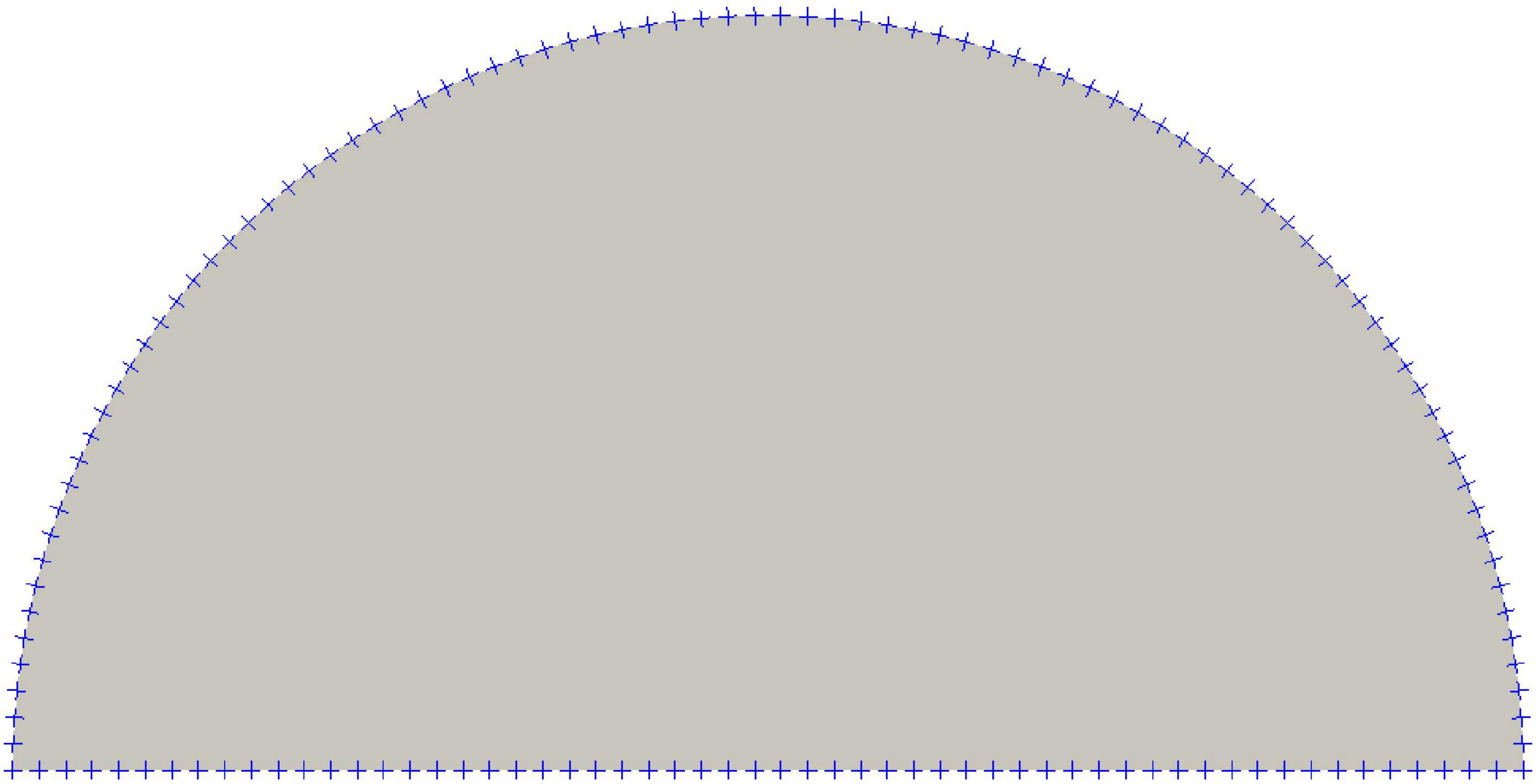
MBO Method for Cross Fields

MBO Method for Cross Fields

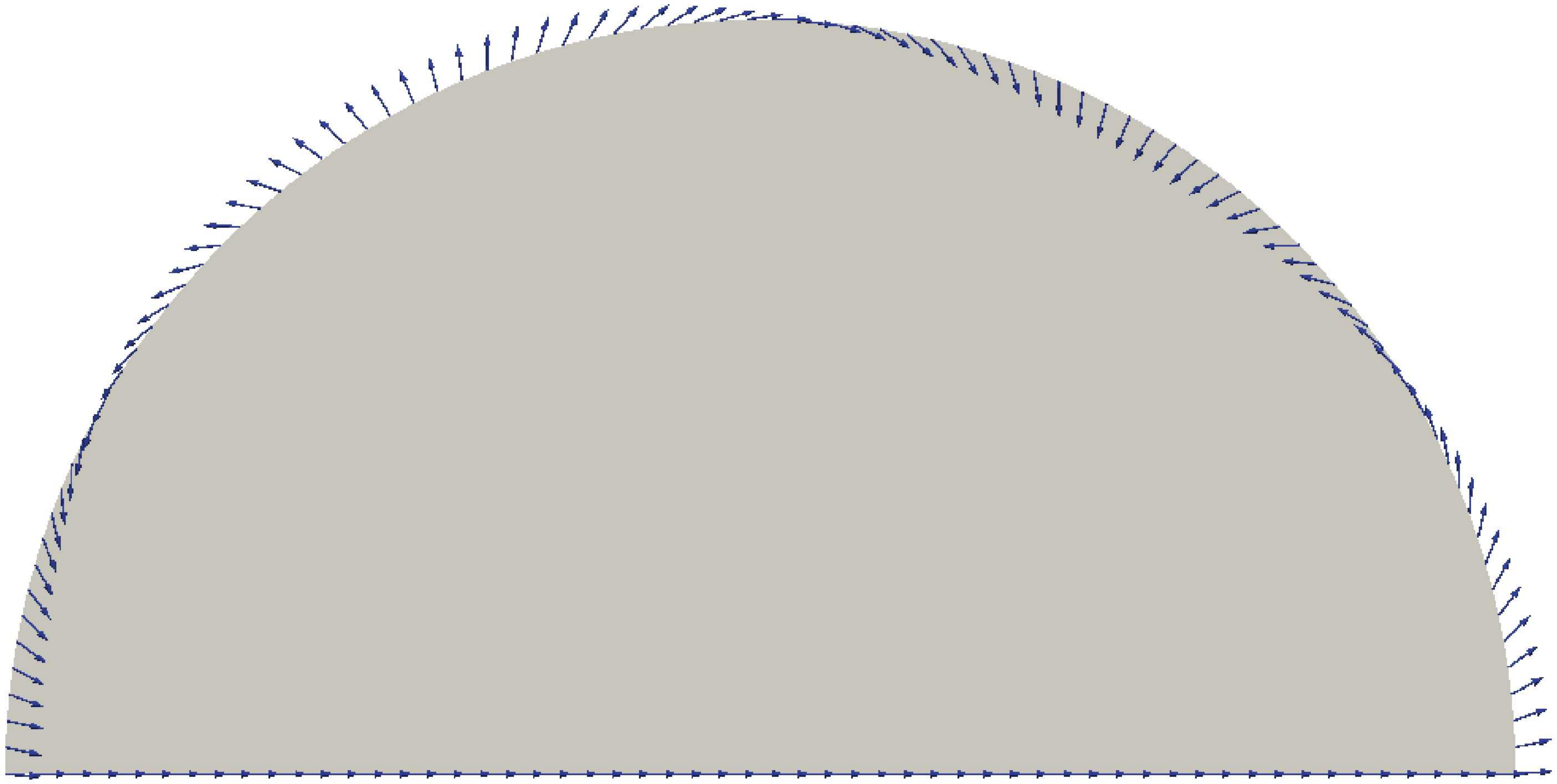


The Representation Map

MBO Method for Cross Fields



MBO Method for Cross Fields



MBO Method for Cross Fields

$$\min_{u \in H^1_g(D,\mathbb{C})} E(u)$$

$$E(u) = \frac{1}{2}\int_D |\nabla u|^2 dA$$

$$u(x) = g(x) \quad \forall x \in \partial D$$

$$|u(x)| = 1 \quad \text{a.e.} \quad x \in D$$

MBO Method for Cross Fields

Ginzburg-Landau Relaxation:

$$E_\epsilon(u) = \frac{1}{2}\int_G |\nabla u|^2 + \frac{1}{4\epsilon^2}\int_G (|u|^2 - 1)^2$$

Iterative method to minimize cross field energy:

1. Until time τ solve

$$u_t(t,x) = \Delta u(t,x) \qquad x \in M$$

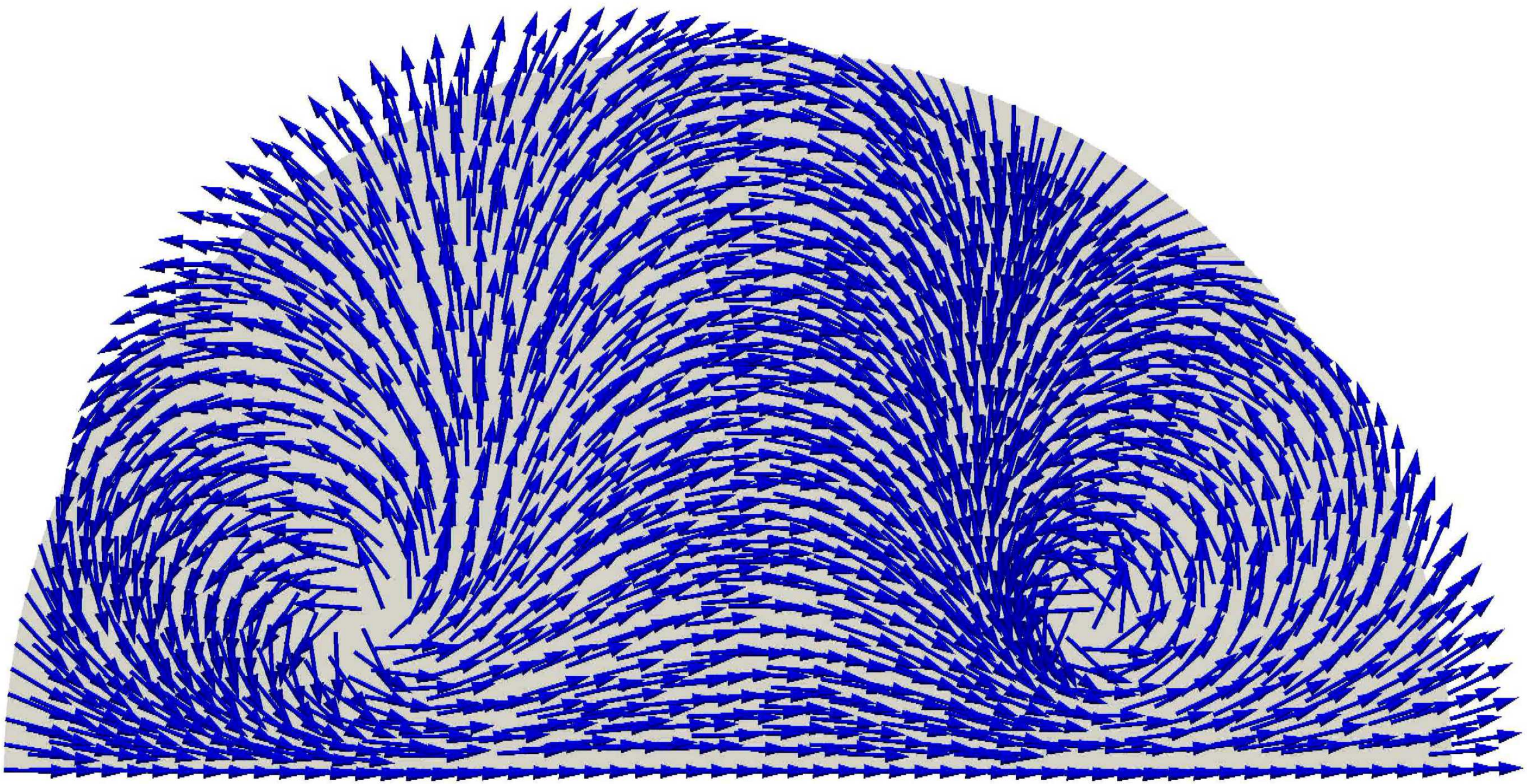
$$u(t,x) = g(x) \qquad x \in \partial M$$

$$u(0,x) = u_0(x) \qquad x \in M$$

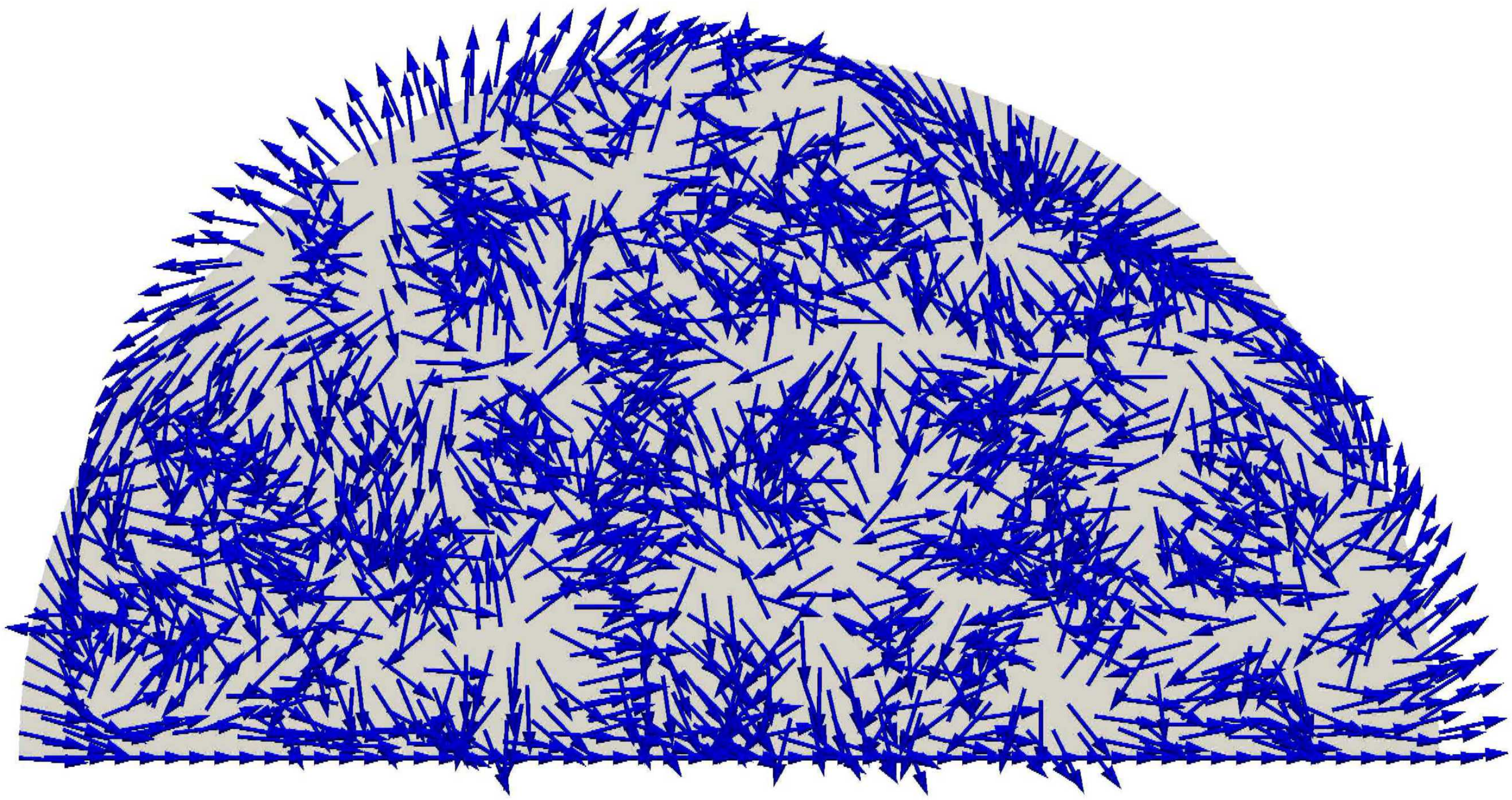
2. Renormalize

$$u(\tau,x) = \frac{u(\tau,x)}{||u(\tau,x)||}$$

MBO Method



MBO Method



MBO on Surfaces

1. Until time τ solve

?

$$u_t(t,x) = \Delta u(t,x) \qquad x \in M$$

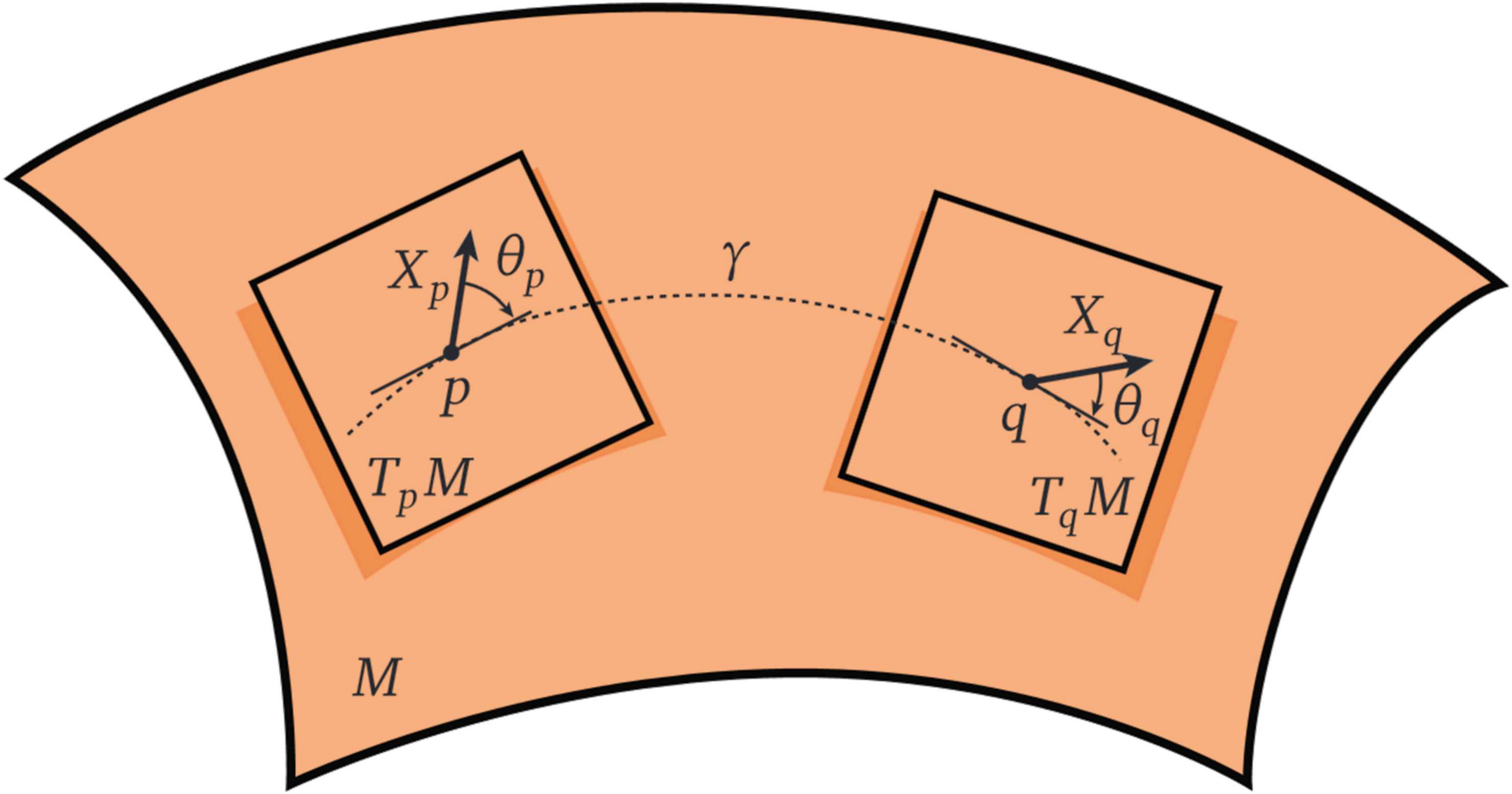
$$u(t,x) = g(x) \qquad x \in \partial M$$

$$u(0,x) = u_0(x) \qquad x \in M$$

2. Renormalize

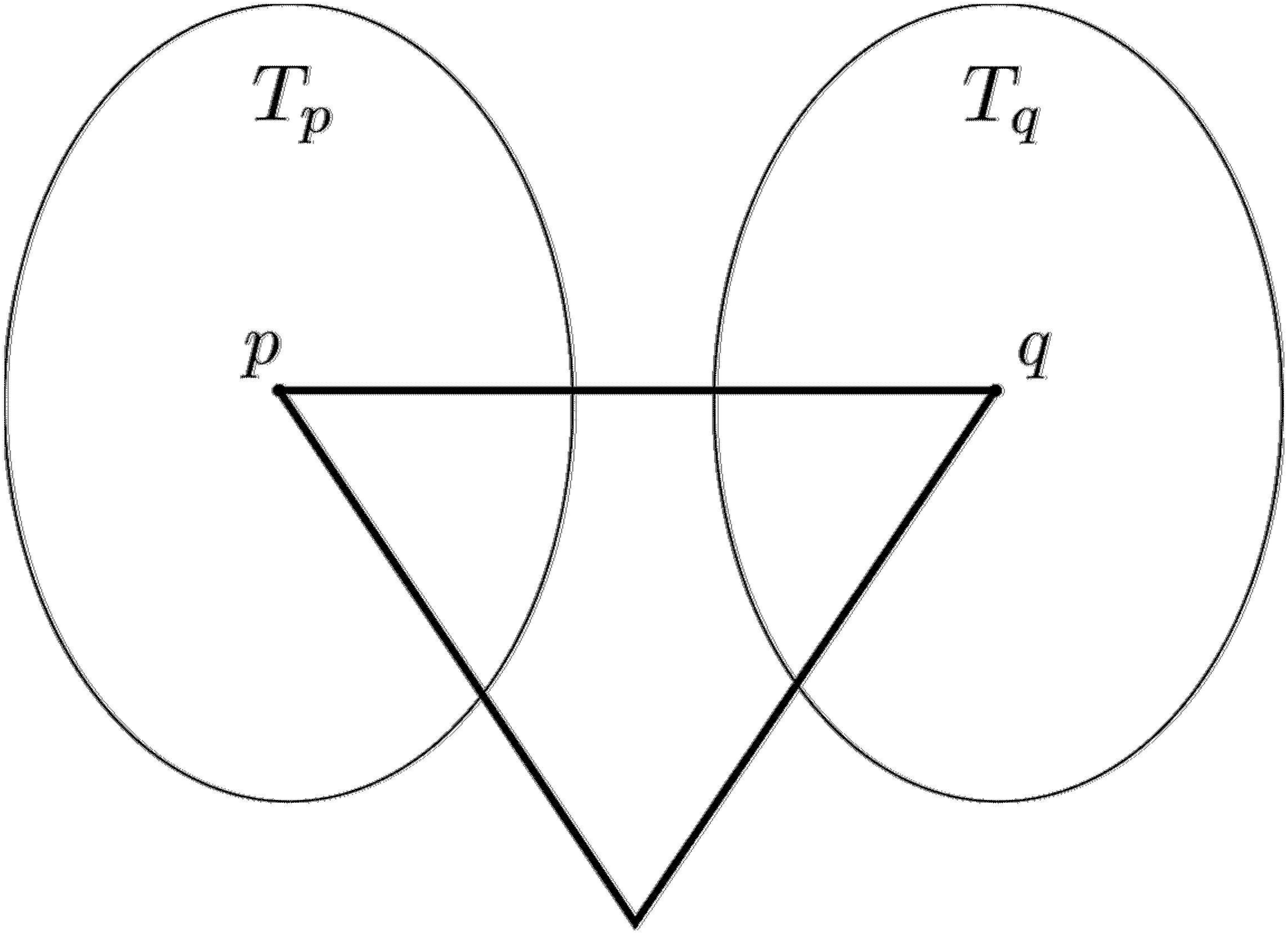
$$u(\tau,x) = \frac{u(\tau,x)}{||u(\tau,x)||}$$

Levi-Civita Connection

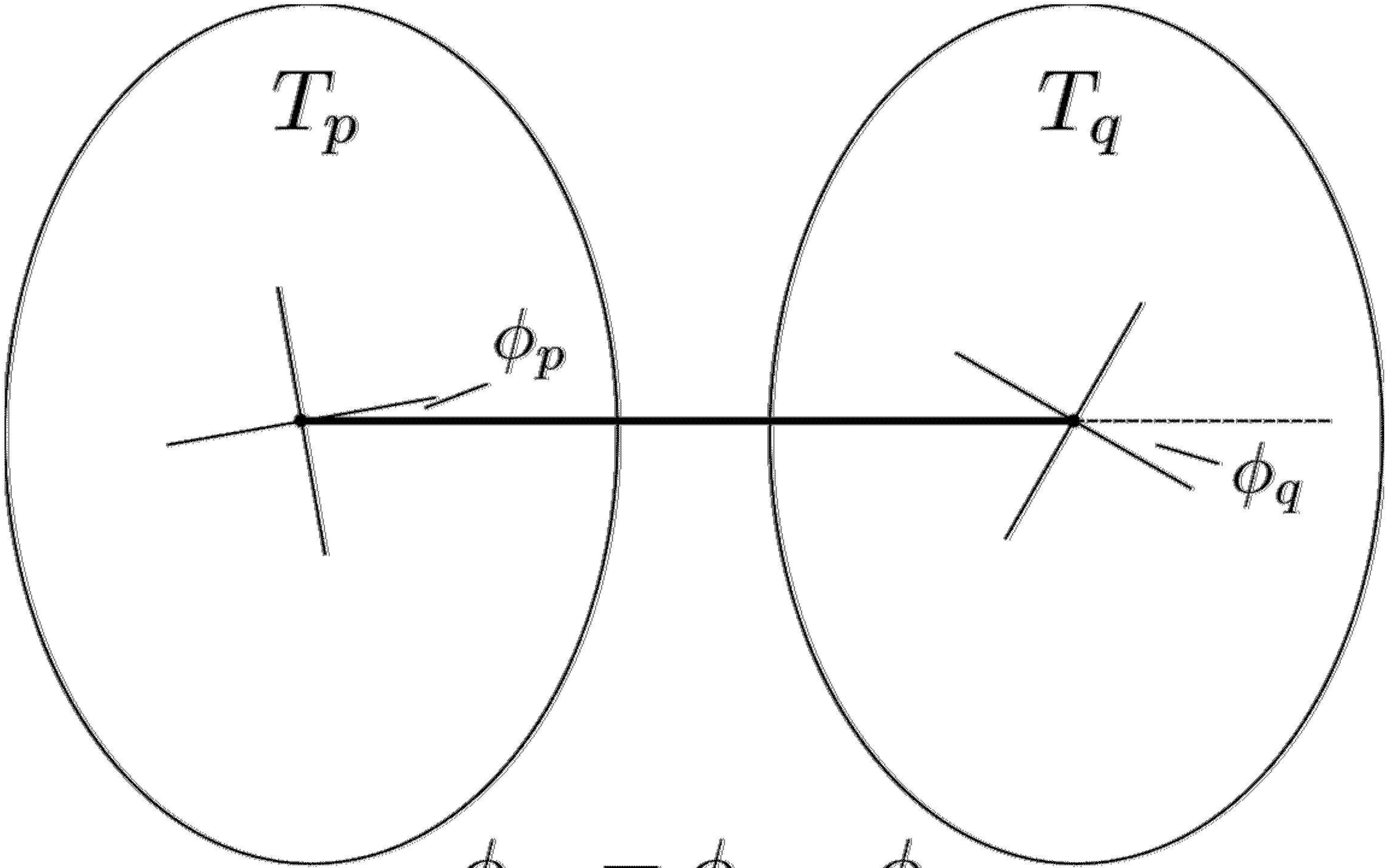


Knoppel et al. 2013

Cross Field Design on Surfaces



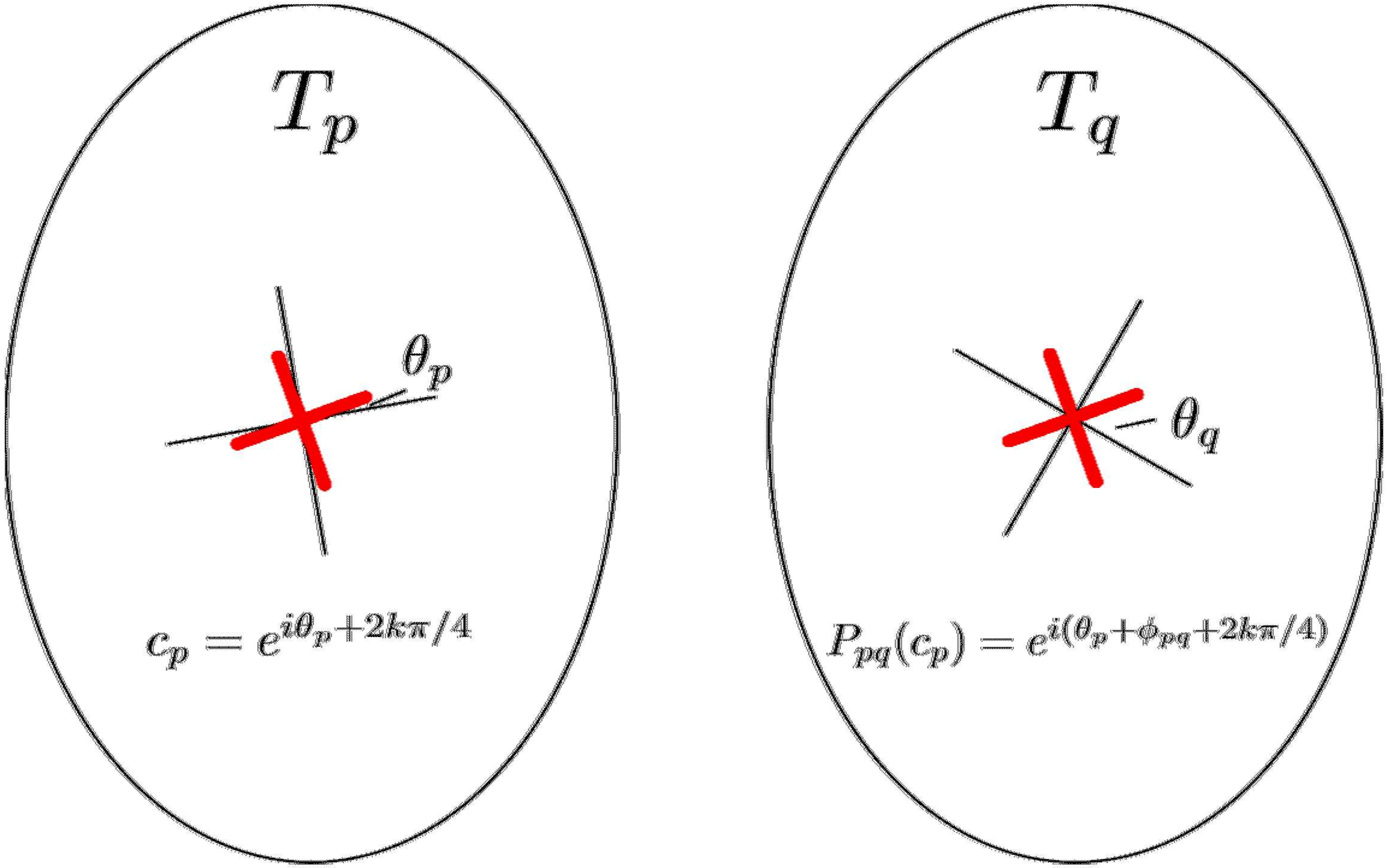
Cross Field Design on Surfaces



$$\phi_{pq} = \phi_p - \phi_q$$

$$P_{pq}(v) = e^{i\phi_{pq}} v$$

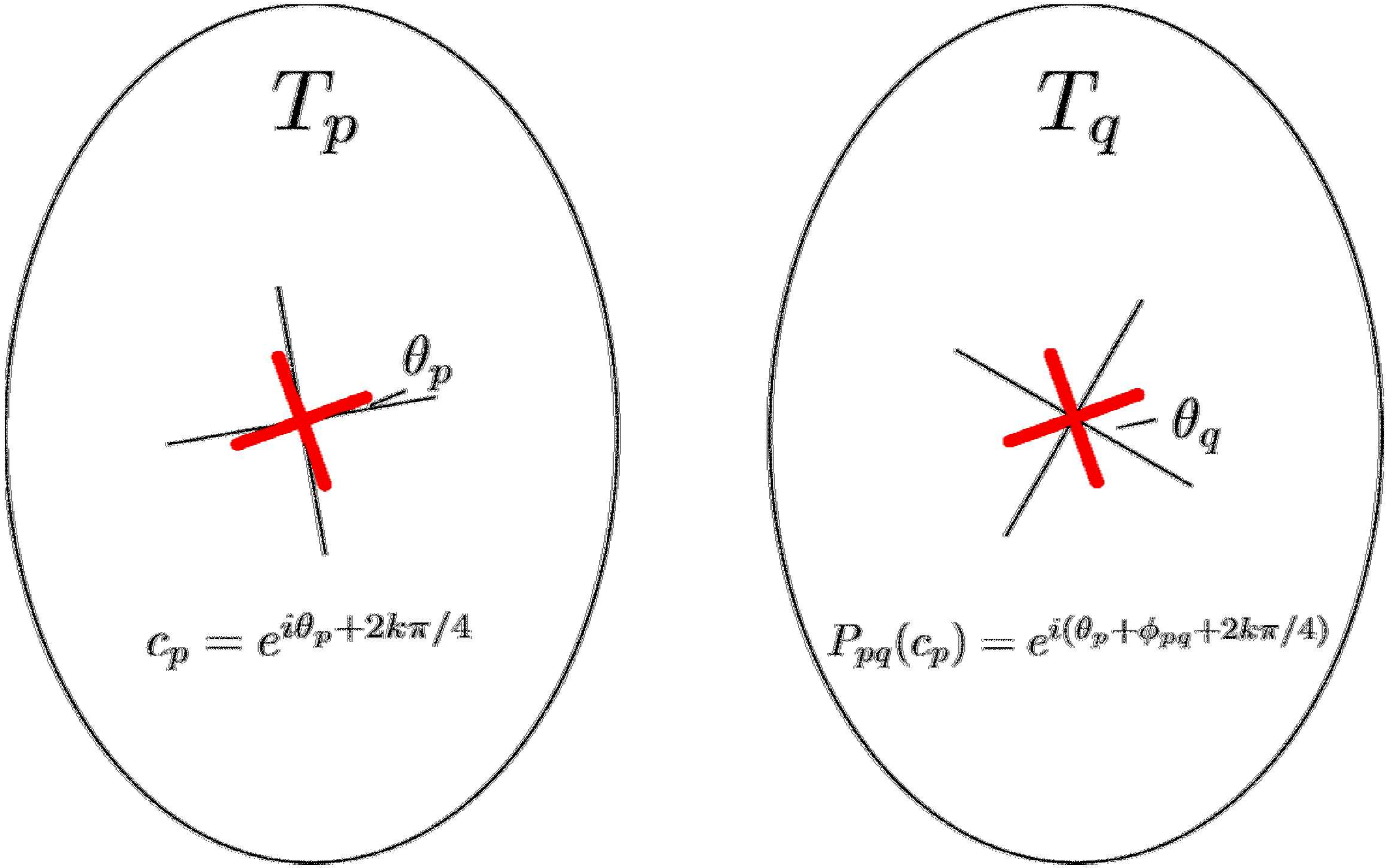
Cross Field Design on Surfaces



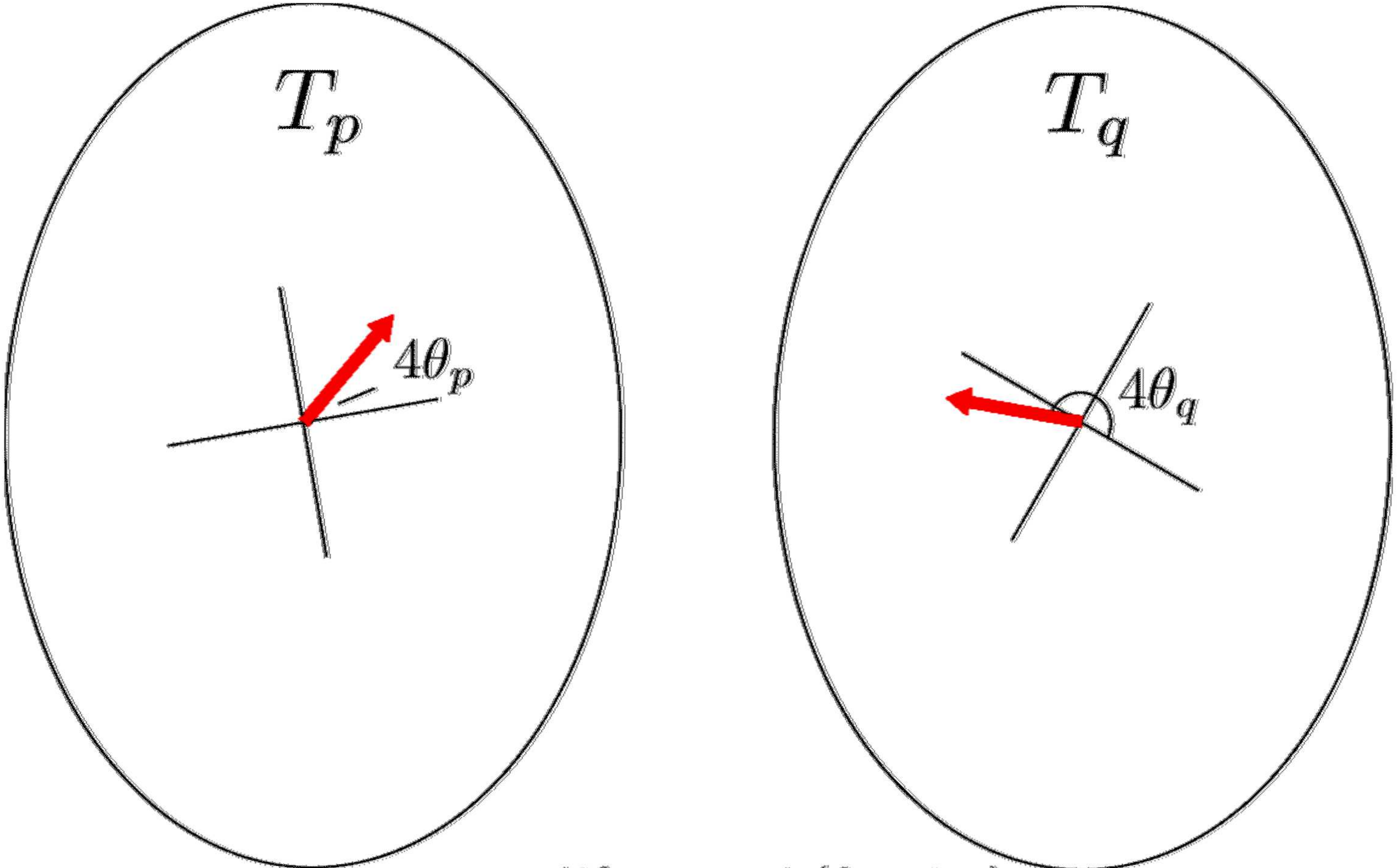
The Representation Map

Palacios et al. 2007

Cross Field Design on Surfaces



Cross Field Design on Surfaces



$$Q_{pq}(e^{4i\theta_p}) = e^{4i(\theta_p + \phi_{pq})}$$
$$\implies Q_{pq}(v) = e^{4i\phi_{pq}} v$$

Discrete Laplacian for Representation Vectors

$$v \in T_q \qquad\qquad u \in T_p$$

$$|v - Q_{pq}u|^2$$

Discrete Laplacian for Representation Vectors

$$v \in T_q \qquad\qquad u \in T_p$$

$$|v - Q_{pq} u|^2$$

$$\Delta_Q(\vec{u})|_i = \frac{1}{|\mathcal{N}(n_i)|} \sum_{n_j \in \mathcal{N}(n_i)} (u_j - Q_{ij}(u_i))$$

Discrete MBO on Surfaces

Algorithm 2 A diffusion generated method for designing smooth cross fields

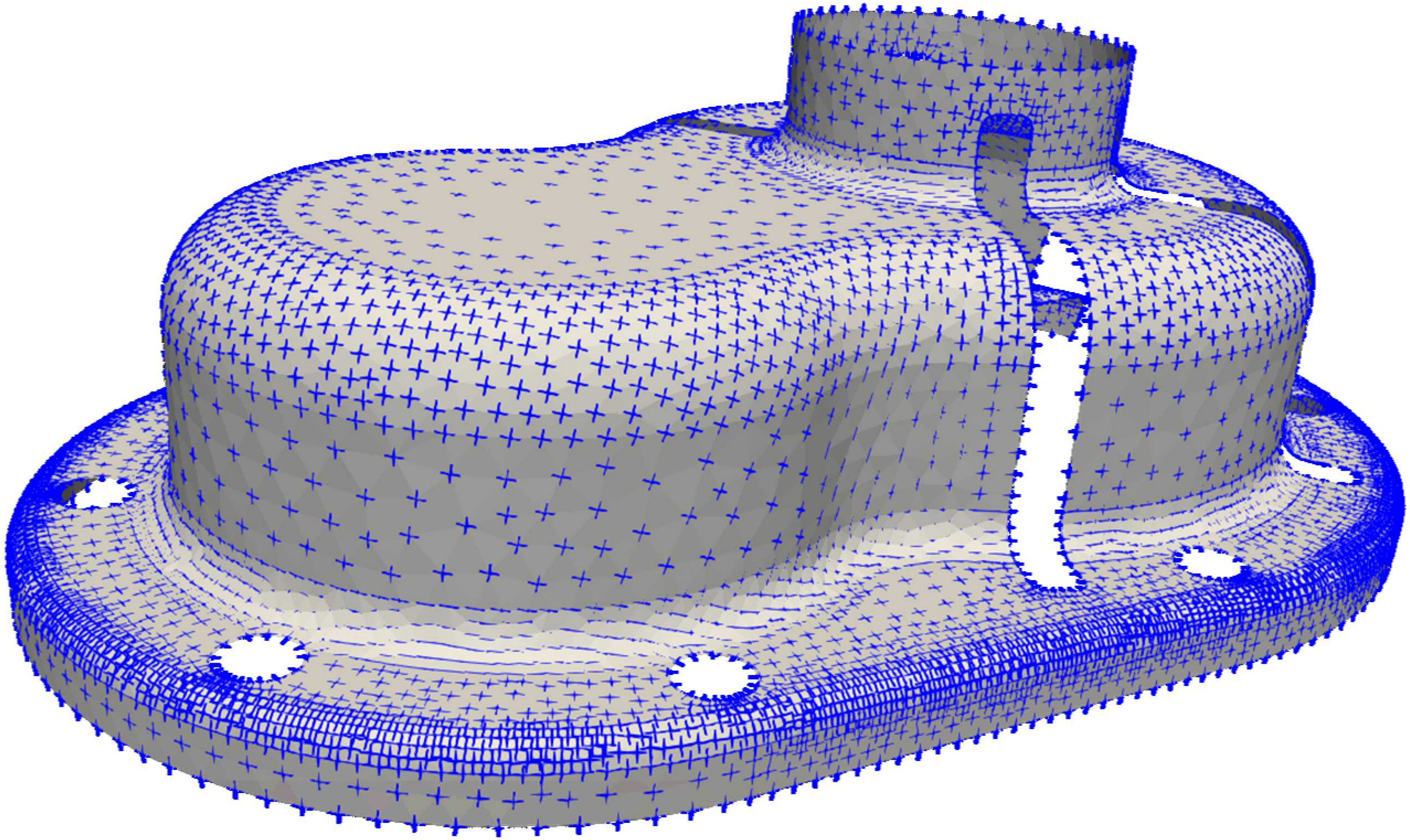
---

Let u^0 be the solution to $Au = b$.
Fix τ, δ, and set $k = 0$.
while $\|u^k - u^{k-1}\| > \delta$, **do**
  Solve the discrete diffusion equation,

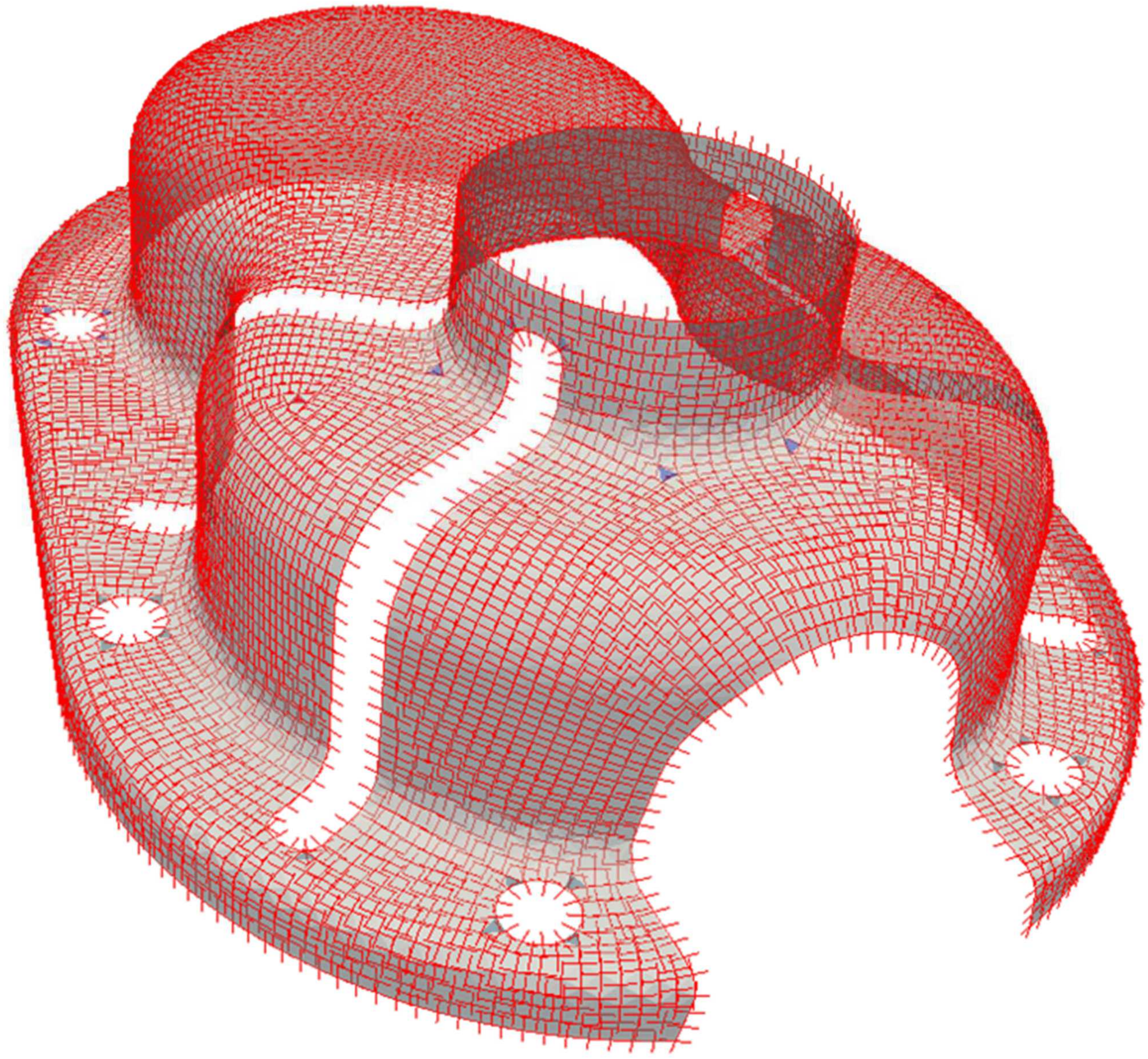
$$(I - \tau A)u^{k+1} = u^k + \tau b \quad (6)$$

  for $j \in [0, n]$ **do**
    Set $u_j^{k+1} = \frac{u_j^{k+1}}{|u_j^{k+1}|}$
  end for
  $k++$
end while

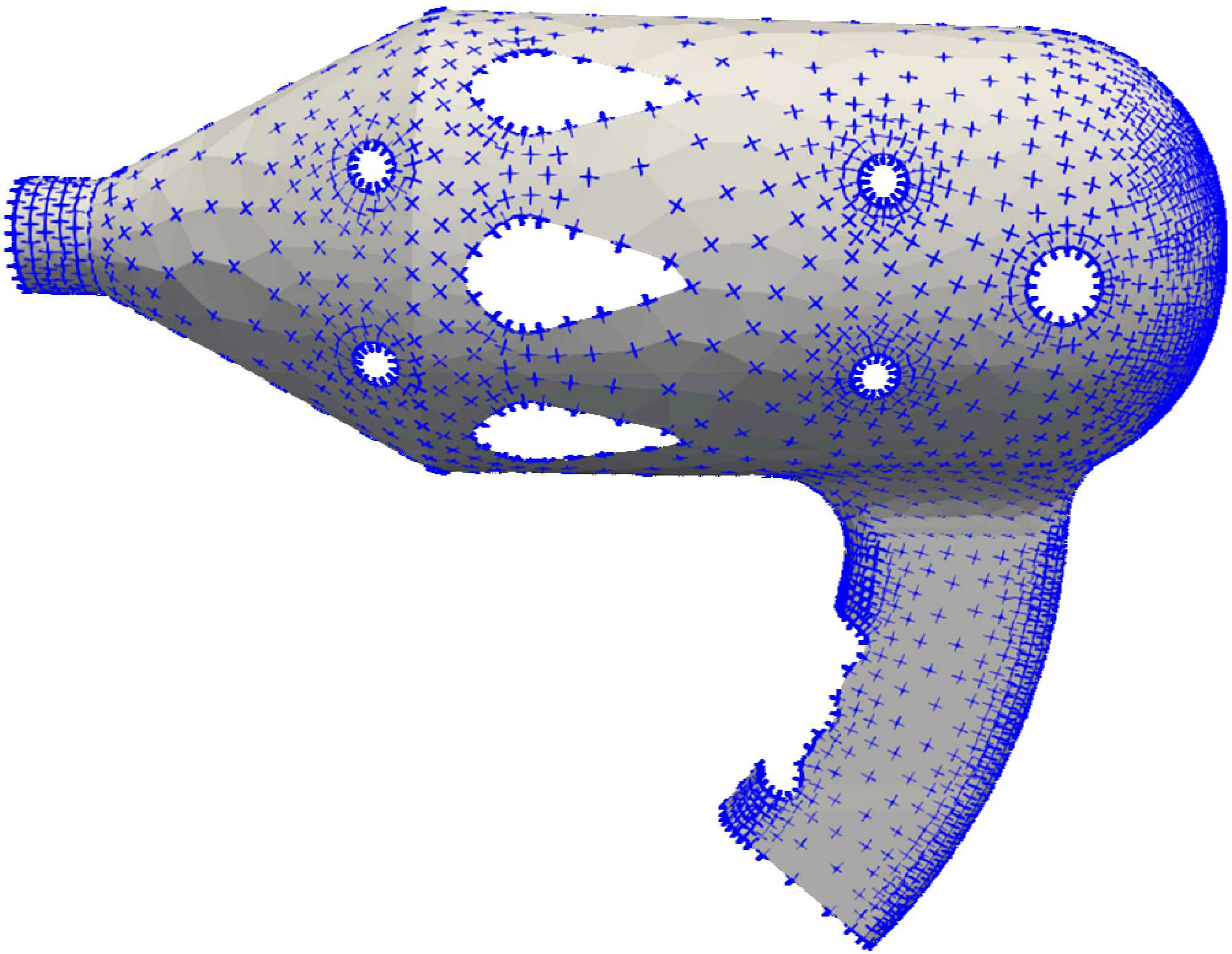
Cross Fields on Surfaces



Curved Surfaces

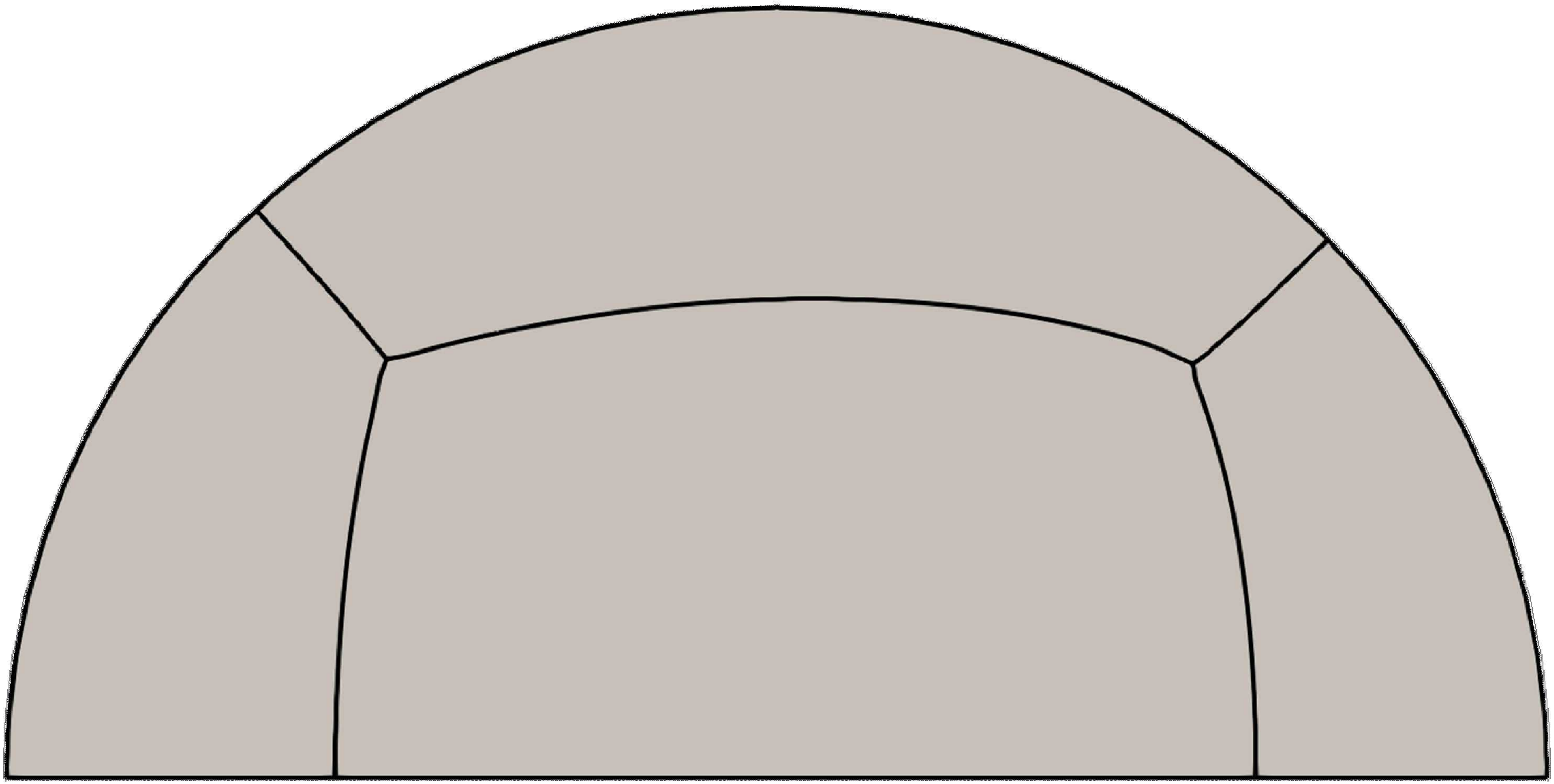


Cross Fields on Surfaces

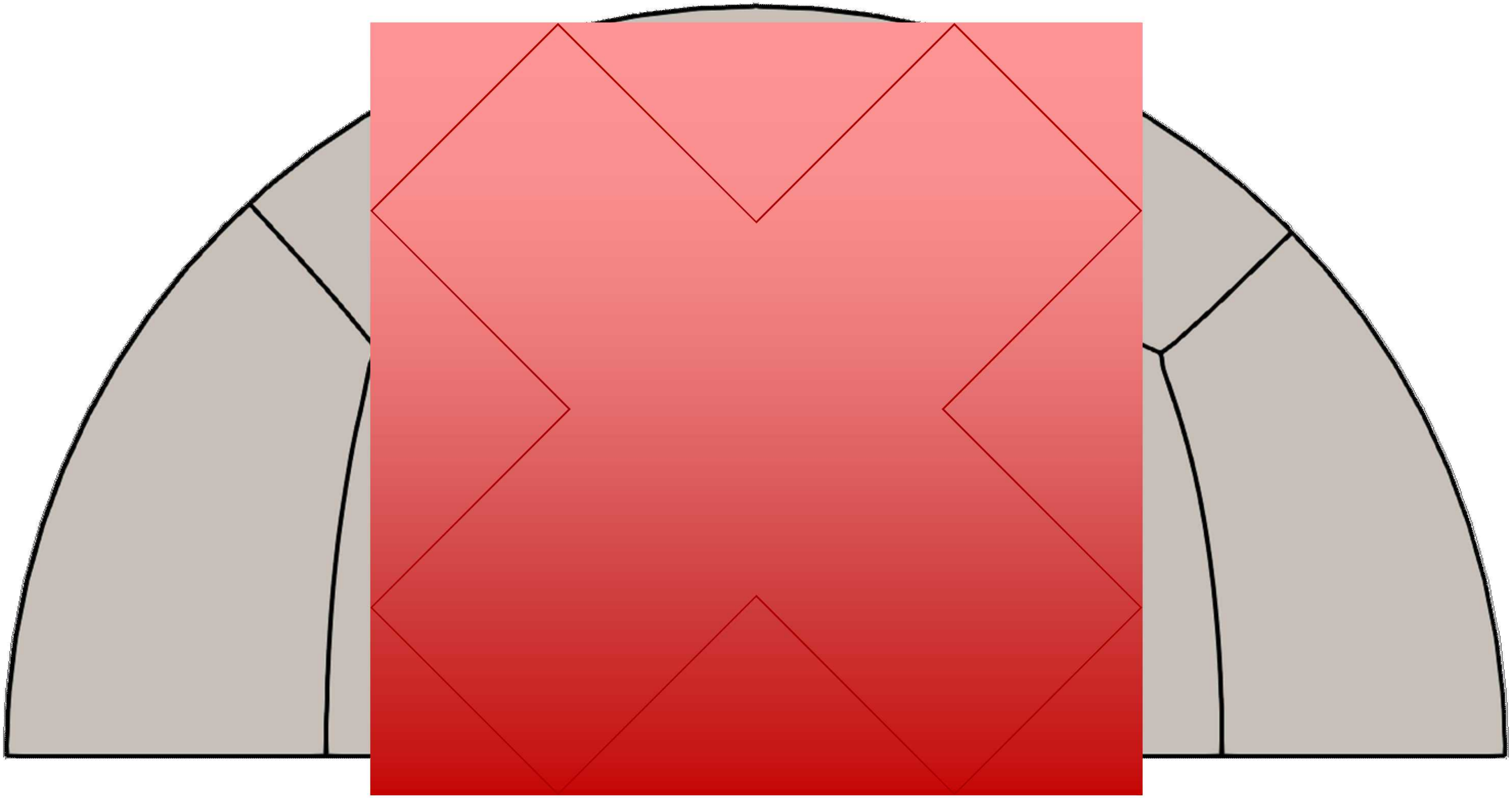


Partition Simplification

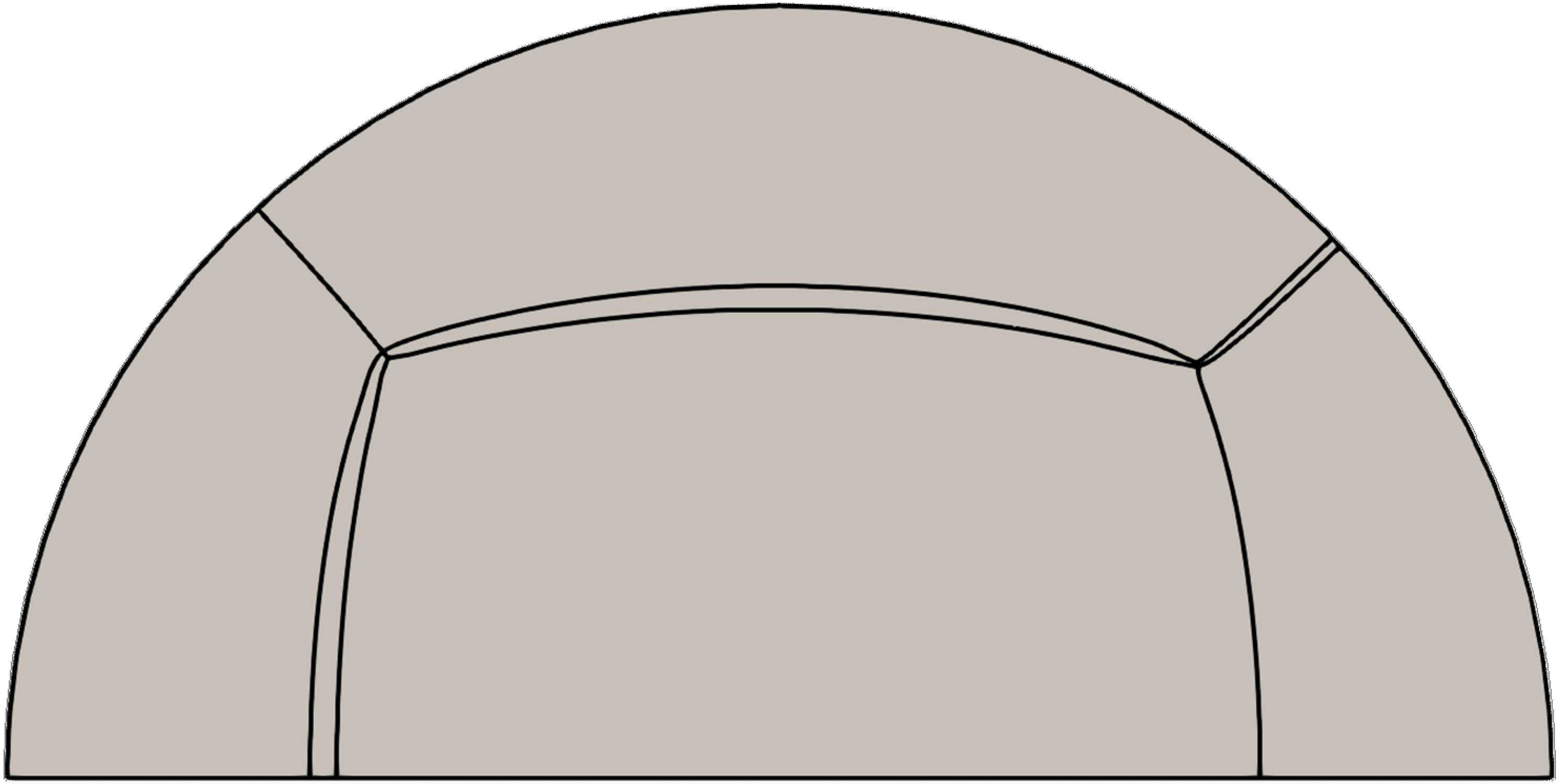
Naive Partition



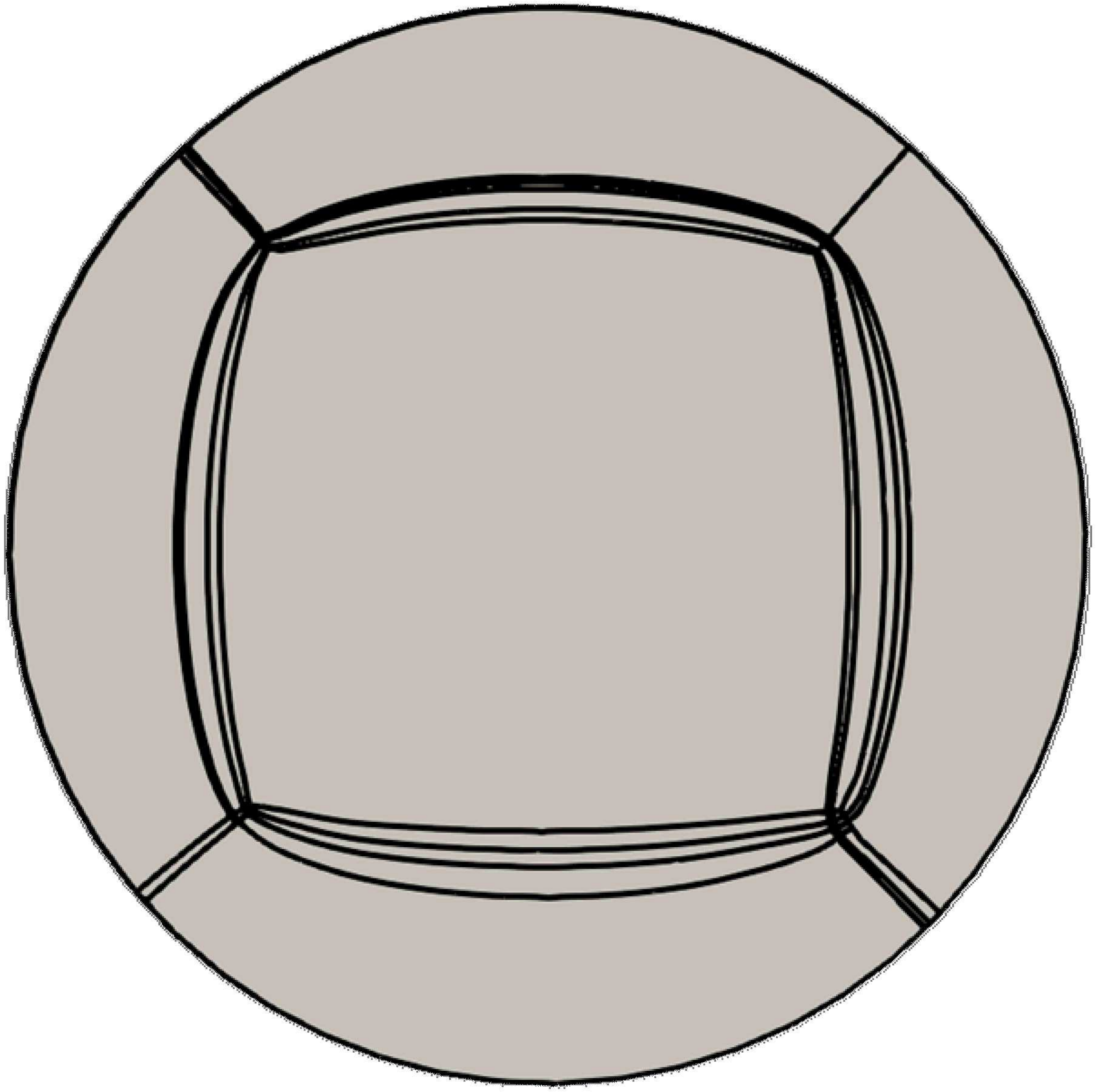
Naive Partition



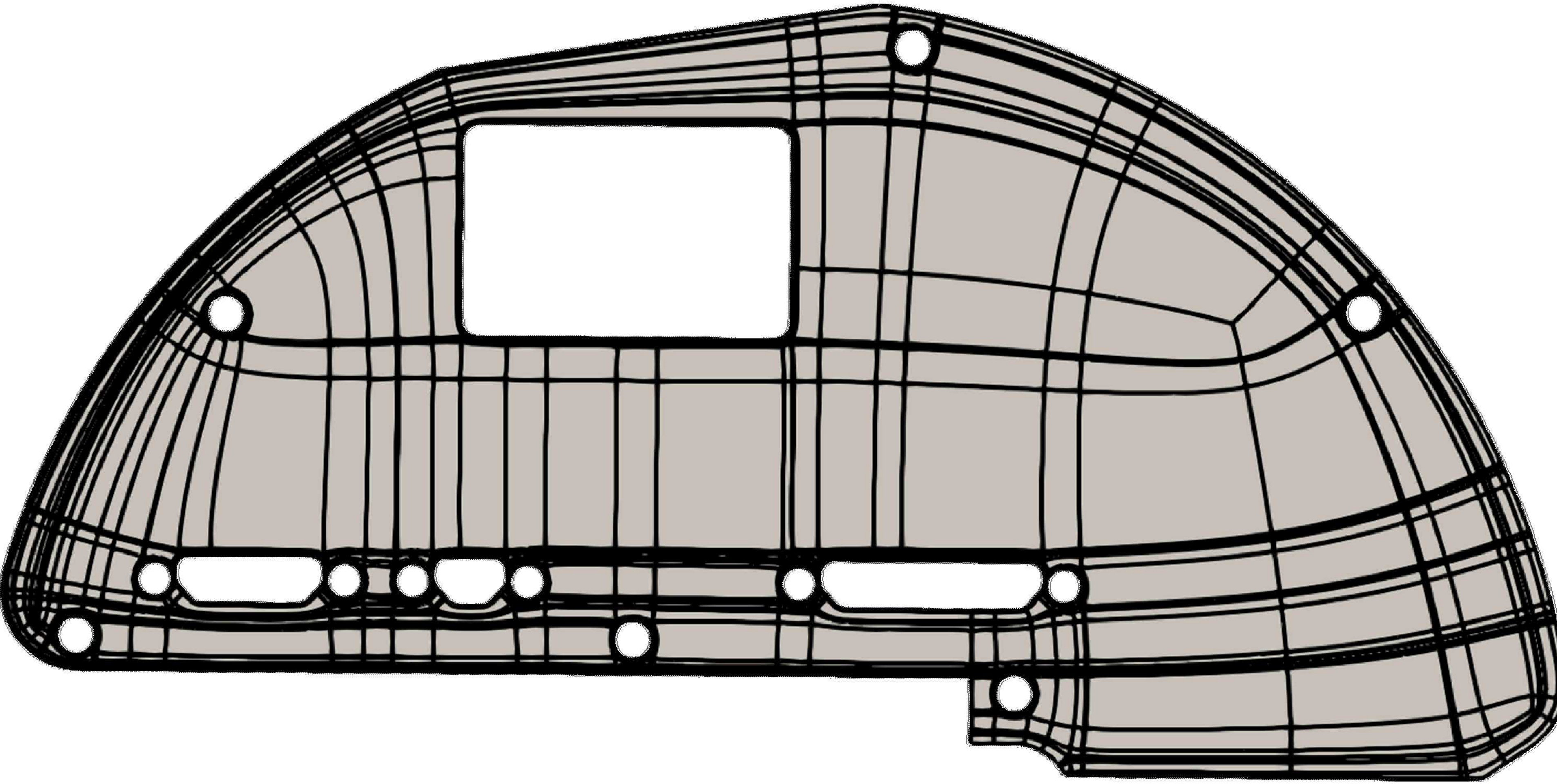
Naive Partition



Limit Cycles Are Common



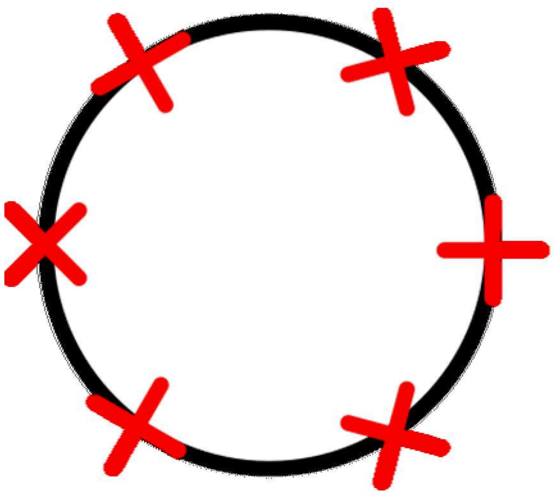
Partition Gets Ugly Quickly

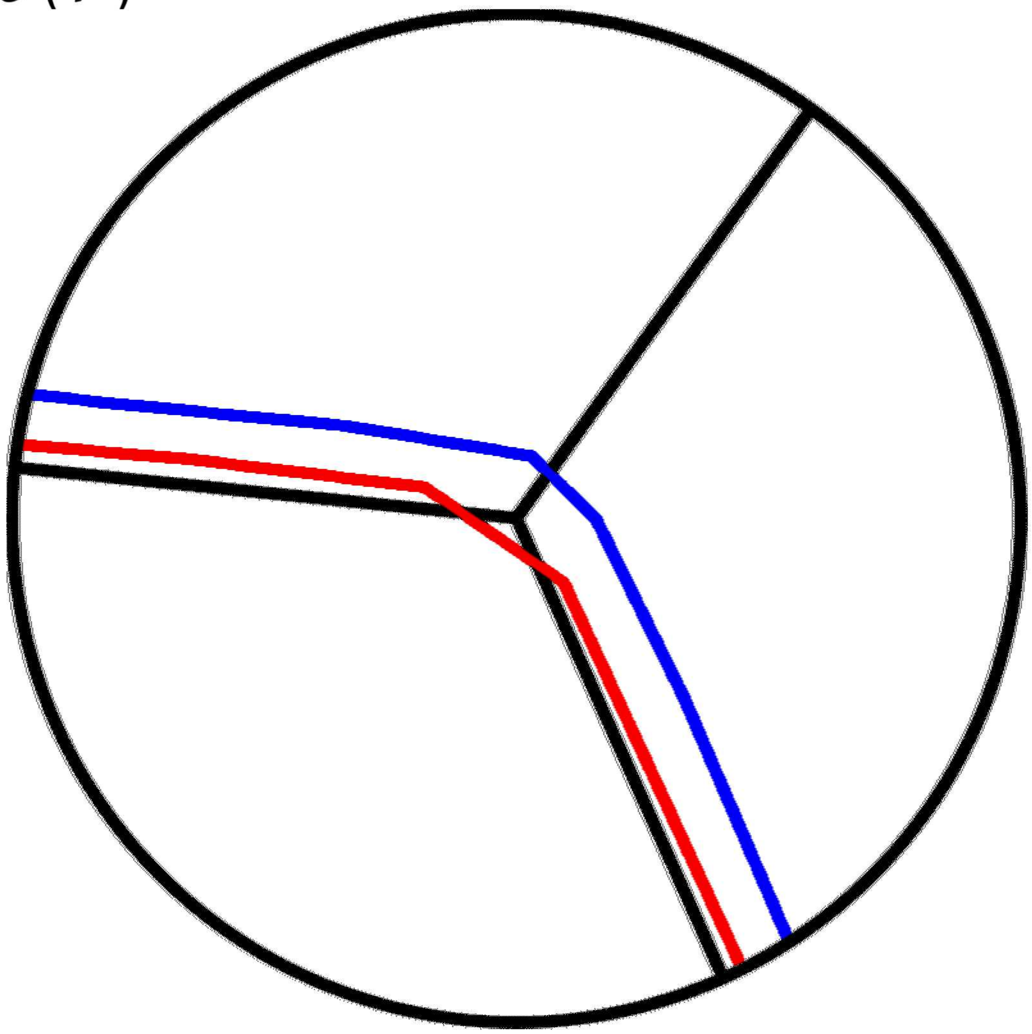


Hyperbolic Trajectory of Streamlines

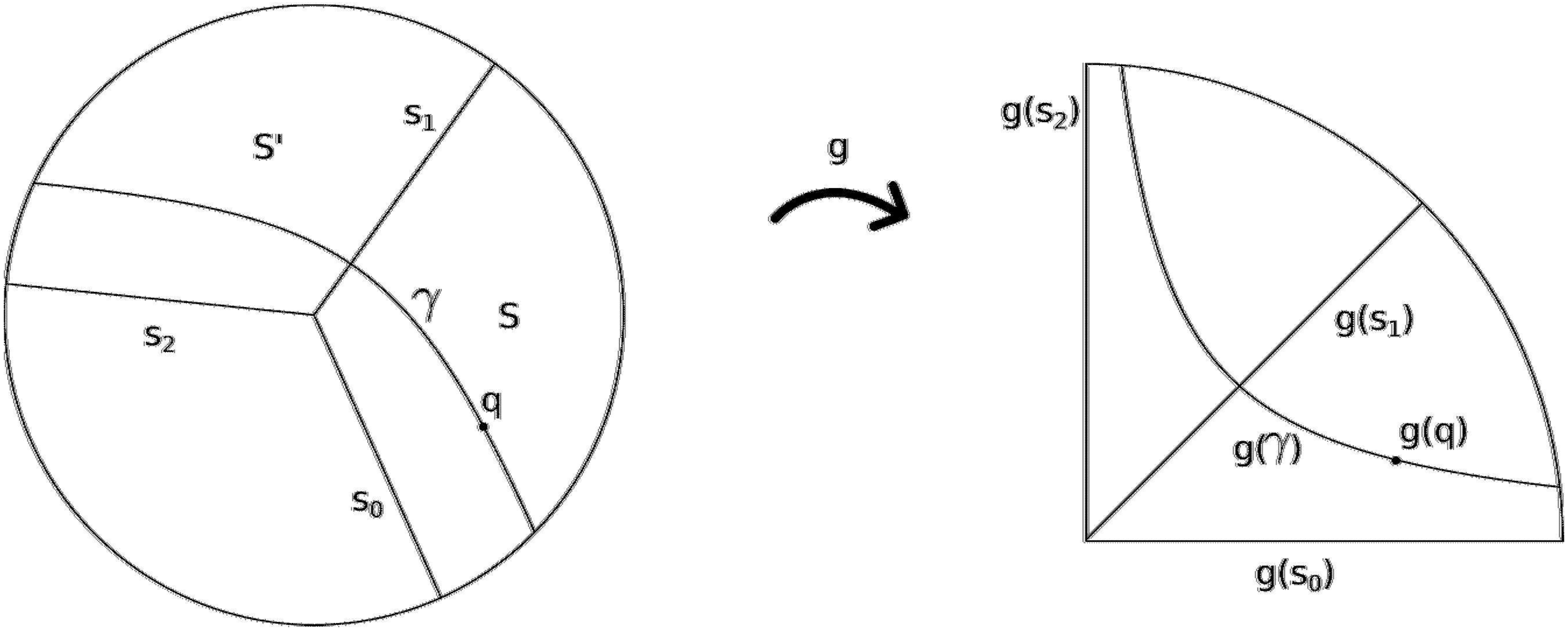
Streamlines Near Singularities

$$f(z) = e^{i\left(\frac{d\theta}{4} + \frac{2k\pi}{4}\right)} + o(r)$$



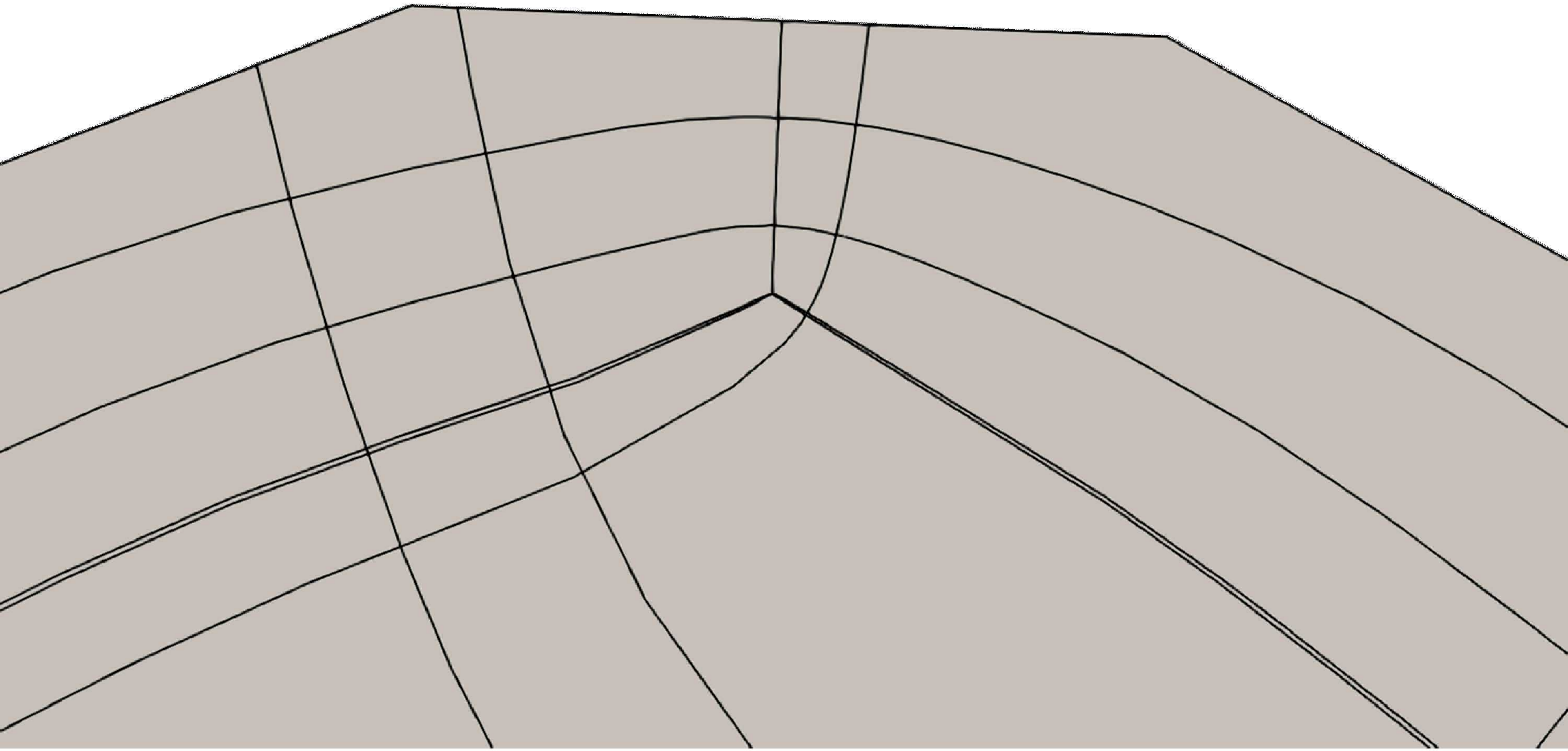


Streamlines Near Singularities



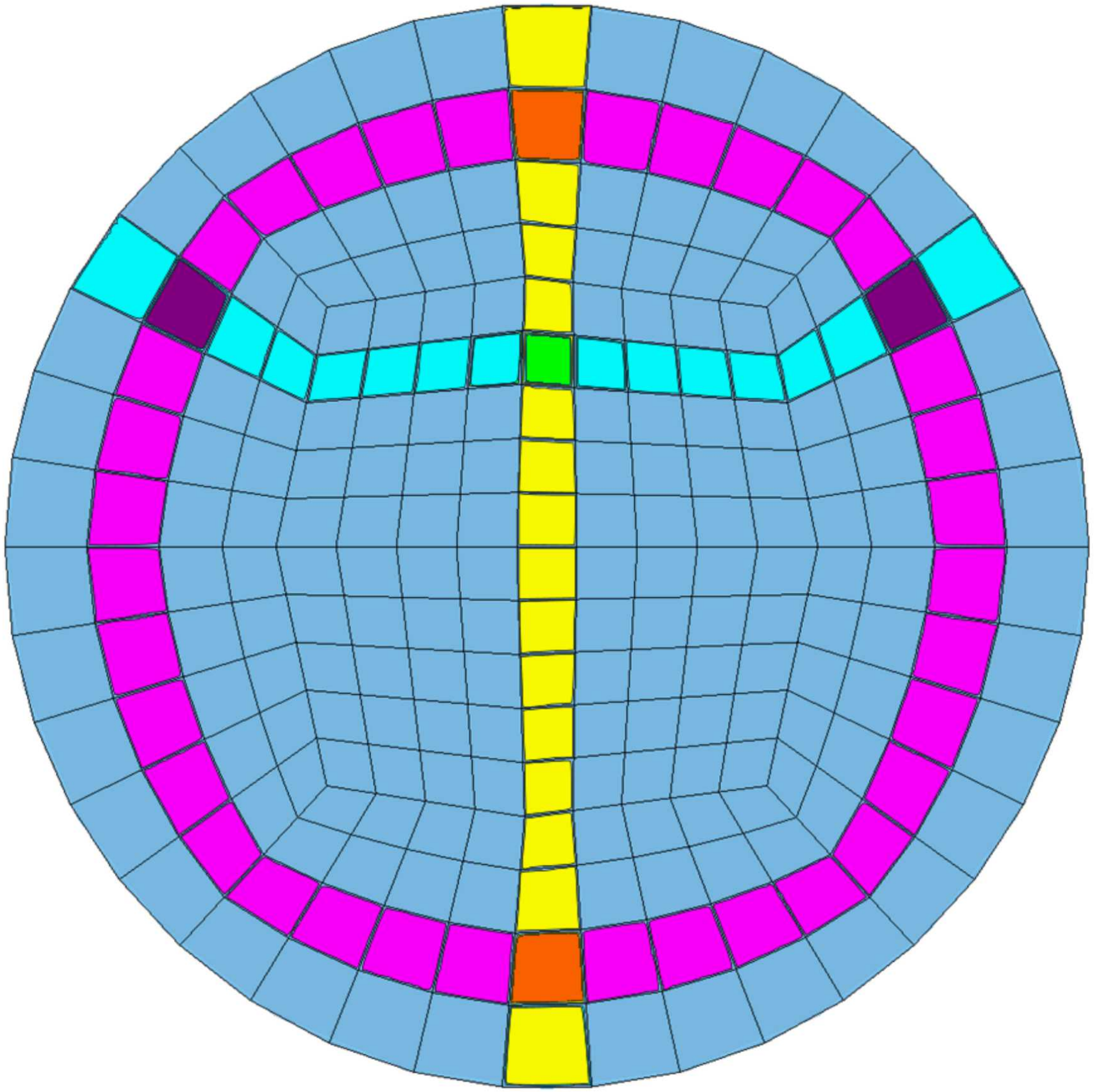
$$g(z) = z^{(4-d)/8}$$

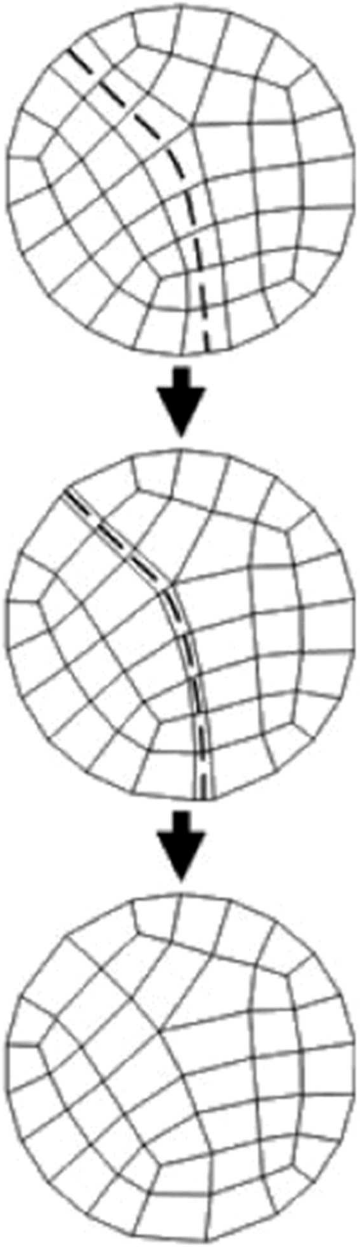
Streamlines Near Singularities



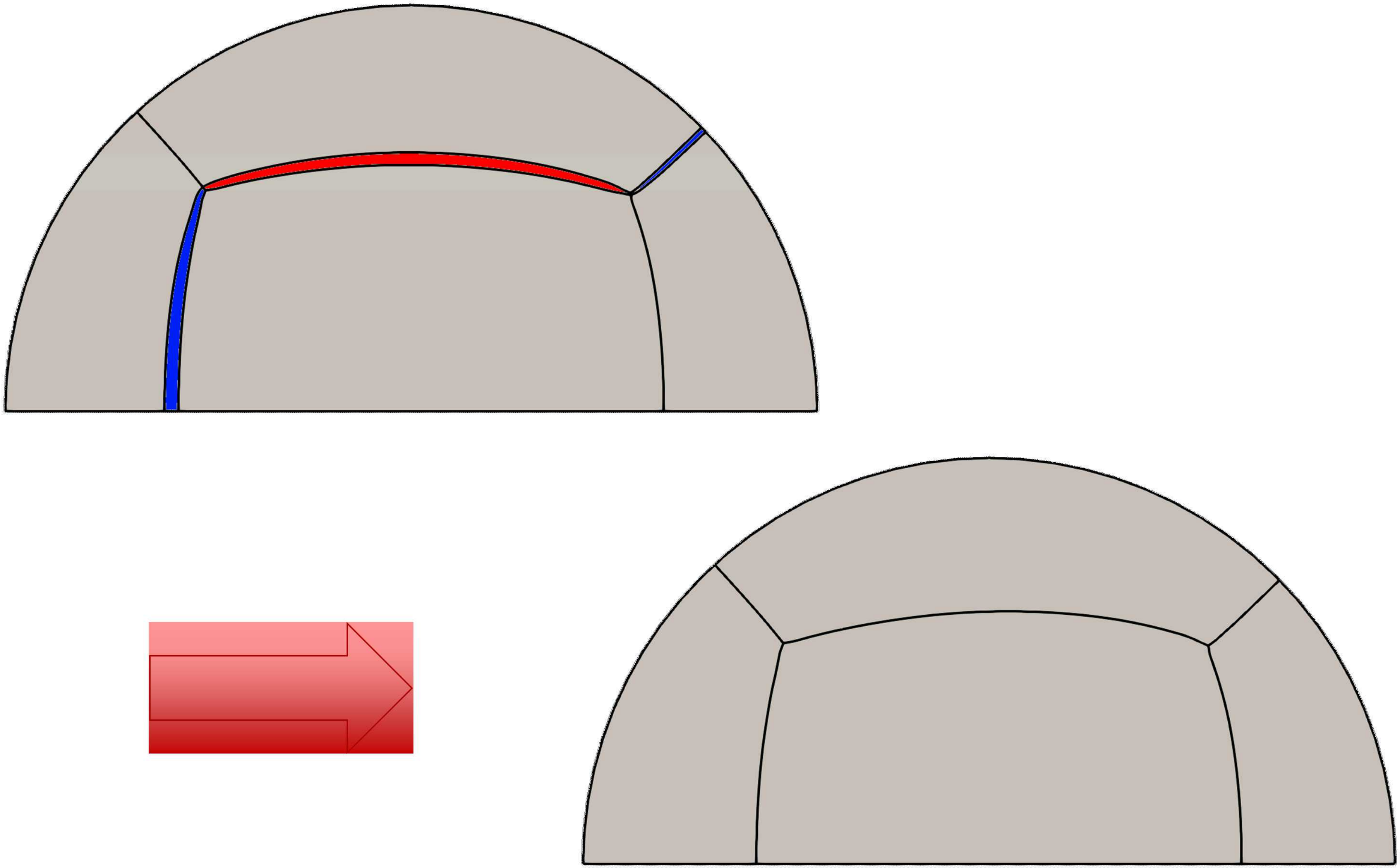
Partition Simplification

Chord Collapse

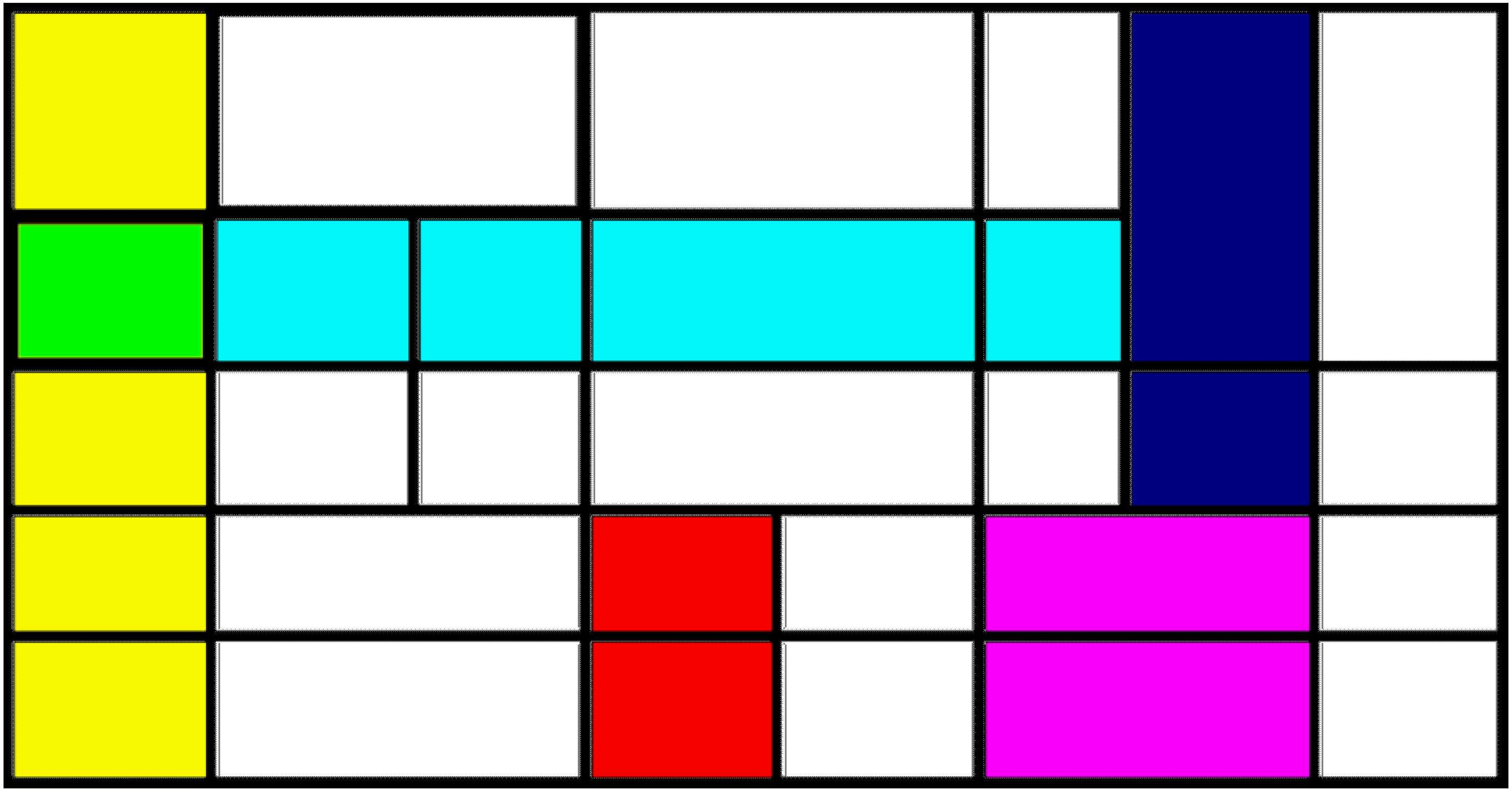




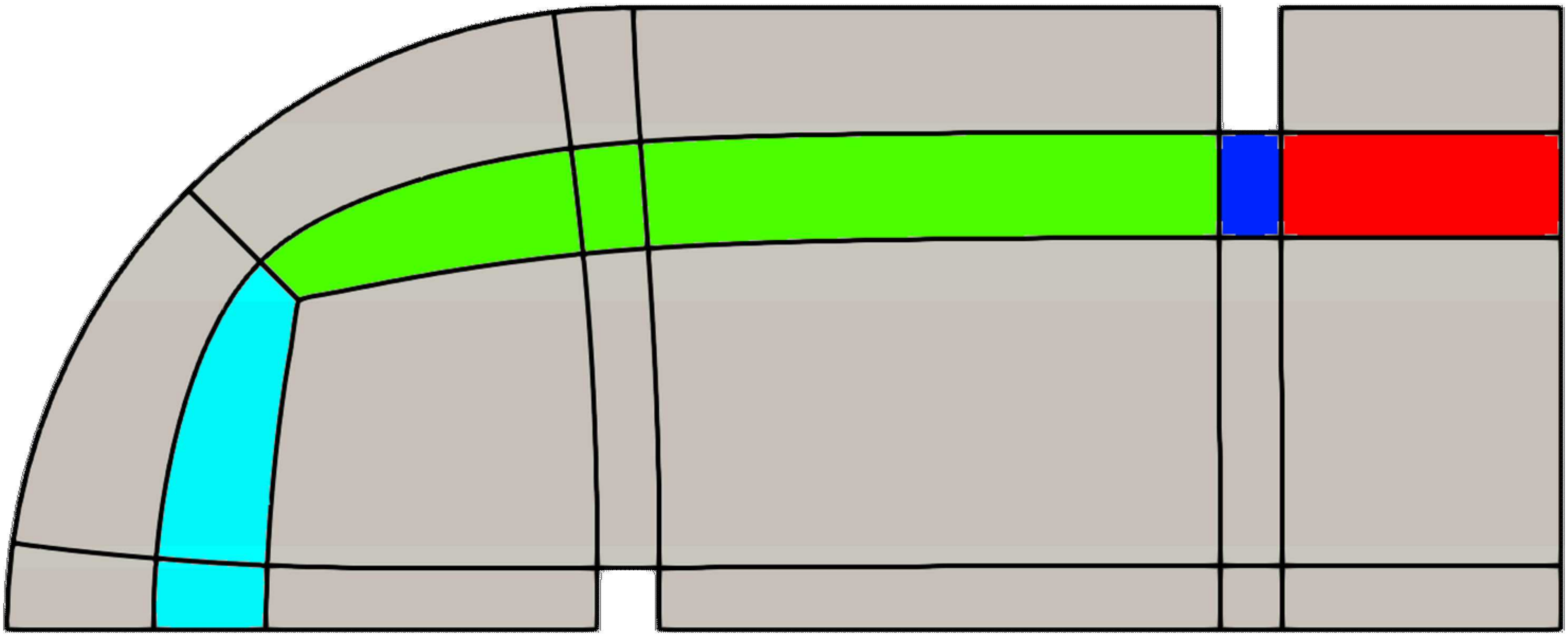
Chord Collapse



Chords of a T-layout

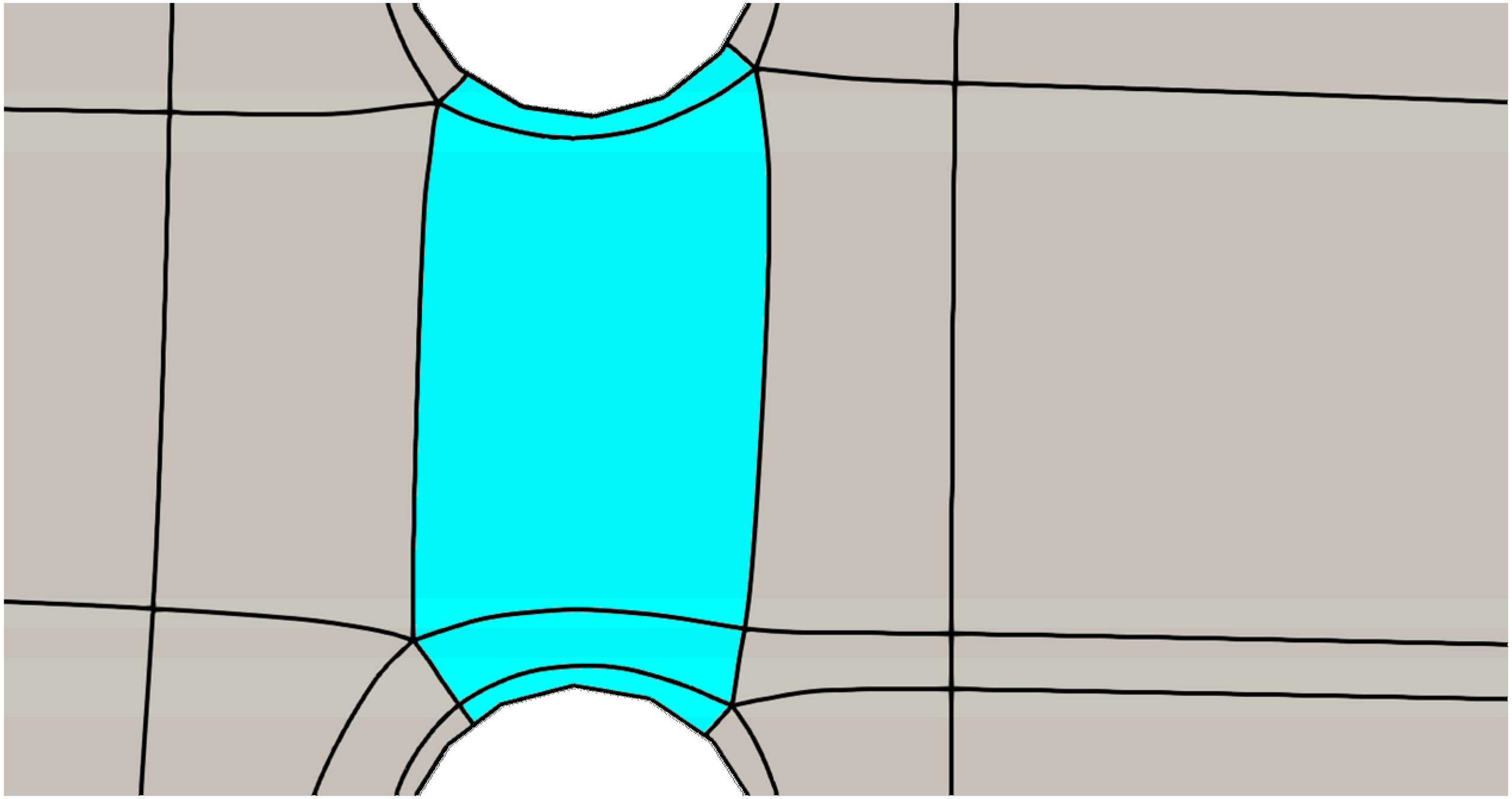


Patches of a Chord



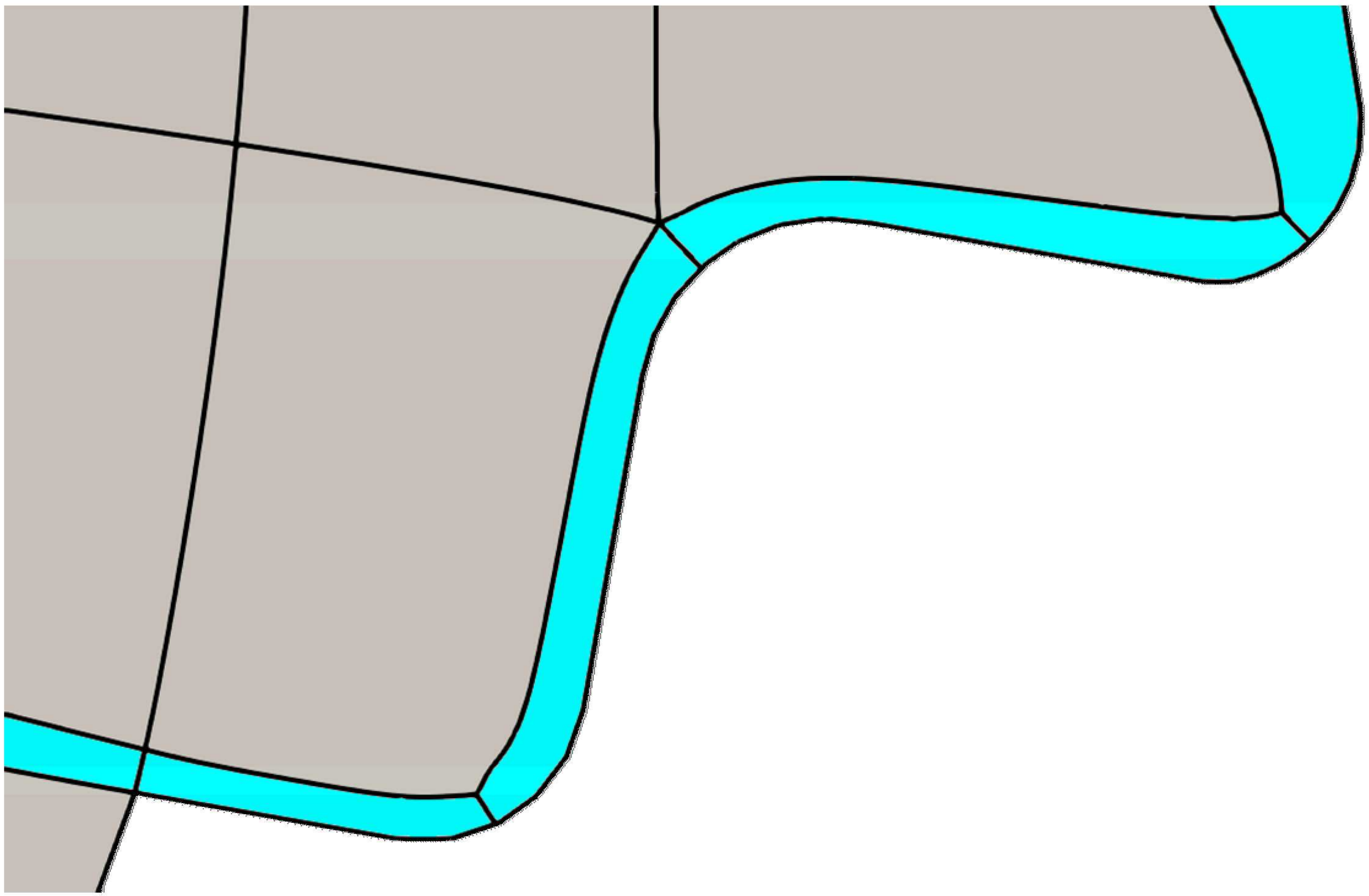
Collapsible Chords

1. No singularities are connected across any transverse rung of the patch.



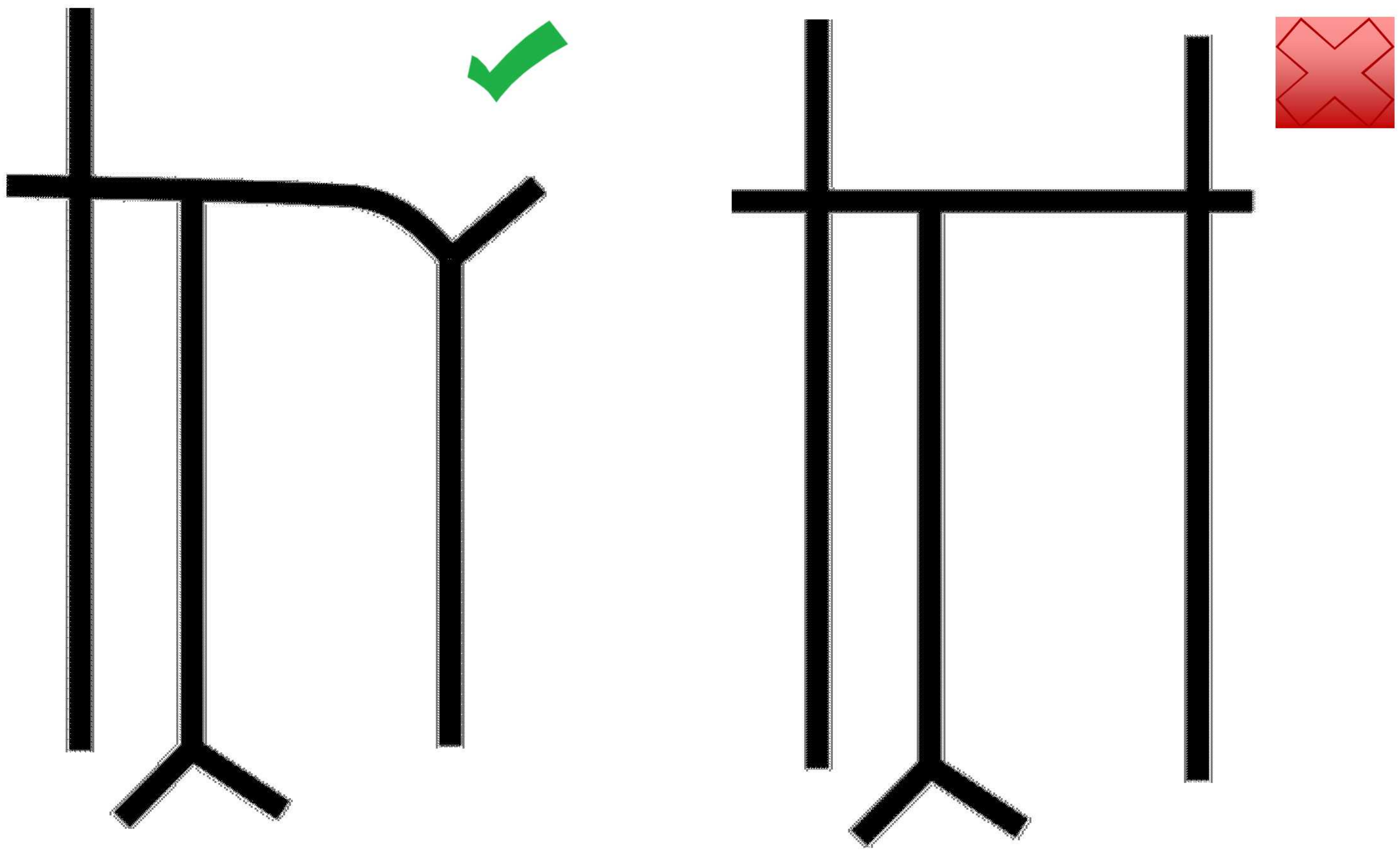
Collapsible Chords

2. No singularity is connected to a boundary across any transverse rung of the patch.



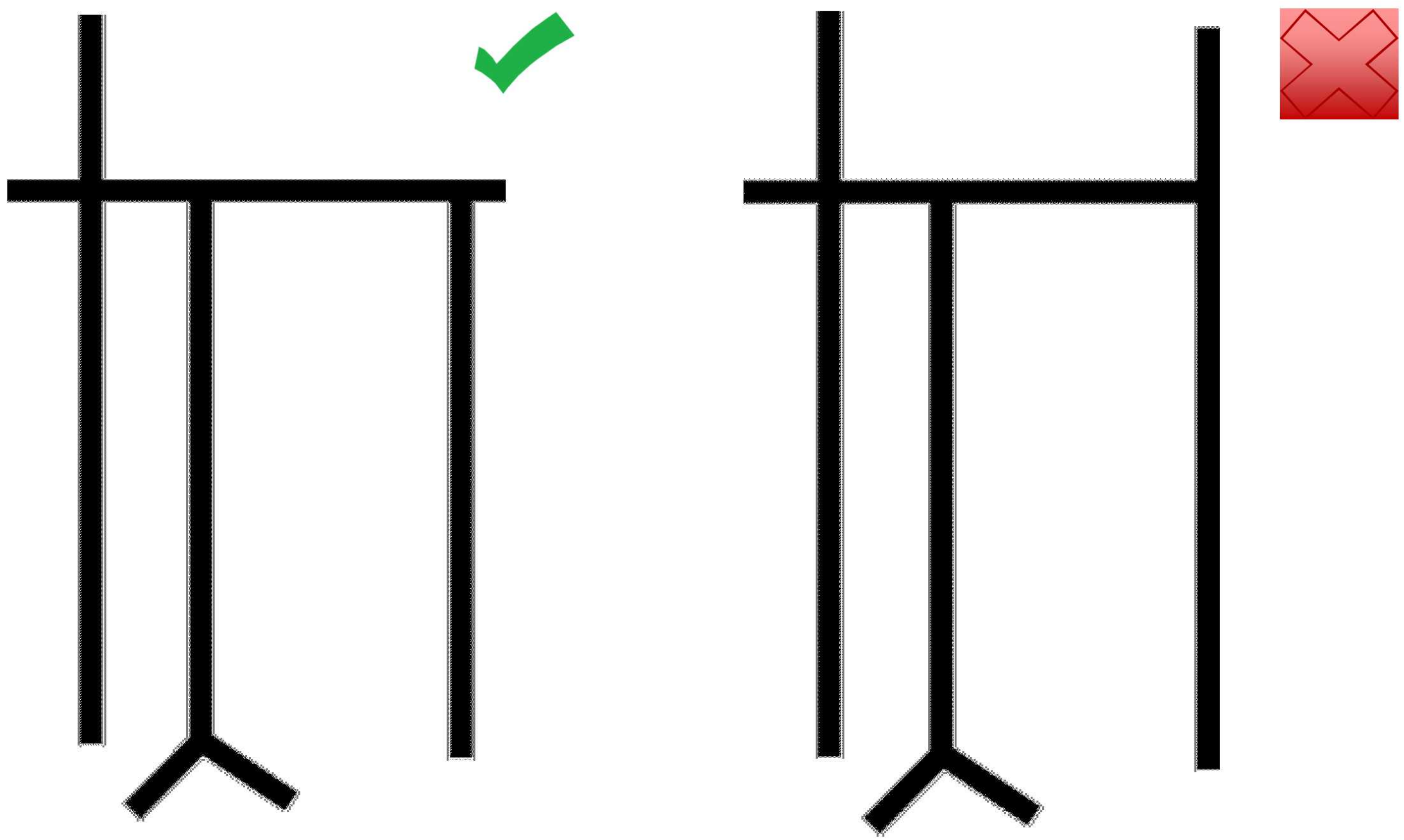
Collapsible Chords

3. If the patch starts or ends at a T-junction:
   (a) The node opposite the T-junction on the same transverse rung is a singularity.



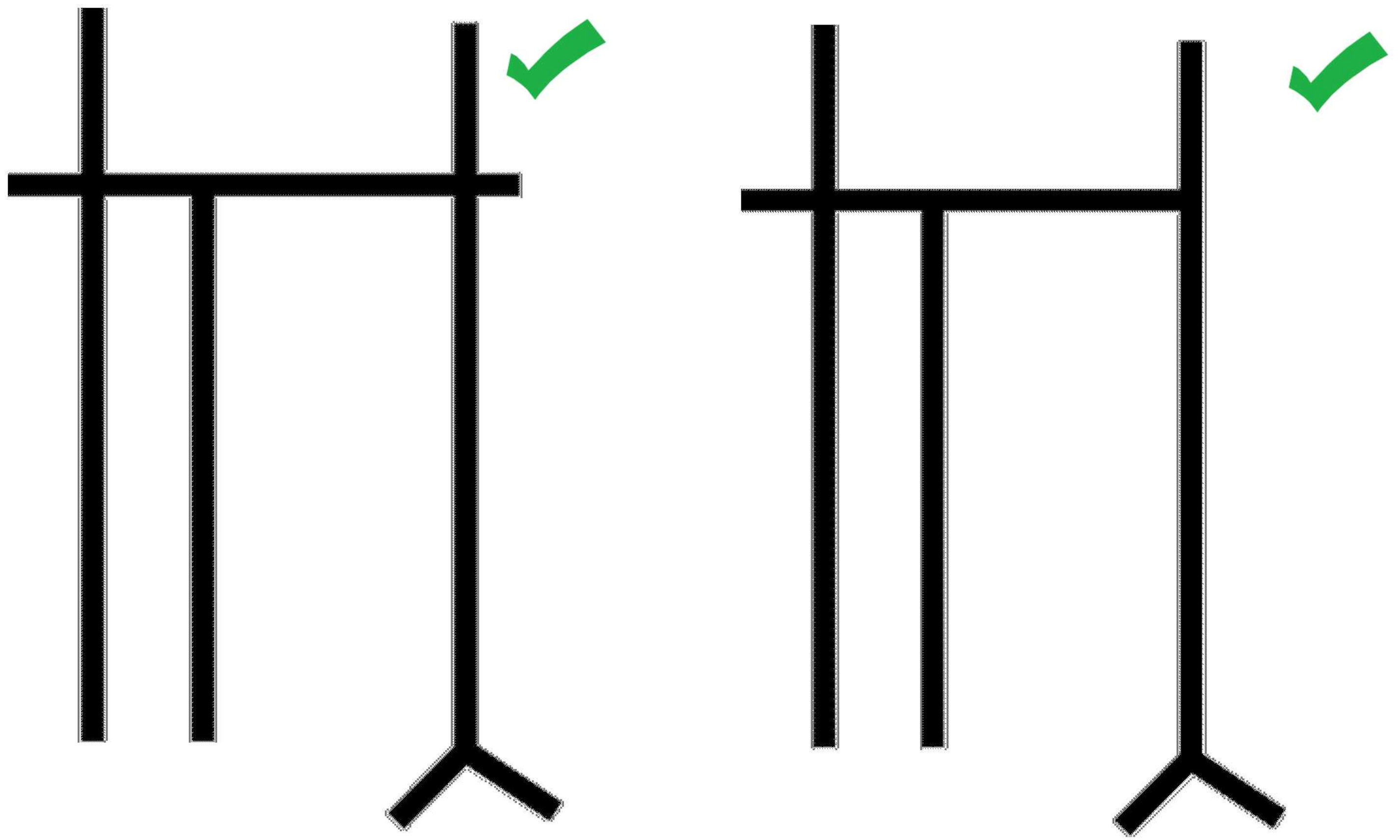
Collapsible Chords

3. If the patch starts or ends at a T-junction:
   (b) The node opposite the T-junction on the same transverse rung is another T-junction with the same orientation.

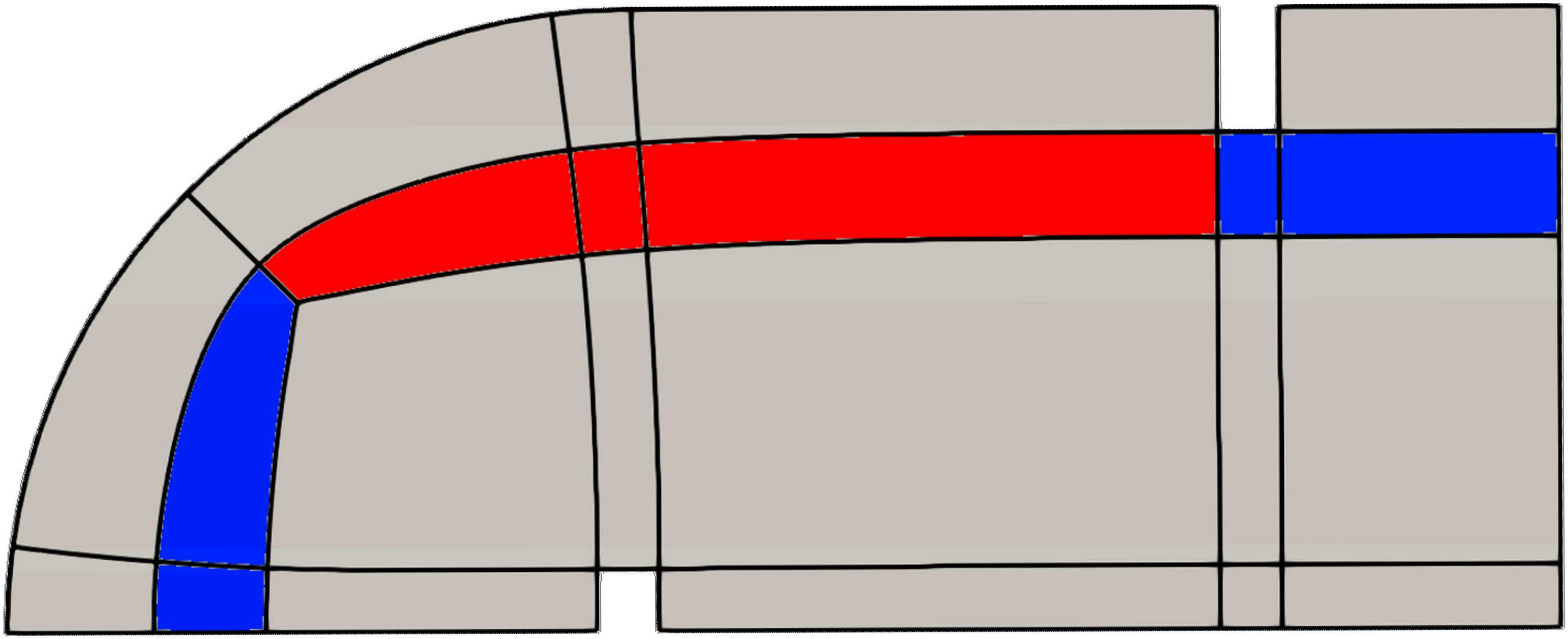


Collapsible Chords

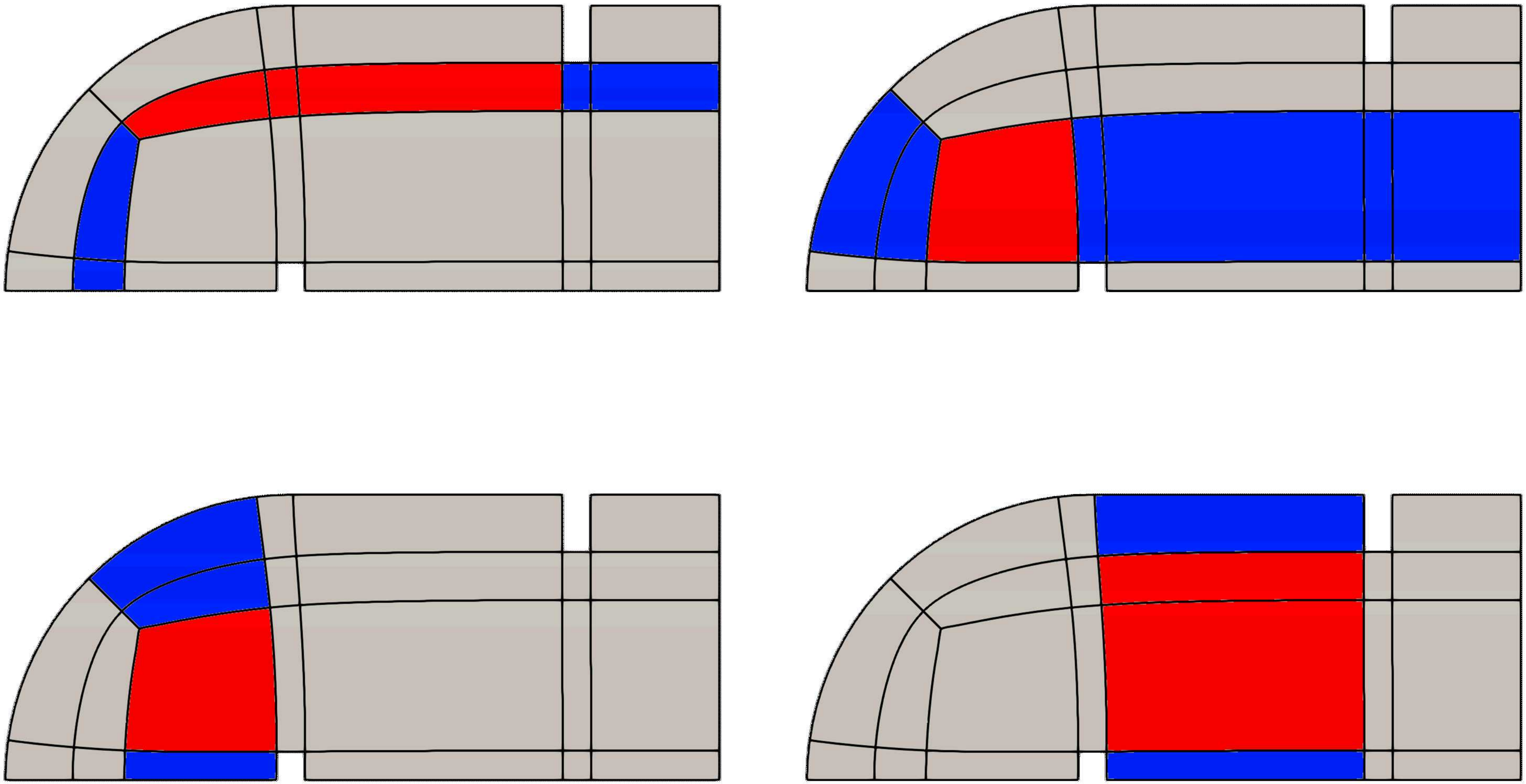
3. If the patch starts or ends at a T-junction:
   (c) The node on the opposite corner of the patch from the T-junction is a singularity.



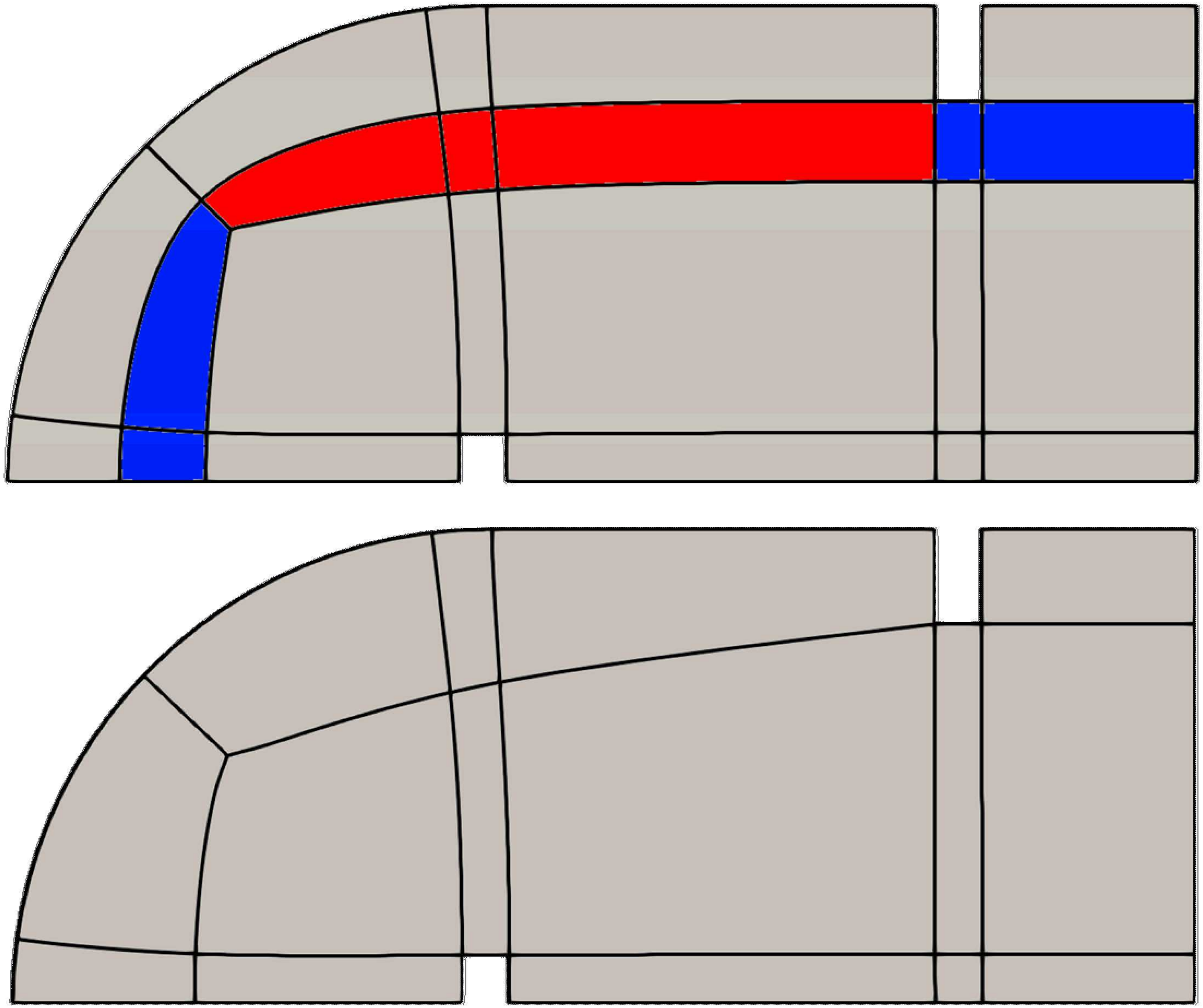
Zip and Non-Zip Patches



Collapsible Chords



Collapse Operation



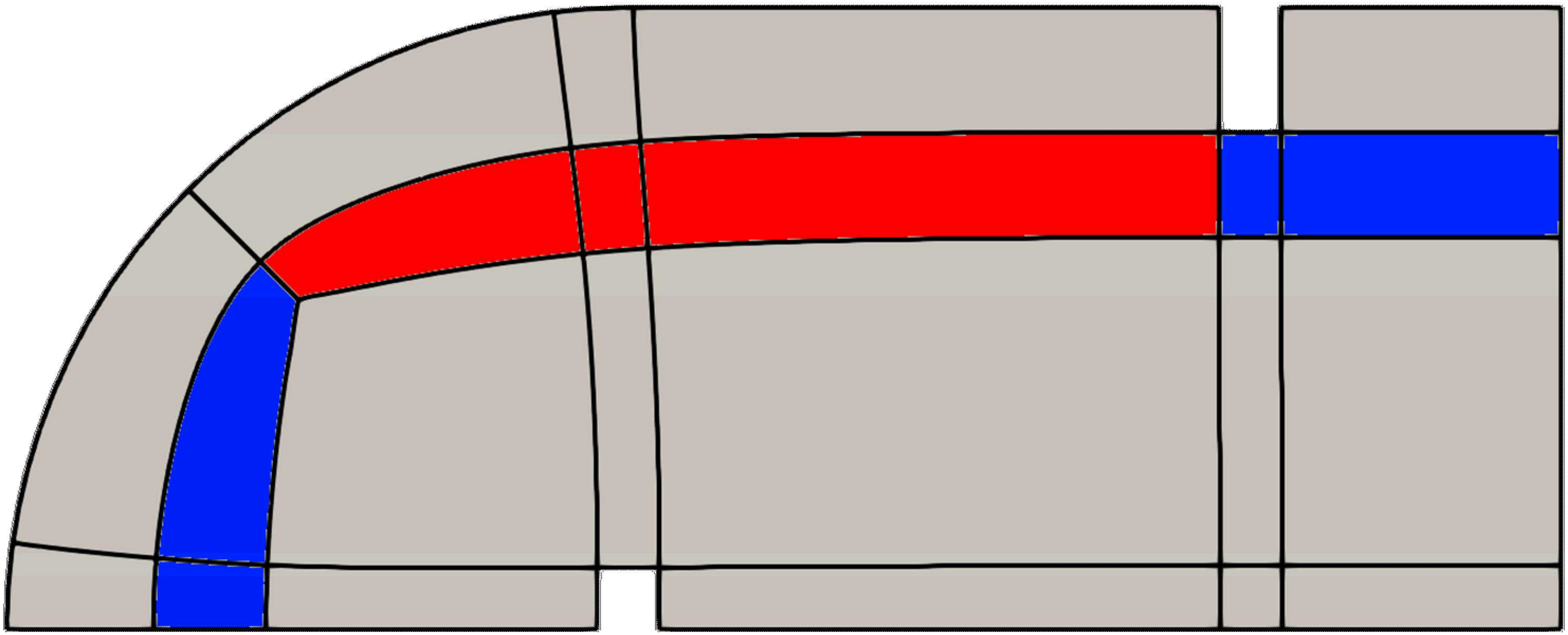
Condition for Collapse

Collapse if: $E > 0$

$$E = \min_{p \in C} E_p$$

$$E_{zip} = \frac{\pi}{8} - \tan^{-1}\frac{w}{l}$$

$$E_{nozip} = 1$$

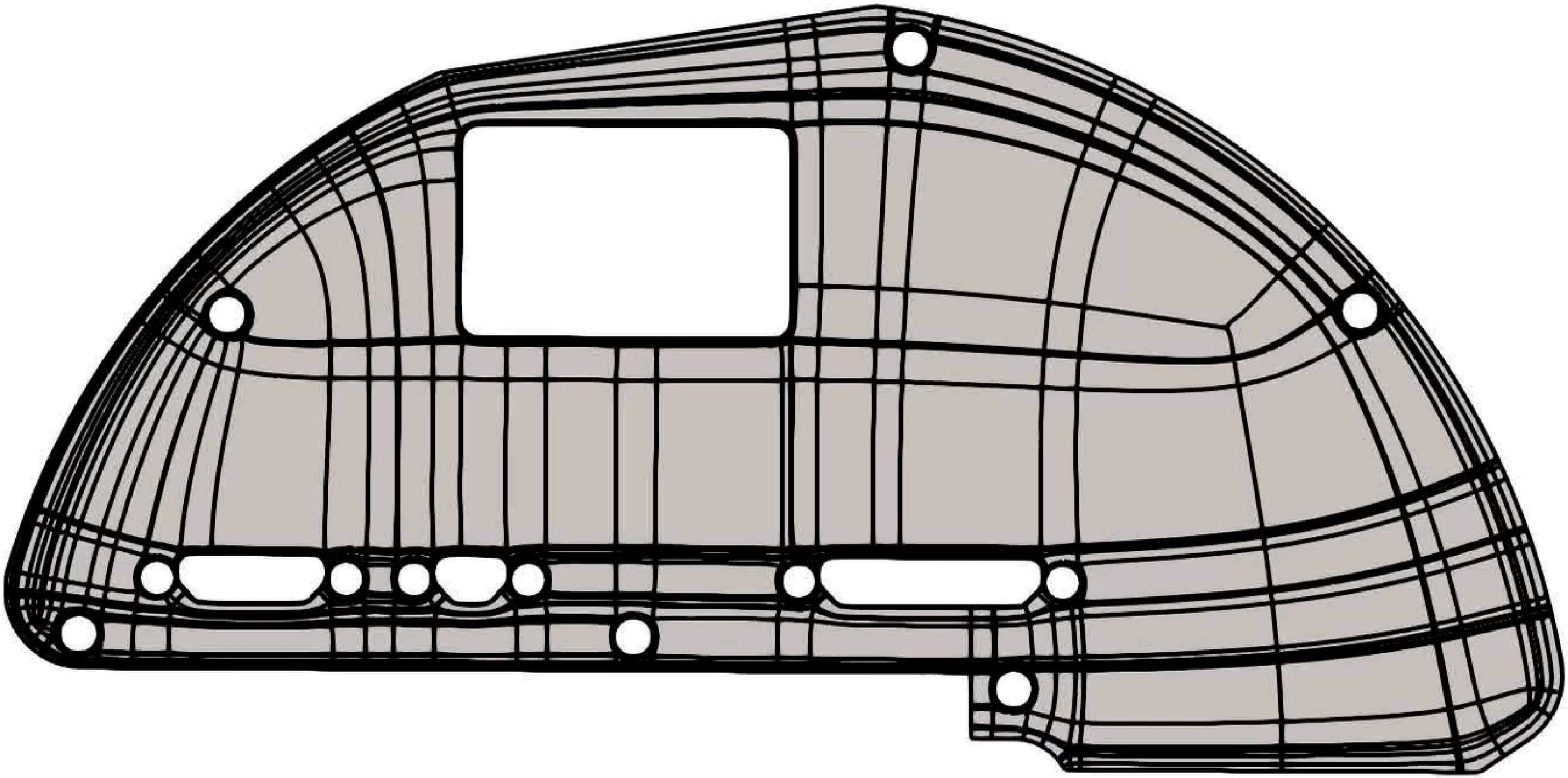


Collapse Algorithm

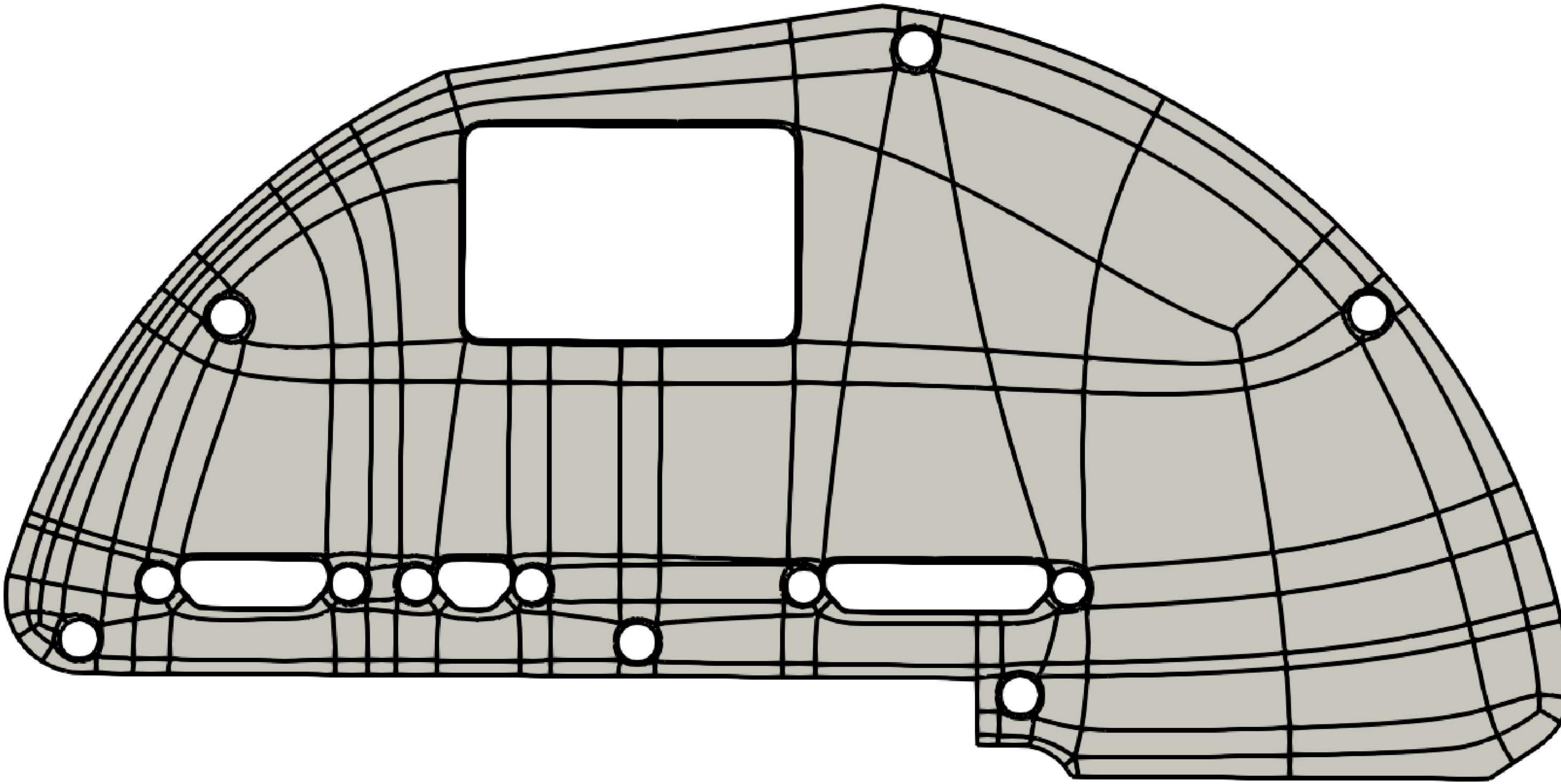
Algorithm 3 Partition Simplification

Let Γ be the set of collapsible chords of the partition
while $|\Gamma| > 0$ **do**
  if No chords meet the conditions for collapse **then**
    Stop.
  else
    Collapse the chord with the smallest minimum width
    Determine new set of collapsible chords Γ
  end if
end while

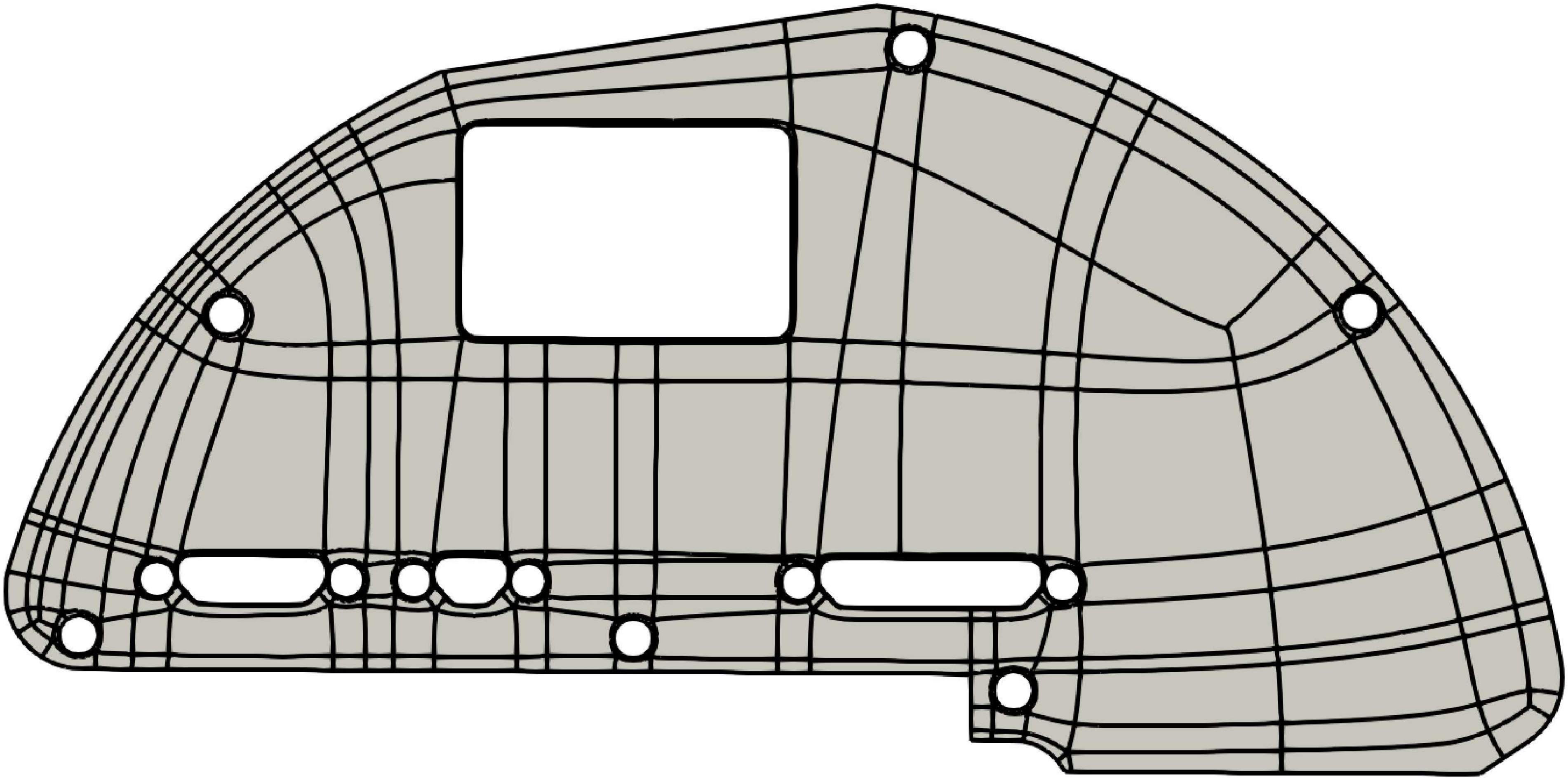
Collapse Algorithm



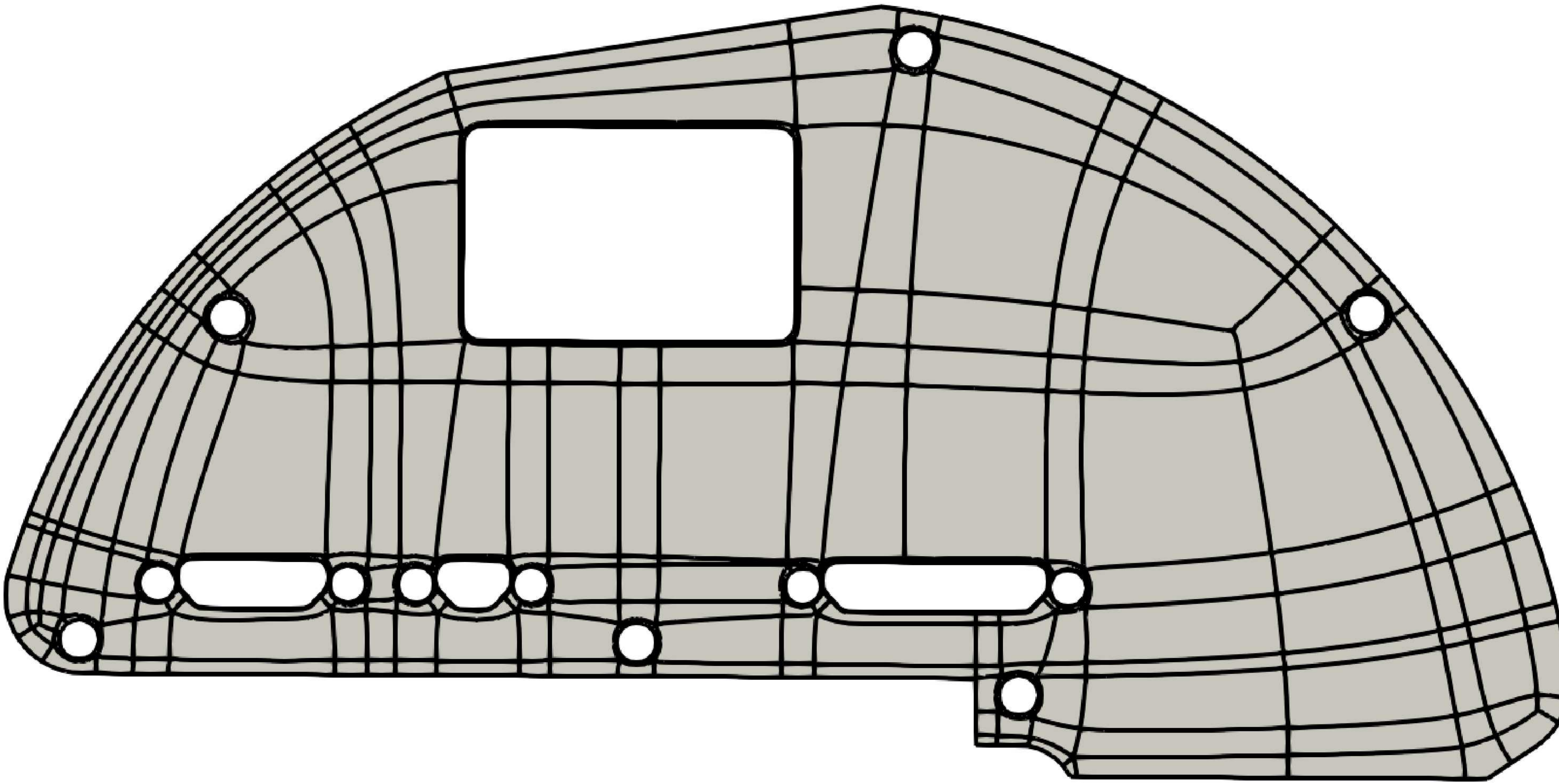
Collapse Algorithm



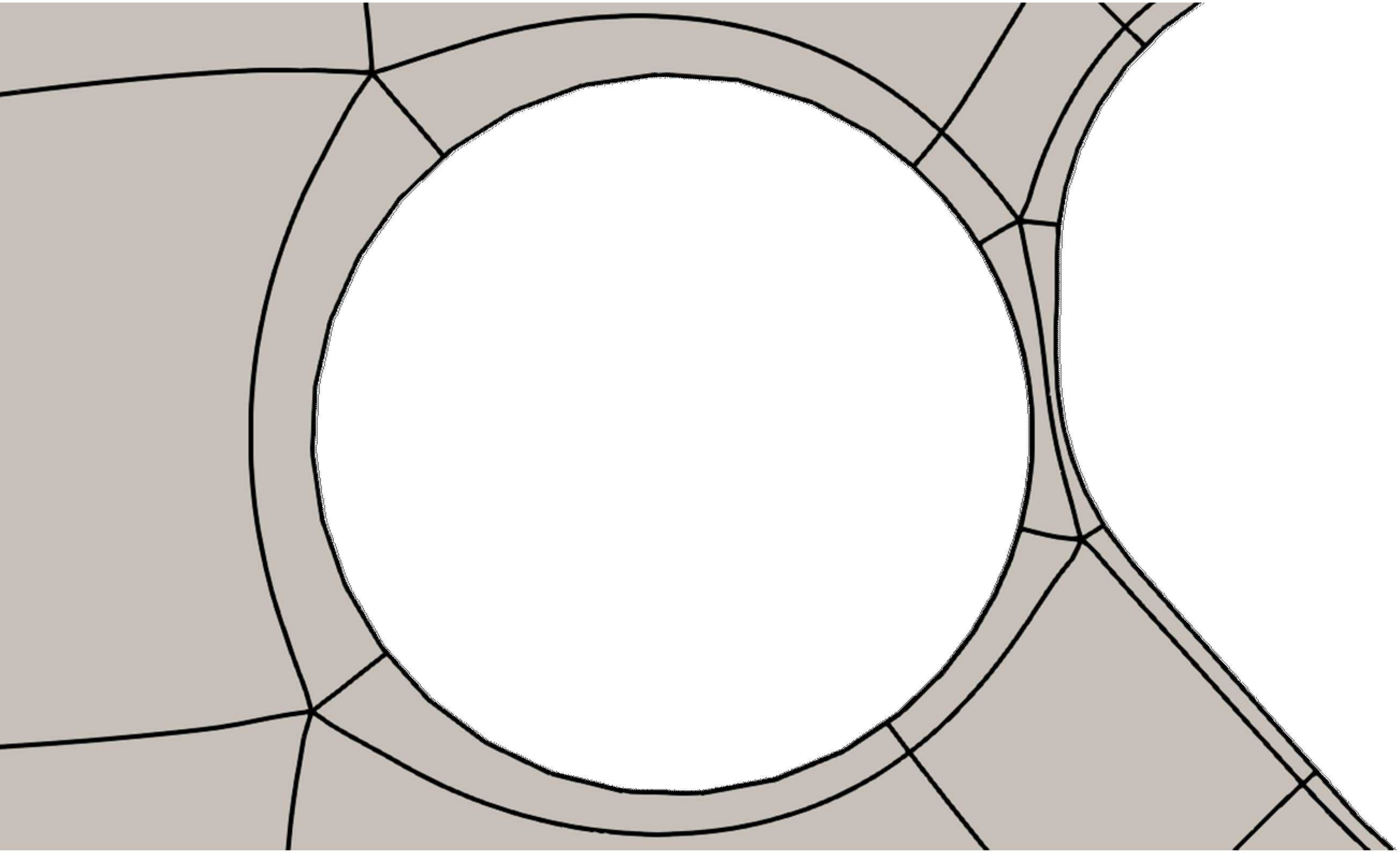
Collapse Algorithm



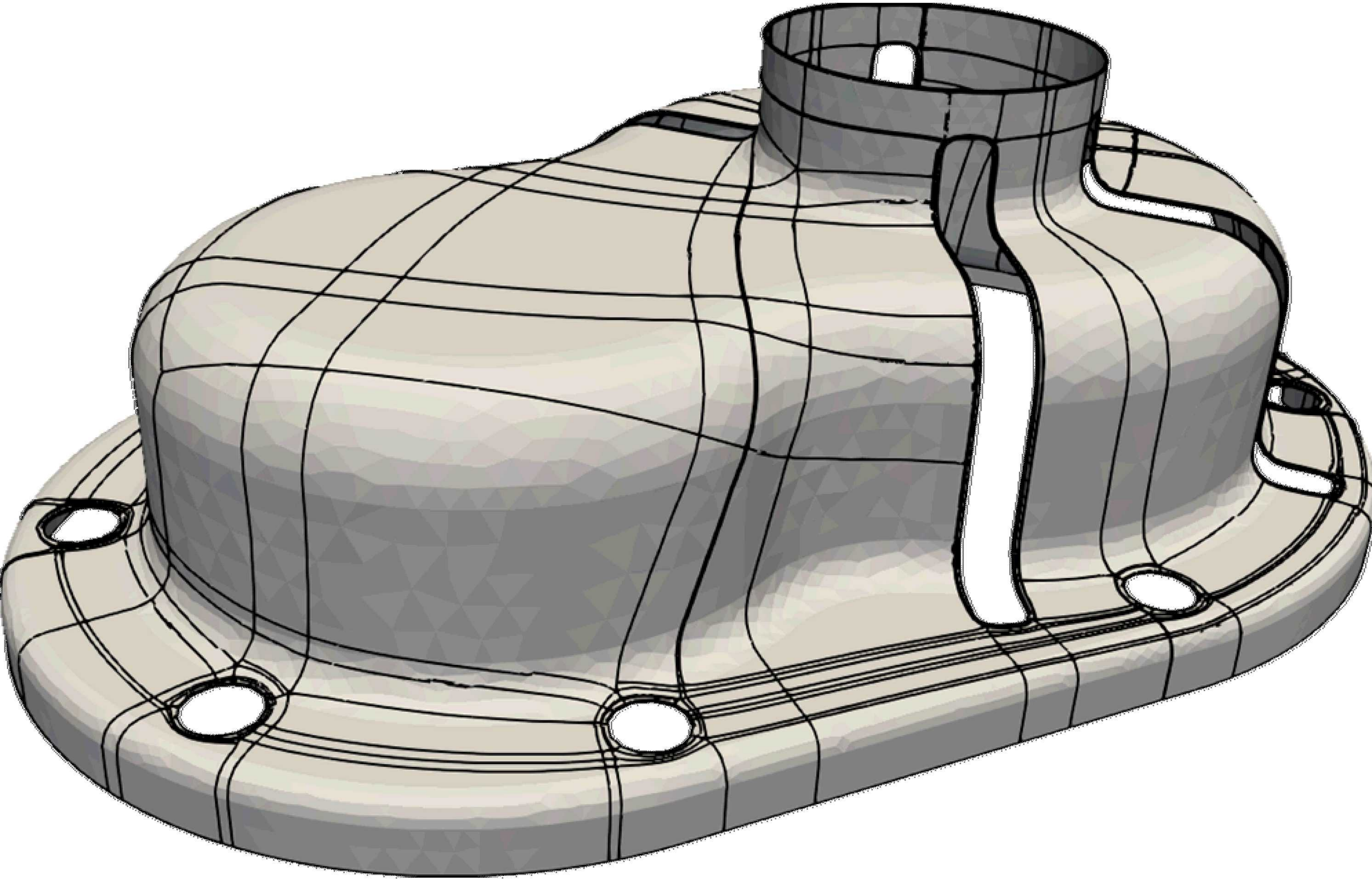
Collapse Algorithm



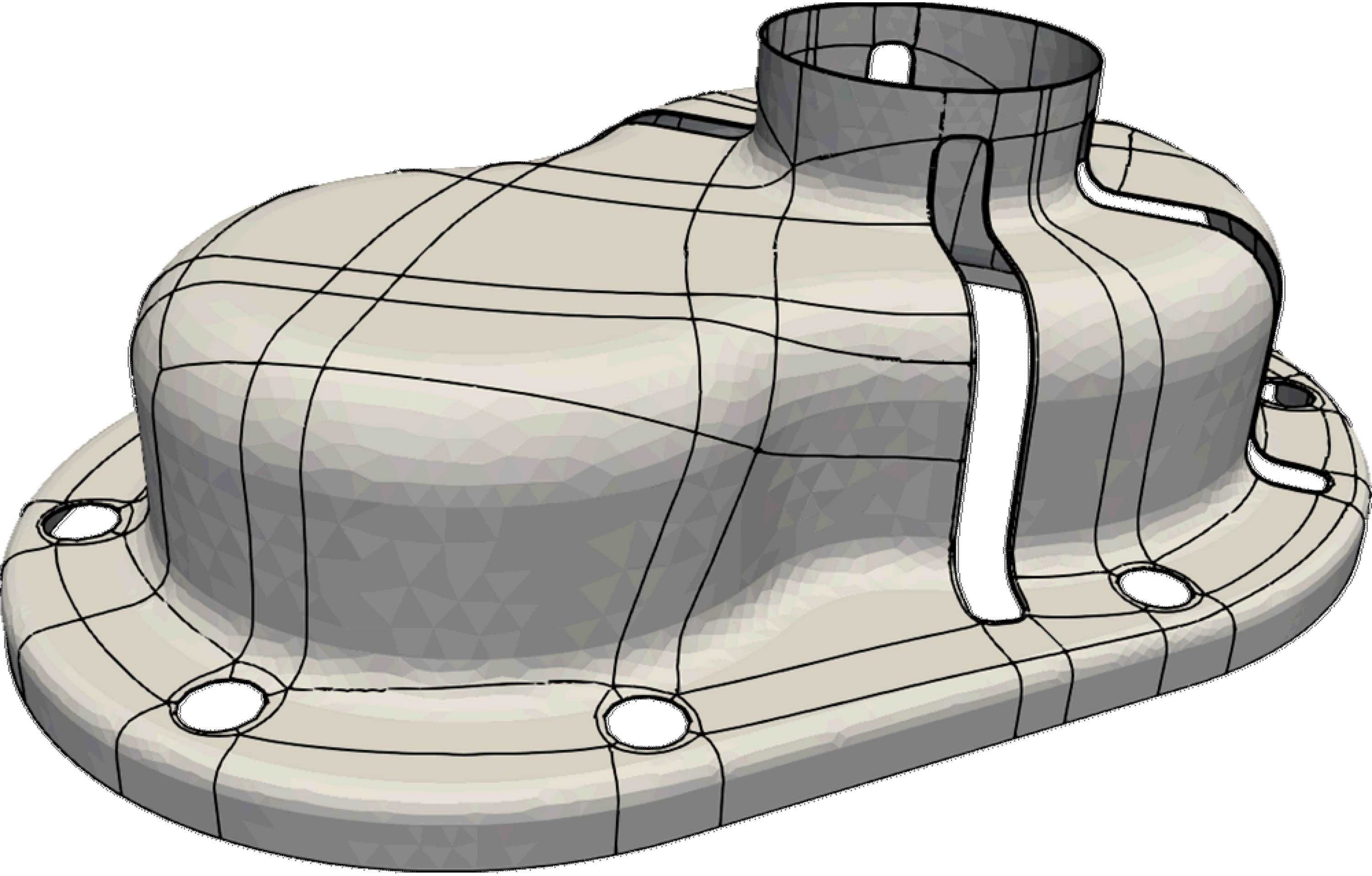
Collapse Algorithm



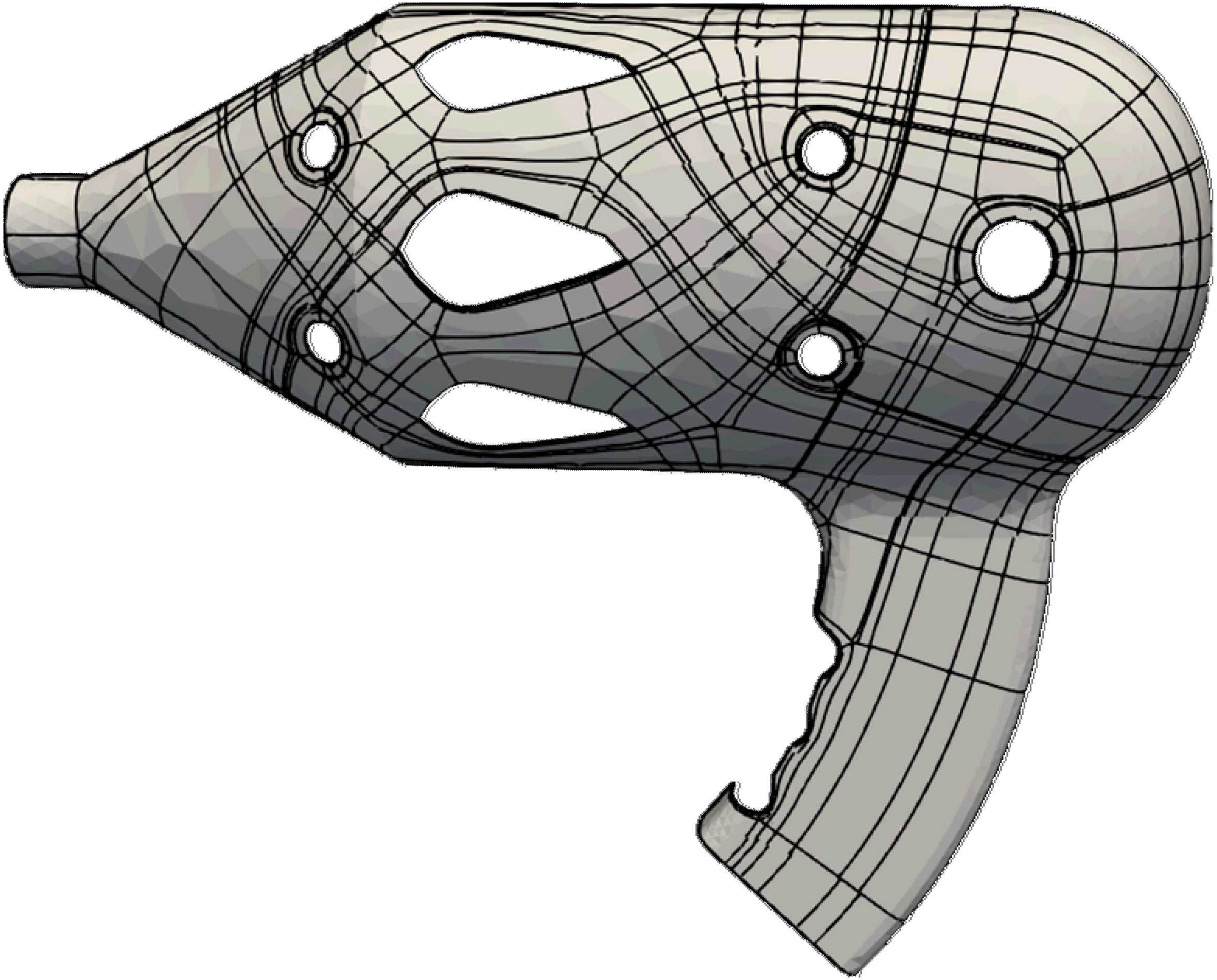
Partition Simplification on Surfaces



Partition Simplification on Surfaces



Partition Simplification on Surfaces



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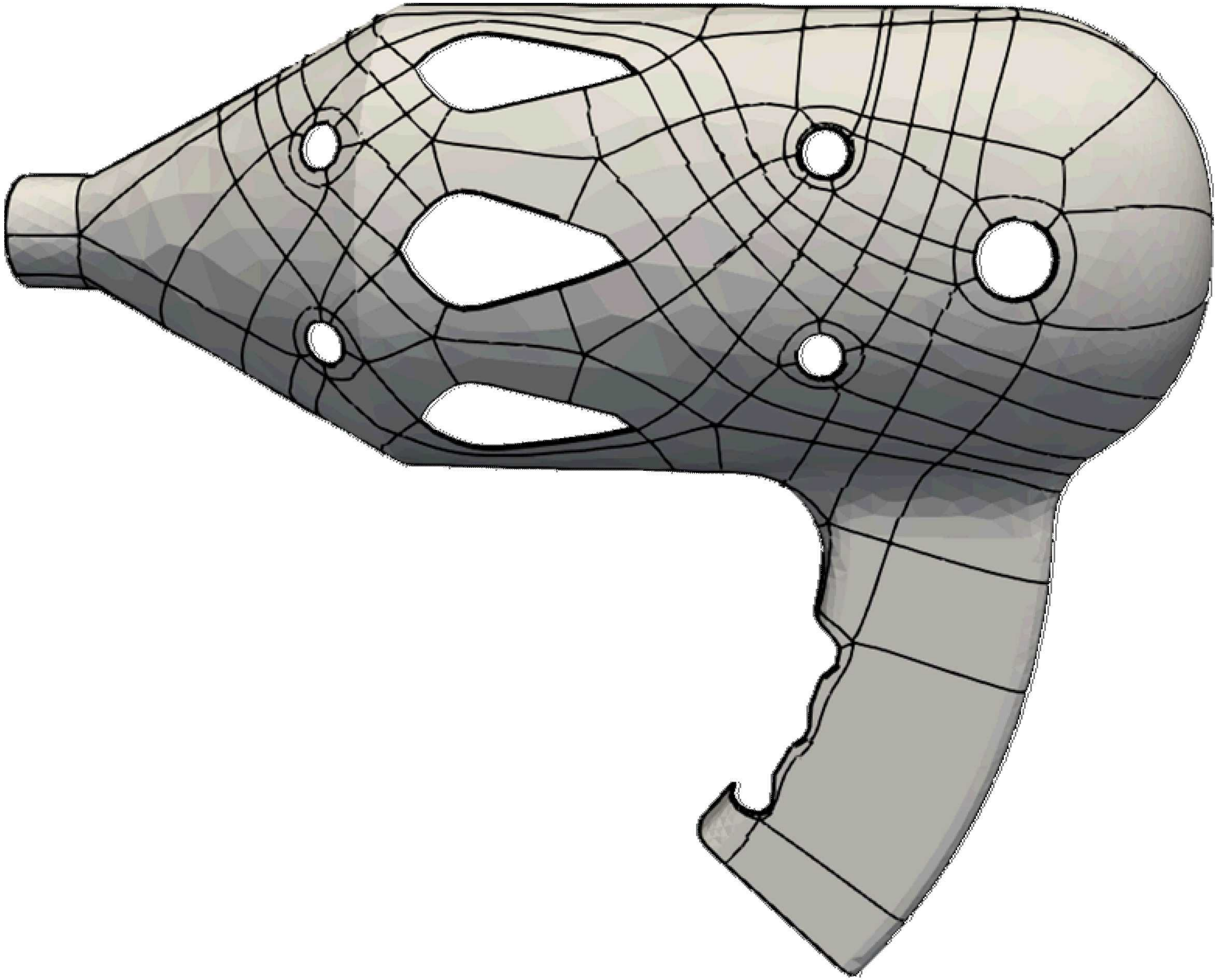

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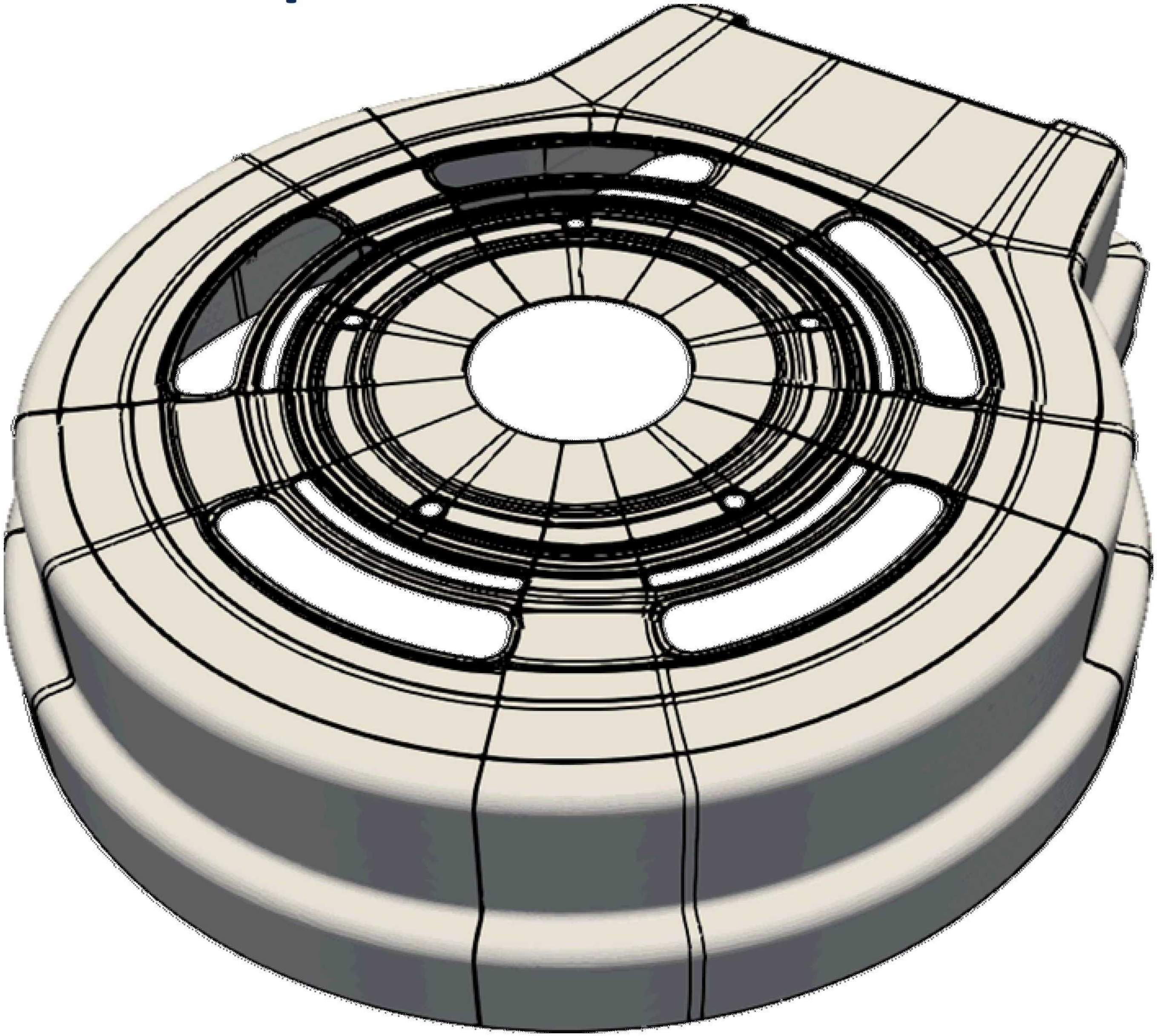

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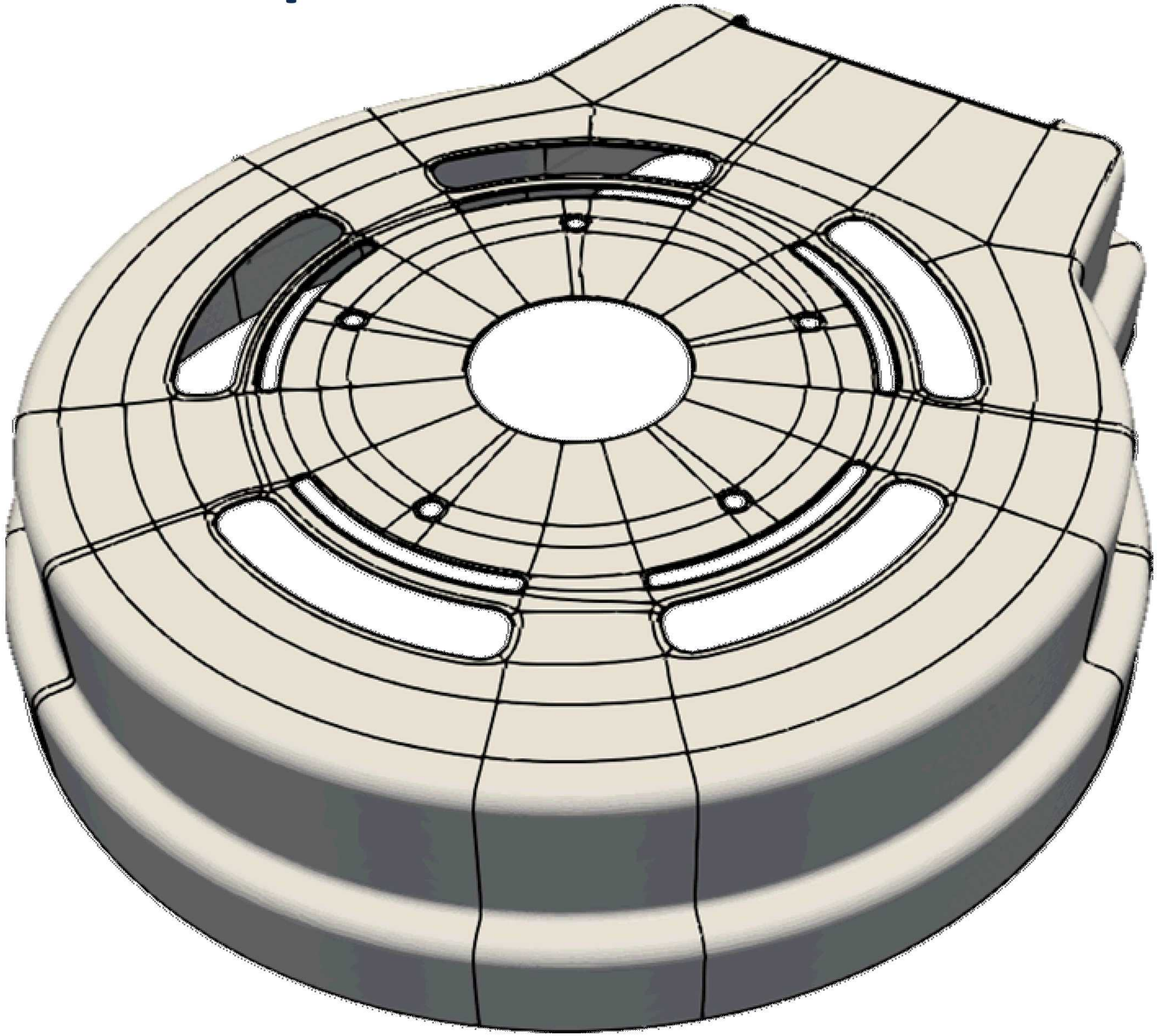

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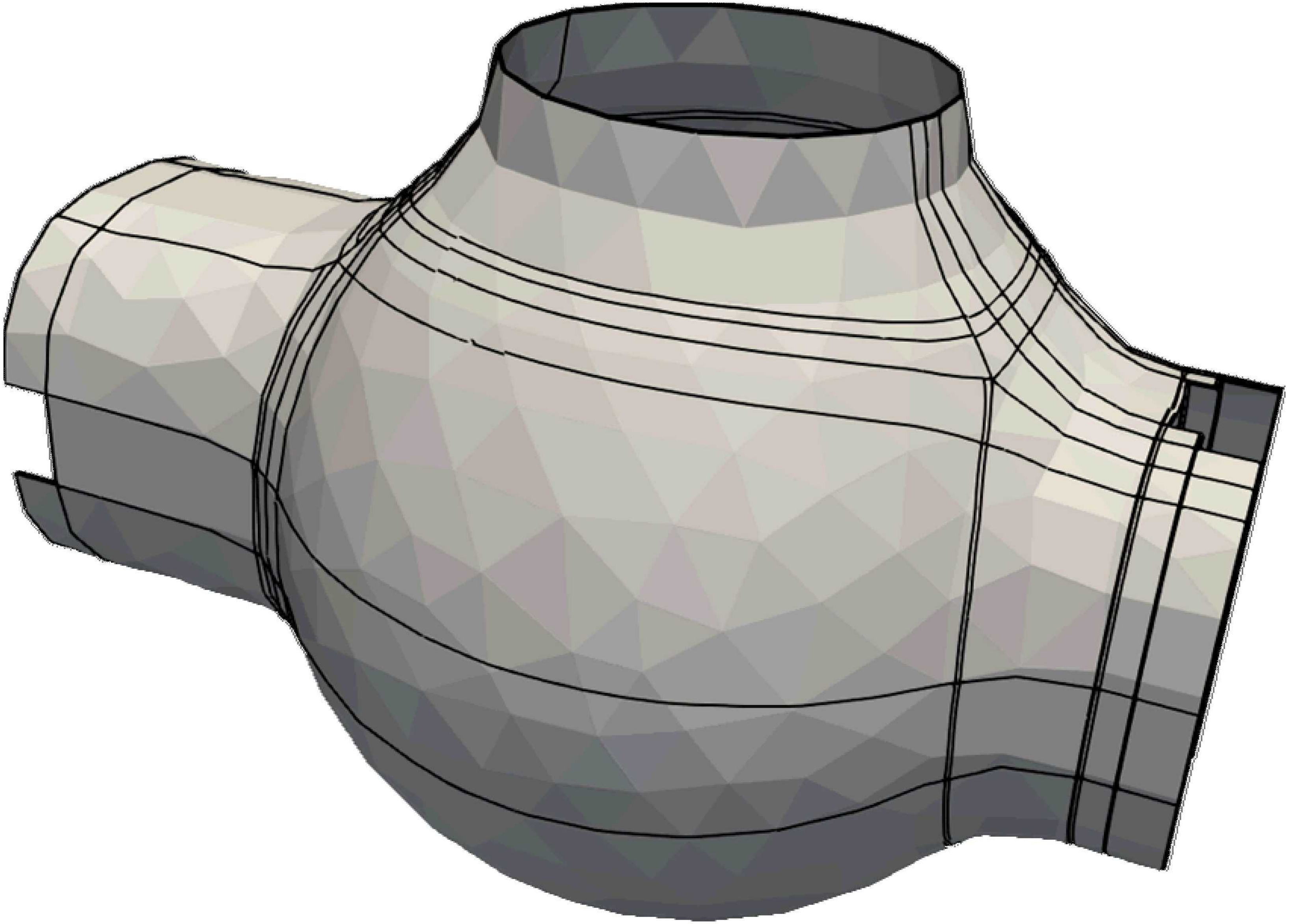
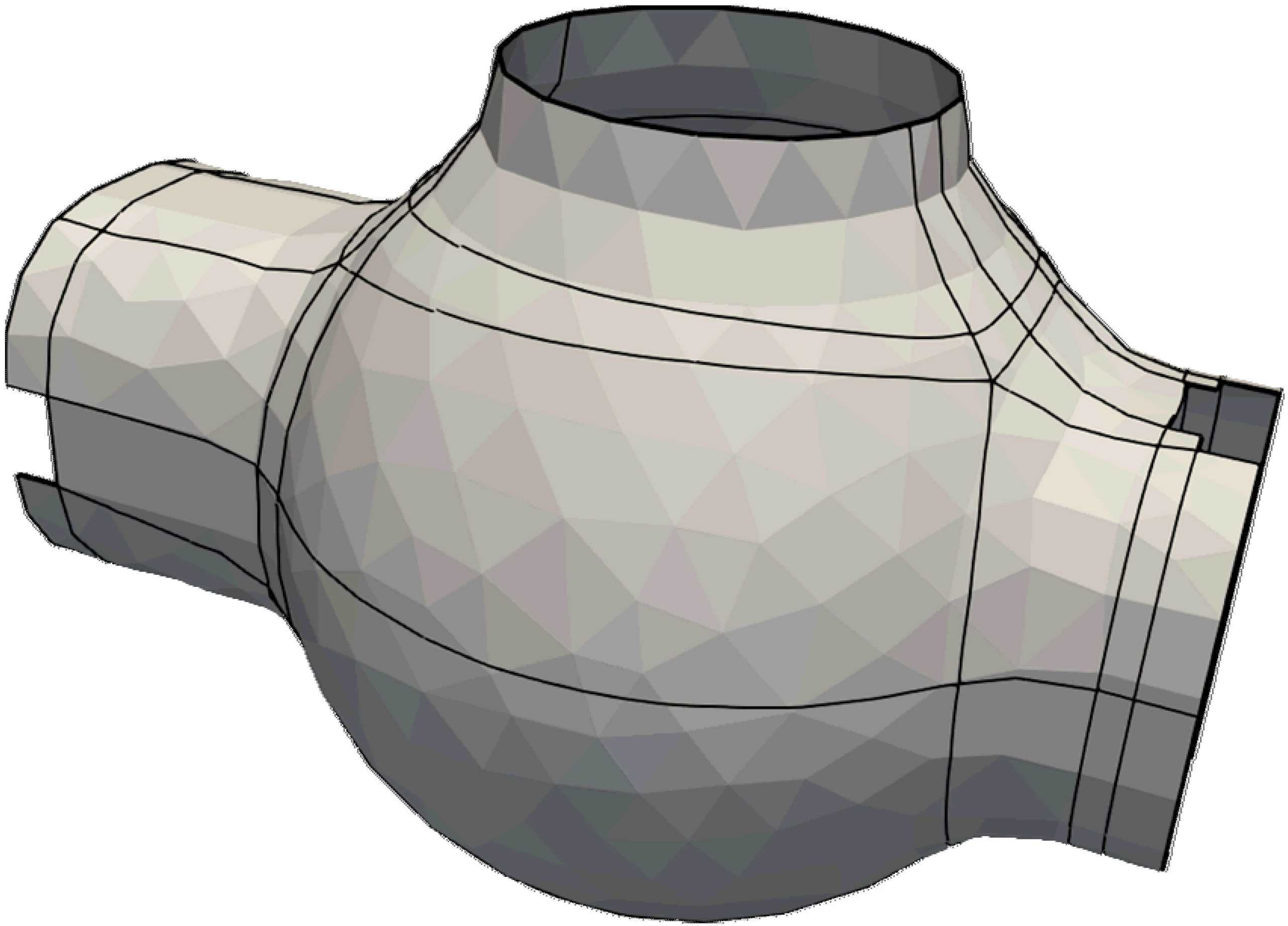


Partition Simplification on Surfaces

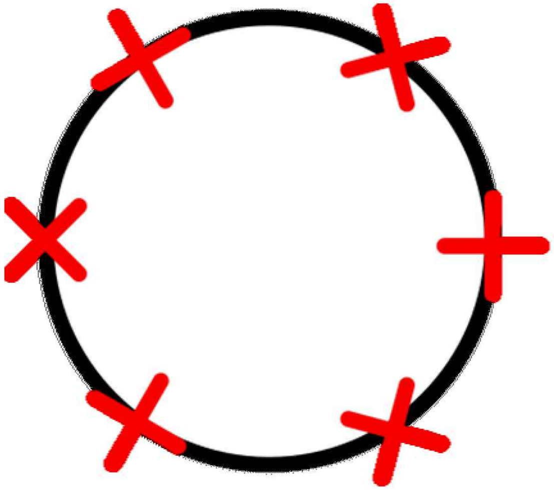


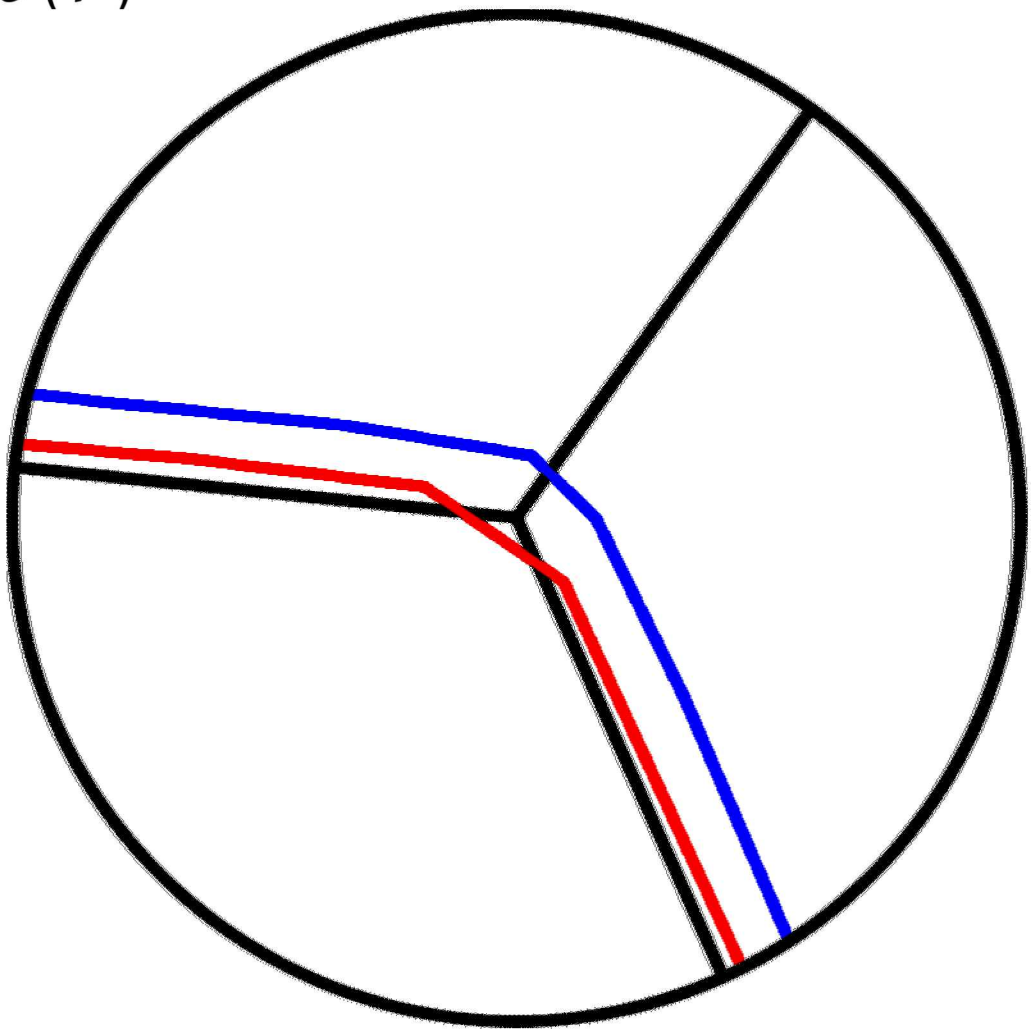
Extra Slides

Hyperbolic Trajectory of Streamlines

Streamlines Near Singularities

$$f(z) = e^{i\left(\frac{d\theta}{4} + \frac{2k\pi}{4}\right)} + o(r)$$





Streamlines Near Singularities

$$f(z) = e^{i\left(\frac{d\theta}{4} + \frac{2k\pi}{4}\right)} + o(r)$$

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WLOG let k = 0, we are looking for the set

$$C = \{z(t) \in B(a, r_0) \mid t \in (t_a, t_b)\}$$

Streamlines Near Singularities

Proposition:

$$C = \{(x+iy)^{-(4-d)/8} \mid xy = A,\ x \in I_x\}$$

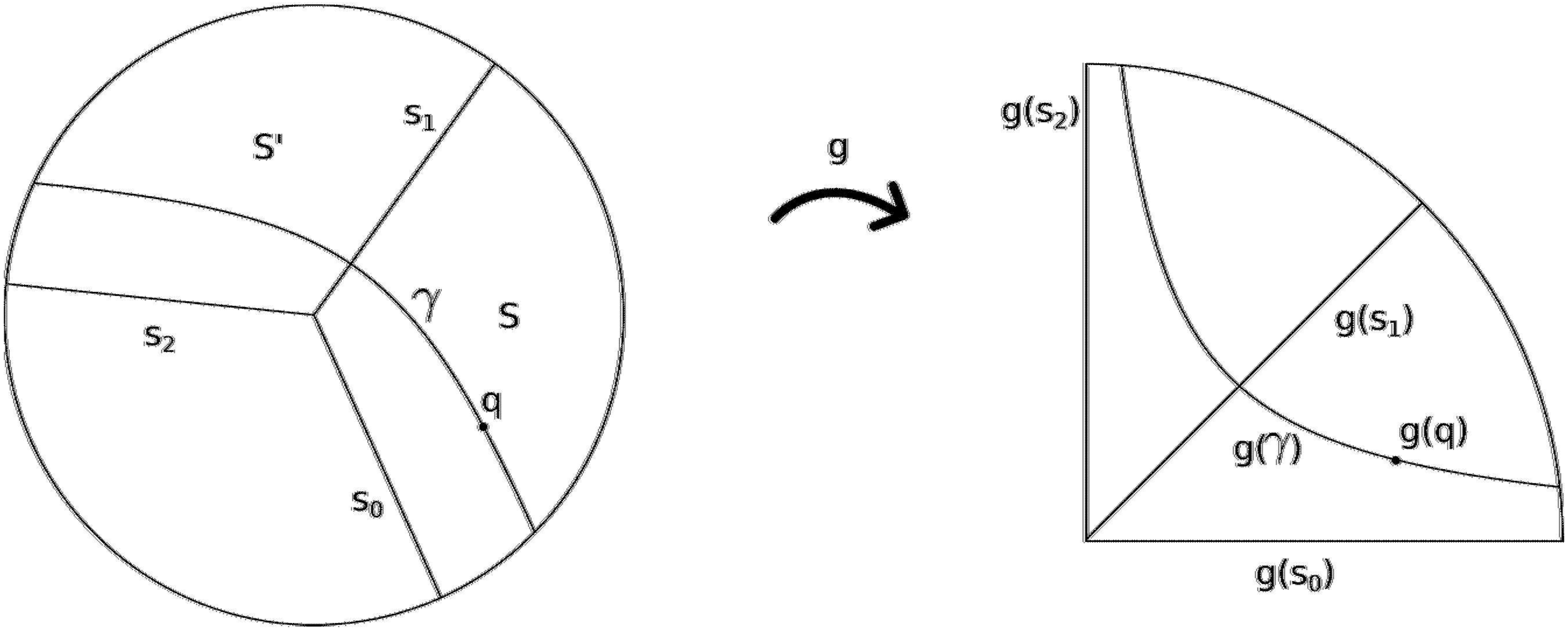
For some constant A on some interval I_x

Proof:

Consider

$$g(z) = z^{(4-d)/8}$$

Streamlines Near Singularities



$$g(z) = z^{(4-d)/8}$$

Streamlines Near Singularities

$$w(t) = g(z(t))$$

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$$g'(z) \neq 0 \text{ in } D$$

$$\arg w'(t) = \arg\left(g'(z(t))z'(t)\right)$$

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Streamlines Near Singularities

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$$\begin{aligned}\arg w'(t) &= \arg\left(g'(z(t))z'(t)\right)\\ &= \arg g'(z(t)) + \arg z'(t)\\ &= \left(\frac{4-d}{8} - 1\right)\theta + \arg(z'(t))\end{aligned}$$

Streamlines Near Singularities

$$\arg w'(t) = \left(\frac{4-d}{8} - 1 \right) \theta + \arg(z'(t))$$

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$$\arg w'(t) = \left(\frac{4-d}{8} - 1\right)\theta + \arg(z'(t))$$

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Streamlines Near Singularities

$$\arg w'(t) = \left(\frac{4-d}{8} - 1\right)\theta + \arg(z'(t))$$

$$= \frac{d\theta}{4} + \left(\frac{4-d}{8} - 1\right)\theta$$

$$= -\frac{(4-d)\theta}{8}$$

Streamlines Near Singularities

$$\arg w'(t) = \left(\frac{4-d}{8} - 1\right)\theta + \arg(z'(t))$$

$$= \frac{d\theta}{4} + \left(\frac{4-d}{8} - 1\right)\theta$$

$$= -\frac{(4-d)\theta}{8}$$

$$= -\varphi$$

Streamlines Near Singularities

$$\Longrightarrow \quad w'(t) = \alpha(t) e^{-i\varphi}$$

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$$w(t) = x(t) + iy(t)$$

$$x'(t) = \alpha(t)\cos(\varphi)$$

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Streamlines Near Singularities

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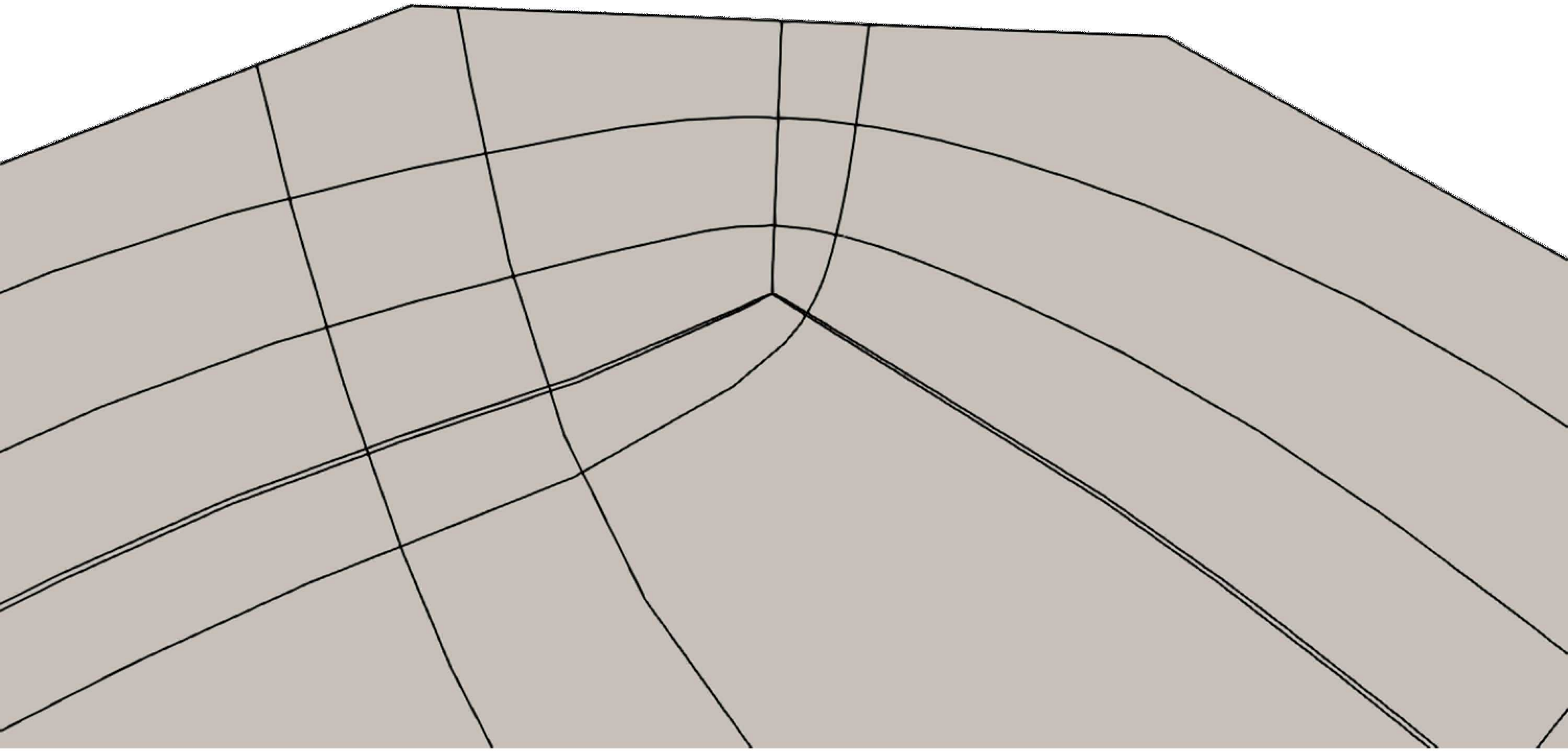
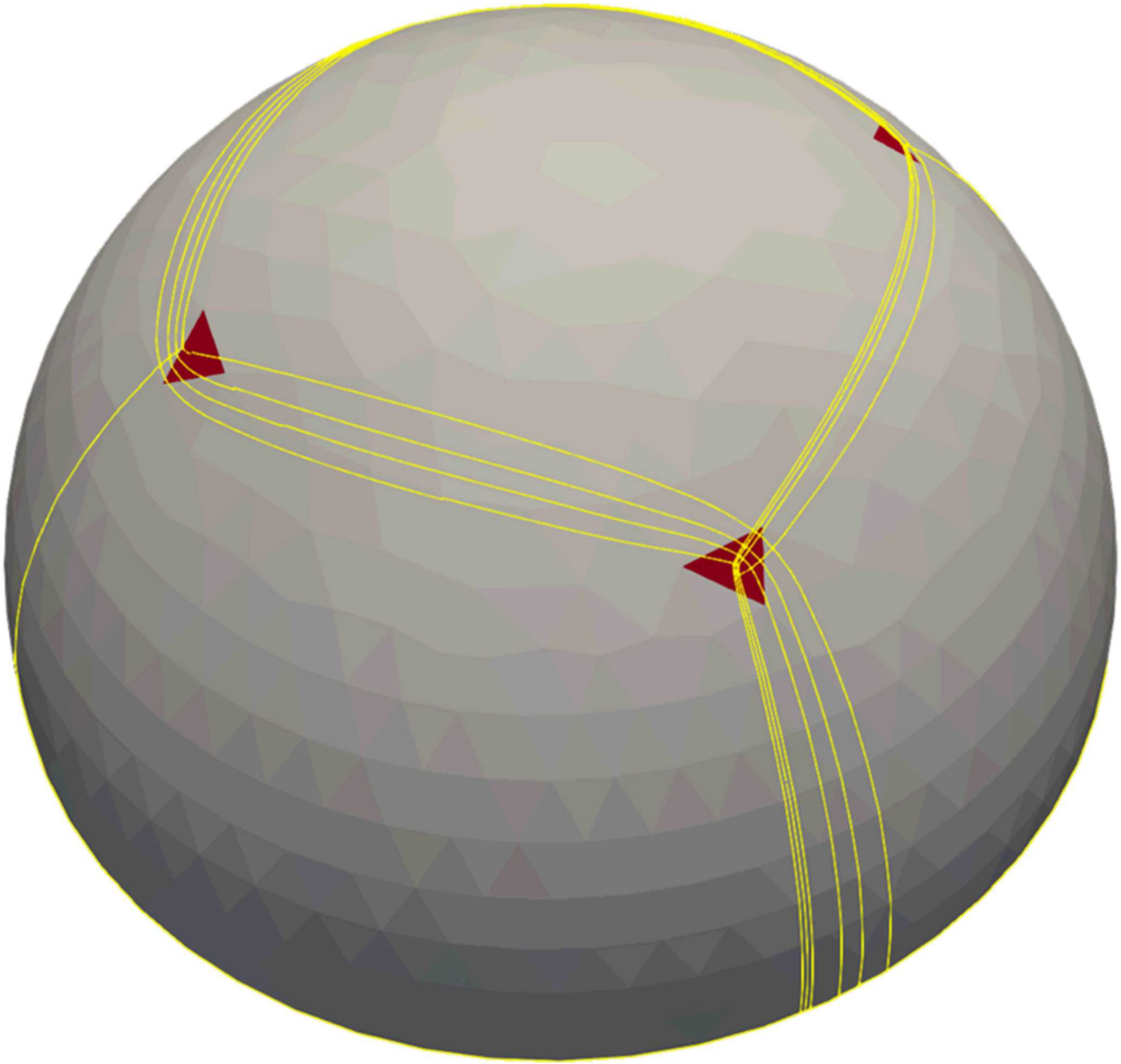
Streamlines Near Singularities



Streamline Tracing



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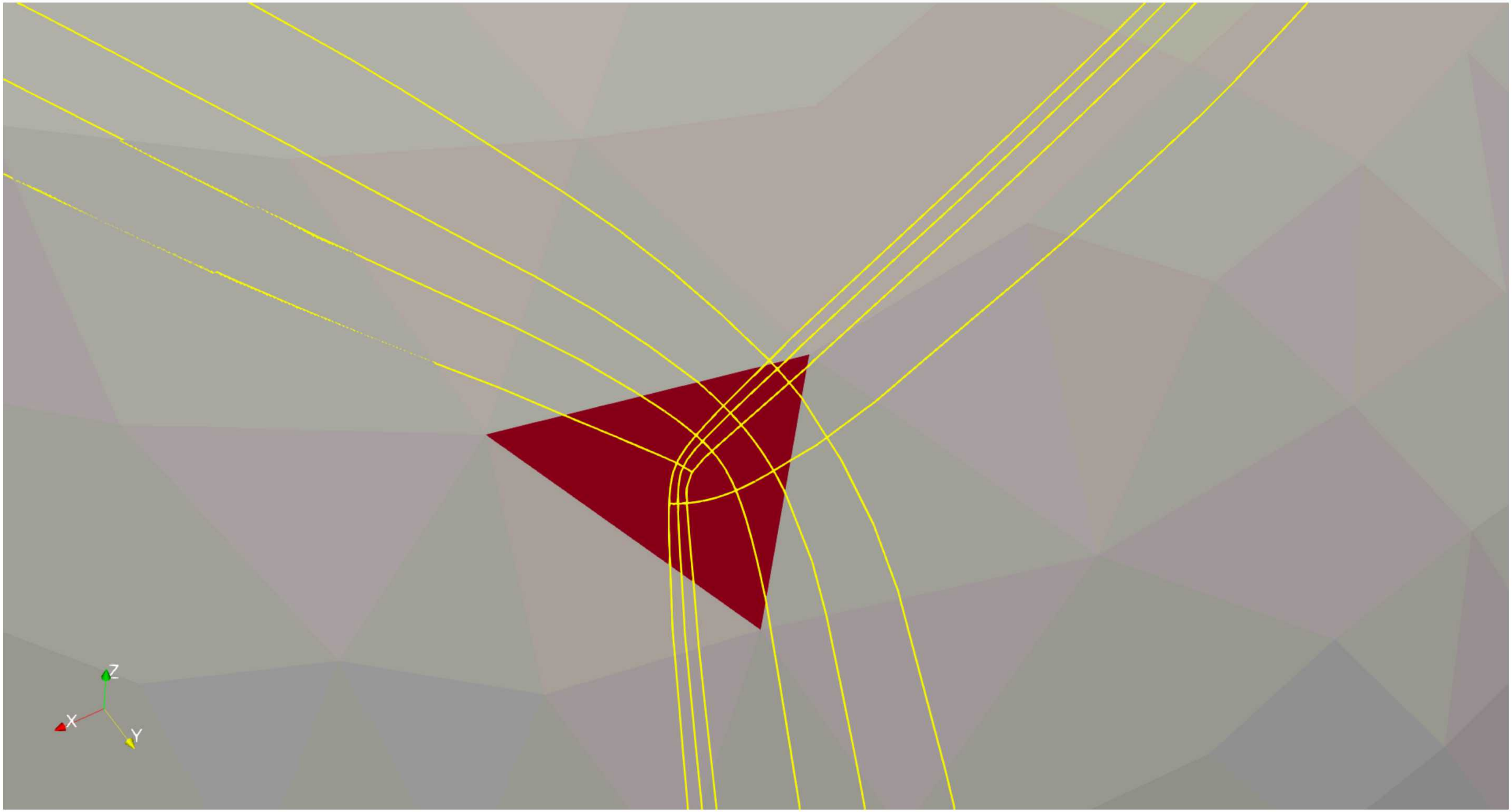

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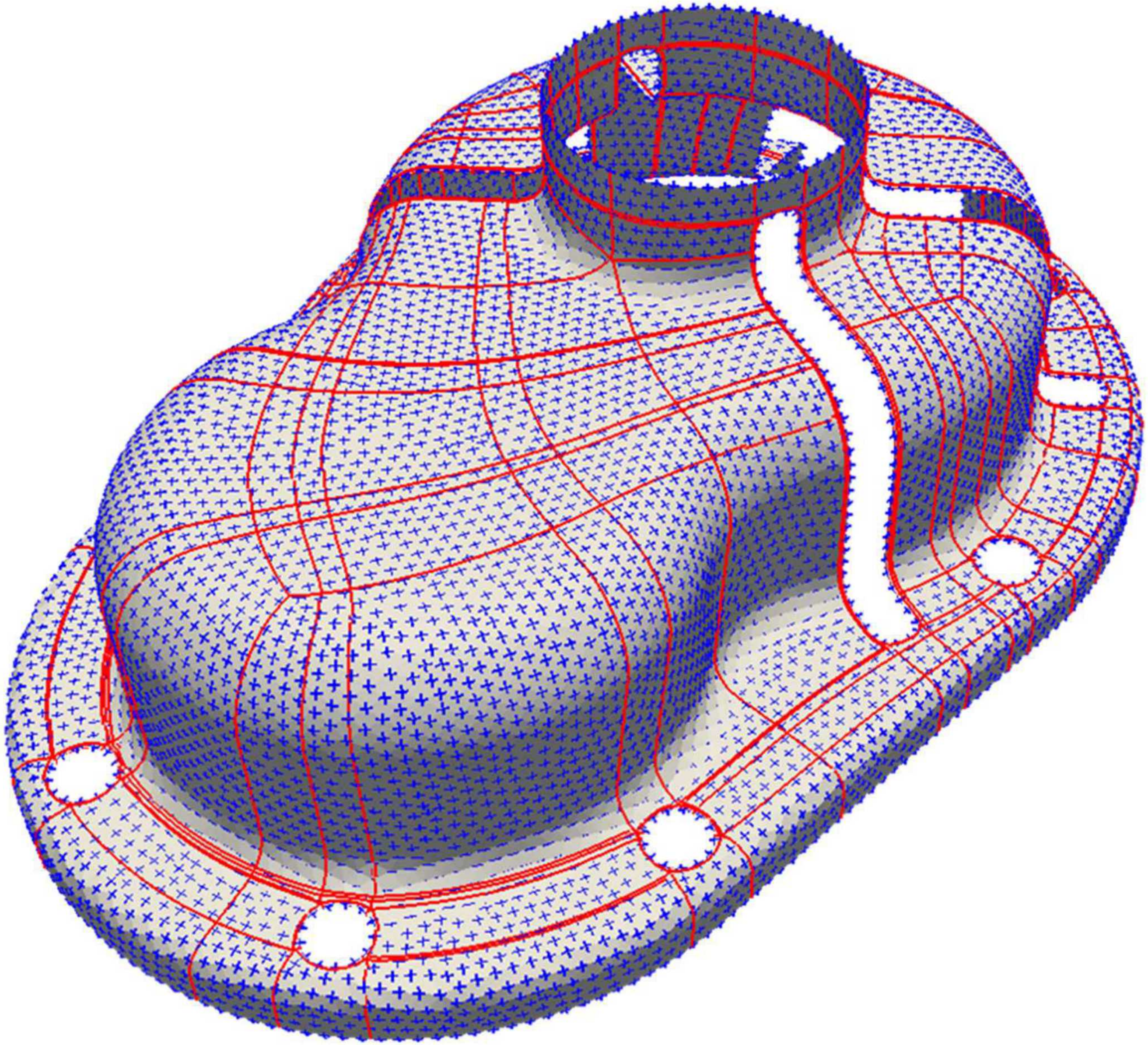


Connection to Ginzburg-Landau Theory

Ginzburg-Landau Functional

Original problem:

$$\min_{u \in H^1_g(D,\mathbb{C})} E(u)$$

$$E(u) = \frac{1}{2}\int_D |\nabla u|^2 dA$$

$$u(x) = g(x) \quad \forall x \in \partial D$$

$$|u(x)| = 1 \quad \text{a.e.} \quad x \in D$$

Relaxed problem:

$$\min_{u \in H^1_g(D,\mathbb{C})} E_\varepsilon(u)$$

$$E_\epsilon(u) = \frac{1}{2}\int_G |\nabla u|^2 + \frac{1}{4\epsilon^2}\int_G (|u|^2 - 1)^2$$

Results of Ginzburg-Landau Theory (Bethuel et al.) and Applications to Cross Fields

Well Defined Limit of Relaxed Problem

Theorem 2.2.2 (Bethuel et al. [4]). *Let $d = deg(g, \partial D)$. Given a sequence $\varepsilon_n \to 0$ there exists a subsequence ε_{n_i} and exactly d points $a_1, a_2, ..., a_d$ in $D \subset \mathbb{C}$ and a smooth harmonic map $u_\star : D \setminus \{a_1, ..., a_d\} \to \mathbb{T}$ with $u_\star = g$ on ∂D such that*

$$\boxed{u_{\varepsilon_{n_i}} \to u_\star} \text{ in } C^k_{loc}(D \setminus \bigcup_i (a_i)) \ \forall k \text{ and in } C^{1,\alpha}(\bar{D} \setminus \bigcup_i (a_i)) \ \forall \alpha < 1$$

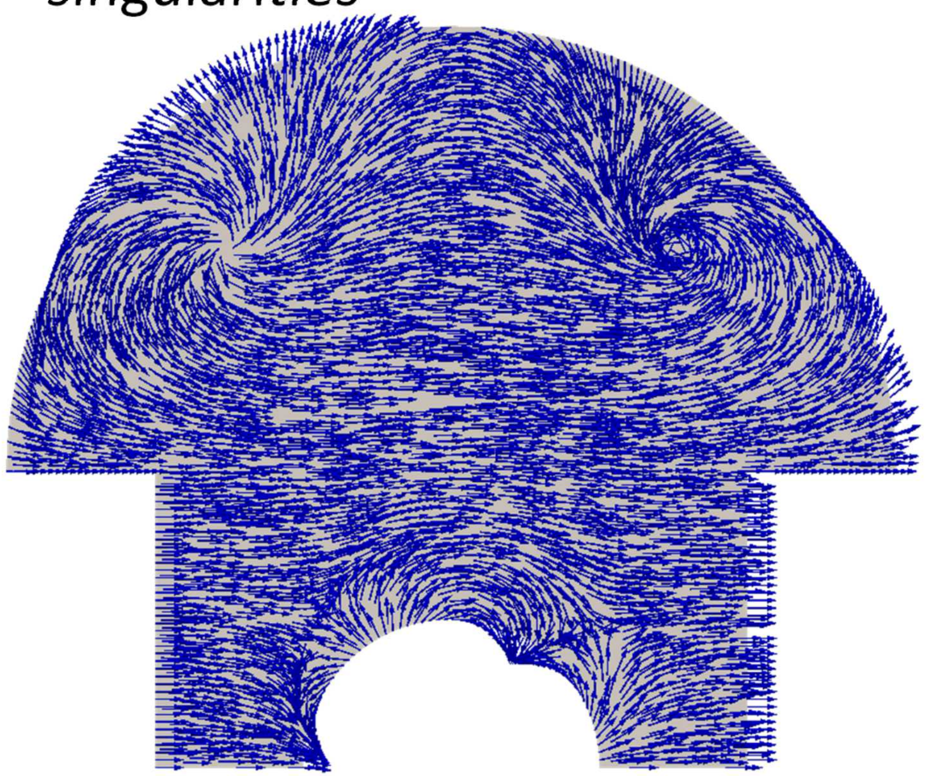
In addition, if $d \neq 0$ each singularity of $u_\star$ has index $\operatorname{sgn}(d)$ and, more precisely, there are complex constants (α_i) with $|\alpha_i| = 1$ such that

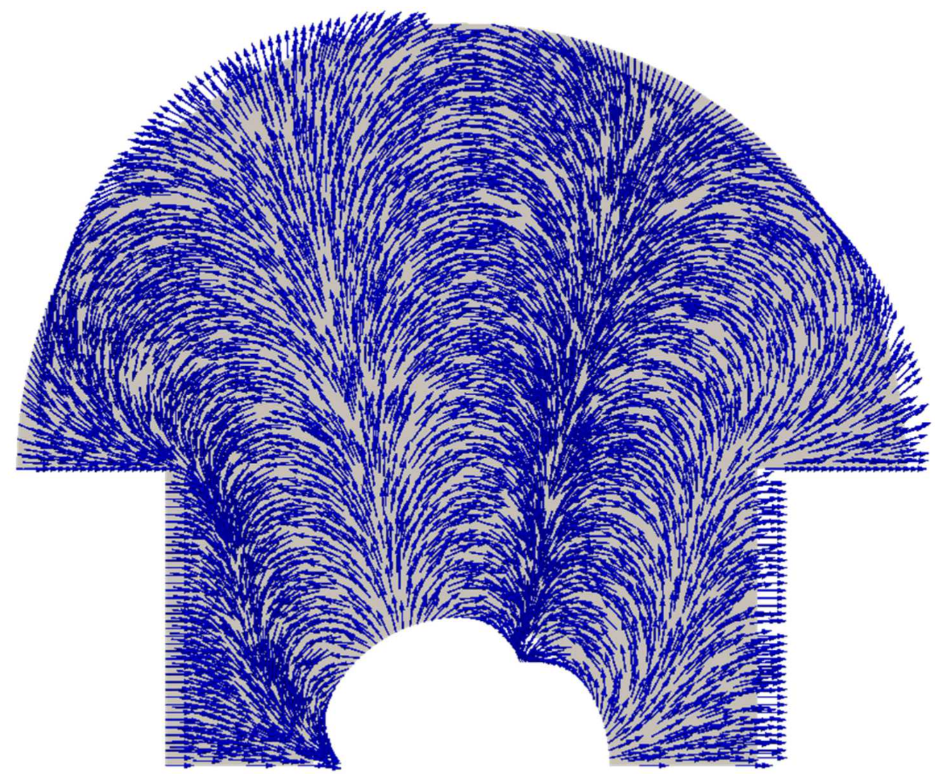
$$\left| u_\star(z) - \alpha_i \frac{z - a_i}{|z - a_i|} \right| \leq C|z - a_i|^2 \text{ as } z \to a_i, \ \forall i$$

This gives us a generalized sense in which to understand the energy minimization problem

Canonical Harmonic Map

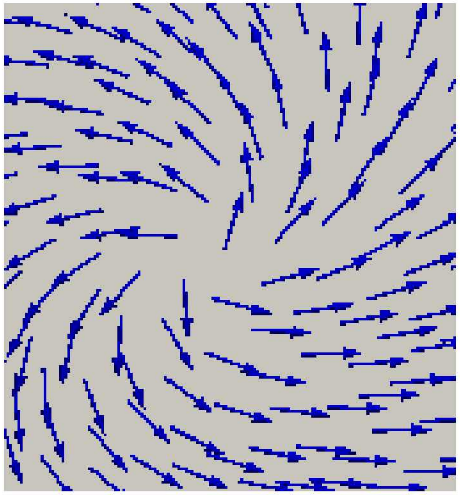
- Harmonic vector field defined everywhere except a finite number of points
- All vectors are unit vectors
- Unique for a given boundary condition and configuration of singularities

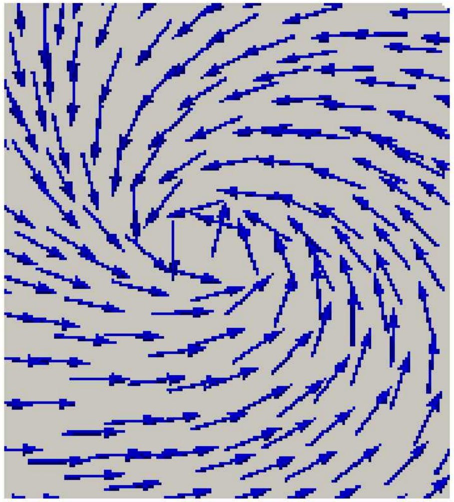




Asymptotic Estimate

$$\left| u_0(z) - \alpha_j \frac{(z-a_j)^{d_j}}{|z-a_j|^{d_j}} \right| \leq C|z-a_j| \quad as \quad z \to a_j$$



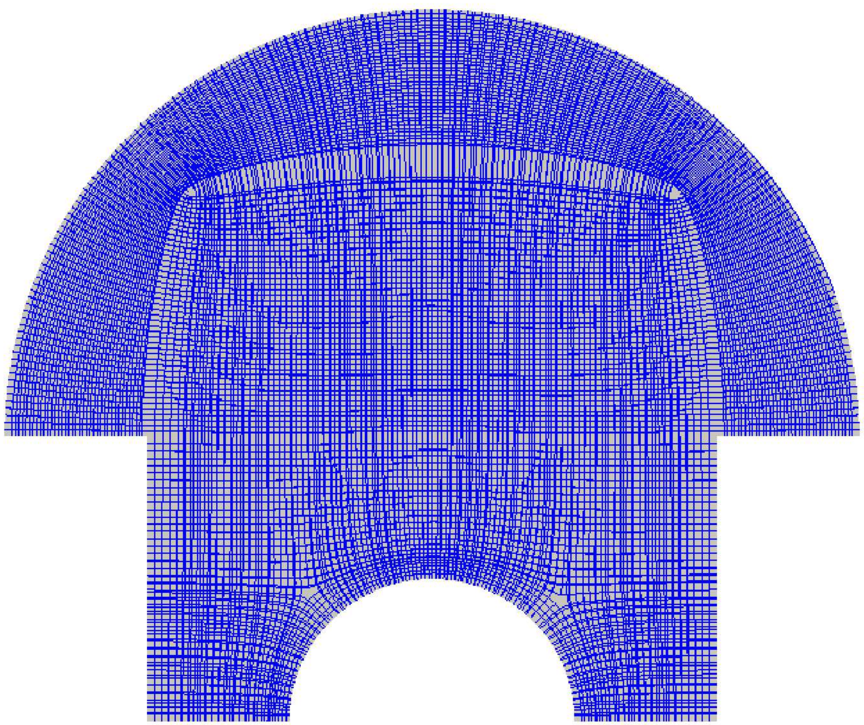


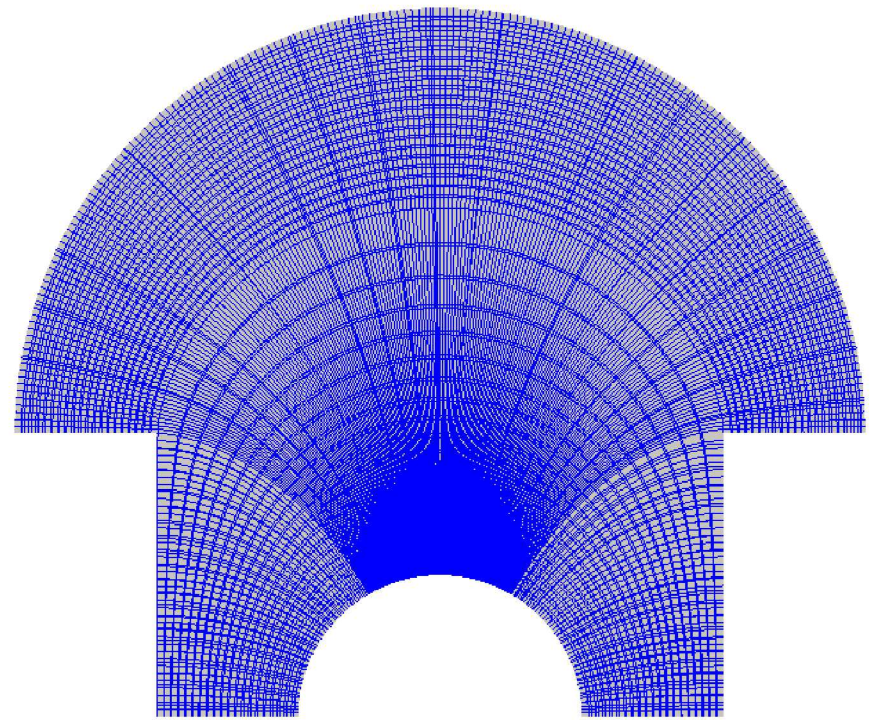
Energy Argument for Local Minimizers

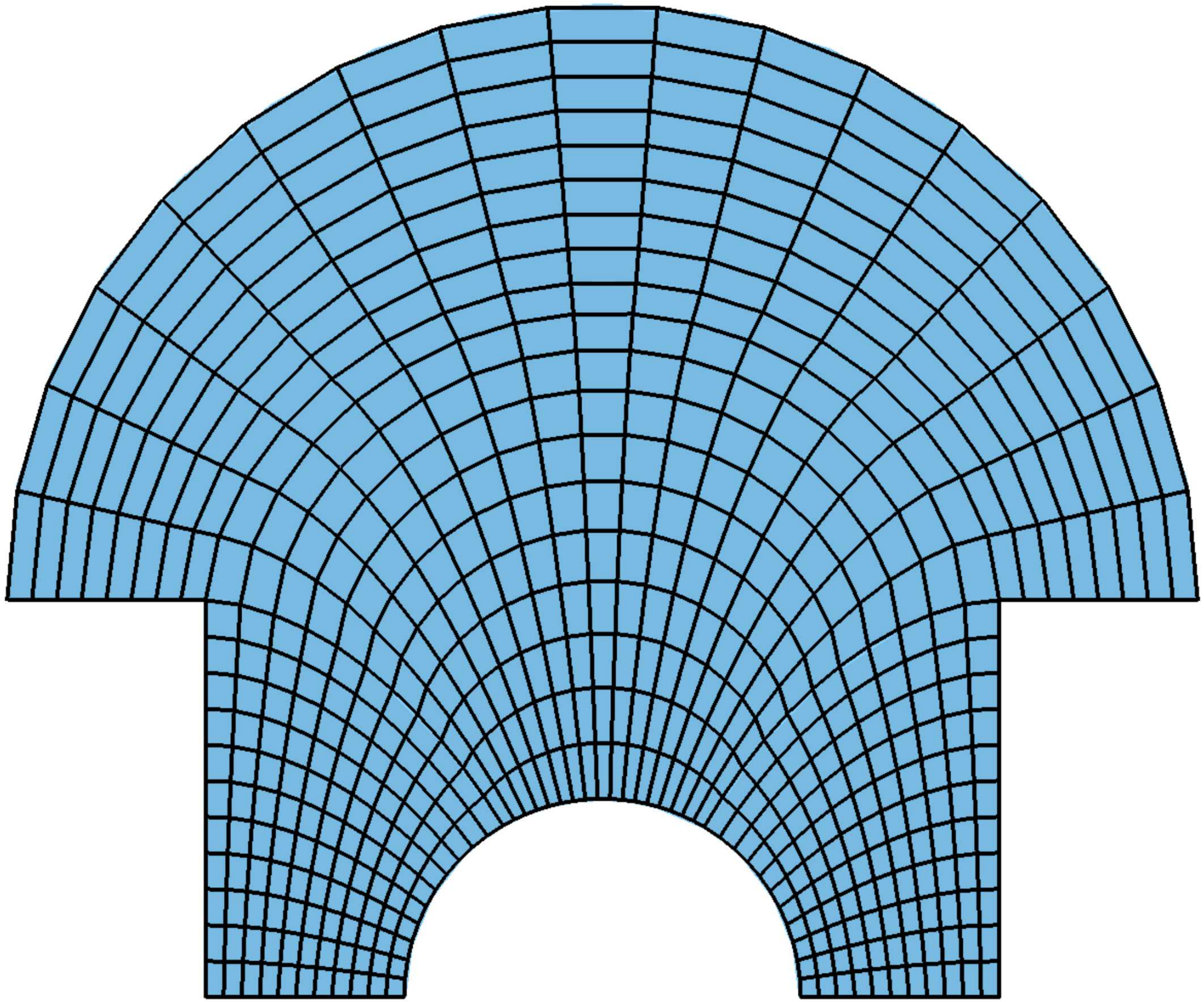
For a local minimizer of the Ginzburg-Landau energy, the singularities are:

- Isolated
- Simple
- Occur on the interior of the domain

Implication for Cross Fields: Strange Minimizer



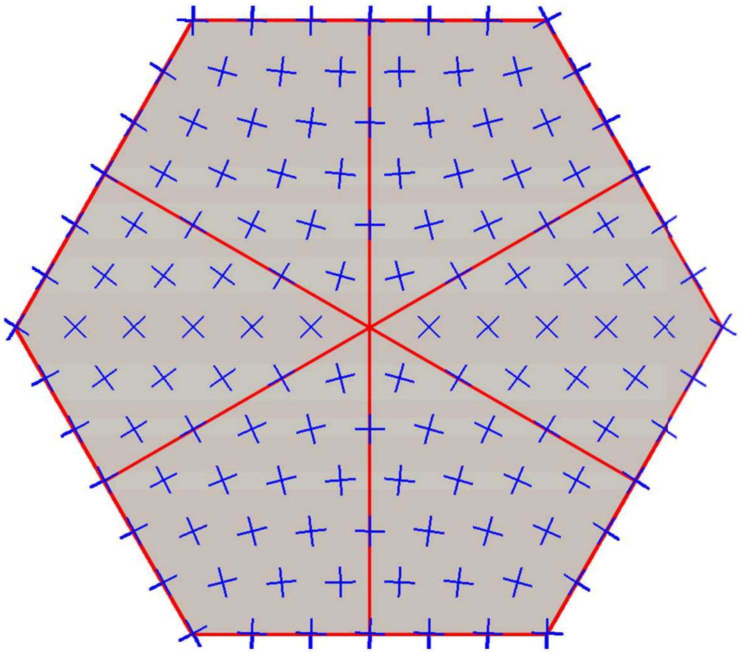


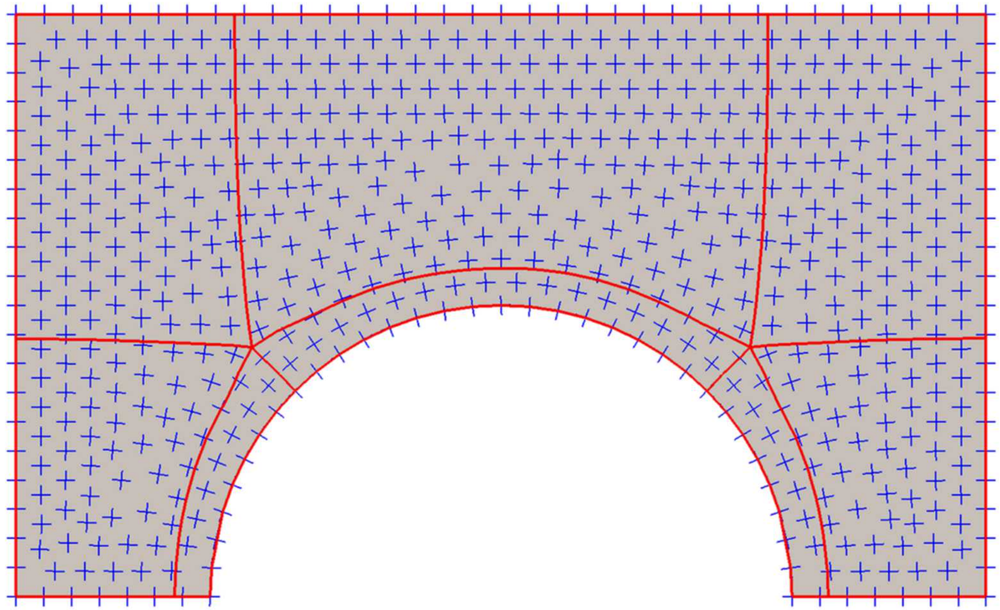


Streamlines and Asymptotic Behavior of Cross Fields Near Singularities

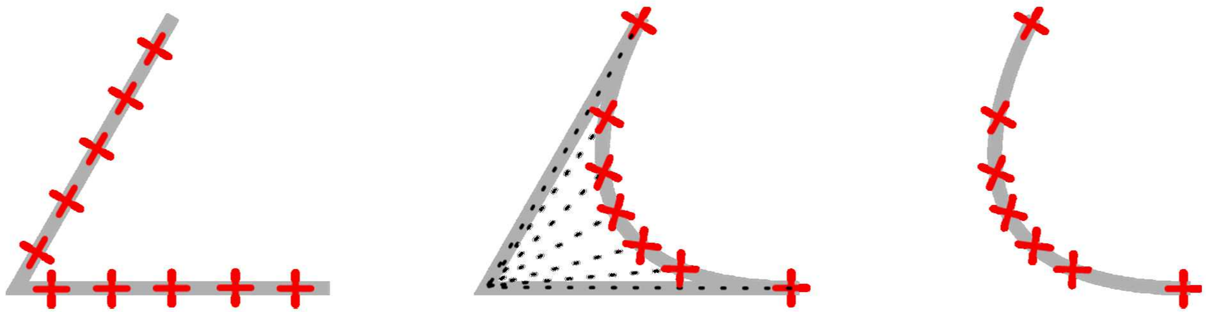
Separatrices

Lemma 5.1. *Let f be a boundary-aligned canonical harmonic cross field on D. Let a be an interior singularity of f of index $d/4$ with $d < 4$. There are exactly $4 - d$ separatrices meeting at a. These separatrices partition a neighborhood of a into $4 - d$ even-angled sectors.*

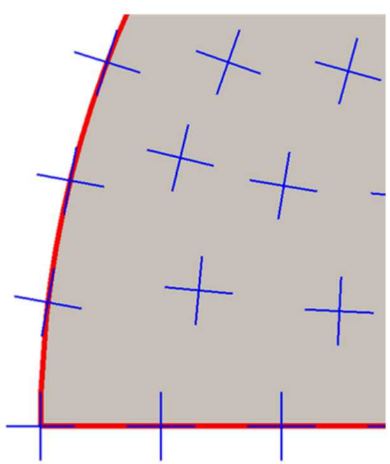


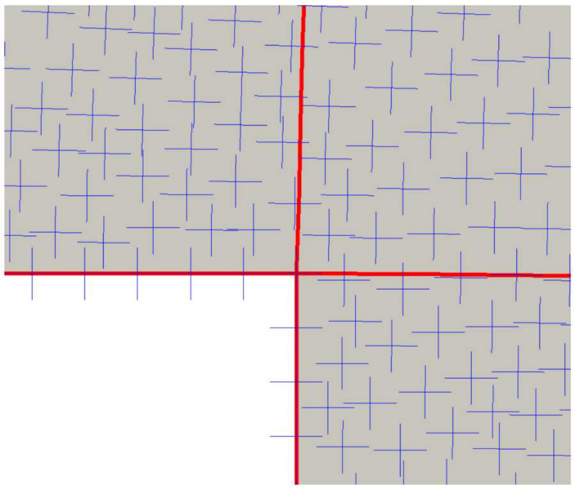


Boundary Singularities



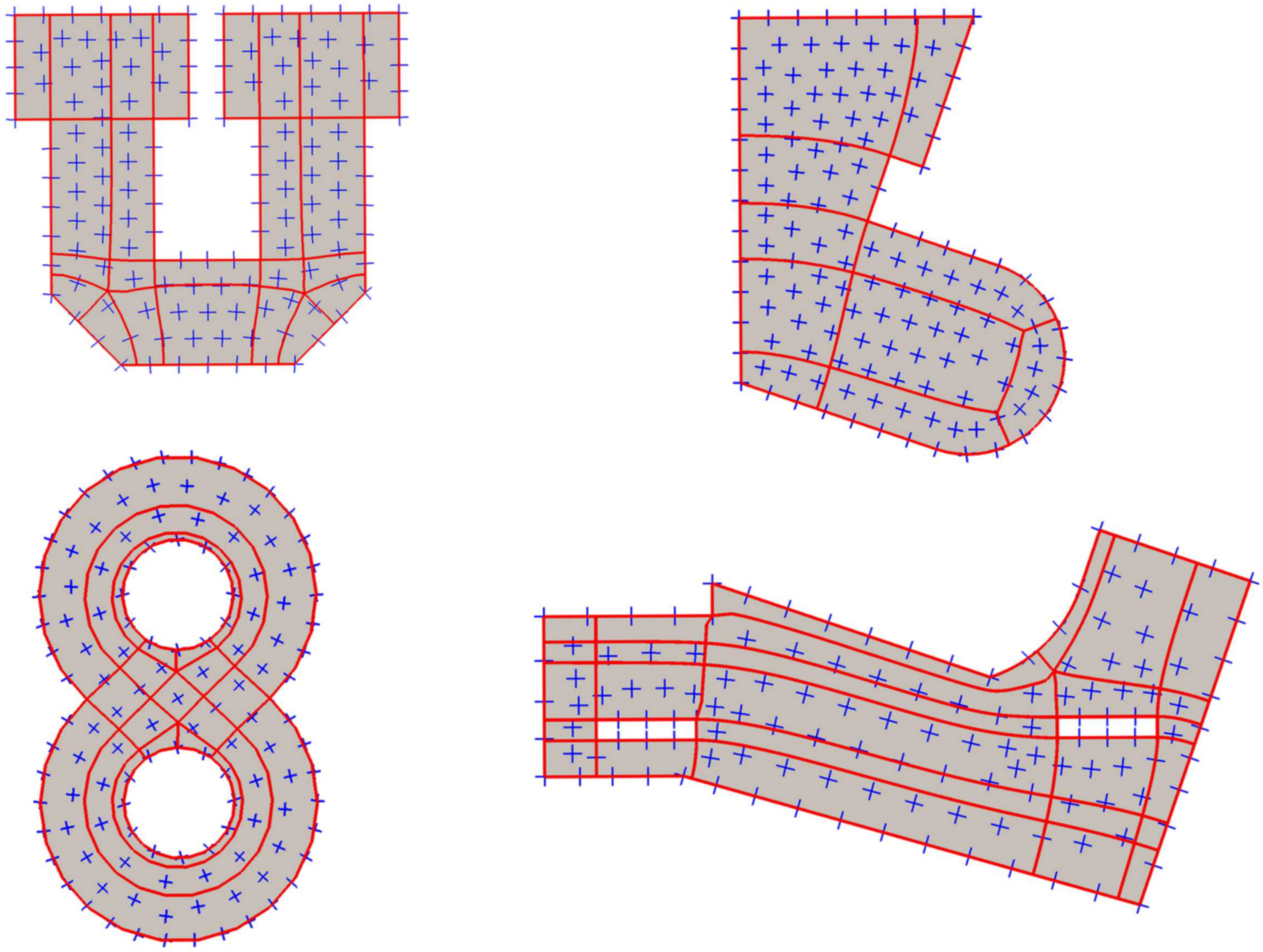
LEMMA 5.4. *Let c be a boundary singularity of f of index d/4 with d < 2. There are exactly 3 − d separatrices meeting at c (including the boundaries themselves). These separatrices partition a neighborhood of c into 2 − d even-angled sectors.*



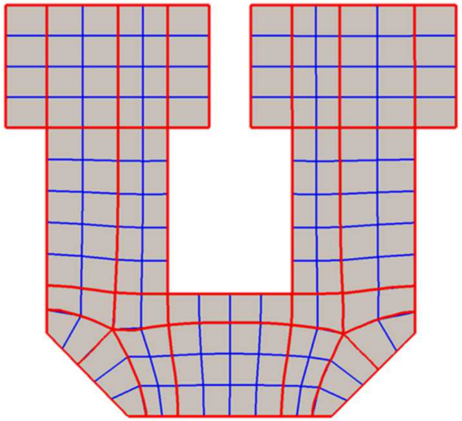


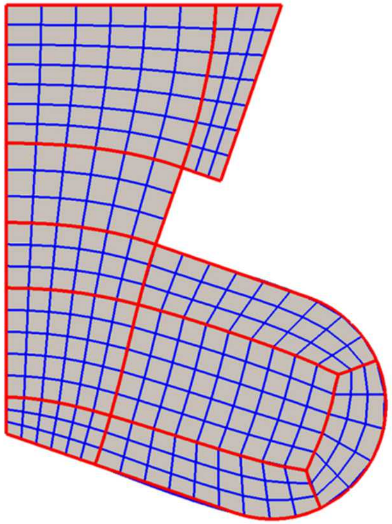
Partitioning Theorem

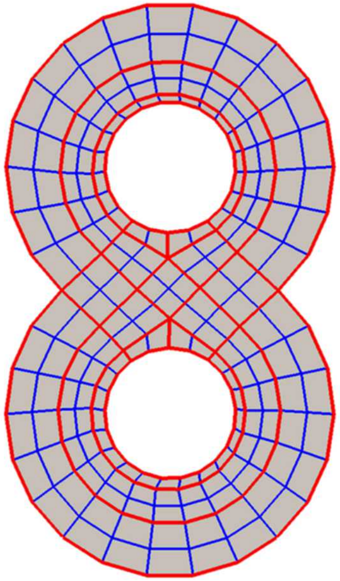
Partition into four-sided regions

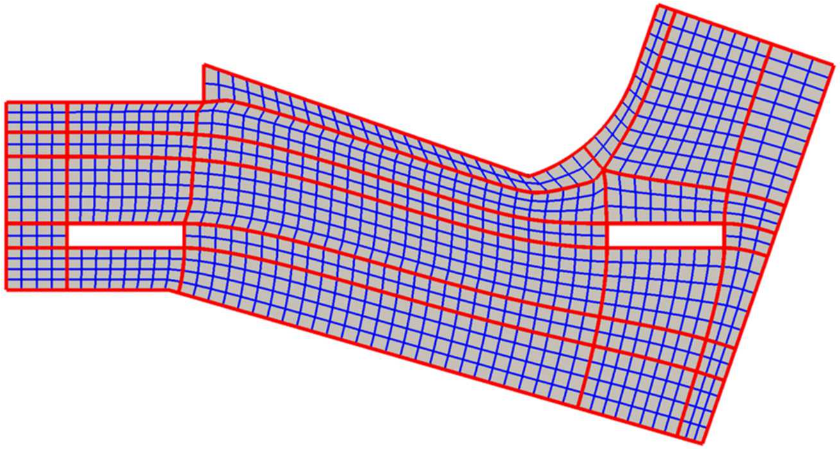


Meshing

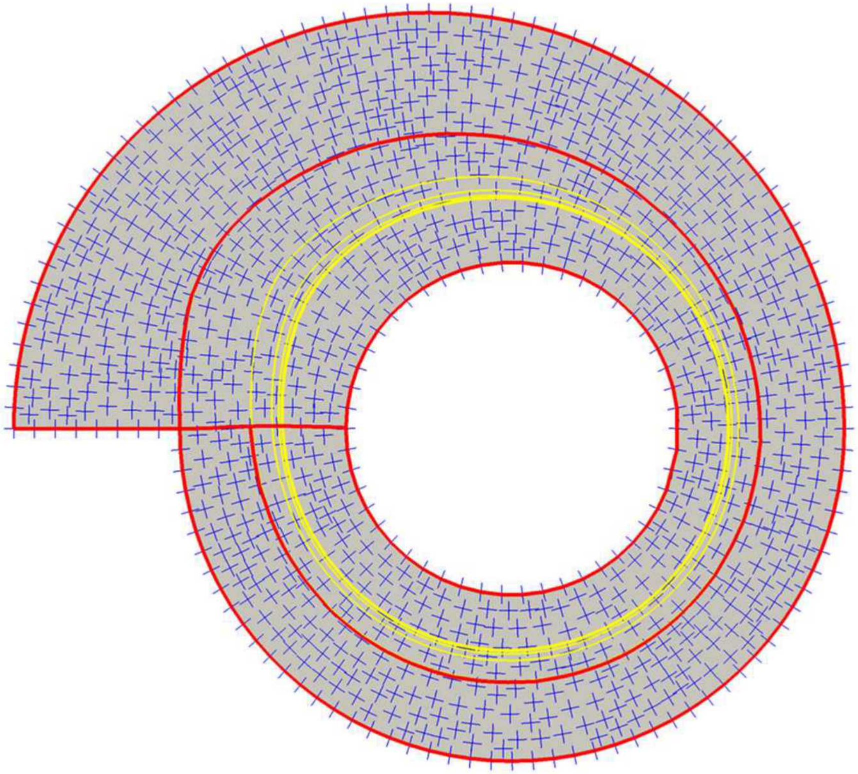


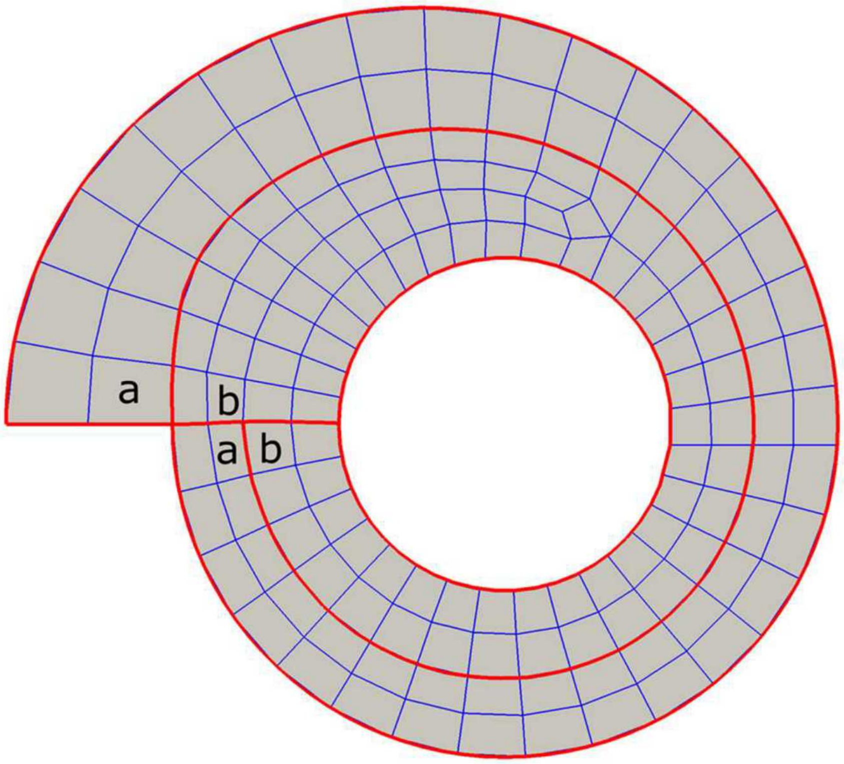




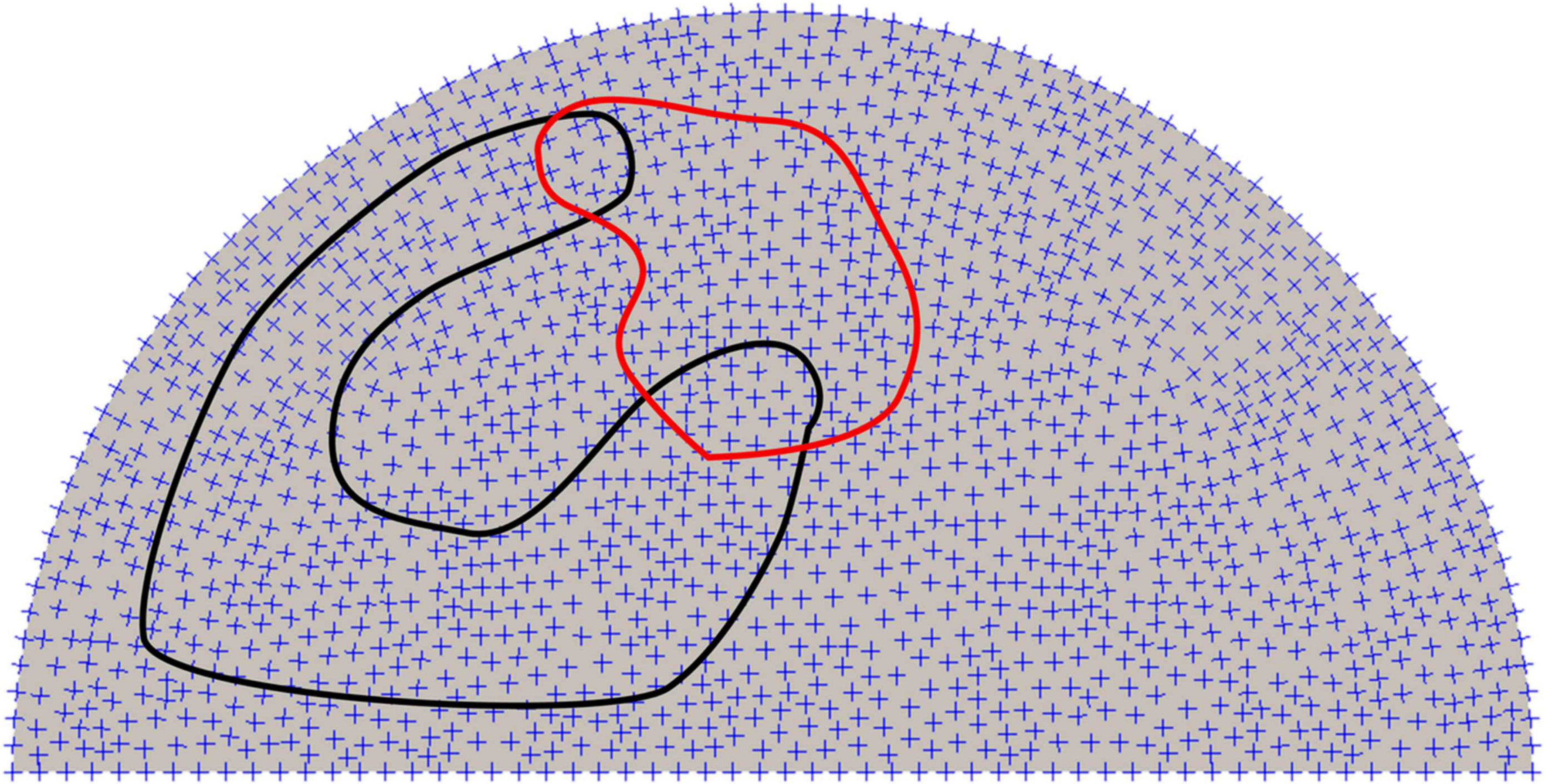


Limit Cycles

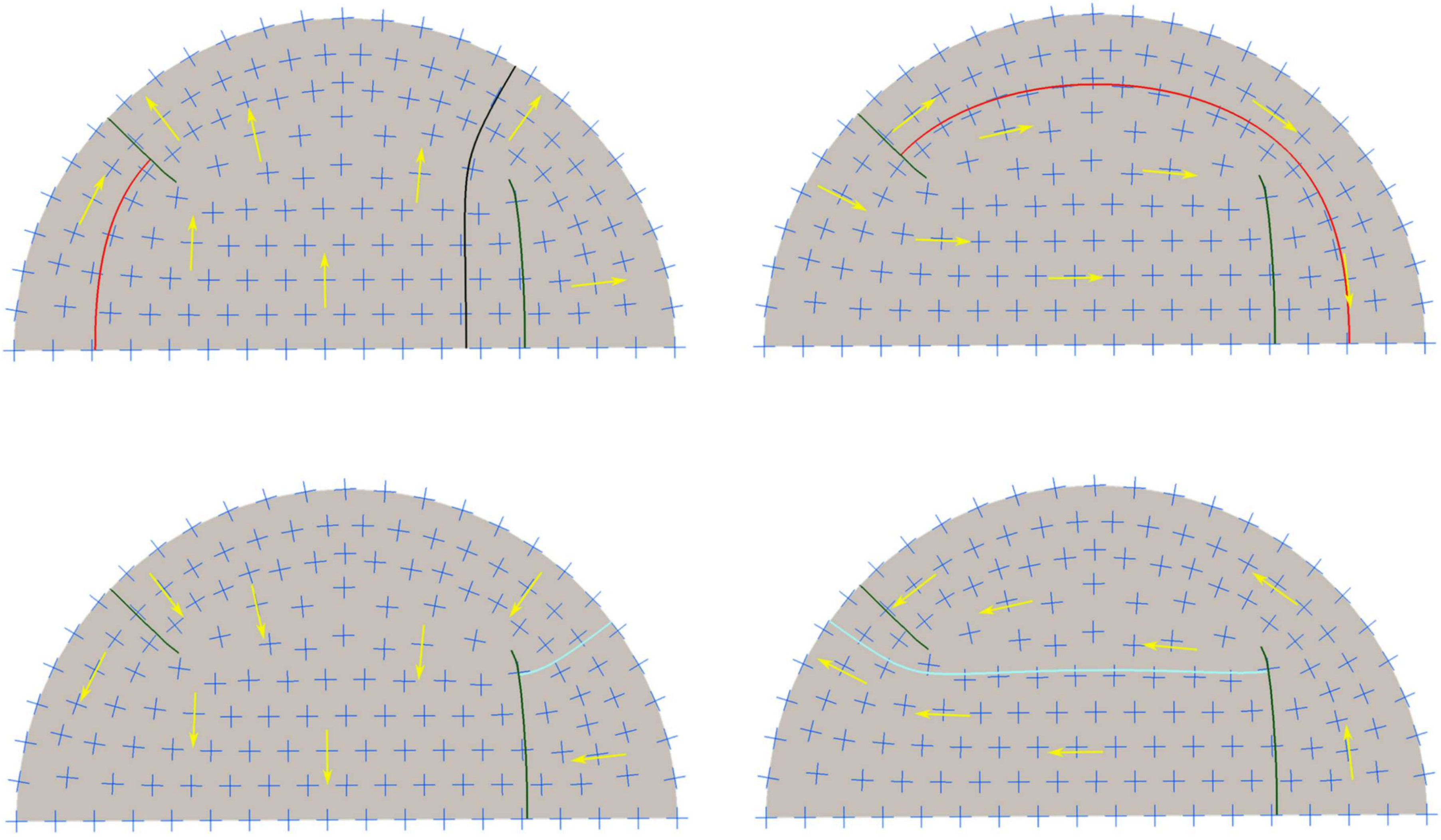




Cross Fields can be Decomposed Locally into 4 Orthogonal Vector Fields



Riemann Surface and Streamlines



Proof

Lemma 5.1. *Let f be a boundary-aligned canonical harmonic cross field on D. Let a be an interior singularity of f of index $d/4$ with $d < 4$. There are exactly $4-d$ separatrices meeting at a. These separatrices partition a neighborhood of a into $4-d$ even-angled sectors.*

$$\left| u_0(z) - \alpha_j \frac{(z-a_j)^{d_j}}{|z-a_j|^{d_j}} \right| \leq C|z-a_j| \quad \text{as} \quad z \to a_j \tag{4}$$

Proof. Let u_0 be the representation vector field for f. Write $z = a + re^{i\theta}$. The estimate (4) gives

$$u_0(z) = \alpha e^{id\theta} + o(r) \qquad \text{for } \theta \in [0, 2\pi). \tag{8}$$

Writing $\alpha = e^{i\theta_0/N}$, the Nth-roots of the $u_0(z)$ are then given by $e^{i(\frac{d\theta+\theta_0}{N} + \frac{2\pi k}{N})}$ for $k \in \mathbb{Z}$.

We seek directions where the vector originating at a and pointing towards z is parallel to a vector originating at the origin and pointing towards any of the *Nth*-roots. Thus we want to solve the equation

$$e^{i\theta} = e^{i(\frac{d\theta+\theta_0}{N} + \frac{2\pi k}{N})} \quad \Longrightarrow \quad \theta = 2\pi k/(N-d) + \theta_0/(N-d) \tag{9}$$

Algorithm

Algorithm 1 Partitioning D into a quad layout with T-junctions.

Input: A domain D satisfying Assumption 3.1, and a boundary-aligned canonical harmonic cross field f with singularities of index $\leq 1/4$.

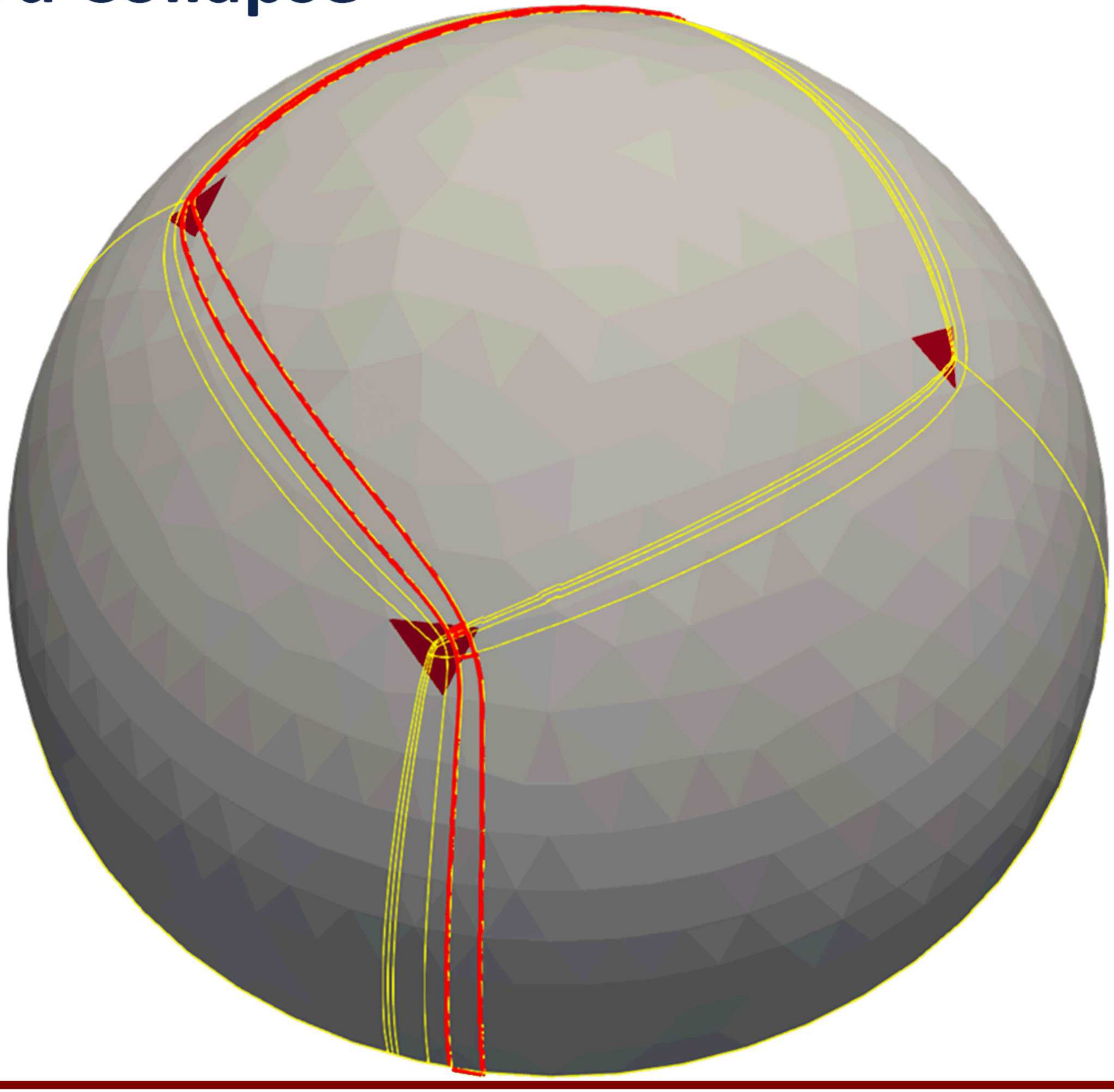
Output: A set $\mathcal{B}$ containing limit cycles and separatrices that define a quad layout with T junctions.

Let $\mathcal{S}$ be the set of separatrices that do not converge to a limit cycle. Let $\mathcal{P}$ be the set of separatrices that do. Let $\mathcal{L}$ be the set of limit cycles.

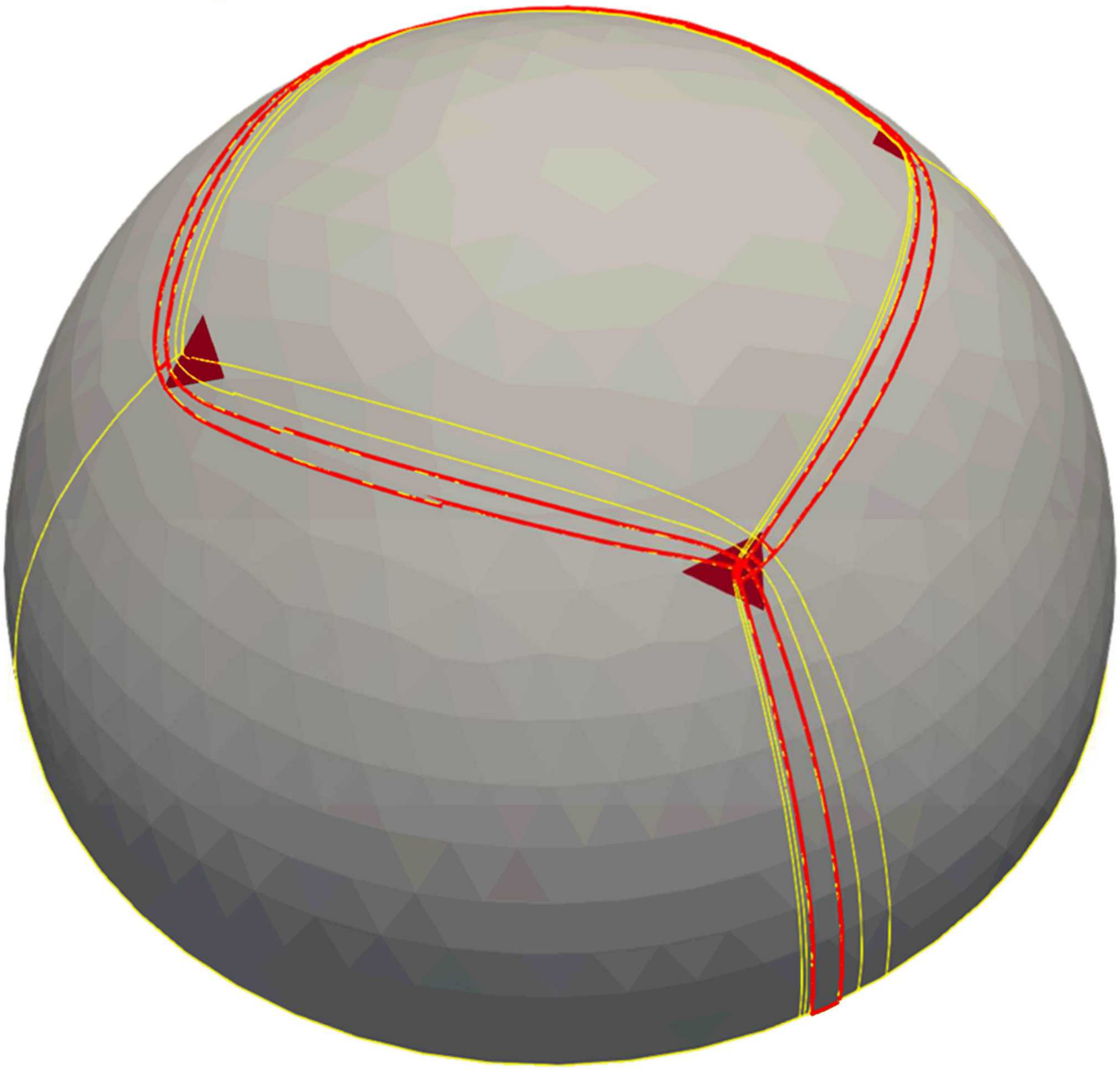
Initialize the set $\mathcal{B} = \mathcal{S}$.

for $l \in \mathcal{L}$ **do**
  if no element of $\mathcal{B}$ intersects l **then**
    (i) Add l to $\mathcal{B}$.
    (ii) By Corollary 5.8, there is an element of $\mathcal{P}$ that intersects l. Let ρ' be the portion of that separatrix beginning at the singularity and ending in a T-junction with l.
    (iii) Add ρ' to $\mathcal{B}$.
    (iv) remove ρ from $\mathcal{P}$.
  end if
end for
for $\rho \in \mathcal{P}$ **do**
  Let ρ' be the curve segment of ρ beginning at the singularity and continuing until it intersects an element of $\mathcal{B}$. Add ρ' to $\mathcal{B}$.
end for

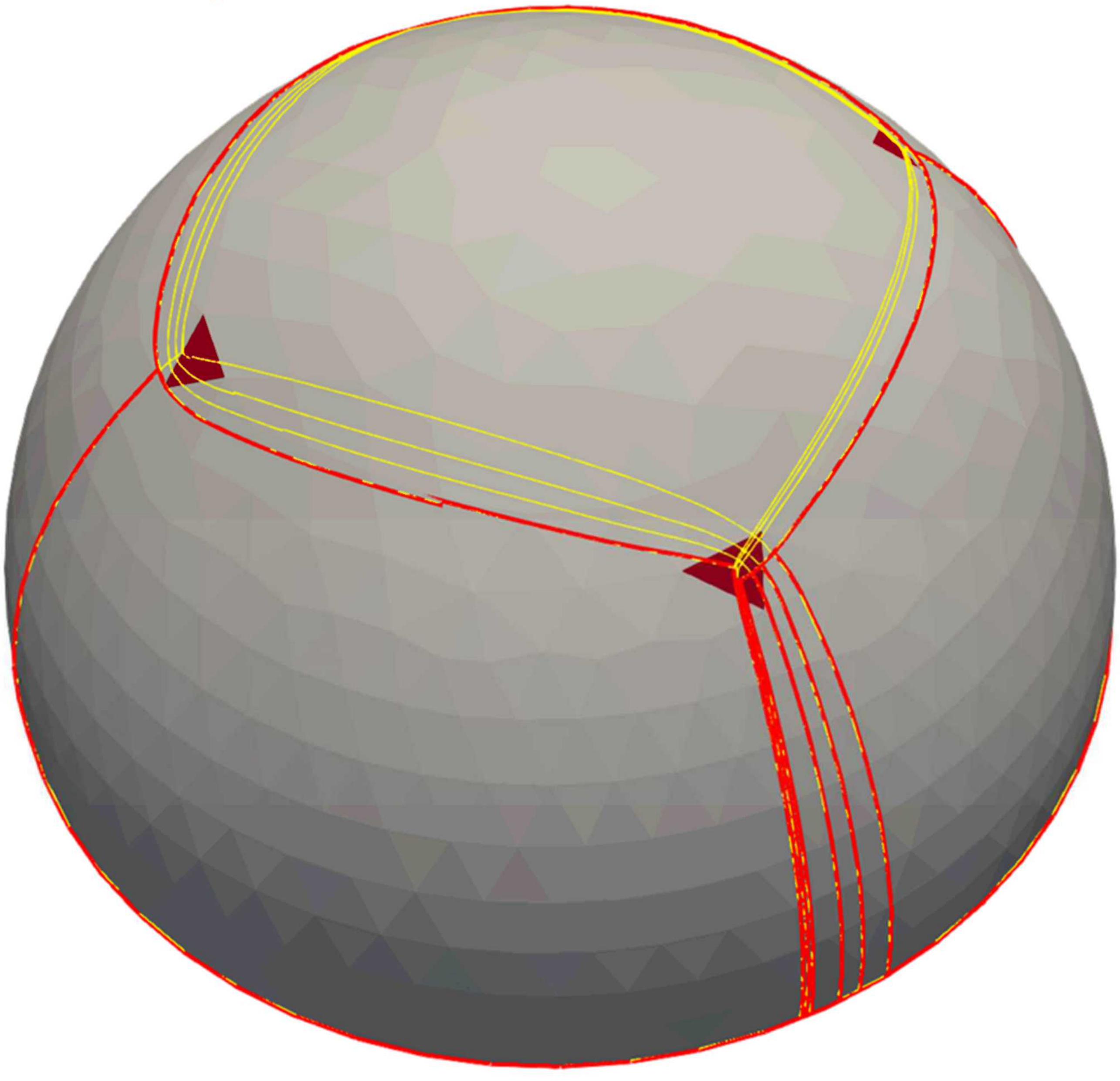
Chord Collapse



Chord Collapse



Chord Collapse



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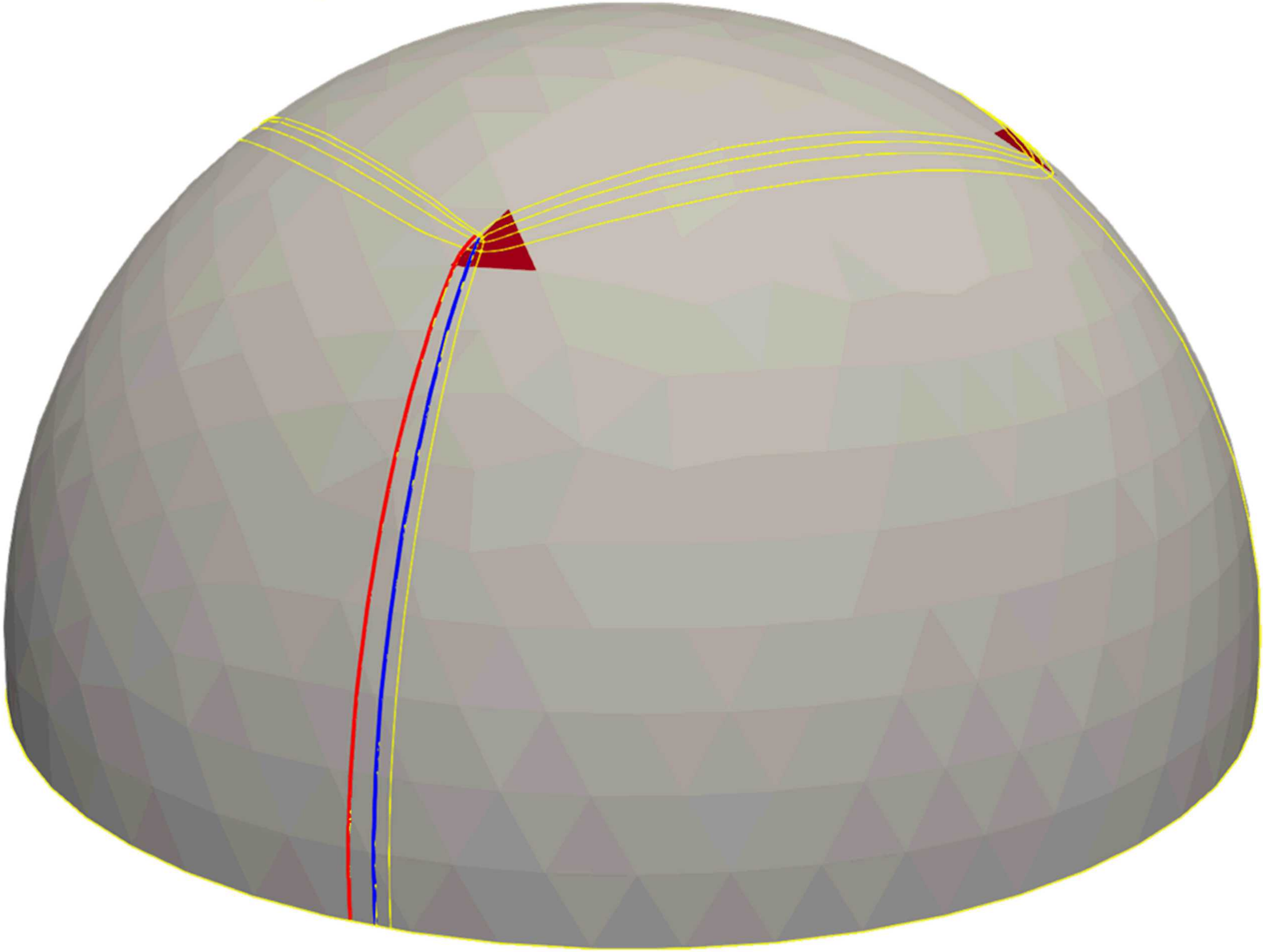

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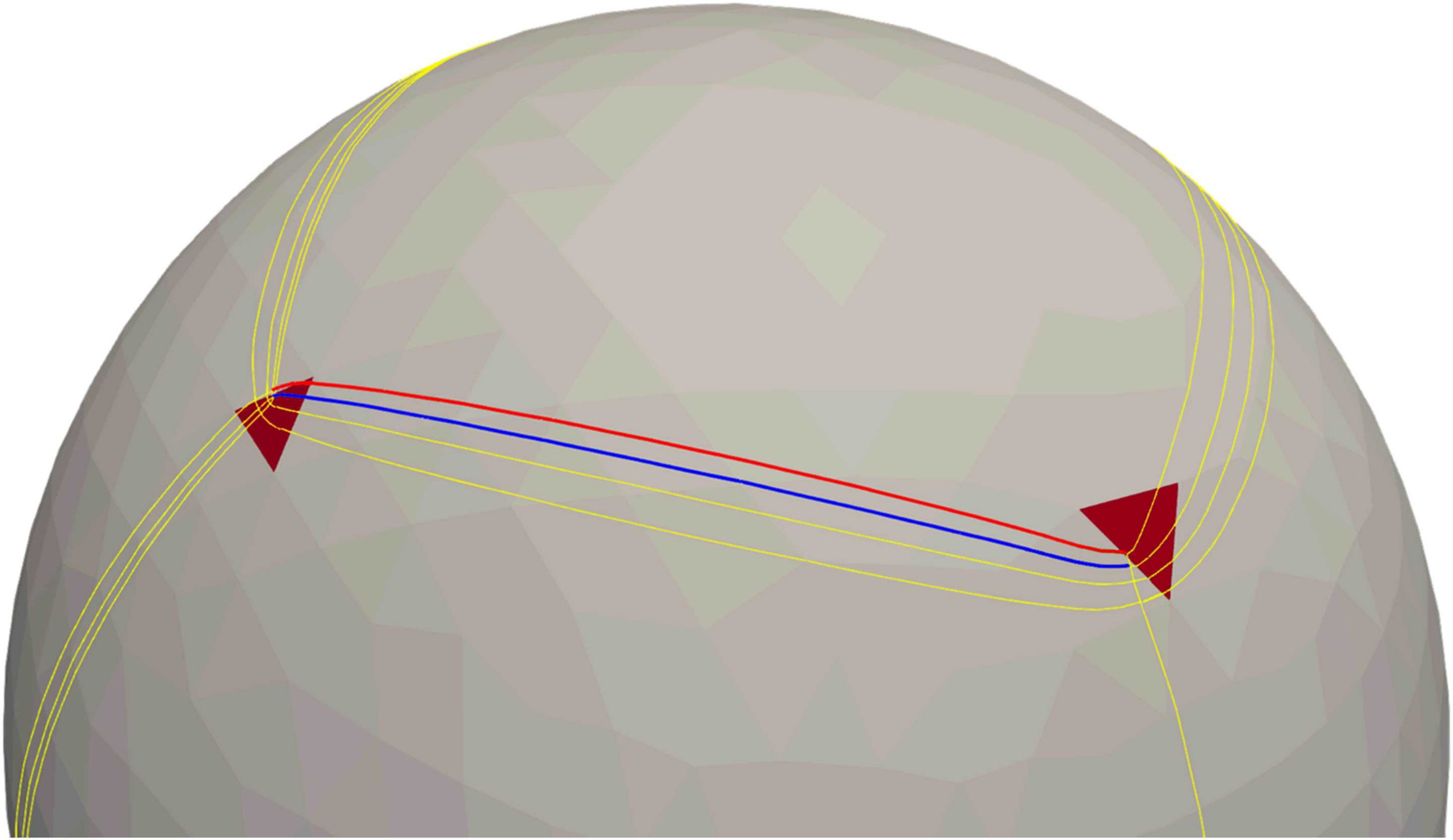
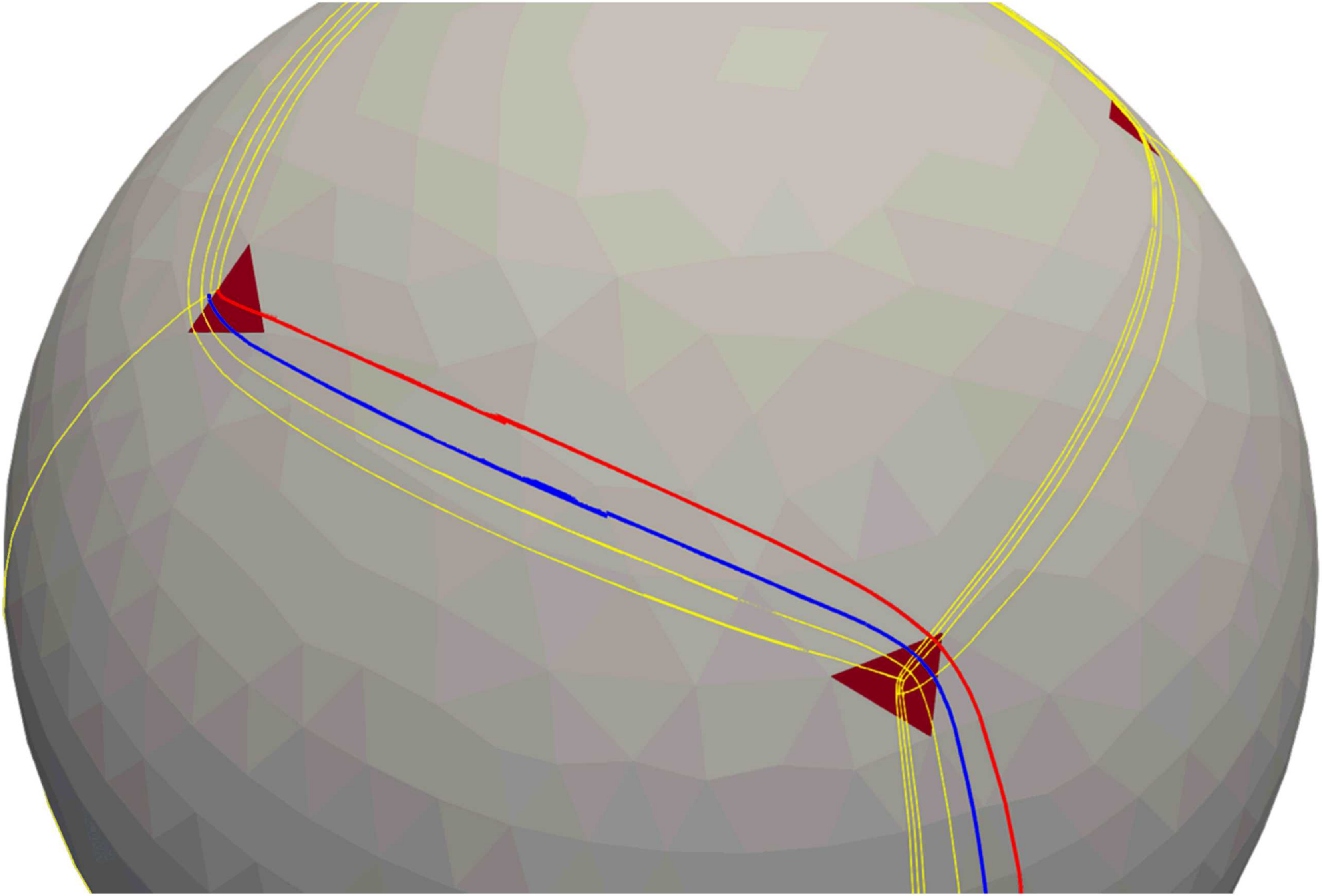
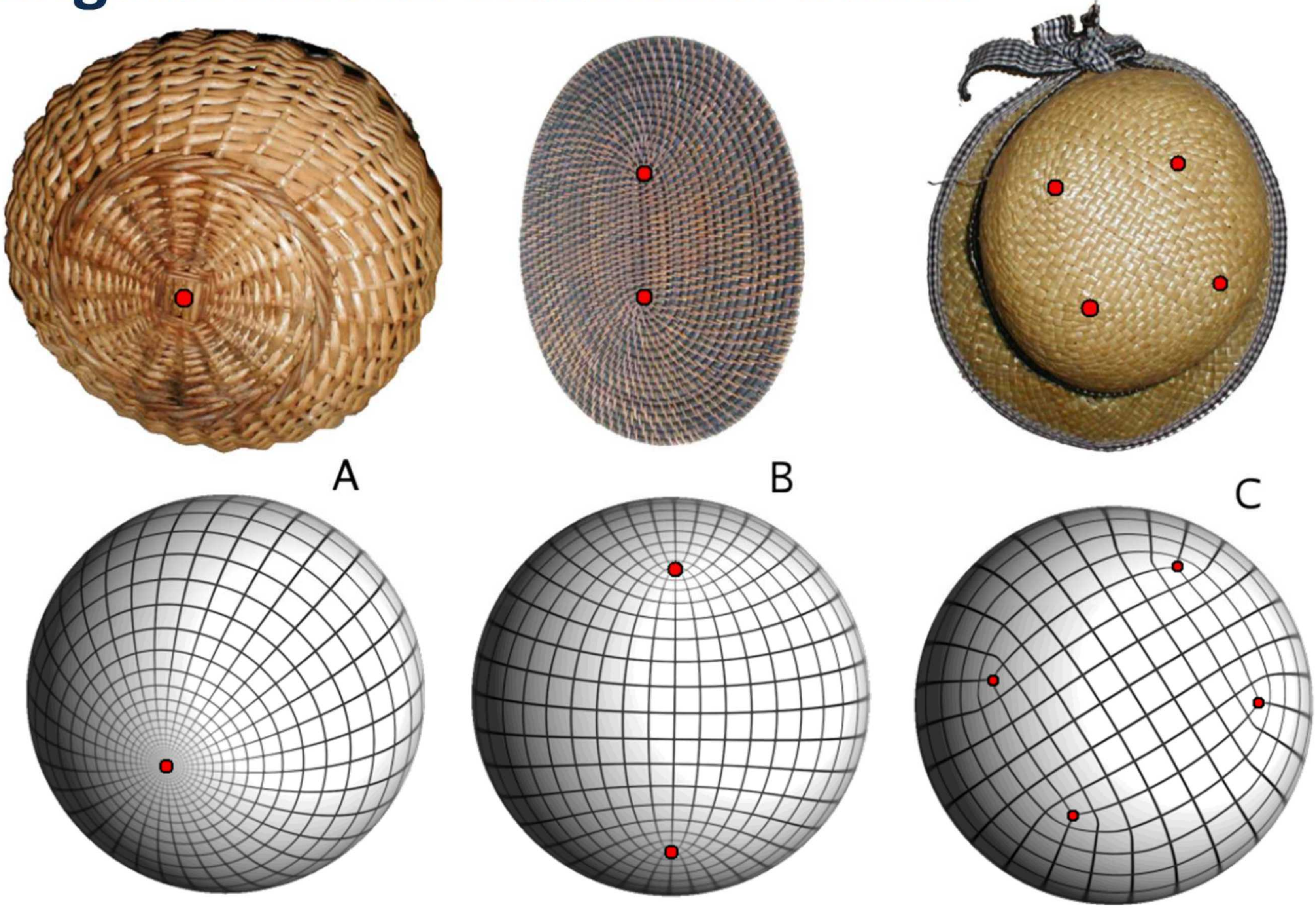


Chord Collapse



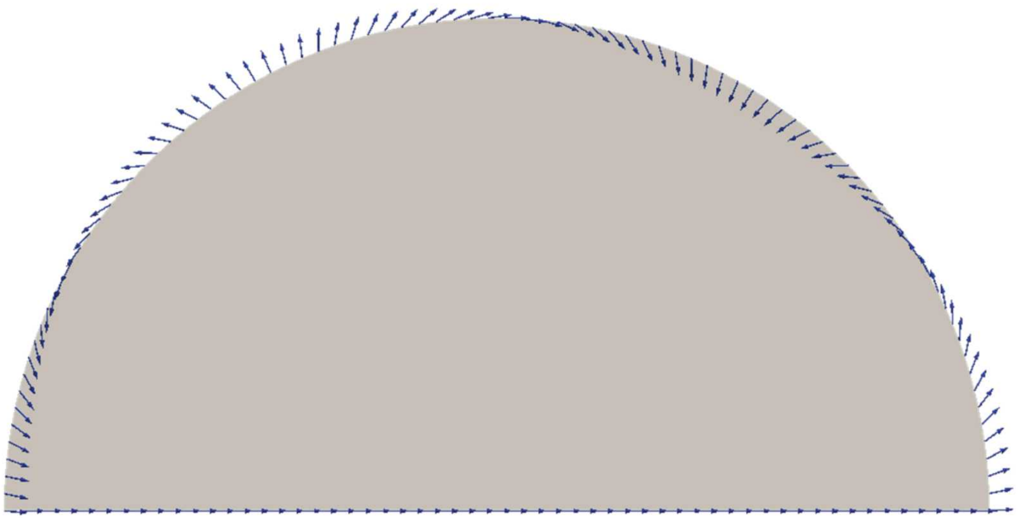
Singularities of Fractional Index



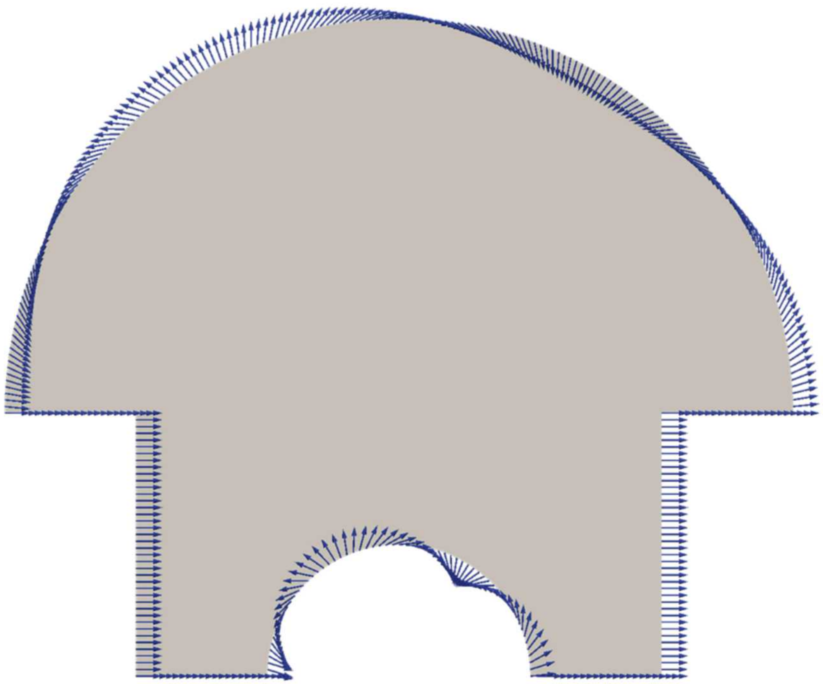
Ray et al. 2006

Brouwer Degree

- Let $g(x)$ be the boundary condition on the domain G.
- Let $d = deg(g, \partial G)$ be the Brouwer degree.



d = 2



d = 0

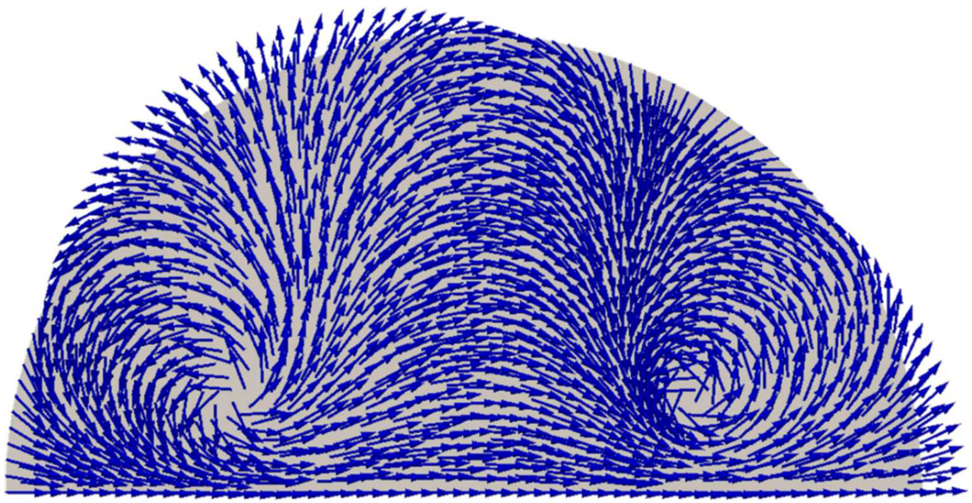
Explicit Formula to Design Field with Fixed Singularities

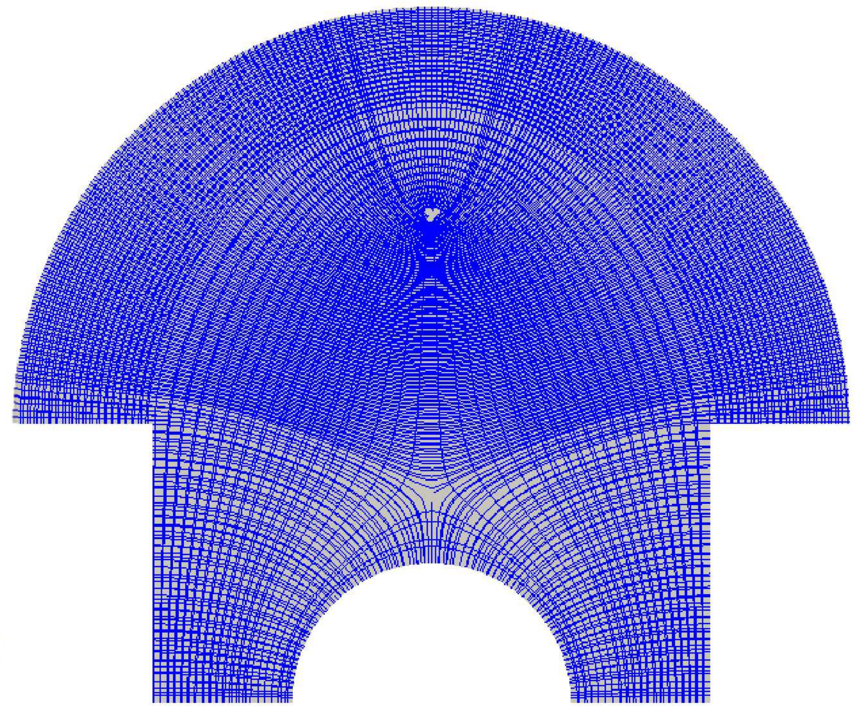
$$e^{i\varphi_0(z)} = g(z)\frac{|z-b_1|^{\alpha_1}}{(z-b_1)^{\alpha_1}}\frac{|z-b_2|^{\alpha_2}}{(z-b_2)^{\alpha_2}}\cdots\frac{|z-b_n|^{\alpha_n}}{(z-b_n)^{\alpha_n}}$$

$$\begin{cases} \Delta\varphi = 0 \text{ in } D \\ \quad \varphi = \varphi_0 \text{ on } \partial D \end{cases}$$

$$u_0 = e^{i\varphi(z)}\frac{(z-b_1)^{\alpha_1}}{|z-b_1|^{\alpha_1}}\frac{(z-b_2)^{\alpha_2}}{|z-b_2|^{\alpha_2}}\cdots\frac{(z-b_n)}{|z-b_n|^{\alpha_n}}$$

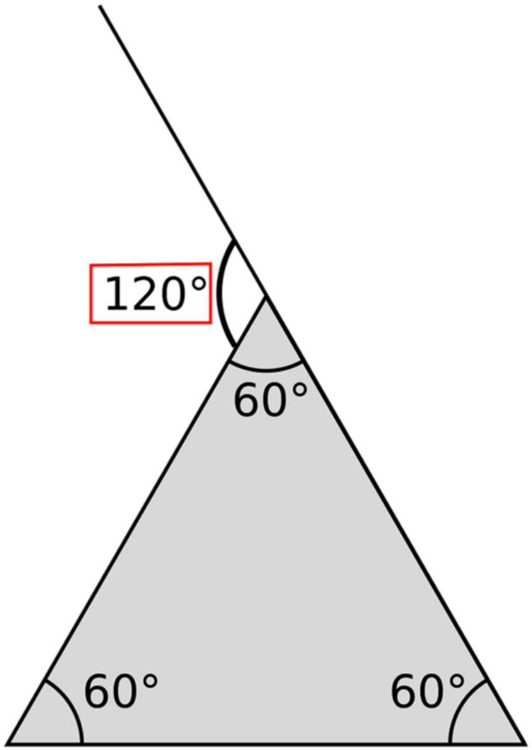
Application: New Cross Field Design Method

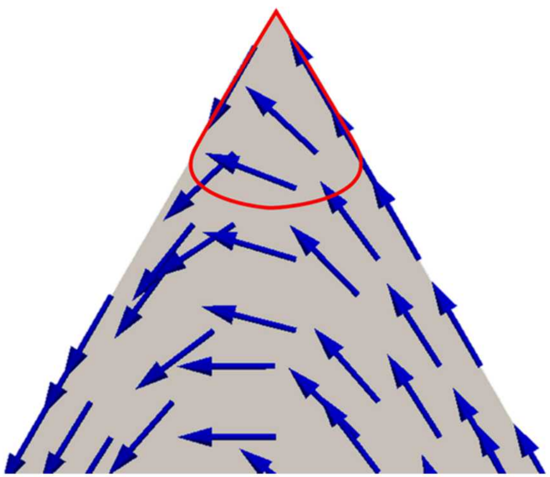


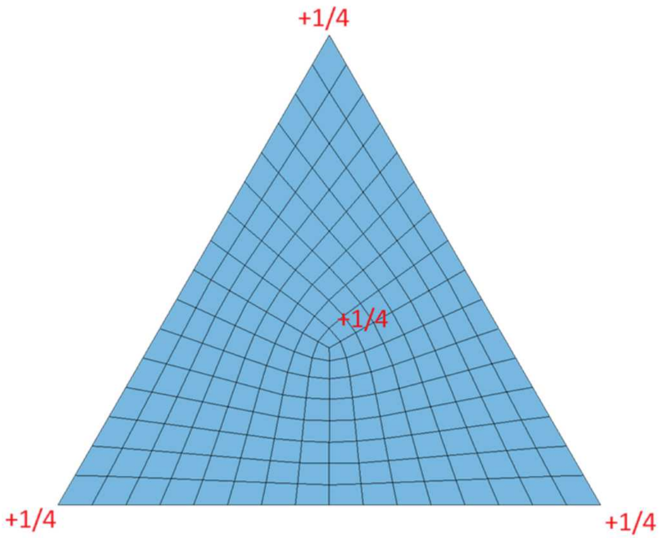


Boundary Singularities

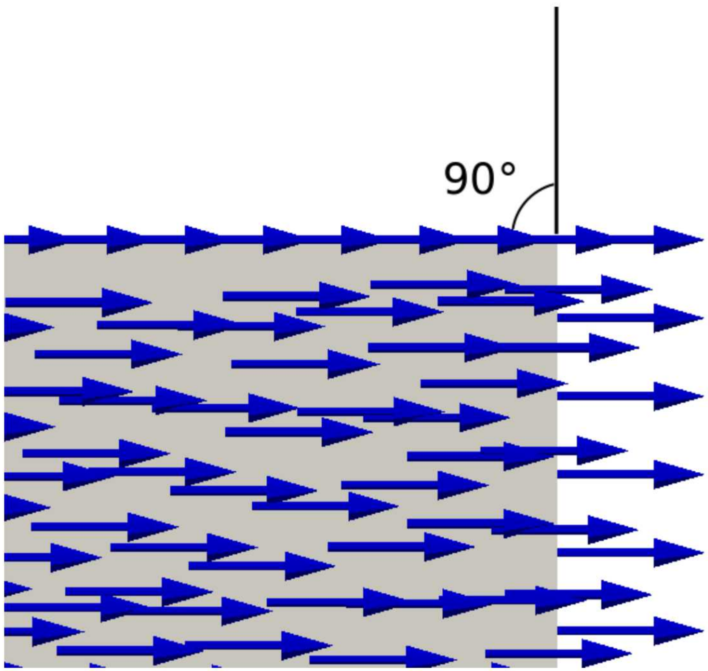
$$\lim_{s \downarrow 0} \frac{\operatorname{dev}(c_i) - \frac{1}{4}\left[\arg R(f(\gamma(s))) - \arg R(f(\gamma(1-s)))\right]}{2\pi}$$

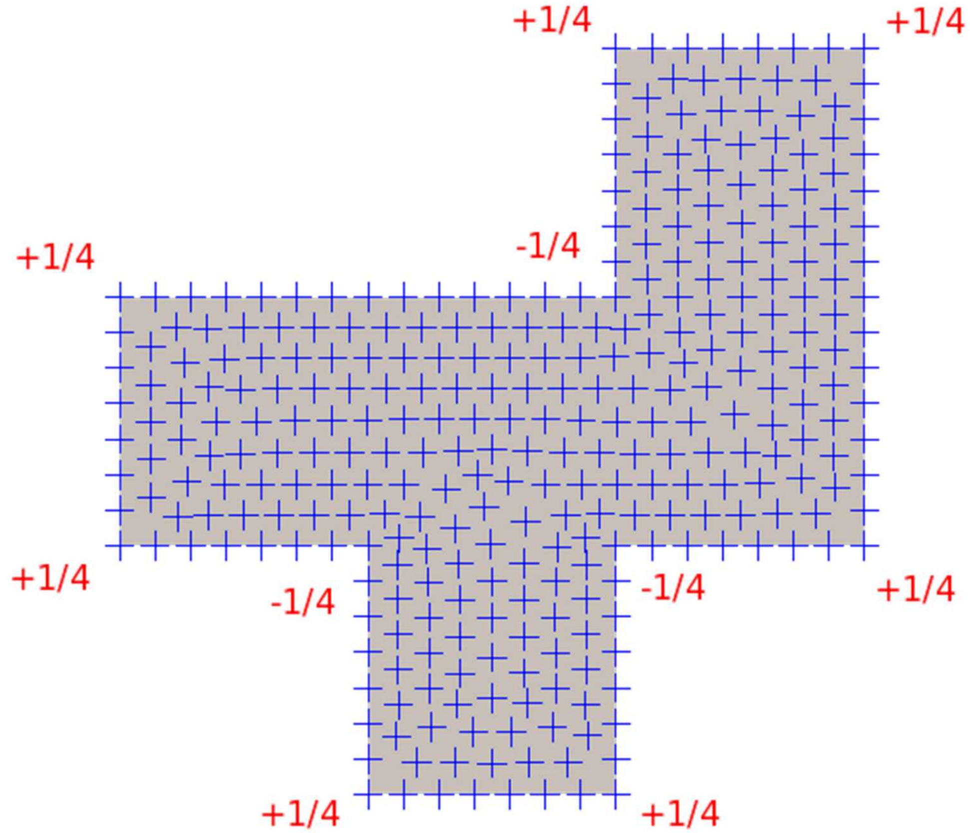






Boundary Singularities





Singularity Indices