SAND2019-8912C

Structural & Material Design for Frequency-Response using Error-in-Constitutive-Equations Approach

Clay Sanders, Timothy Walsh, Julián Norato, Wilkins Aquino

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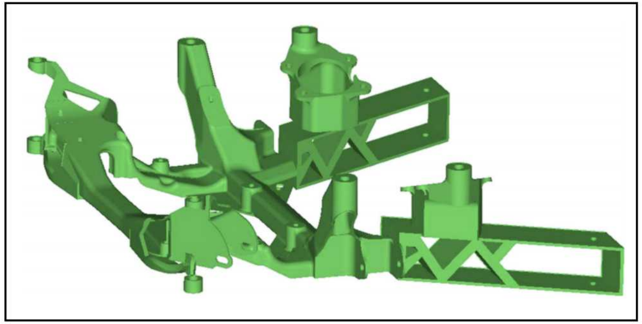
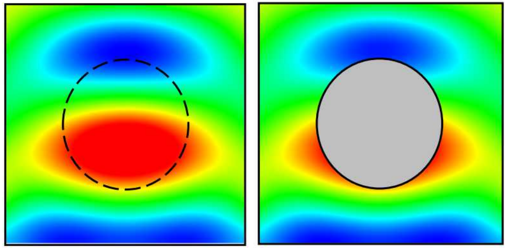
Design of structures & materials for dynamic response

Automotive engine cradle design



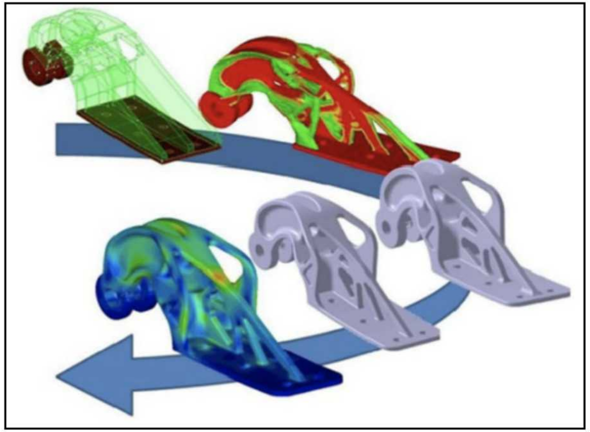
Li et al. 2015

Acoustic Cloak Design



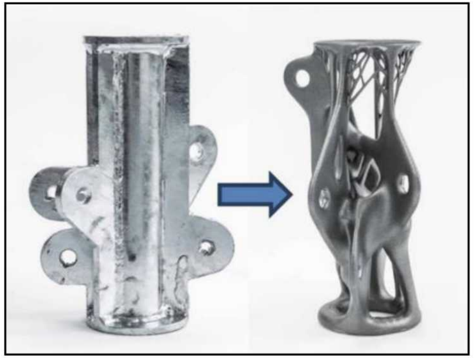
Sanders et al. 2018

Topology optimization (TO) of aircraft bracket



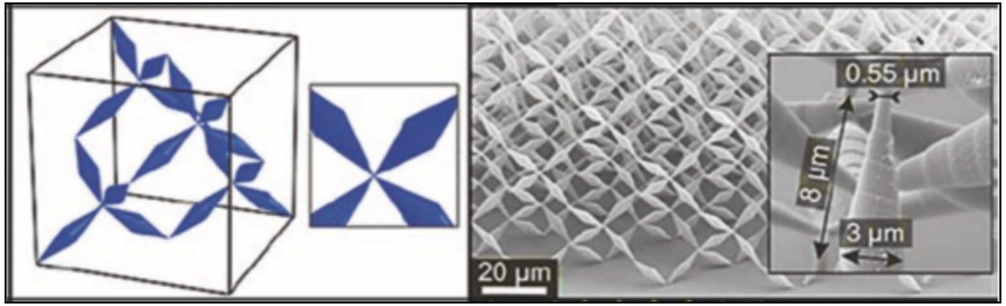
Shapiro et al, 2016

Complex 3D printed part, designed via TO



Shapiro et al, 2016

Pentamode mechanical metamaterials



Tong et al 2018

Overview

- Introduction & Motivation
- Formulation:
  - PDE-constrained optimization for frequency-response design
  - Modified Error-In-Constitutive-Equations Approach for Frequency-Response Design
- Design Results
  - Parameter selection
  - Vibration Isolation
  - Frequency-response matching
- Future Directions

Designing for dynamic response

Dynamic Response Design

- Spectral properties, transient response
- Frequency-domain steady-state response (FRF)

Components of a PDE-constrained design framework:

- Governing PDE constraints
- Design representation
- Objective and/or constraints dependent on state variable
- Algorithmic approach

PDE-CONSTRAINED OPTIMIZATION FOR DESIGN

Frequency Domain Elastodynamics

Governing Elastodynamic Equations in Frequency Domain

$$-\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{\gamma} - \boldsymbol{b} = \boldsymbol{0} \text{ in } \Omega$$

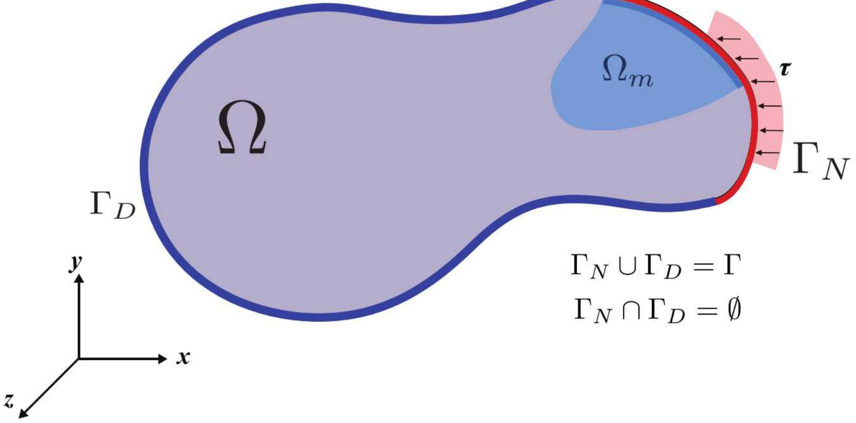
$$\boldsymbol{u} = \boldsymbol{g} \text{ on } \Gamma_D$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{\tau} \text{ on } \Gamma_N$$

$$\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\epsilon}$$ Constitutive Equation

$$\boldsymbol{\epsilon}[\boldsymbol{u}] = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$$

$$\boldsymbol{\gamma} = -\rho\omega^2 \boldsymbol{u}$$ Inertial Constitutive Equation



Variational Weak-Form

Find $\boldsymbol{u} \in \mathcal{U}$ s.t.:

$$\langle \boldsymbol{u}, \boldsymbol{w} \rangle_{\mathbb{C}} - \omega^2(\rho \boldsymbol{u}, \boldsymbol{w}) = \boldsymbol{f}(\boldsymbol{w}) \;\; \forall \boldsymbol{w} \in \mathcal{W}$$

Relevant Function Spaces

$$\mathcal{U} := \{\boldsymbol{u} : \boldsymbol{u} \in H^1(\Omega), \boldsymbol{u} = \boldsymbol{g} \text{ on } \Gamma_D\}$$

$$\mathcal{M} := \{\boldsymbol{\gamma} : \boldsymbol{\gamma} \in H^1(\Omega), \boldsymbol{\gamma} = \boldsymbol{g}_m \text{ on } \Gamma \text{ in } \Omega\}$$

$$\mathcal{S} := \{\boldsymbol{\sigma} : \boldsymbol{\sigma} \in H_{div}, \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{\gamma} \text{ in } \Omega, \boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{\tau} \text{ on } \Gamma_N\}$$

$$\mathcal{W} := \{\boldsymbol{w} : \boldsymbol{w} \in H^1(\Omega), \boldsymbol{w} = \boldsymbol{0} \text{ on } \Gamma_D\}$$

How can we evaluate dynamic behavior?

- **Structural objective**, $\mathcal{D}(\boldsymbol{u})$, dependent on the state (displacements) and quantifies structural performance
- **Frequency response matching**: least-squares misfit between calculated and targeted displacements

$$\begin{aligned}\mathcal{D}(\boldsymbol{u}) &:= \|\boldsymbol{u}-\boldsymbol{u}^t\|^2_{L_2(\Omega_m)} = \int_{\Omega_m} |\boldsymbol{u}-\boldsymbol{u}^t|^2 \, d\Omega_m \\ &= (\boldsymbol{u}-\boldsymbol{u}^t, \boldsymbol{u}-\boldsymbol{u}^t)_{\Omega_m}\end{aligned}$$

- **Vibration isolation**: minimized vibrational amplitude in subdomain, target displacements set to zero
- Multiple frequency cases represented as sum of individual frequency responses
- Static stability enforced through additional load case or inequality constraint

How are designs represented?

We utilize typical topology optimization strategies for density-based design parameterization

- **Density design variable**: pseudo-density field β
- "Analysis density", $\phi(\beta)$, produced through series of **filtering** and **Heaviside-projection operations**
- **Solid Isotropic Microstructure with Penalization** (SIMP): Interpolation between material phases using $\phi(\beta)$, penalizing intermediate densities (Bendsøe 1989)

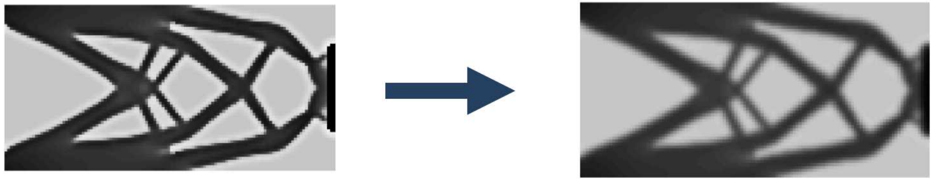
$$
\begin{aligned}
\mathbb{C}(\beta) &:= \mathbb{C}_0 + \Delta\mathbb{C}\,\phi(\beta)^p \\
\rho(\beta) &:= \mathbb{C}_0 + \Delta\rho\,\phi(\beta)^{p_m} \\
p \geq 3&,\ p_m = 1 \\
\Delta\mathbb{C} = (\mathbb{C}_1 - \mathbb{C}_0)&,\ \Delta\rho = (\rho_1 - \rho_0)
\end{aligned}
$$

Filtering & Projection Operations

Density-based design requires additional operations upon design variable to have well-posed problem while obtaining black-white design.

$$\beta \longrightarrow \tilde{\beta}$$

FILTER



Helmholtz PDE Filter (Lazarov 2011):

- Ensure well-posedness, eliminate checkerboarding, mesh-dependency
- Filtered design is solution to "Helmholtz" PDE
- Requires two linear solves per iteration, but utilizes original FE mesh

$$-r^2 \Delta \tilde{\beta} + \tilde{\beta} = \beta \text{ in } \Omega$$

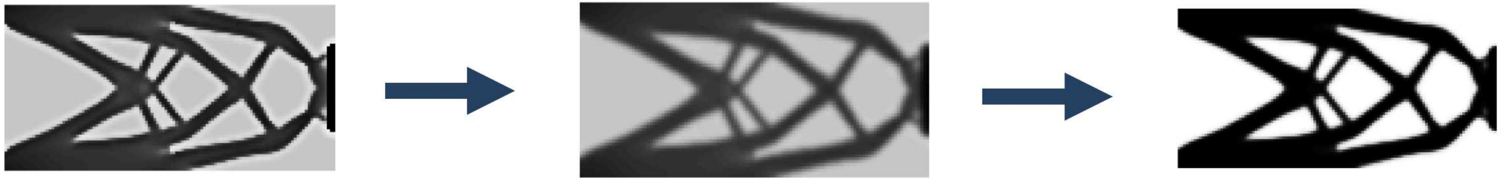
$$\nabla \tilde{\beta} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma$$

Filtering & Projection Operations

Density-based design requires additional operations upon design variable to have well-posed problem while obtaining black-white design

$$\beta \longrightarrow \tilde{\beta} \longrightarrow \phi \longrightarrow \mathbb{C}(\phi(\beta)),\ \rho(\phi(\beta))$$

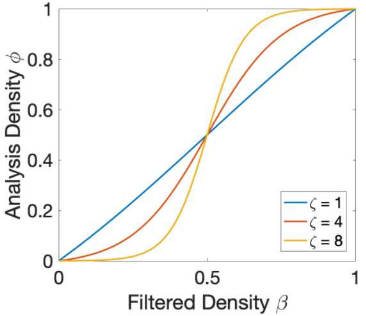
FILTER — HEAVISIDE PROJECTION — MATERIAL MODEL



$$\phi(\tilde{\beta}) = \frac{\tanh(\zeta\eta) + \tanh(\zeta(\tilde{\beta} - \eta))}{\tanh(\zeta\eta) + \tanh(\zeta(1 - \eta))}$$

Heaviside Projection:

- Encourage black/white design
- Project filtered field onto smooth-approximation of Heaviside function (Guest 2011)



PDE-Constrained Design Problem

Define the conventional Least-Squares design problem:

(L2)

$$\min_{\boldsymbol{u}\in\mathcal{U},\beta\in\mathcal{B}} \mathcal{D}(\boldsymbol{u};\beta)$$

$$\text{subject to: Governing PDE + boundary conditions}$$

$$\text{Design Variable Constraints (volume, etc)}$$

Motivation for new design approach…

Specific challenges in FRF design methods:

- L2 objective functions possess singularities, creating local minima
- Solvers face challenges for large, ill-conditioned elastodynamic systems
- Achieving both static stiffness & dynamic behavior can be challenging

Issues faced by design methods for dynamic response are similar to issues in material inversion

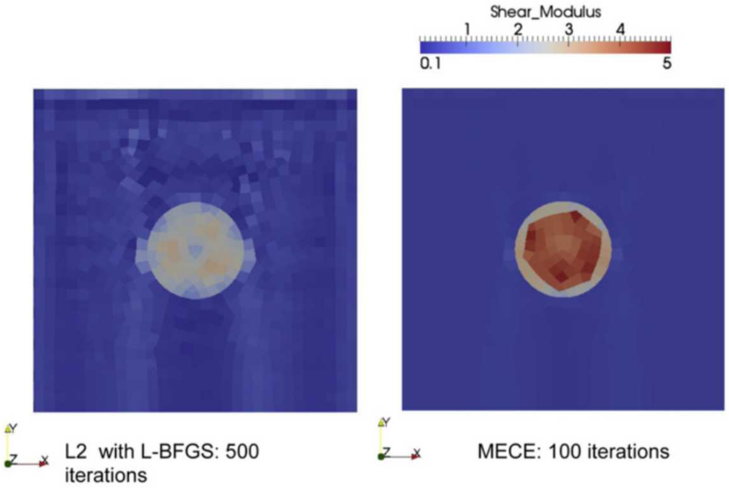
Motivation for new design approach…

In design, we want to match computed behavior **to *desired response***

Modified Error in Constitutive Equations (MECE):

- **Main Idea**: Relax enforcement of constitutive equations, use objective term to enforce relationships
  - Previous applications in material ID for elasticity, ultrasound elastography, viscoelasticity, acoustic structural interaction

- Extend MECE strategy to topology optimization of structures for frequency-domain response



Banerjee et al. (2013)

MECE Objective

Formulation to permit violation of **both constitutive equations** and **dynamical equation**

- Both constitutive parameters & mass density are unknowns

Objective Function Components:

- Sum of ECE, EDE terms, weighted by scalar $\alpha \in (0,1)$ (default, select $\alpha = 0.5$)
- Objective function of state variable, weighted by $\boldsymbol{\kappa}$:

Error in Constitutive Equations (ECE)

$$\mathcal{E}(\boldsymbol{u}, \boldsymbol{\sigma}, \beta) := \int_\Omega (\boldsymbol{\sigma} - \mathbb{C}(\beta) : \epsilon[\boldsymbol{u}]) : \mathbb{C}(\beta)^{-1} : \overline{(\boldsymbol{\sigma} - \mathbb{C}(\beta) : \epsilon[\boldsymbol{u}])} d\Omega$$

Error in Dynamical Equation (EDE)

$$\mathcal{I}(\boldsymbol{u}, \boldsymbol{\gamma}, \beta) := \int_\Omega \frac{1}{\rho(\beta)\omega^2} |\boldsymbol{\gamma} + \rho(\beta)\omega^2 \boldsymbol{u}|^2 d\Omega$$

$$\Lambda(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\gamma}, \beta; \kappa) := \frac{\alpha}{2}\mathcal{E}(\boldsymbol{u}, \boldsymbol{\sigma}, \beta) + \frac{(1-\alpha)}{2}\mathcal{I}(\boldsymbol{u}, \boldsymbol{\gamma}, \beta) + \frac{\kappa}{2}\mathcal{D}(\boldsymbol{u})$$

State objective, weighted by parameter $\boldsymbol{\kappa}$

MECE Design Problem

- Define the minimization problem:

(MECE) $$\boldsymbol{u}^*, \boldsymbol{\sigma}^*, \boldsymbol{\gamma}^*, \beta^* = \underset{\boldsymbol{u}\in\mathcal{U}, \boldsymbol{\sigma}\in\mathcal{S}, \boldsymbol{\gamma}\in\mathcal{M}, \beta\in\mathcal{B}}{\arg\min} \Lambda(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\gamma}, \beta; \kappa)$$ *MECE objective function*

subject to:

$$\frac{1}{V_\Omega}\int_\Omega \phi(\beta) d\Omega - v_{max} \leq 0$$ *s.t. optional volume constraints*

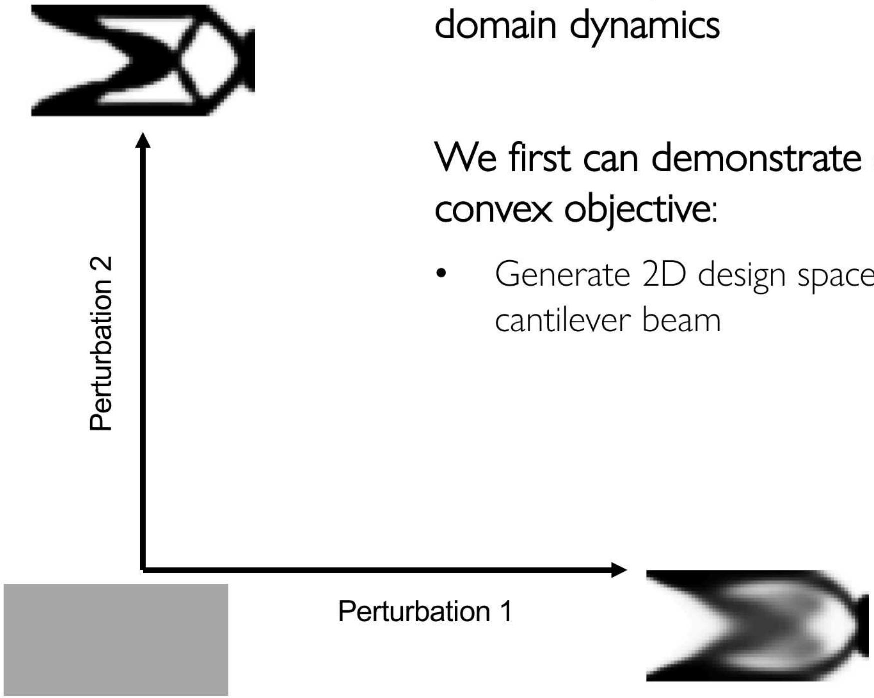
- Constitutive equations ***not explicitly enforced*** as equality constraints, enforced through objective minimization

Motivation for MECE in Design

MECE has key mathematical properties for optimization considering frequency-domain dynamics

We first can demonstrate qualitatively that MECE features "smoother", more-convex objective:

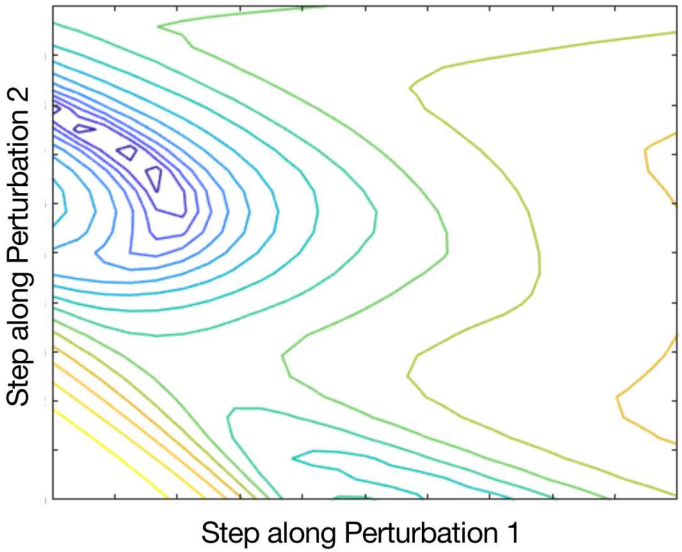
- Generate 2D design space, characterized by two perturbations of design variable for a cantilever beam



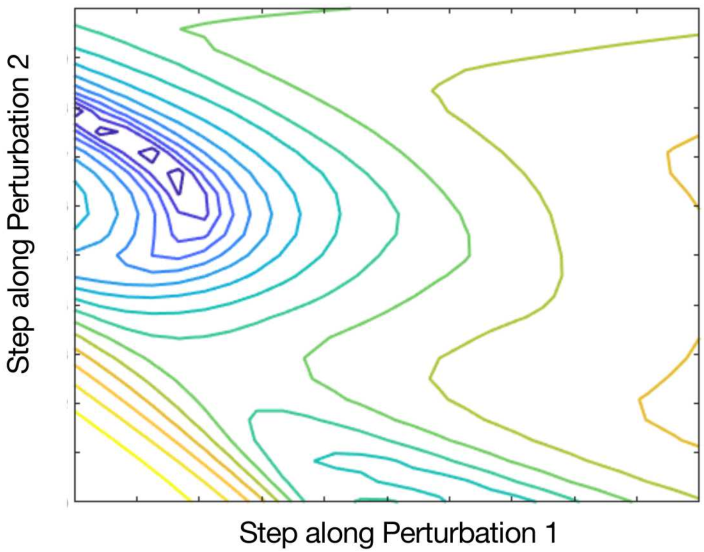
Motivation for MECE in Design

With low $\boldsymbol{\kappa}$ parameter magnitudes, the MECE objective resembles L2 counterpart:

Least-Squares Objective



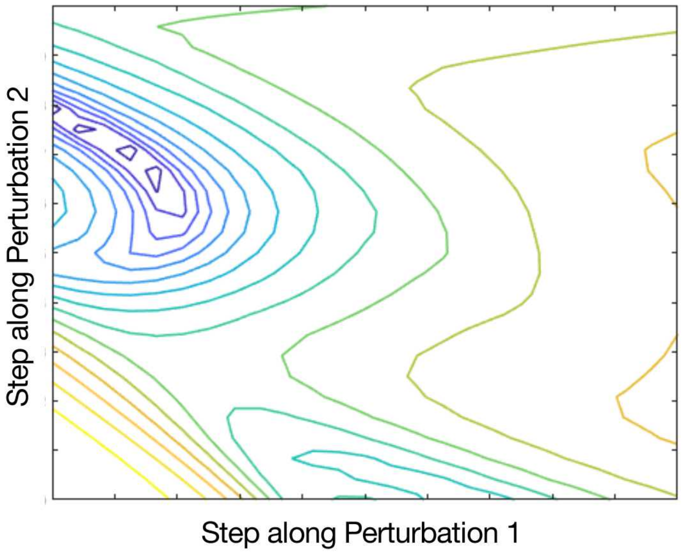
MECE Objective, $\boldsymbol{\kappa} = \mathbf{1e-5}$



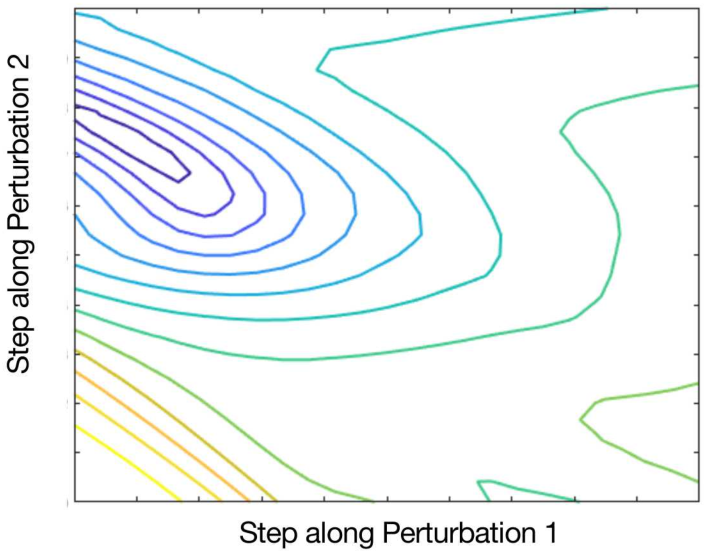
Motivation for MECE in Design

With larger $\boldsymbol{\kappa}$ parameters, the objective becomes smoother and more convex:

Least-Squares Objective



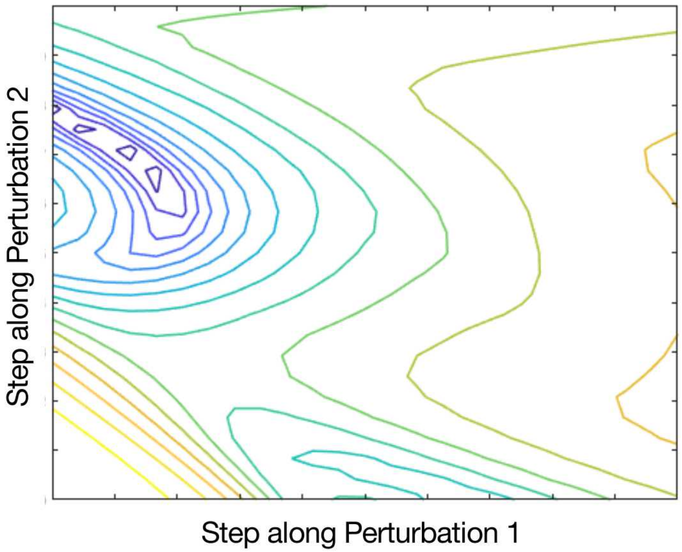
MECE Objective, $\boldsymbol{\kappa} = \mathbf{1e-1}$



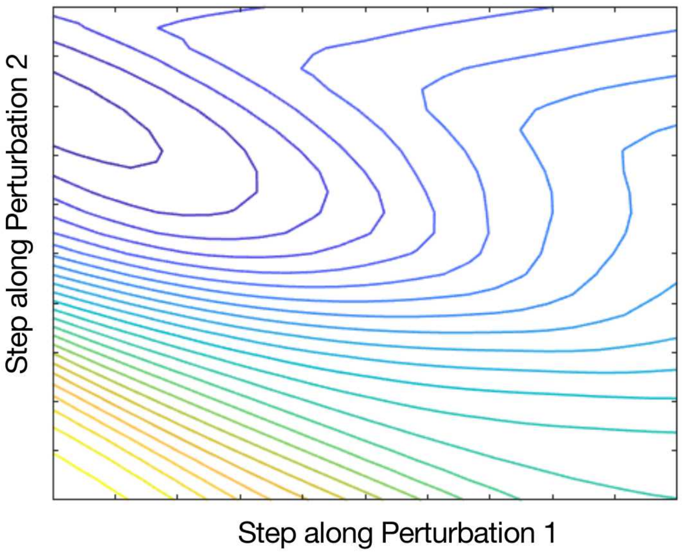
Motivation for MECE in Design

MECE thus exhibits reduced risk of convergence to poor local minima, though can introduce intolerable error for large $\boldsymbol{\kappa}$.

Least-Squares Objective



MECE Objective, $\boldsymbol{\kappa} = \mathbf{1e+3}$



Optimality Conditions

First order optimality conditions require the solution of a pair of coupled variational problems, fundamental to MECE

Define Lagrangian: $\mathcal{L}(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\gamma}, \beta, \boldsymbol{w}) := \Lambda(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\gamma}, \beta) - \mathbb{R}((\boldsymbol{\sigma}, \boldsymbol{\epsilon}[\boldsymbol{w}]) + (\boldsymbol{\gamma}, \boldsymbol{w}) - F(\boldsymbol{w}))$

where $\boldsymbol{w} \in \mathcal{W} := \{\boldsymbol{w} : \boldsymbol{w} \in H^1(\Omega), \boldsymbol{w} = \mathbf{0} \text{ on } \Gamma_D\}$

First-order necessary Karush-Kahn-Tucker (KKT) optimality conditions require:

$$\mathcal{L}'_\sigma[\hat{\boldsymbol{\sigma}}] = \mathcal{L}'_\gamma[\hat{\boldsymbol{\gamma}}] = \mathcal{L}'_u[\hat{\boldsymbol{u}}] = \mathcal{L}'_w[\hat{\boldsymbol{w}}] = \mathcal{L}'_\beta[\hat{\beta}] = 0$$

$$\text{where: } \mathcal{L}'_\sigma[\hat{\boldsymbol{\sigma}}] := \frac{d}{dt}\mathcal{L}(\boldsymbol{u}, \boldsymbol{\sigma} + t\hat{\boldsymbol{\sigma}}, \boldsymbol{\gamma}, \beta, \boldsymbol{w})|_{t=0}$$

Optimality Conditions

Evaluating derivatives with respect to stresses and inertial forces:

$$\mathcal{L}'_\sigma[\hat{\boldsymbol{\sigma}}] = 0 \ \forall \hat{\boldsymbol{\sigma}} \in \mathcal{S},\ \mathcal{L}'_\gamma[\hat{\gamma}] = 0 \ \forall \hat{\gamma} \in \mathcal{M}$$

yields identities:

$$\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\epsilon}[\boldsymbol{u} + \frac{1}{\alpha}\boldsymbol{w}]$$

$$\boldsymbol{\gamma} = -\rho\omega^2 \left(\boldsymbol{u} - \frac{1}{(1-\alpha)}\boldsymbol{w}\right)$$

Optimality Conditions

Setting $\mathcal{L}'_u[\hat{\boldsymbol{u}}] = 0 \ \forall \hat{\boldsymbol{u}} \in \mathcal{U}, \ \mathcal{L}'_w[\hat{\boldsymbol{w}}] = 0 \ \forall \hat{\boldsymbol{w}} \in \mathcal{W}$ and substituting the expressions for stresses and inertial forces yields a **coupled system of variational equations:**

$$\begin{aligned} a(\hat{\boldsymbol{u}}, \boldsymbol{w}) \quad & -\kappa d(\boldsymbol{u}, \hat{\boldsymbol{u}}) && = -\kappa d(\boldsymbol{u}^t, \hat{\boldsymbol{u}}), \ \forall \hat{\boldsymbol{u}} \in \mathcal{W} \\ a_m(\boldsymbol{w}, \hat{\boldsymbol{w}}) \quad & +a(\boldsymbol{u}, \hat{\boldsymbol{w}}) && = \boldsymbol{f}(\hat{\boldsymbol{w}}), \ \forall \hat{\boldsymbol{w}} \in \mathcal{W} \end{aligned}$$

where:

$$a(\boldsymbol{v}, \boldsymbol{v}) := \langle \boldsymbol{v}, \boldsymbol{v} \rangle_{\mathbb{C}} - \omega^2 (\rho \boldsymbol{v}, \boldsymbol{v})$$

$$a_m(\boldsymbol{v}, \boldsymbol{v}) := \frac{1}{\alpha} \langle \boldsymbol{v}, \boldsymbol{v} \rangle_{\mathbb{C}} + \frac{\omega^2}{(1-\alpha)} (\rho \boldsymbol{v}, \boldsymbol{v})$$

Finally, setting $\mathcal{L}'_\beta[\hat{\beta}] = 0, \forall \hat{\beta} \in \mathcal{B}$ yields:

$$\mathcal{L}'_\beta[\hat{\beta}] = -(\boldsymbol{\sigma}, \mathbb{C}^{-1} : \hat{\mathbb{C}} : \mathbb{C}^{-1} : \boldsymbol{\sigma}) + (\boldsymbol{\epsilon}[\boldsymbol{u}], \hat{\mathbb{C}} : \boldsymbol{\epsilon}[\boldsymbol{u}]) - \frac{1}{\omega^2} \left(\frac{\hat{\rho}}{\rho^2} \boldsymbol{\gamma}, \boldsymbol{\gamma} \right) + \omega^2 (\hat{\rho} \boldsymbol{u}, \boldsymbol{u})$$

$$\hat{\mathbb{C}} := \mathbb{C}'_\phi[\hat{\phi}], \hat{\rho} := \rho'_\phi[\hat{\phi}]$$

Reduced Optimization Problem

We use a **reduced-space formulation** to solve the optimization problem:

$$\min_{\beta \in \mathcal{B}} \tilde{\Lambda}(\beta) := \Lambda(\boldsymbol{u}_\beta, \boldsymbol{\sigma}(\boldsymbol{u}_\beta, \boldsymbol{w}_\beta), \boldsymbol{\gamma}(\boldsymbol{u}_\beta, \boldsymbol{w}_\beta), \beta; \kappa)$$

- **Reduced Objective**: Substitution of stress, mass inertia expressions yield ECE functionals dependent only on $\boldsymbol{w}_{\boldsymbol{\beta}}$:

$$\tilde{\Lambda}(\beta) := \frac{1}{2\alpha}\tilde{\mathcal{E}}(\boldsymbol{w}_\beta, \beta) + \frac{1}{2(1-\alpha)}\tilde{\mathcal{I}}(\boldsymbol{w}_\beta, \beta) + \frac{\kappa}{2}\mathcal{D}(\boldsymbol{u}_\beta)$$

$$\tilde{\mathcal{E}}(\beta) := \langle \boldsymbol{w}_\beta, \boldsymbol{w}_\beta \rangle_{\mathbb{C}(\beta)}$$

$$\tilde{\mathcal{I}}(\beta) := \omega^2(\rho(\beta)\boldsymbol{w}_\beta, \boldsymbol{w}_\beta)$$

Discretization

- Introduce a Galerkin approximation of the displacements and Lagrange multipliers:

$$\boldsymbol{u}^h = [N]\{u\}, \ \boldsymbol{w}^h = [N]\{w\}$$
$$\hat{\boldsymbol{u}}^h = [N]\{\hat{u}\}, \ \hat{\boldsymbol{w}}^h = [N]\{\hat{w}\}$$

- Element-wise design variable. $\{\beta\}$ and analysis density $\{\phi(\beta)\}$
- Substitution of Galerkin approx. and discretization yields <u>2N x 2N coupled system</u> :

$$\begin{bmatrix} A & -\kappa Q \\ A_m & A \end{bmatrix} \left\{ \begin{array}{c} \{w^h\} \\ \{u^h\} \end{array} \right\} = \left\{ \begin{array}{c} -\kappa\{R\} \\ \{F\} \end{array} \right\}$$

$$[A] := [K] - \omega^2[M], \ [A_m] := \frac{1}{\alpha}[K] + \frac{\omega^2}{(1-\alpha)}[M]$$

$$\{R\} := [Q]\{\hat{u}^t\}$$

Reduced Objective & Gradient Discretization

- Discrete reduced objective:

$$\tilde{\Lambda}(\{\beta\}) = \frac{1}{2\alpha}\{w^h\}^T[K]\{w^h\} + \frac{\omega^2}{2(1-\alpha)}\{w^h\}^T[M]\{w^h\} + \frac{\kappa}{2}\{u^h - u^t\}^T[Q]\{u^h - u^t\}$$

- Reduced gradient: element-wise calculation

$$\begin{aligned}\tilde{\Lambda}'_e(\{\phi\}) := &- \{u^h\}_e^T[K'_e]\{w^h\}_e - \frac{1}{2\alpha}\{w^h\}_e^T[K'_e]\{w^h\}_e \\ &+ \omega^2\{u^h\}_e^T[M'_e]\{w^h\}_e - \frac{\omega^2}{2(1-\alpha)}\{w^h\}_e^T[M'_e]\{w^h\}_e\end{aligned}$$

- Heaviside projection gradient: element-wise calculation
- Adjoint PDE-filter linear solve: $\tilde{\Lambda}'_e(\{\beta\}) := [T]^T[H_f]^{-1}[T]\{\phi'_{\tilde{\beta}}\}(\tilde{\Lambda}'_e(\{\phi\})$

Key Mathematical Properties of MECE

The coupled system is well-posed, even at resonant frequencies:

– We assume that that modes excited in structure are "measured" by structural objective.

MECE objective has *improved* convexity:

- For linear-elastic material inversion, MECE objective is globally convex as $\kappa \to +\infty$, with a strictly positive-definite Hessian
- MECE Hessian using SIMP can be sign-indefinite due to high-order material model derivatives & non-convex Heaviside operations

Key Asymptotic Properties of MECE

Analysis of the MECE problem in asymptotic cases $\boldsymbol{\kappa} \to +\infty$, $\boldsymbol{\kappa} \to 0$ illuminates key properties:

- Large-$\boldsymbol{\kappa}$:
  - Displacements are enforced to match target behavior: $\boldsymbol{u} = \boldsymbol{u}^t \text{ on } \Omega_m$
  - The objective is dominated by ECE terms: $\tilde{\Lambda}(\beta) = \frac{1}{\alpha}\tilde{\mathcal{E}}(\boldsymbol{w}_\beta, \beta) + \frac{1}{(1-\alpha)}\mathcal{I}(\boldsymbol{w}_\beta, \beta)$
- Small-$\boldsymbol{\kappa}$:
  - The objective is dominated by the original least-squares structural objective:

$$\boldsymbol{w} \to 0,\ \tilde{\Lambda}(\beta) = \frac{1}{2}\kappa\mathcal{D}(\boldsymbol{u}_\beta) + o(\kappa)$$

  - Displacements $\boldsymbol{u}$ solve original elastodynamic forward problem:

$$a(\boldsymbol{u}, \hat{\boldsymbol{u}}) = f(\hat{\boldsymbol{u}})\ \forall \hat{\boldsymbol{u}} \in \mathcal{W}$$

EXAMPLES

- PARAMETER SELECTION
- VIBRATION ISOLATION
- FREQUENCY RESPONSE MATCHING

$\boldsymbol{\kappa}$ Parameter Selection

Penalty parameter $\boldsymbol{\kappa}$ controls level of constitutive equation violation

- In material ID, selected based upon level of measurement noise (e.g. using Morozov Principle)

Proposed Approach: Error-Balance

- Select optimal parameter which balances error, state-objective terms, measured by error-balance objective:

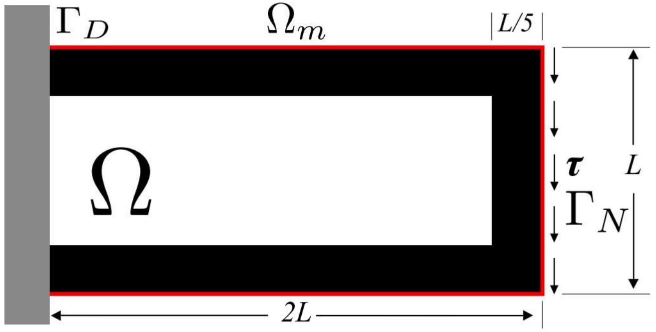
$$B(\kappa, \beta_\kappa^*) = \left[\alpha \mathcal{E}(\kappa, \beta_\kappa^*) + (1 - \alpha)\mathcal{I}(\kappa, \beta_\kappa^*)\right]^2 + (\mathcal{D}(\kappa, \beta_\kappa^*))^2$$

- Solve series of optimization problems for optimal $\boldsymbol{\kappa}$ minimizing error-balance objective

  - Approximate minimizer occurs when $\kappa = \dfrac{\mathcal{D}(\boldsymbol{u}_\kappa^*)}{\mathcal{E}(\beta_\kappa^*, \boldsymbol{u}_\kappa^*)}$

FRF Match: Design Reconstruction

TARGET STRUCTURE



Goal:

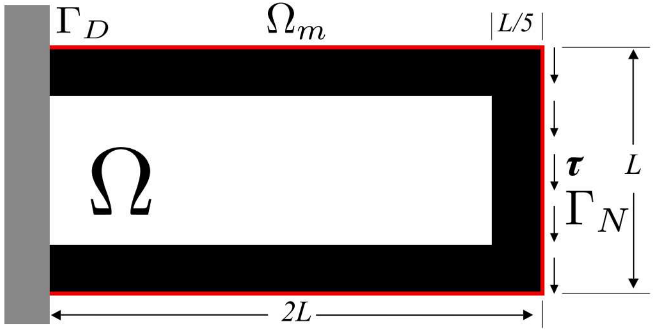
Demonstrate ability of MECE to design a structure which can match a target FRF

Problem Objective:

Match exterior measurements of the target "frame" structure with the designed structure's displacements under static and multiple dynamic loading conditions.

FRF Match: Design Reconstruction

TARGET STRUCTURE



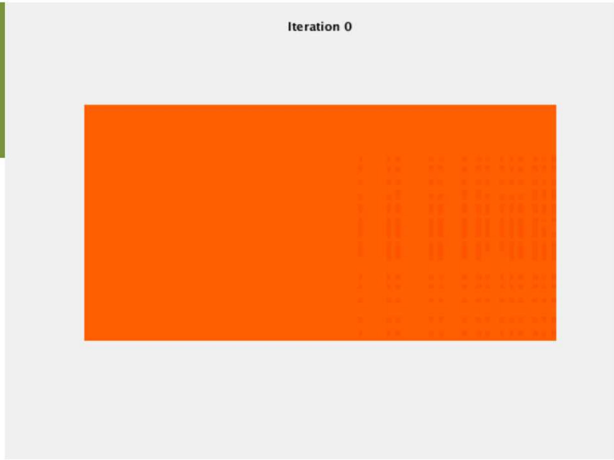
Goal:

Demonstrate ability of MECE to design a structure which can match a target FRF

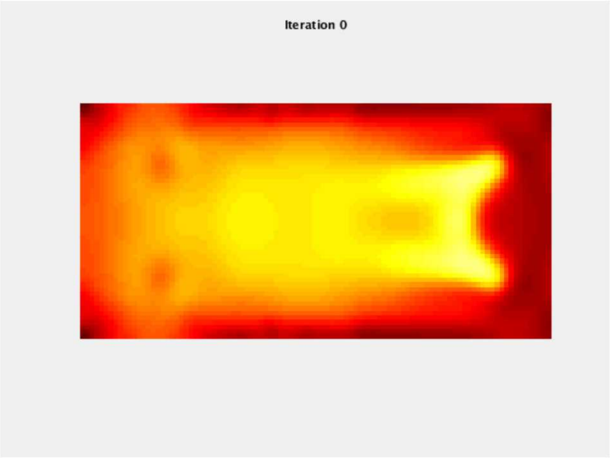
Problem Objective:

Match static and dynamic exterior measurements of the target "frame" structure with the designed structure's displacements at multiple frequencies

LEAST-SQUARES DESIGNED STRUCTURE

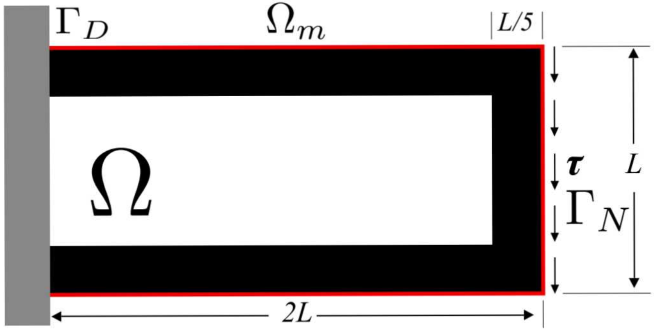


MECE DESIGNED STRUCTURE



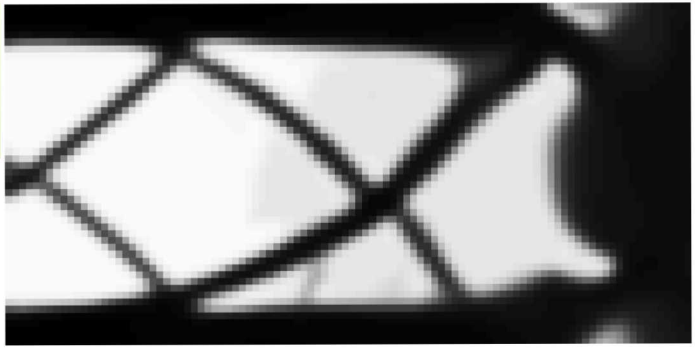
FRF Match: Design Reconstruction

TARGET STRUCTURE

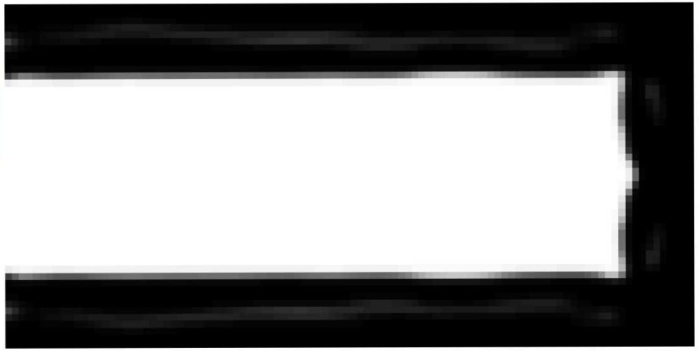


Value	L2	MECE
Mean Relative Disp. Mismatch	4.216e-01	1.147e-03
Relative Design Accuracy	36%	83%
Gray Fraction	29.9%	9.8%

LEAST-SQUARES DESIGNED STRUCTURE



MECE DESIGNED STRUCTURE



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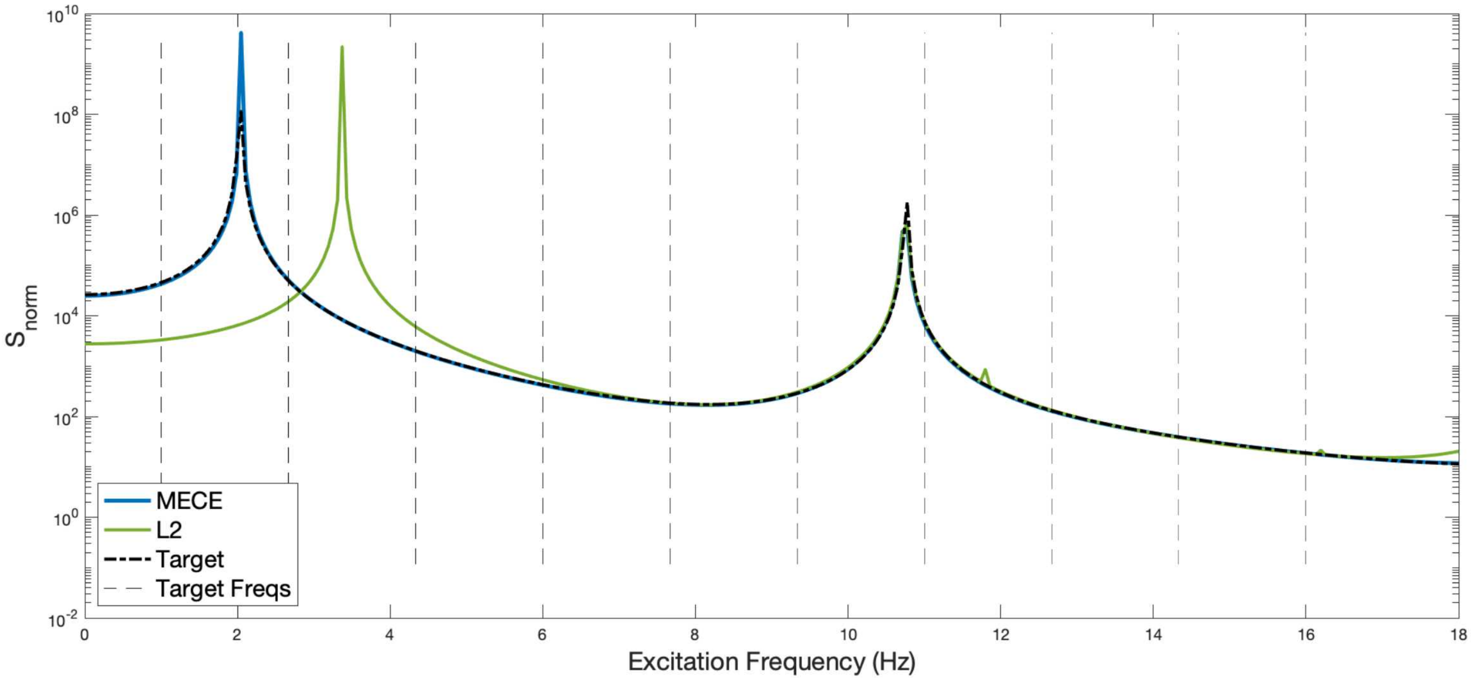
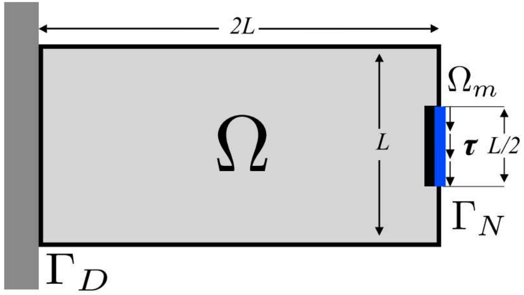
FRF Match: Design Reconstruction



- Comparisons for FRF's of MECE and L2 designs vs. FRF of target structure
- Dashed lines indicate target excitation frequencies

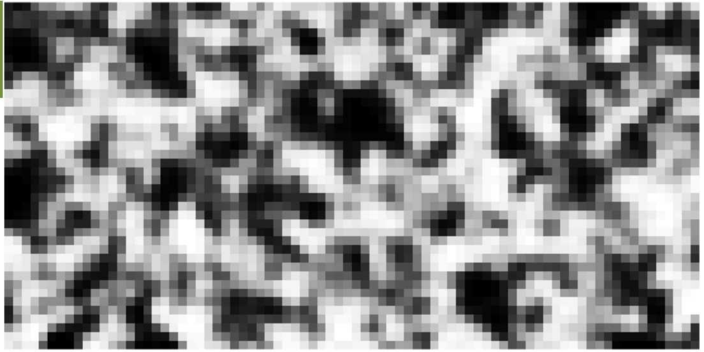
FRF Match Design



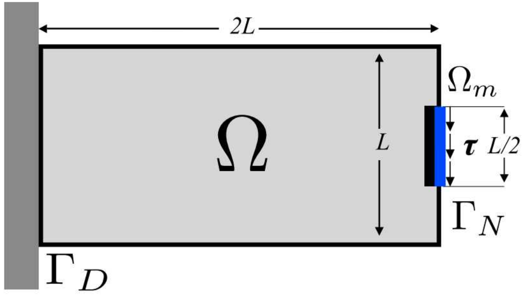
Problem Objective:

- Design a structure that matches a target FRF
- Match dynamic measurements of a target structure made of lighter, less-stiff material
  - Shear Modulus: $G_1 = 5G_0$
  - Bulk Modulus: $b_1 = 3b_0$
  - Mass Density: $\rho_1 = 4\rho_0$
- Use poor, random initial guess

Random Initial Guess



FRF Match Design



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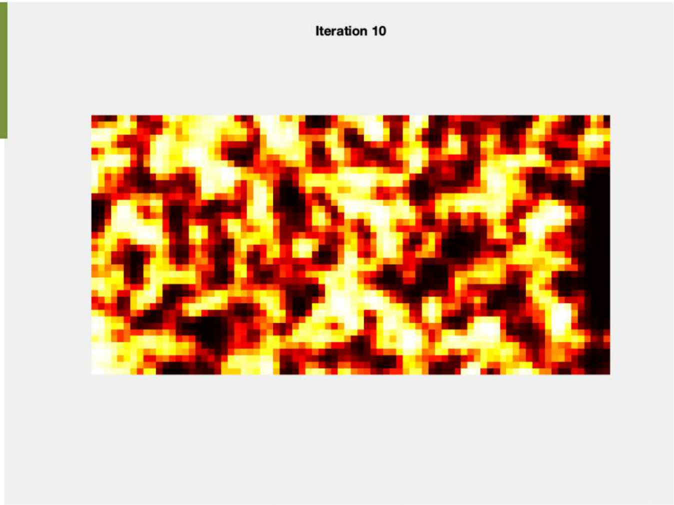
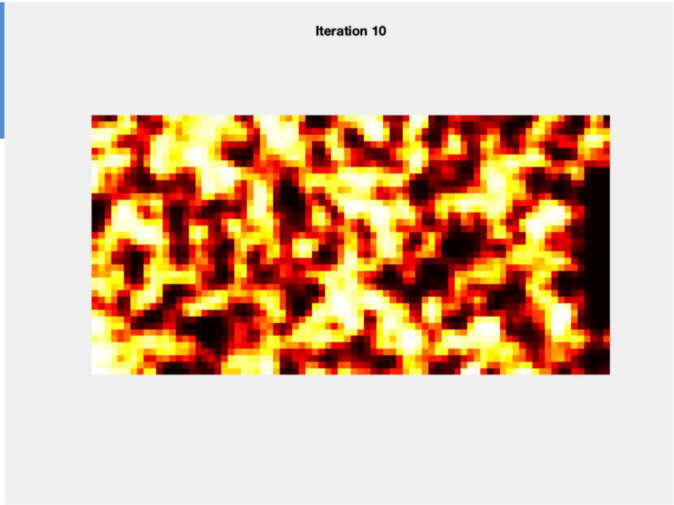
LEAST SQUARES Design



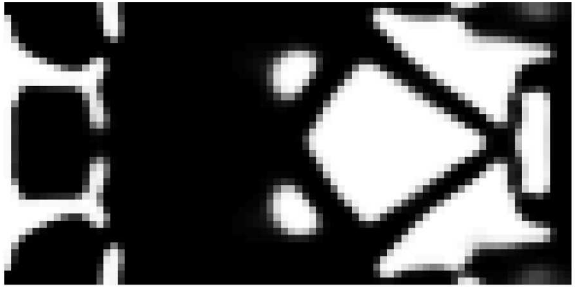
MECE Design



FRF Match Design

LEAST SQUARES Design

MECE Design

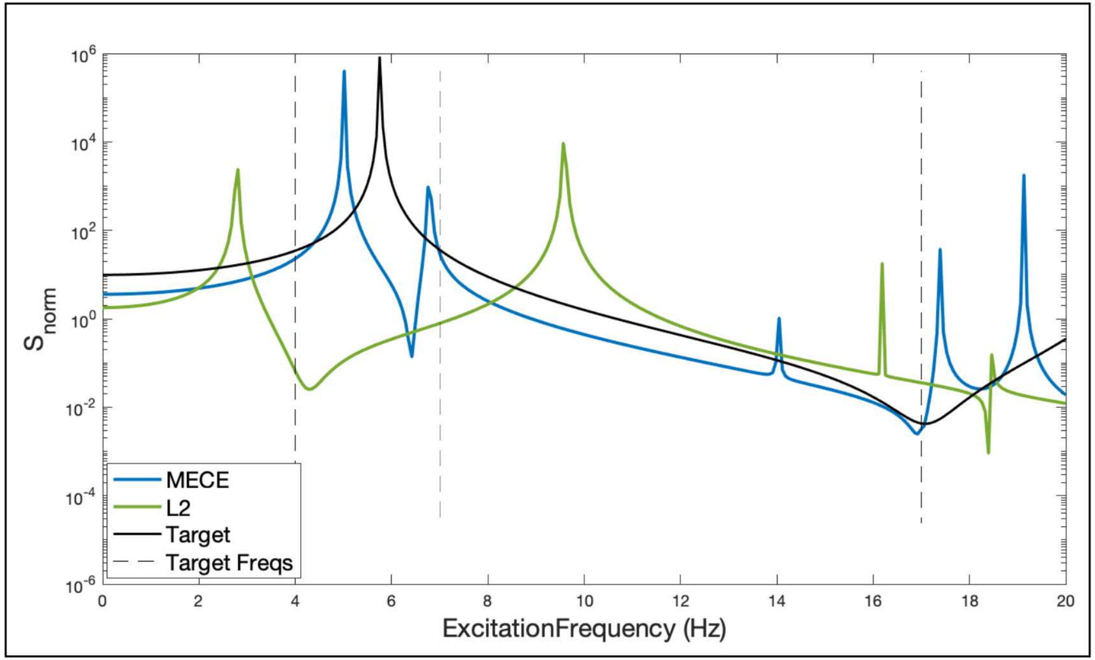


Value	Frequency (Hz)	Value
Static Disp Norm	0	1.785e+00
Mean Relative Displacement Misfit	4, 7, 17	3.511e+00
Gray Fraction	-	29.9%

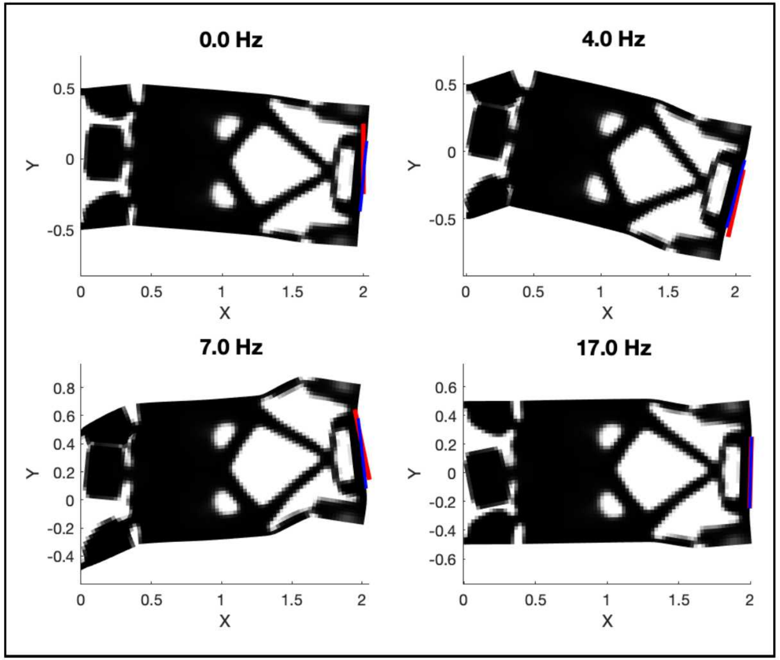
Value	Frequency (Hz)	Value
Static Disp Norm	0	3.571e+00
Mean Relative Displacement Misfit	4, 7, 17	7.989e-02
Gray Fraction	-	15.6%

FRF Match Design

Frequency response shows near match at target frequencies



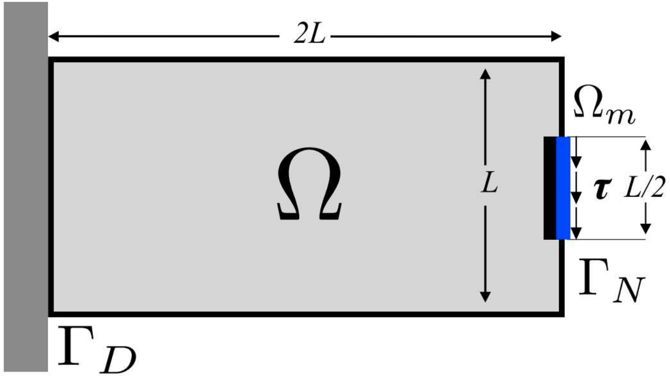
Displacements (blue) vs targets (red)



Multifrequency Vibration Isolation Design

- **Goal:** Design stable structures with minimized vibration in target frequency ranges
- **Problem Objective**: Minimize static and dynamic displacements at free end of cantilever beam at low and/or high frequency ranges

Cantilever beam design domain

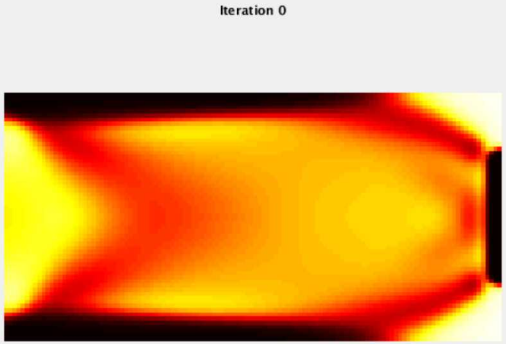


Multifrequency Vibration Isolation Design

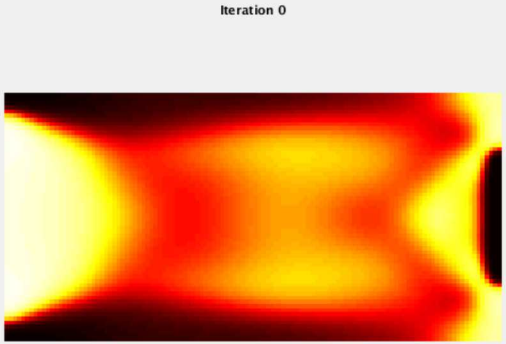
- **Goal:** Design stable structures with minimized vibration in target frequency ranges
- **Problem Objective**: Minimize static and dynamic displacements at free end of cantilever beam at low and/or high frequency ranges

Case 1 : Low frequencies (5.7-5.8 Hz)

Case 2 : High frequencies (11.4-11.6 Hz)



Case 3 : Low & High frequencies

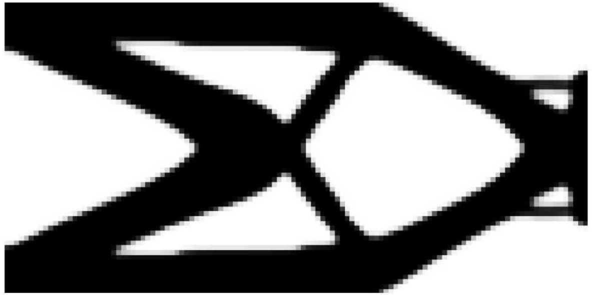


Multifrequency Vibration Isolation Design

Case 1 : Low frequencies (5.7-5.8 Hz)

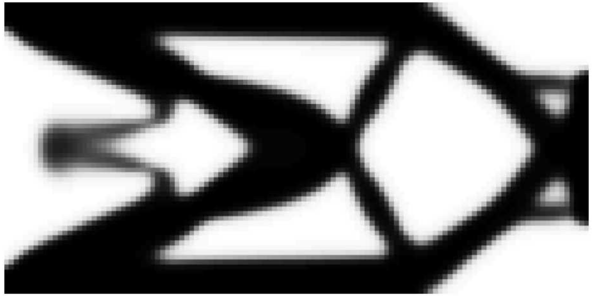
Value		Freq (Hz)	Mean Squared Disp. Norm
Static		0	6.052e+03
Dynamic		Low	8.461e+01
		High	4.429e+03
Gray Fraction		-	8.4%

Case 2 : High frequencies (10-10.1 Hz)



Value		Freq (Hz)	Mean Squared Disp. Norm
Static		0	4.719e+03
Dynamic		Low	2.486e+05
		High	5.377e+00
Gray Fraction		-	9.8 %

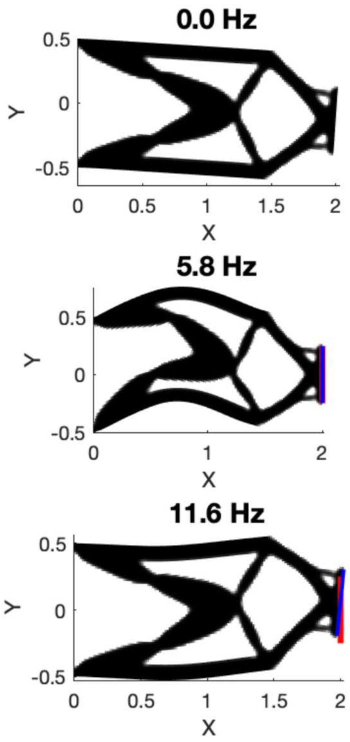
Case 3 : Low & High frequencies



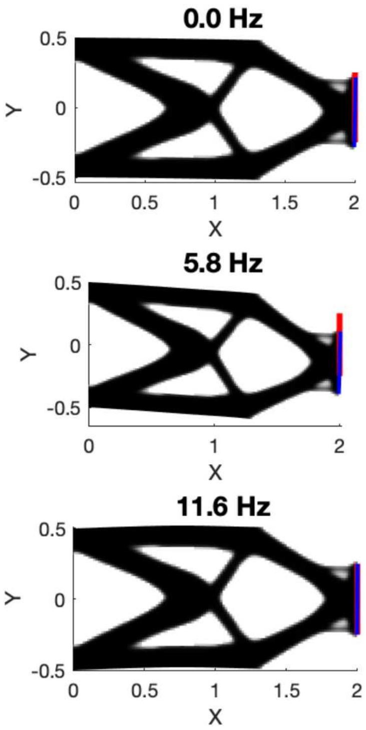
Value		Freq (Hz)	Mean Squared Disp. Norm
Static		0	5.979e+03
Dynamic		Low	7.311e+01
		High	5.271e+01
Gray Fraction		-	20.1%

Multifrequency Vibration Isolation: Displacements

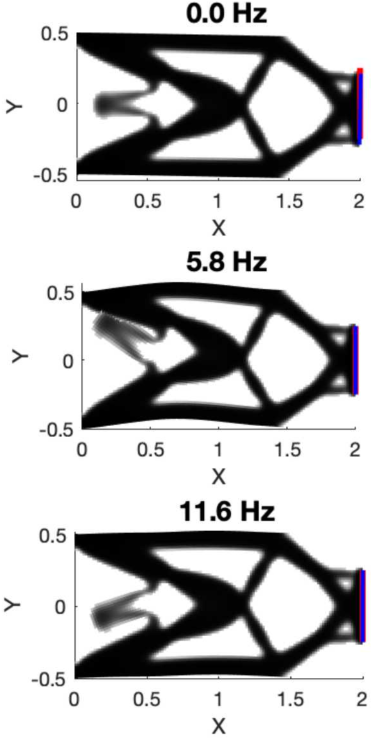
Case 1 : Low frequencies



Case 2 : High frequencies

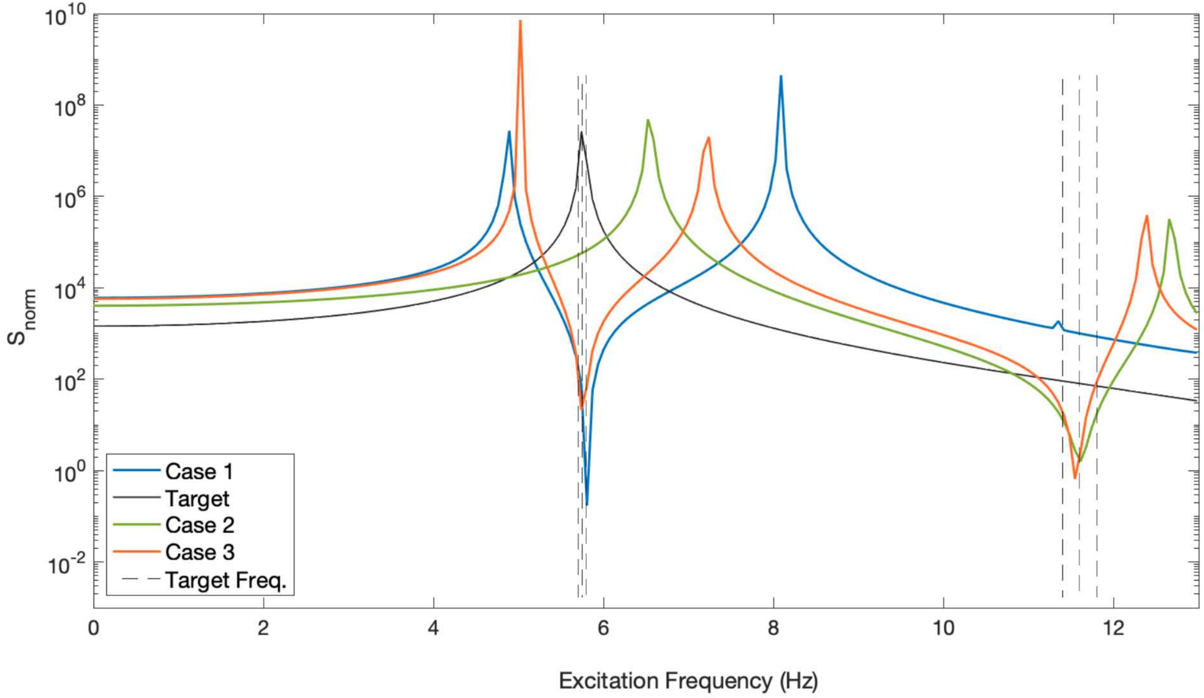


Case 3 : Low & High frequencies



Multifrequency Vibration Isolation: Frequency Response

MECE designs achieve vibration isolation at target frequencies through complicated modal behavior



Future Directions

Develop preconditioner for coupled system, for use in parallel iterative solver.

Construct Reduced Order Modeling strategies for state and design spaces to accelerate optimization solution

Demonstrate MECE in large-scale design problems

Preconditioner Development

- Coupled stationarity system poses computational bottleneck and conditioning challenges:
  - 2N x 2N dimension, poor conditioning due to material contrast & ill-conditioned blocks

- Construct preconditioner for coupled stationarity system for use in parallelized, iterative solver
  - Shifted Laplacian: use block diagonal matrix, with complex-shifted Laplacian blocks

$$P_D = \begin{bmatrix} [G] & 0 \\ 0 & [G] \end{bmatrix} \;,\; \hat{P_D}^{-1} \begin{bmatrix} [A] & -\kappa[Q] \\ A_m & [A] \end{bmatrix} = \hat{P_D}^{-1} \begin{Bmatrix} \{R\} \\ \{F\} \end{Bmatrix}$$

where: $[G] := [K] + (\alpha_1 + i\beta_1)[M]$

Design Dimensionality Reduction

- Early-stage work investigating reduced-basis approximations of density field
- Solve series of TO problems, adapting basis each pass to better represent solution
- Benefits:
  - Reduces dimensionality of design problem
  - Removes constraints
  - Does not require filter
  - Converges quickly to 0-1 designs

Conclusions

MECE strategy improves upon conventional approaches to structural FRF design.

- MECE can superiorly handle multiple frequency cases, poor initial guesses.
- Demonstrated successful designs using MECE to match target behavior, minimize FRF.

MECE suggests significant potential in new design applications.

To leverage MECE effectively, we must address its **computational efficiency**.

Relaxation of constitutive equations may prove useful in **stress-constrained TO**

KEY REFERENCES

- Aquino, W., & Bonnet, M. (2019). Analysis of the error in constitutive equation approach for time-harmonic elasticity imaging. *SIAM Journal on Applied Mathematics*, *79*(3), 822-849.
- Banerjee, B., Walsh, T. F., Aquino, W., & Bonnet, M. (2013). Large scale parameter estimation problems in frequency-domain elastodynamics using an error in constitutive equation functional. *Computer methods in applied mechanics and engineering*, *253*, 60-72.
- Bendsøe, M. P. (1989). Optimal shape design as a material distribution problem. *Structural optimization*, *1*(4), 193-202.
- Guest, J. K., Asadpoure, A., & Ha, S. H. (2011). Eliminating beta-continuation from heaviside projection and density filter algorithms. *Structural and Multidisciplinary Optimization*, *44*(4), 443-453.
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Acknowledgements

- The generous support of the DOE CSGF program, provided through grant DE-FG02-97ER25308, is gratefully acknowledged.

THANK YOU!

- Implementation:

- Prototyping code implemented in MATLAB, using `fmincon` interior-point method
- Large scale problems solved with Sierra-SD, coupled with Rapid Optimization Library (ROL)
  - Object oriented, massively-parallelized software for structural dynamics problems
- Optimization performed using constrained-optimization methods from ROL
  - Operator framework to interface with gradient-based optimization methods

Reduced Optimization Problem

We use a **reduced-space formulation** to solve the optimization problem:

- Given well-posedness of coupled system (Aquino 2018), there exist unique solutions $\{\boldsymbol{u}_\beta, \boldsymbol{w}_\beta\}$ for given design variable value. Solution of CS represents partial minimization w.r.t. independent variables $\{\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\gamma}\}$:

$$\boldsymbol{u}_\beta, \boldsymbol{\sigma}_\beta, \boldsymbol{\gamma}_\beta = \underset{\boldsymbol{u}\in\mathcal{U}, \boldsymbol{\sigma}\in\mathcal{S}, \boldsymbol{\gamma}\in\mathcal{M}}{\arg\min} \Lambda(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\gamma}, \beta)$$

- We may formulate reduced problem:

$$\min_{\beta\in\mathcal{B}} \tilde{\Lambda}(\beta) := \Lambda(\boldsymbol{u}_\beta, \boldsymbol{\sigma}(\boldsymbol{u}_\beta, \boldsymbol{w}_\beta), \boldsymbol{\gamma}(\boldsymbol{u}_\beta, \boldsymbol{w}_\beta), \beta; \kappa)$$

- **Reduced Objective**: Substitution of stress, mass inertia expressions yield ECE functionals dependent only on $\boldsymbol{w}_\beta$:

$$\tilde{\Lambda}(\beta) := \frac{1}{2\alpha}\tilde{\mathcal{E}}(\boldsymbol{w}_\beta, \beta) + \frac{1}{2(1-\alpha)}\tilde{\mathcal{I}}(\boldsymbol{w}_\beta, \beta) + \frac{\kappa}{2}\mathcal{D}(\boldsymbol{u}_\beta)$$

$$\tilde{\mathcal{E}}(\beta) := \langle \boldsymbol{w}_\beta, \boldsymbol{w}_\beta \rangle_{\mathbb{C}(\beta)}$$

$$\tilde{\mathcal{I}}(\beta) := \omega^2(\rho(\beta)\boldsymbol{w}_\beta, \boldsymbol{w}_\beta)$$

Key Mathematical Properties of MECE

The coupled system is well-posed, even at resonant frequencies:

- We assume that that modes excited in structure are "measured" by structural objective.
- Helmholtz operator $\boldsymbol{A}$ may have null space, but elements of N(A) must yield non-zero value for structural objective operator $\boldsymbol{D}$:

$$N(A) \cap N(D) = \emptyset$$

MECE objective has "improved" convexity:

- For linear-elastic material inversion, MECE objective is globally convex as $\kappa \rightarrow +\infty$, with a strictly positive-definite Hessian
- MECE Hessian using SIMP model can be sign-indefinite due to high-order material model derivatives & non-convex Heaviside operations

κ Parameter Selection

Penalty parameter κ controls level of constitutive equation violation

- In material ID, selected based upon level of measurement noise, using Morozov Principle, for example

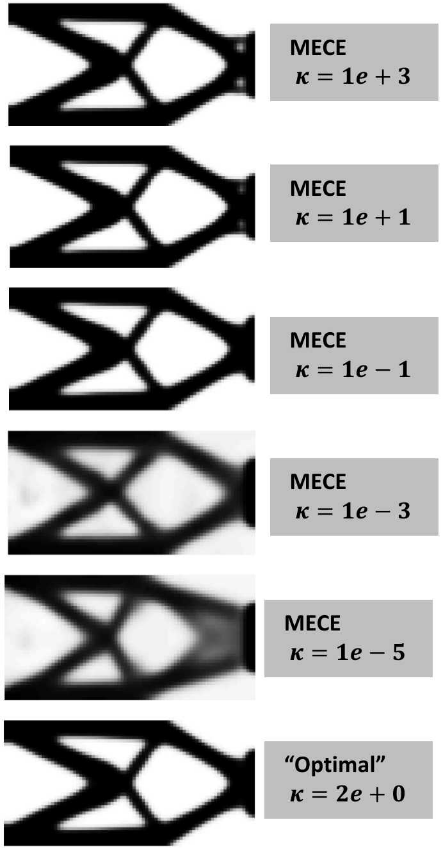
Approach1: Error-Balance

- Select optimal parameter which balances error, state-objective terms, measured by error-balance objective:

$$B(\kappa, \beta_\kappa^*) = \left[\alpha \mathcal{E}(\kappa, \beta_\kappa^*) + (1-\alpha)\mathcal{I}(\kappa, \beta_\kappa^*)\right]^2 + \left(\mathcal{D}(\kappa, \beta_\kappa^*)\right)^2$$

- Solve for optimal κ minimizing error-balance objective
  - Approximate minimizer occurs when $\kappa = \dfrac{\mathcal{D}(\boldsymbol{u}_\kappa^*)}{\mathcal{E}(\beta_\kappa^*, \boldsymbol{u}_\kappa^*)}$

Error Balance Example



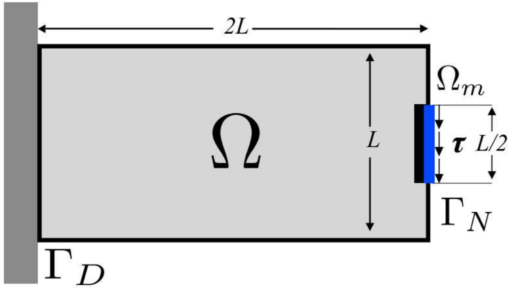
κ Parameter Selection

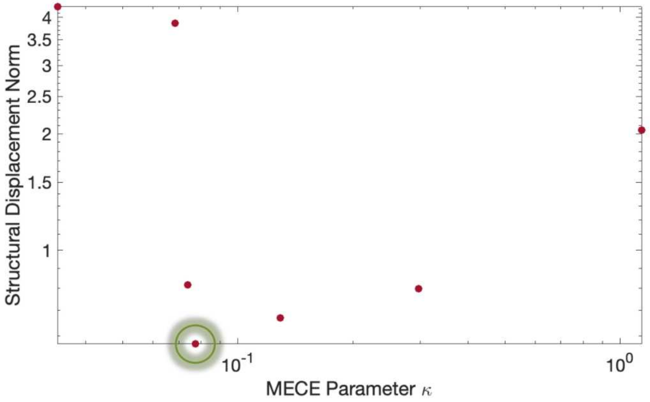
Approach 2: Structural response minimization

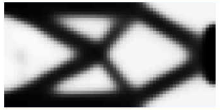
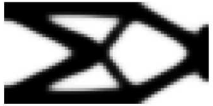
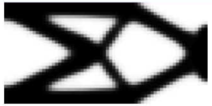
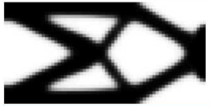
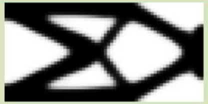
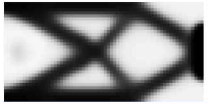
- Determine κ_{opt} which yields "optimal" minimizer $\beta^*_{\kappa_{opt}}$ that:
  - Obtains minimized structural response
  - Possesses adequately low gray fraction
- Solve 1D minimization problem to determine κ_{opt}
  - Solve original forward problem, using system constructed with β^*_κ
  - Evaluate original structural objective
  - Use golden sections/bisection/etc method to determine the minimizer κ_{opt} over range of κ values.

κ Parameter Selection Example

Cantilever beam design domain





Iter	MECE κ	DESIGN	Gray Fraction	Objective Decrease
1	$\kappa = 3.3718e-02$		4.0259e-01	4.265e+00
2	$\kappa = 2.9658e-01$		1.9325e-01	7.973e-01
3	$\kappa = 1.137e+00$		2.149e-01	2.045e+00
4	$\kappa = 1.293e-01$		2.128e-01	6.700e-01
5	$\kappa = 7.736e-02$		1.845e-01	5.749e-01
6	$\kappa = 6.854e-02$		4.917e-01	3.857e+00
7	$\kappa = 7.387e-02$		3.349e-01	8.148e-01