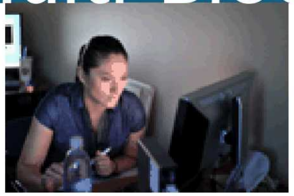
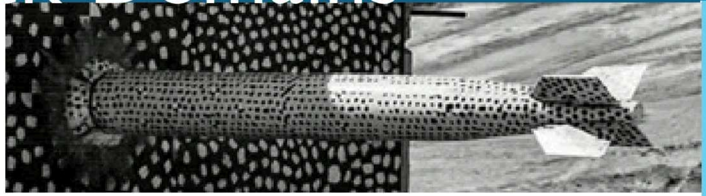
SAND2019-8911C

Generalized Entropy Stable Weighted Essentially Non-Oscillatory Finite Difference Scheme in Multi-Block Domains





PRESENTED BY

Jungyeoul Brad Maeng, Travis Fisher, Mark Carpenter

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Motivation

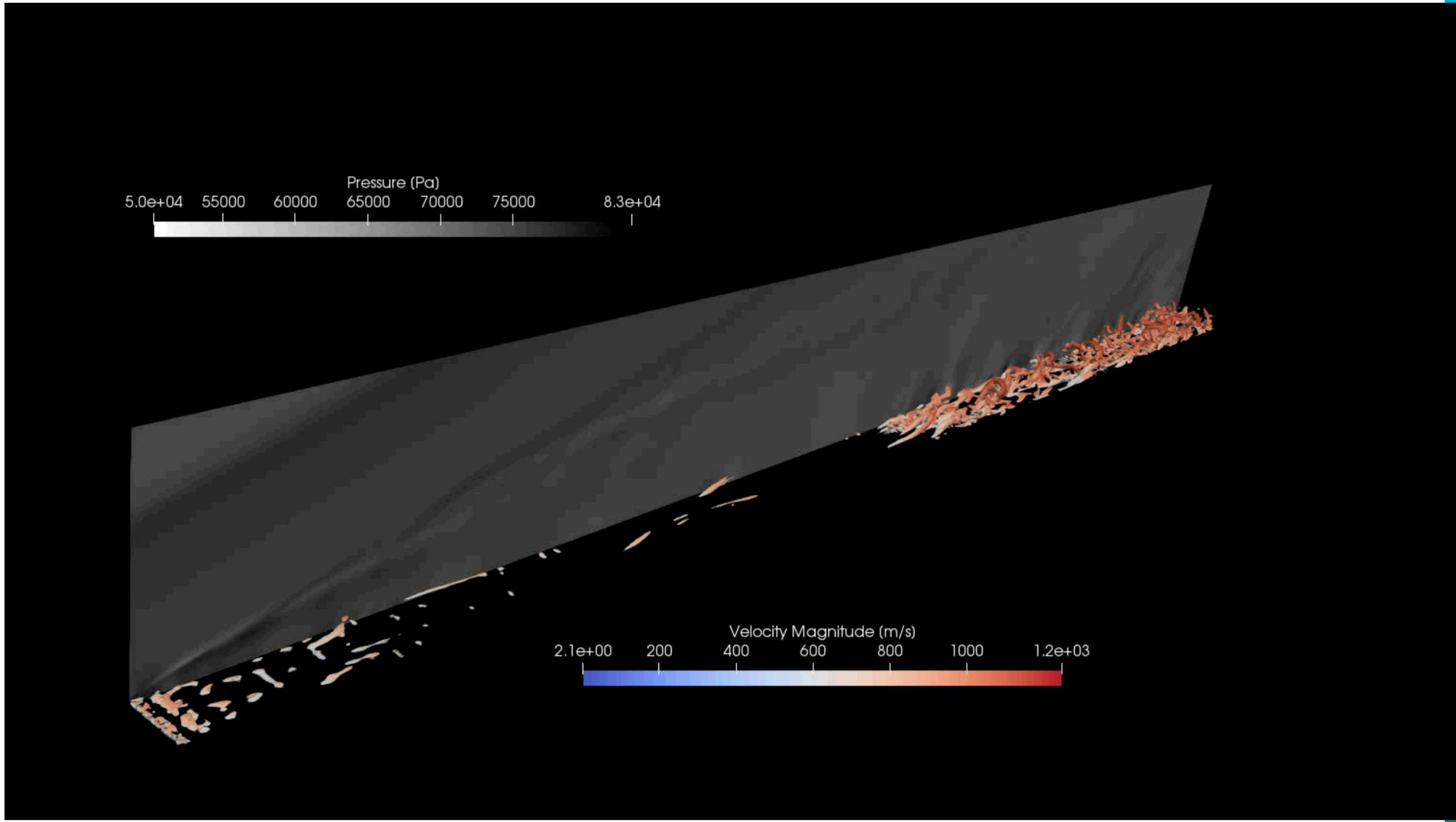
High-fidelity simulations of 3D turbulent compressible flows

Challenges

- Resolve various fluid scales
- Accuracy vs. stability
- Large-scale
- Complex geometry

Numerical requirements

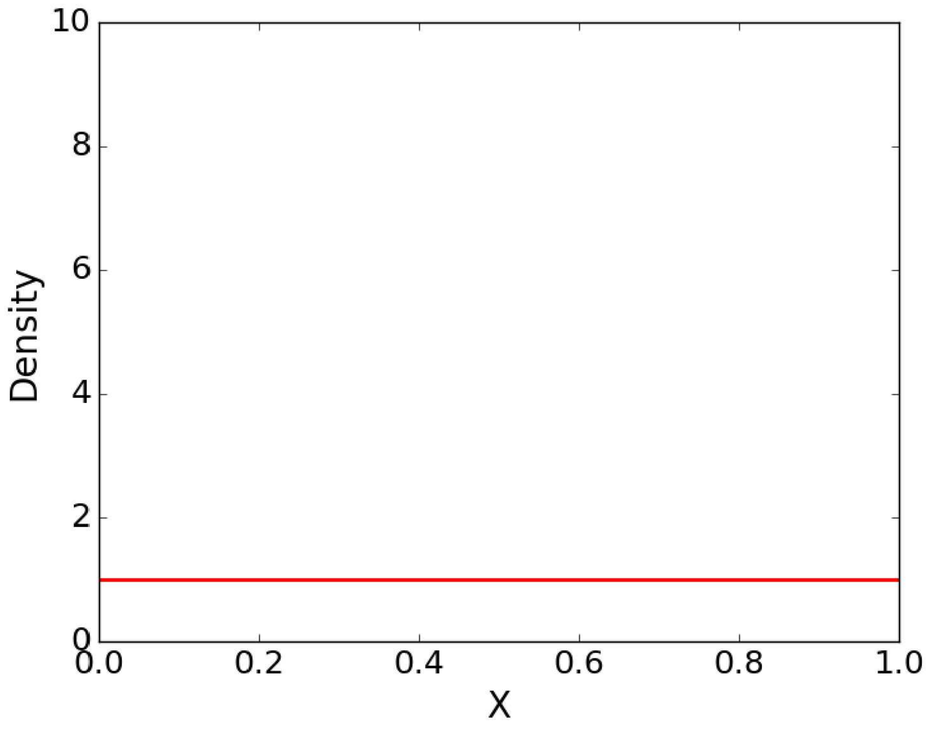
- High-order
- Low dissipation
- Handle shocks
- Multi-block capable



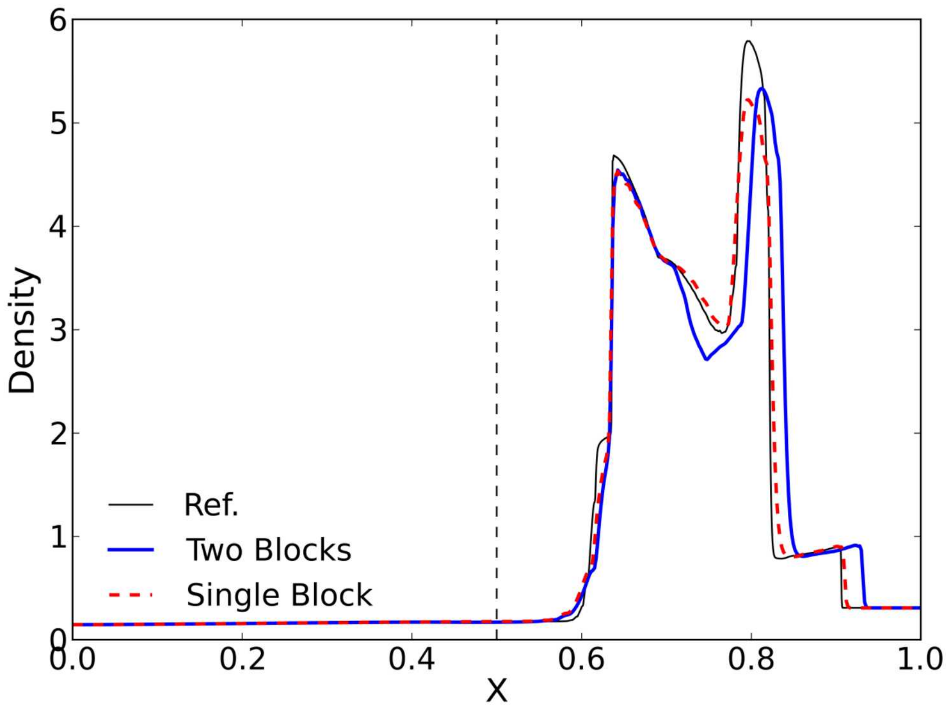
Mach 3.5 flat plate boundary layer DNS

Challenge: Robust Shock Capturing in Multi-Block Domains

Simplified example of a real world shock capturing in curvilinear multi-block domain



Solution in time



N=512, $t_f = 0.04$, $N_{ref} = 2000$

Woodward Colella with **conventional** high-order finite difference

Generalized Summation-By-Parts

Nonlinear conservation laws

$$\mathbf{u}_t + (\mathbf{f}_k)_{x_k} = 0, \; x_k \in \Omega, \; t \in [0,\infty),$$
$$\mathbb{B}(\mathbf{u}) = \mathbf{g}^{bnd}, \; x_k \in \partial\Omega, \; t \in [0,\infty),$$
$$\mathbf{u}(x,0) = \mathbf{g}_0(x_k), \; x_k \in \Omega,$$

$$\mathbf{u}_t + \mathcal{D}_k \mathbf{f}_k = \mathcal{P}_k^{-1}\mathbf{g}_k^{int} + \mathcal{P}_k^{-1}\mathbf{g}_k^{bnd}, \quad k = 1,2,3$$

Generalized summation-by-parts (SBP) operator (Del Rey Fernandez, JCP 2014)

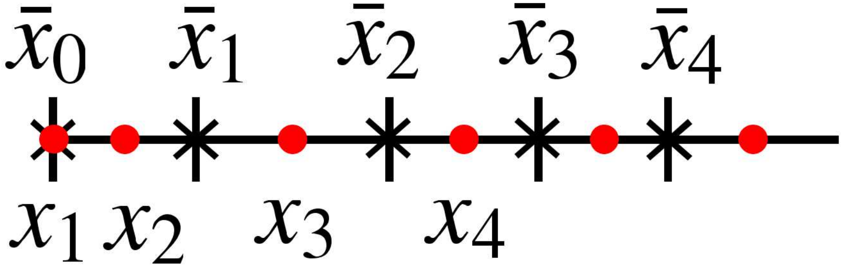
- Discrete analogue of **integration-by-parts**

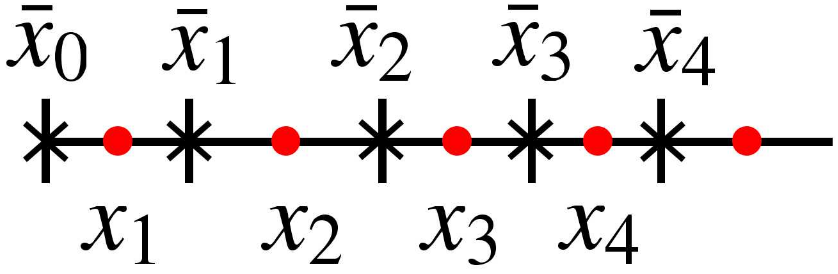
$$\mathcal{D} = \mathcal{P}^{-1}\mathcal{Q}, \quad \mathcal{P} = \mathcal{P}^T, \quad \boldsymbol{\xi}^T\mathcal{P}\boldsymbol{\xi} > 0, \quad \boldsymbol{\xi} \neq 0$$
$$\mathcal{Q}^T = \mathcal{B} - \mathcal{Q}. \qquad \mathcal{B} = \mathbf{b}_1\mathbf{b}_1^T - \mathbf{b}_{-1}\mathbf{b}_{-1}^T,$$

- Generalization of boundary solution points

$$\mathbf{b}_{-1} = (1, 0, 0, \ldots, 0)^T$$

$$\mathbf{b}_{-1} = \left(\frac{35}{16}, -\frac{35}{16}, \frac{21}{16}, -\frac{5}{16}, 0, \ldots, 0\right)^T$$





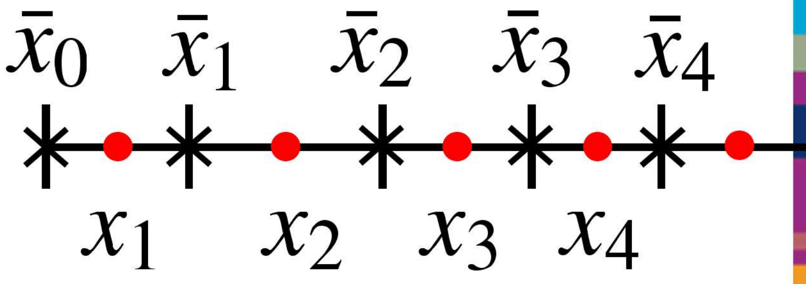
Entropy Stable Cell-Centered High-Order Finite Difference

$$\mathbf{u}_t + \mathcal{P}^{-1}[2\mathcal{Q} \circ \mathcal{F}]\mathbf{1} = \mathcal{P}^{-1}\mathbf{g}^{int}$$

Complementary grid enables us to recast gradient form to **flux form**

- Important for entropy stable WENO flux (Fisher, JCP 2013)

$$\mathbf{f}(\mathbf{u})_x = \mathcal{P}^{-1}[2\mathcal{Q} \circ \mathcal{F}]\mathbf{1} = \mathcal{P}^{-1}\Delta\bar{\mathbf{f}} \quad \Longrightarrow \quad \mathbf{u}_t + \mathcal{P}^{-1}\Delta\bar{\mathbf{f}} = \mathcal{P}^{-1}\mathbf{g}^{int}$$



$$\mathcal{P}_{ii} = \bar{x}_{i+1} - \bar{x}_i$$

Entropy stable two-point nonlinear flux

$$\frac{d}{dt}\mathbf{1}^T \mathcal{P} S + \sum_{k=1}^{N}\sum_{l=1}^{N} b_{1,k}b_{1,l}\bar{F}(u_l,u_k) - b_{-1,k}b_{-1,l}\bar{F}(u_l,u_k) = 0$$

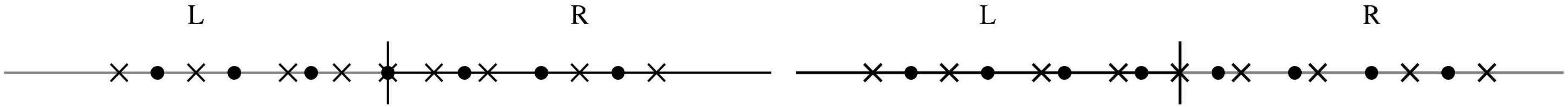
$$\frac{d}{dt}\mathbf{1}^T \mathcal{P} S + \bar{F}|_1 - \bar{F}|_{-1} = 0$$

$$\bar{f}_i^S = \sum_{k=i}^{N}\sum_{l=1}^{i} 2\hat{q}_{(l,k)}\bar{f}(u_l,u_k) + \sum_{k=i+1}^{N}\sum_{l=1}^{N} -b_{-1,l}b_{-1,k}\bar{f}(u_l,u_k) + \sum_{k=1}^{i}\sum_{l=1}^{N} b_{1,l}b_{1,k}\bar{f}(u_l,u_k), \quad 1 \le i \le N-1,$$

Benefits of cell-centered approach

- Similar to finite volume and satisfies telescoping flux property
- Stronger coupling across multi-block interface
- Better shock capturing

Generalized Entropy Stable Interface Penalty



Conventional — Cell-centered

Two-domain finite difference in flux form

$$\mathbf{u}_t + \mathcal{P}^{-1}\Delta\bar{\mathbf{f}} = \mathcal{P}^{-1}\mathbf{g}^{int}$$

$$\Delta\bar{\mathbf{f}} = (\mathcal{Q} + \mathcal{G})\begin{bmatrix}\mathbf{f}_L \\ \mathbf{f}_R\end{bmatrix} \qquad \mathcal{Q} = \begin{bmatrix} Q_L & 0 \\ 0 & Q_R \end{bmatrix} \quad \mathcal{G} = \begin{bmatrix} -\frac{1}{2}\mathbf{b}_1^L\mathbf{b}_1^{L^T} & \frac{1}{2}\mathbf{b}_1^L\mathbf{b}_{-1}^{R^T} \\ -\frac{1}{2}\mathbf{b}_{-1}^R\mathbf{b}_1^{L^T} & \frac{1}{2}\mathbf{b}_{-1}^R\mathbf{b}_{-1}^{R^T} \end{bmatrix}$$

Generalized entropy stable interface penalty

$$\begin{aligned}\mathbf{g}^{int} = &\left\{\left(\mathbf{b}_1^L\mathbf{b}_1^{L^T} \circ \mathbf{f}(\mathbf{u},\mathbf{u}) - \mathbf{b}_1^L\mathbf{b}_{-1}^{R^T} \circ \mathbf{f}(\mathbf{u},\mathbf{u})\right)\right. \\ &\left. -\frac{1}{2}\mathbf{b}_1^L\mathbf{R}|\Lambda|\mathbf{R}^T\left(\mathbf{b}_1^{L^T}\mathbf{w} - \mathbf{b}_{-1}^{R^T}\mathbf{w}\right)\right\} \\ -&\left\{\left(-\mathbf{b}_{-1}^R\mathbf{b}_1^{L^T} \circ \mathbf{f}(\mathbf{u},\mathbf{u}) + \mathbf{b}_{-1}^R\mathbf{b}_{-1}^{R^T} \circ \mathbf{f}(\mathbf{u},\mathbf{u})\right)\right. \\ &\left. -\frac{1}{2}\mathbf{b}_{-1}^R\mathbf{R}|\Lambda|\mathbf{R}^T\left(\mathbf{b}_{-1}^{R^T}\mathbf{w} - \mathbf{b}_1^{L^T}\mathbf{w}\right)\right\}\end{aligned}$$

Shock Capturing with Weighted Essentially Non-Oscillatory

Entropy stable WENO (Fisher and Carpenter, JCP 2013)

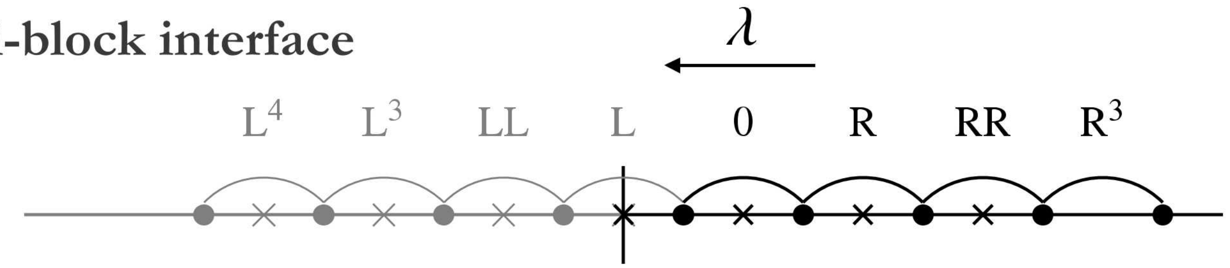
$$\bar{f}_i^W = \sum_{l=1}^{n_s} \bar{\omega}_i^l \bar{f}_i^{S_l}. \qquad \bar{\omega}_i^l = \frac{\bar{\alpha}_i^l}{\sum_j \bar{\alpha}_i^j}, \quad \bar{\alpha}_i^l = \bar{d}_i^l \left(1 + \frac{\bar{\tau}_i^l}{\bar{\beta}_i^l + \bar{\epsilon}_i}\right), \quad l = 1, \dots, n_s$$

$$\bar{f}_i^{SSW} = \bar{f}_i^W + \delta(\bar{f}_i^S - \bar{f}_i^W), \quad \delta = \frac{\sqrt{b^2 + c^2} - b}{\sqrt{b^2 + c^2}}, \quad b = (w_{i+1} - w_i)^T (\bar{f}_i^S - \bar{f}_i^W), \quad c = 10^{-12}$$

- Entropy stability condition is satisfied with entropy stable WENO

$$(w_{i+1} - w_i)^T (\bar{f}_i^{SSW} - \bar{f}_i^S) \le 0, \quad 0 \le i \le N - 1,$$

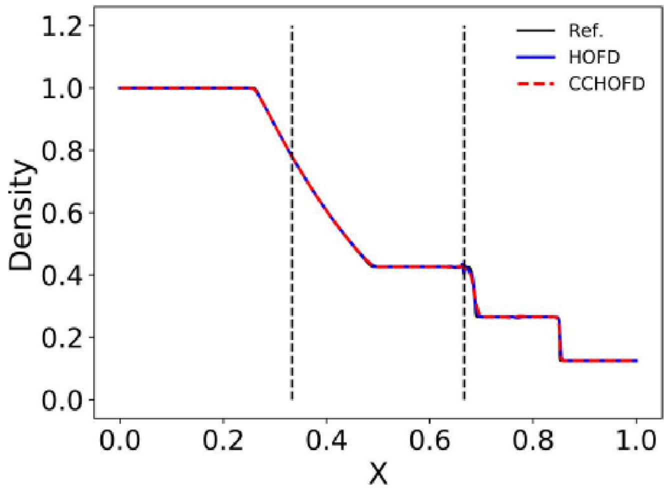
WENO across multi-block interface



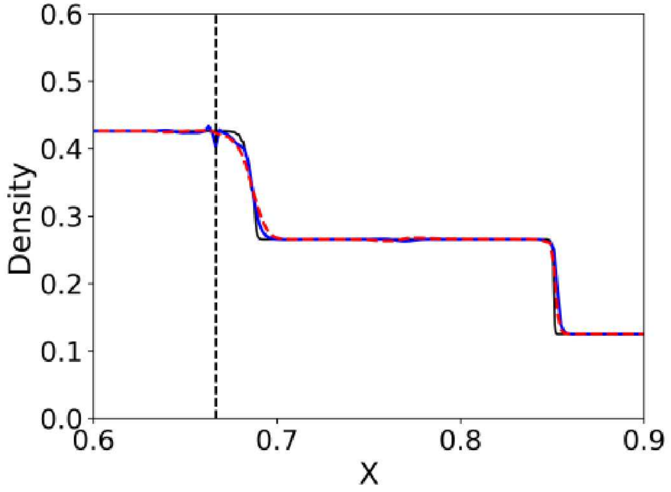
- Cell-centered SBP operator gives a strong coupling between blocks
- WENO target flux, weight, candidate stencil based on non-dissipative interface operator
- Need a different biasing due to larger stencil width

$$\bar{\alpha}_i^l = \begin{cases} \bar{d}_i^l \left(1 + \frac{\bar{\tau}_i^l}{\bar{\beta}_i^l + \bar{\epsilon}_i}\right), & \text{if } l \in [0, R, RR, R^3] \\ \gamma \bar{d}_i^l \left(1 + \frac{\bar{\tau}_i^l}{\bar{\beta}_i^l + \bar{\epsilon}_i}\right), & \text{if } l \in [L, LL, L^3, L^4] \end{cases}, \quad l = 1, \dots, n_s,$$

Multi-Block Shock Capturing WENO: 1D Shock Examples



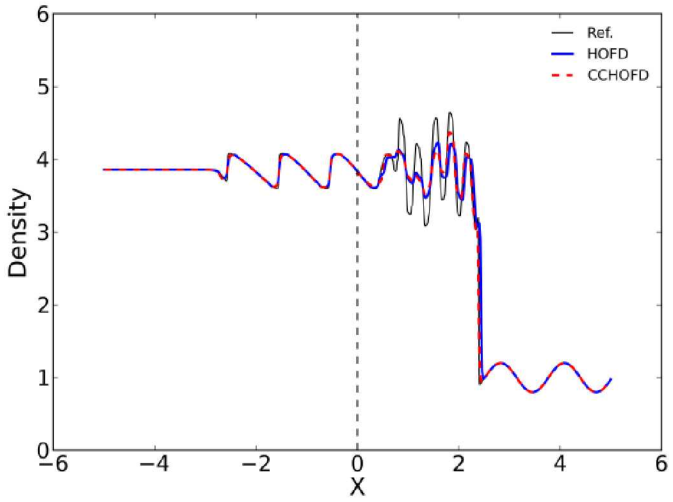
(a) Density



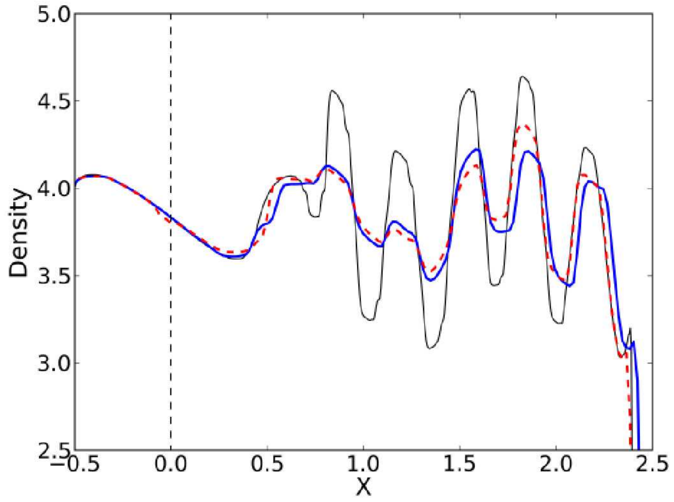
(b) Density close up near the contact discontinuity

$$(\rho, u, p) = \begin{cases} (1, 0, 1), & \text{if } x < 0.5 \\ (0.125, 0, 0.1), & \text{if } x \geq 0.5, \end{cases}$$

N=512, t_f = 0.25
Three-block



(a) Density

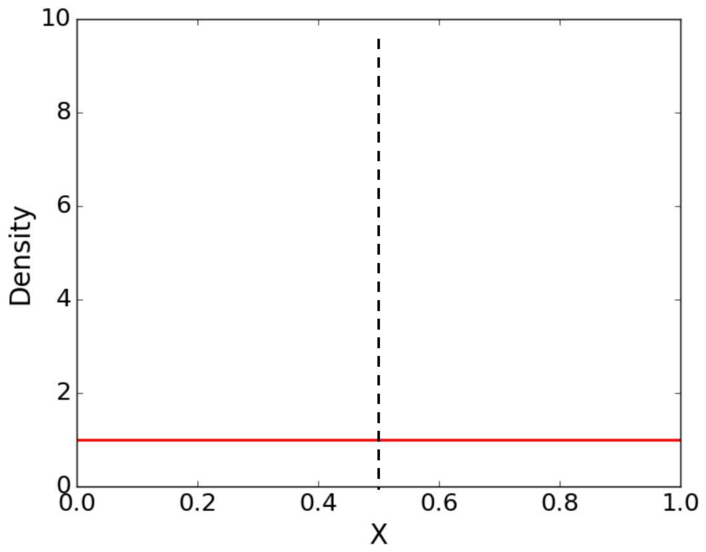


(b) Density close up

$$(\rho, u, p) = \begin{cases} (3.857143, 2.629369, 10.3333), & \text{if } x < -4.0 \\ (1 + 0.2\sin(5x), 0, 1), & \text{if } x \geq -4.0. \end{cases}$$

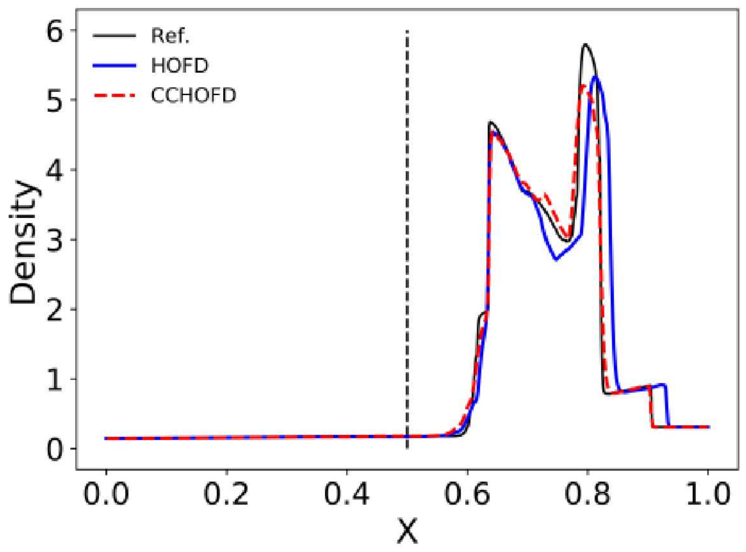
N=512, t_f = 1.8, N_{ref} = 2000
Two-block

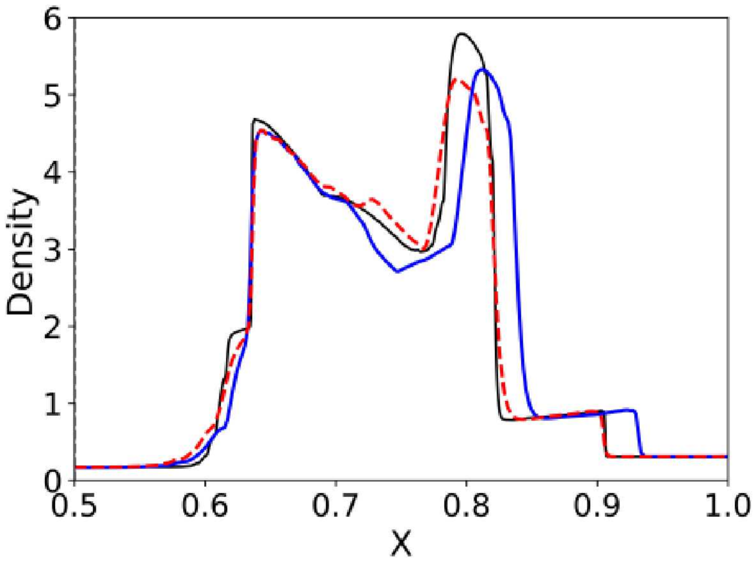
Strong Shock Across Interface: Woodward Colella



Initial condition

$$(\rho, u, p) = \begin{cases} (1, 0, 1000), & \text{if } x < 0.1 \\ (1, 0, 100), & \text{if } 0.1 \le x < 0.9 \\ (1, 0, 0.01), & \text{if } x \ge 0.9. \end{cases}$$





N=512, $t_f = 0.04$, $N_{ref} = 2000$
Two-block

Turbulence and Shocks Have Different Characteristics

Cell-centered high-order finite difference
- Non-dissipative
- Can't handle shocks

Shock capturing method
- Dissipative
- Destroys turbulent structures

Hybrid scheme
- Blend CCHOFD and shock capturing method
  - Non-dissipative + Handle shocks
- Shock sensor (Larsson, CTR 2011)
  - Activate shock capturing method when

$$-\nabla \cdot \mathbf{v} > \max\left(A\sqrt{\omega \cdot \omega},\ B\frac{c}{h}\right) \quad \text{where } A > 1,\ B < 1$$

- Artificial dissipation (Mattsson, JSC 2004)
  - Un/Under-resolved flow region where shock capturing is off

Entropy Stable Artificial Dissipation Operator

Entropy stable CCHOFD is inherently non-dissipative

- Other schemes introduce numerical dissipation through proper upwinding

Artificial dissipation in flux form

$$\mathbf{u}_t + \mathcal{P}_k^{-1}\Delta_k\bar{\mathbf{f}}_k = \mathcal{P}_k^{-1}\mathbf{g}_k^{int} + \mathcal{P}_k^{-1}\Delta_k\bar{\mathbf{f}}_k^{ad}$$

$$\mathcal{P}_k^{-1}\Delta_k\bar{\mathbf{f}}_k^{ad} = \mathcal{D}_2\,|\Lambda|\,\frac{\partial\mathbf{u}}{\partial\mathbf{w}}\mathcal{D}_2\mathbf{w} \quad \text{where } \mathcal{D}_2 = \Delta\Delta^T$$

- Can be used with a different shock capturing method (e.g. artificial viscosity)
- Applies dissipation based on wave speed
  - Can introduce extra dissipation even in freestream regions
  - Can utilize an adaptive mechanism to reduce dissipation, may increase tuning parameters

Turbulence Dissipation Characterization

Taylor Green vortex

- 3D periodic domain $[-\pi L \leq (x, y, z) \leq \pi L]$
- Initial condition

$$M = 0.1 \quad Re = 1600$$

$$\rho(x, y, z, 0) = 1.0$$

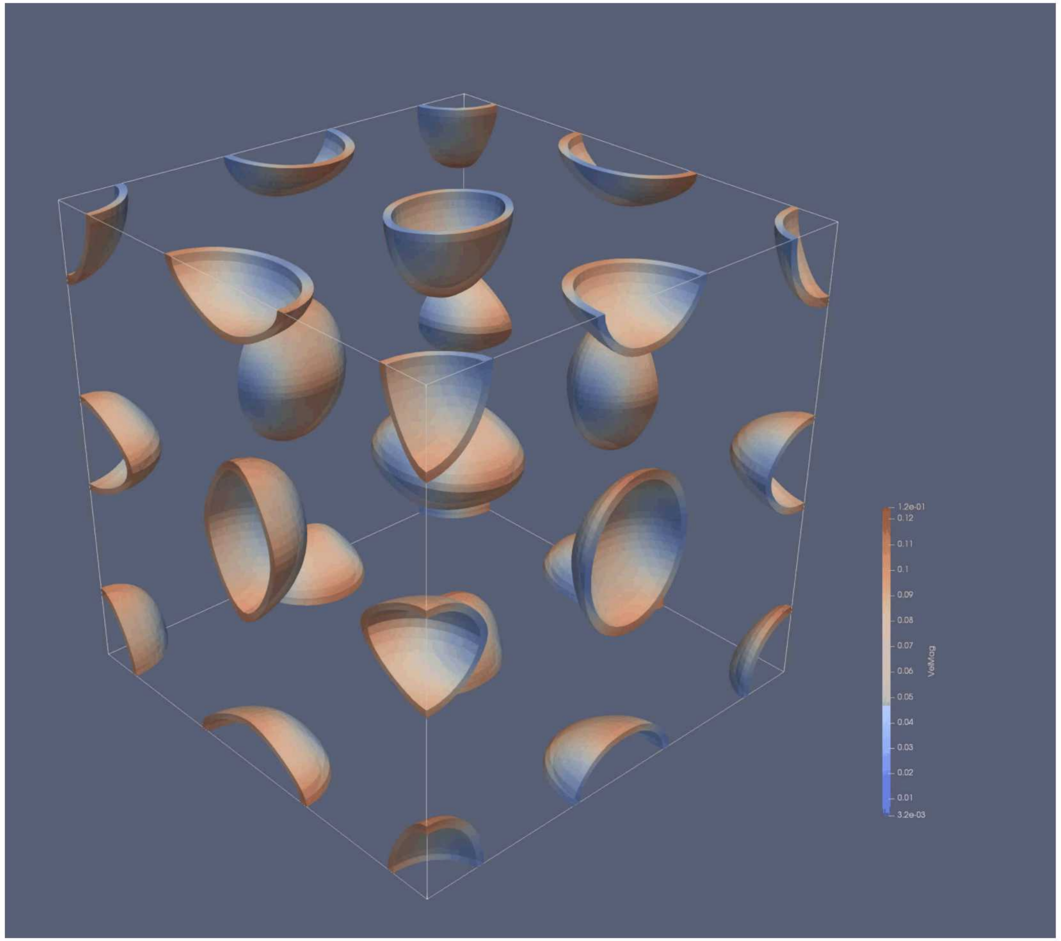
$$u(x, y, z, 0) = V_0 \sin\left(\frac{x}{L}\right) \cos\left(\frac{y}{L}\right) \cos\left(\frac{z}{L}\right)$$

$$v(x, y, z, 0) = -V_0 \cos\left(\frac{x}{L}\right) \sin\left(\frac{y}{L}\right) \cos\left(\frac{z}{L}\right)$$

$$w(x, y, z, 0) = 0$$

$$p(x, y, z, 0) = 1.0 + \frac{\gamma M_0^2}{16}\left(\cos\left(\frac{2x}{L}\right) + \cos\left(\frac{2y}{L}\right)\right)\left(\cos\left(\frac{2z}{L}\right) + 2\right),$$

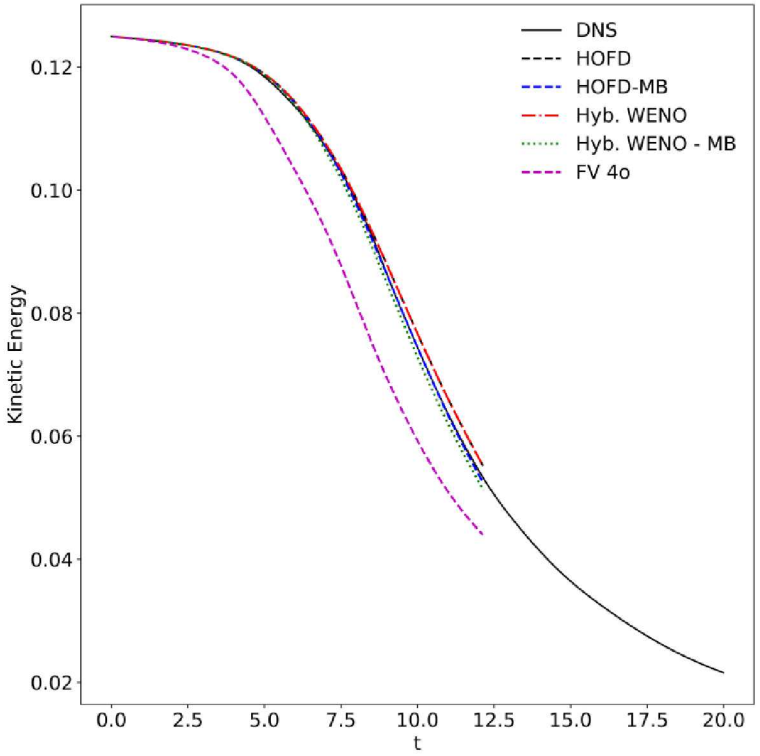
- Benchmark schemes
  - Cell-centered HOFD
  - Hybrid cell-centered HOFD (no AD)
- Assess multi-block capability
  - 8 32^3 blocks



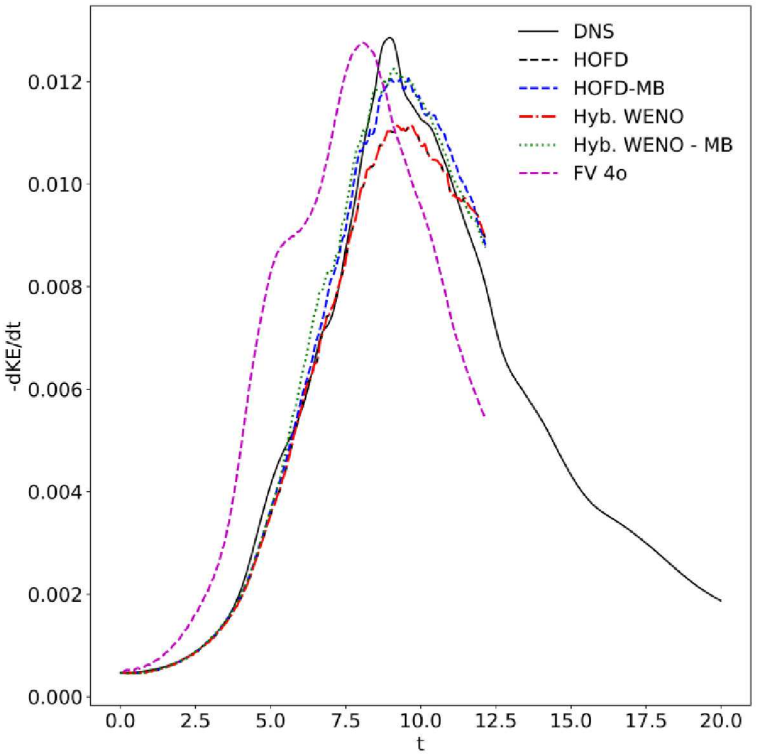
Vorticity visualization using cell-centered HOFD 64^3

Quantitative Evaluation of Cell-Centered HOFD

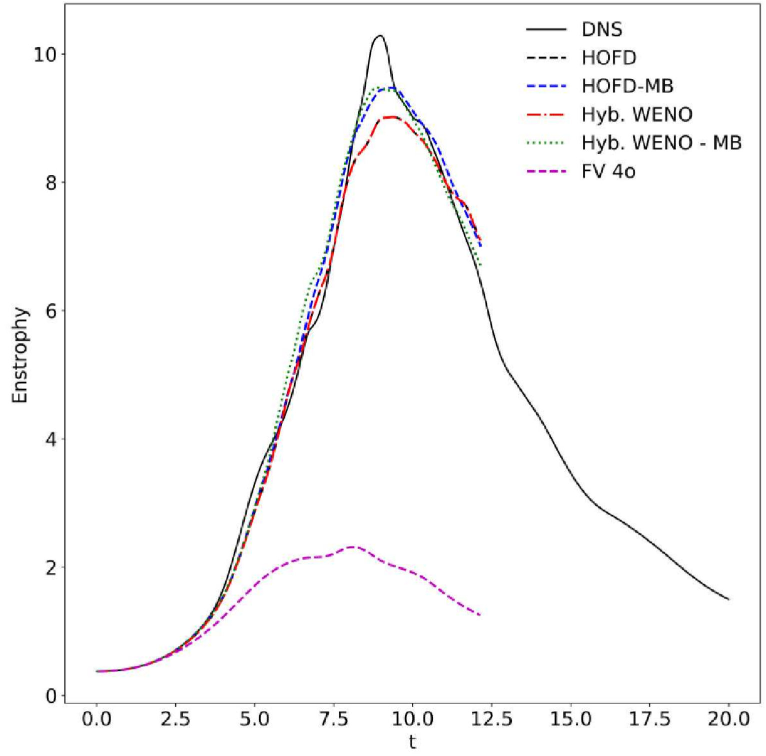
- DNS: 512^3 spectral method
- Cell-centered HOFD and Hybrid WENO: demonstrate multi-block structured mesh
- Popular CCFV : Subbareddy and Candler low dissipation method with Mach shock sensor



Total kinetic energy



Total kinetic energy dissipation rate



Enstrophy

Dof. = 64^3

Future Works

Demonstrate full scale multi-block LES/DNS with generalized interface penalty

Improve shock capturing scheme

- Interface WENO stencil biasing
- Develop an alternative shock capturing scheme, e.g. artificial viscosity

Improve hybrid scheme

- Investigate a hybrid scheme that treats under-resolved and well-resolved flow regions differently
- Adaptive artificial dissipation for entropy stable CCHOFD not based on flow parameter.

Acknowledgements

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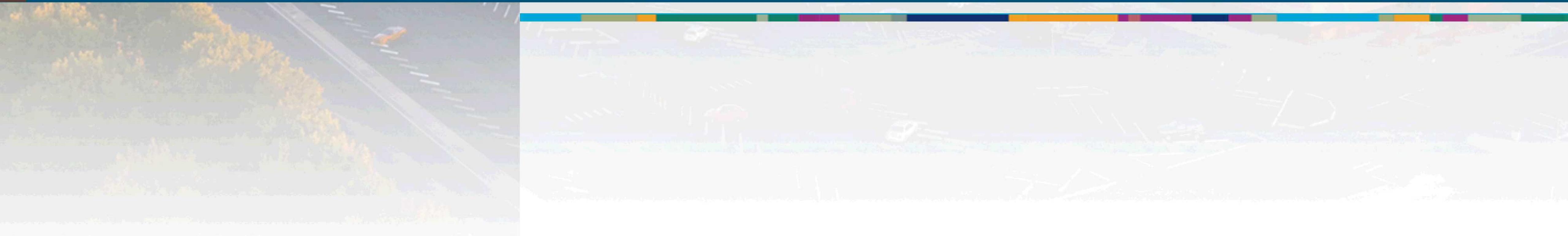
- NASA: Mark Carpenter

Questions?



Extra Slides

Entropy Stable Cell-Centered High-Order Finite Difference

Entropy stability

- Continuous entropy stability analysis

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} = 0$$

$$\mathbf{w}^T \frac{\partial \mathbf{u}}{\partial t} + \mathbf{w}^T \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial x} = 0 \quad \text{where } \mathbf{w}^T = \frac{\partial S}{\partial \mathbf{u}}$$

$$\frac{\partial S}{\partial t} + \frac{\partial F}{\partial x} = 0$$

- Global entropy consistency

$$\frac{d}{dt}\mathbf{1}^T \mathcal{P} S + \sum_{k=1}^{N} \sum_{l=1}^{N} b_{1,k} b_{1,l} \bar{F}(u_l, u_k) - b_{-1,k} b_{-1,l} \bar{F}(u_l, u_k) = 0$$

$$\frac{d}{dt}\mathbf{1}^T \mathcal{P} S + \bar{F}|_1 - \bar{F}|_{-1} = 0$$

- Entropy stable condition (Fisher, JCP 2013)
  - Numerical flux satisfies condition for entropy conservation/stability

$$(\mathbf{w}_{i+1} - \mathbf{w}_i)^T \bar{\mathbf{f}}^S(\mathbf{u}_i, \mathbf{u}_{i+1}) = \bar{\psi}_{i+1} - \bar{\psi}_i, \quad 1 \le i \le N-1$$

Generalized Entropy Stable Interface Penalty

Entropy stability for interface SAT penalty

$$\frac{d}{dt}\mathbf{1}^T\mathcal{P}S = \bar{F}_L|_{-1} - \bar{F}_L|_{(-)} + \bar{F}_R|_{(+)} - \bar{F}_R|_{-1} + \mathbf{w}^T\mathbf{g}^{int}$$

$$-\bar{F}_L|_{(-)} + \bar{F}_R|_{(R)} + \mathbf{w}^T\left(\mathbf{b}_1^L\mathbf{b}_1^{L^T}\circ\mathbf{f}(\mathbf{u},\mathbf{u}) - \mathbf{b}_1^L\mathbf{b}_{-1}^{R^T}\circ\mathbf{f}(\mathbf{u},\mathbf{u}) - \frac{1}{2}\mathbf{b}_1^L\mathbf{R}|\Lambda|\mathbf{R}^T\left(\mathbf{b}_1^{L^T}\mathbf{w} - \mathbf{b}_{-1}^{R^T}\mathbf{w}\right)\right)$$
$$-\mathbf{w}^T\left(-\mathbf{b}_{-1}^R\mathbf{b}_1^{L^T}\circ\mathbf{f}(\mathbf{u},\mathbf{u}) + \mathbf{b}_{-1}^R\mathbf{b}_{-1}^{R^T}\circ\mathbf{f}(\mathbf{u},\mathbf{u}) - \frac{1}{2}\mathbf{b}_1^R\mathbf{R}|\Lambda|\mathbf{R}^T\left(\mathbf{b}_{-1}^{R^T}\mathbf{w} - \mathbf{b}_1^{L^T}\mathbf{w}\right)\right) = 0$$

$$\psi|_{(-)} - \psi|_{(+)} + \mathbf{w}^T\left(-\mathbf{b}_1^L\mathbf{b}_{-1}^{R^T}\circ\mathbf{f}(\mathbf{u},\mathbf{u}) - \frac{1}{2}\mathbf{b}_1^L\mathbf{R}|\Lambda|\mathbf{R}^T\left(\mathbf{b}_1^{L^T}\mathbf{w} - \mathbf{b}_{-1}^{R^T}\mathbf{w}\right)\right)$$
$$+\mathbf{w}^T\left(\mathbf{b}_{-1}^R\mathbf{b}_1^{L^T}\circ\mathbf{f}(\mathbf{u},\mathbf{u}) + \frac{1}{2}\mathbf{b}_1^R\mathbf{R}|\Lambda|\mathbf{R}^T\left(\mathbf{b}_{-1}^{R^T}\mathbf{w} - \mathbf{b}_1^{L^T}\mathbf{w}\right)\right) = 0$$

$$\mathbf{w}^T\left(\mathbf{b}_{-1}^R\mathbf{b}_1^{L^T}\circ\mathbf{f}(\mathbf{u},\mathbf{u}) + \frac{1}{2}\mathbf{b}_1^R\mathbf{R}|\Lambda|\mathbf{R}^T\left(\mathbf{b}_{-1}^{R^T}\mathbf{w} - \mathbf{b}_1^{L^T}\mathbf{w}\right)\right) - \mathbf{w}^T\left(\mathbf{b}_1^L\mathbf{b}_{-1}^{R^T}\circ\mathbf{f}(\mathbf{u},\mathbf{u}) + \frac{1}{2}\mathbf{b}_1^L\mathbf{R}|\Lambda|\mathbf{R}^T\left(\mathbf{b}_1^{L^T}\mathbf{w} - \mathbf{b}_{-1}^{R^T}\mathbf{w}\right)\right)$$
$$= \psi|_{(+)} - \psi|_{(-)}$$

Shock Capturing with Artificial Viscosity

$$\mathbf{u}_t + \mathcal{P}_k^{-1}\Delta_k\bar{\mathbf{f}}_k = \mathcal{P}_k^{-1}\mathbf{g}_k^{int} + \mathcal{P}_k^{-1}\Delta_k\bar{\mathbf{f}}_k^{ad} + \mathcal{P}_k^{-1}\Delta_k\bar{\mathbf{f}}_k^{av}$$

Artificial viscosity method (Shakib, CM 1991)

- Formulated using entropy variables

$$\hat{\mu} = \max\left[\left(\frac{(L\mathbf{u})^T\mathbf{w}_{\mathbf{u}}(L\mathbf{u})}{\phi + (\mathbf{w}_{x_i})^T g_{ij}\mathbf{u}_{\mathbf{w}}(\mathbf{w}_{x_j})}\right)^{1/2}, c_{ref}\frac{|u|+c}{h}\right], \quad L\mathbf{u} = \frac{\partial \mathbf{f}_k}{\partial \mathbf{v}}\frac{\partial \mathbf{v}}{\partial x_k} - \frac{\partial \mathbf{f}_k}{\partial x_k}$$

$$\mathcal{P}_k^{-1}\Delta_k\bar{\mathbf{f}}_k^{av} = \mathcal{D}_i g_{ij}\frac{\partial \mathbf{u}}{\partial \mathbf{w}}\hat{\mu}\mathcal{D}_j\mathbf{w}$$

$$\mathcal{P}_k^{-1}\Delta_k\bar{\mathbf{f}}_k^{ad} = \mathcal{D}_2|\Lambda|\frac{\partial \mathbf{u}}{\partial \mathbf{w}}\mathcal{D}_2\mathbf{w} \quad \text{where} \quad \mathcal{D}_2 = \Delta\Delta^T$$

- CCHOFD requires artificial dissipation due to lack of dissipation
- Design issues and concerns
  - Artificial dissipation introduces extra dissipation even in freestream regions
  - Shock sensor type adaptive mechanism is utilized to reduce dissipation

2D Curvilinear Problem

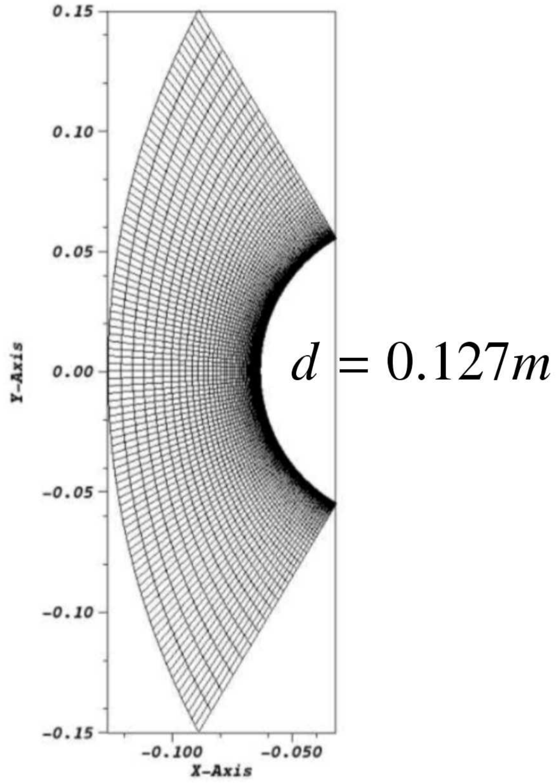
Blottner cylinder

- Perfect gas law
- Sutherland viscosity
- Inflow conditions

$$M = 5.0 \quad Re = 1.8875 \times 10^6$$

Property	Value
Density, $\rho\,[kg/m^3]$	8.788×10^{-2}
Velocity, $u[m/s]$	871.47
Temperature, $T[K]$	75.85

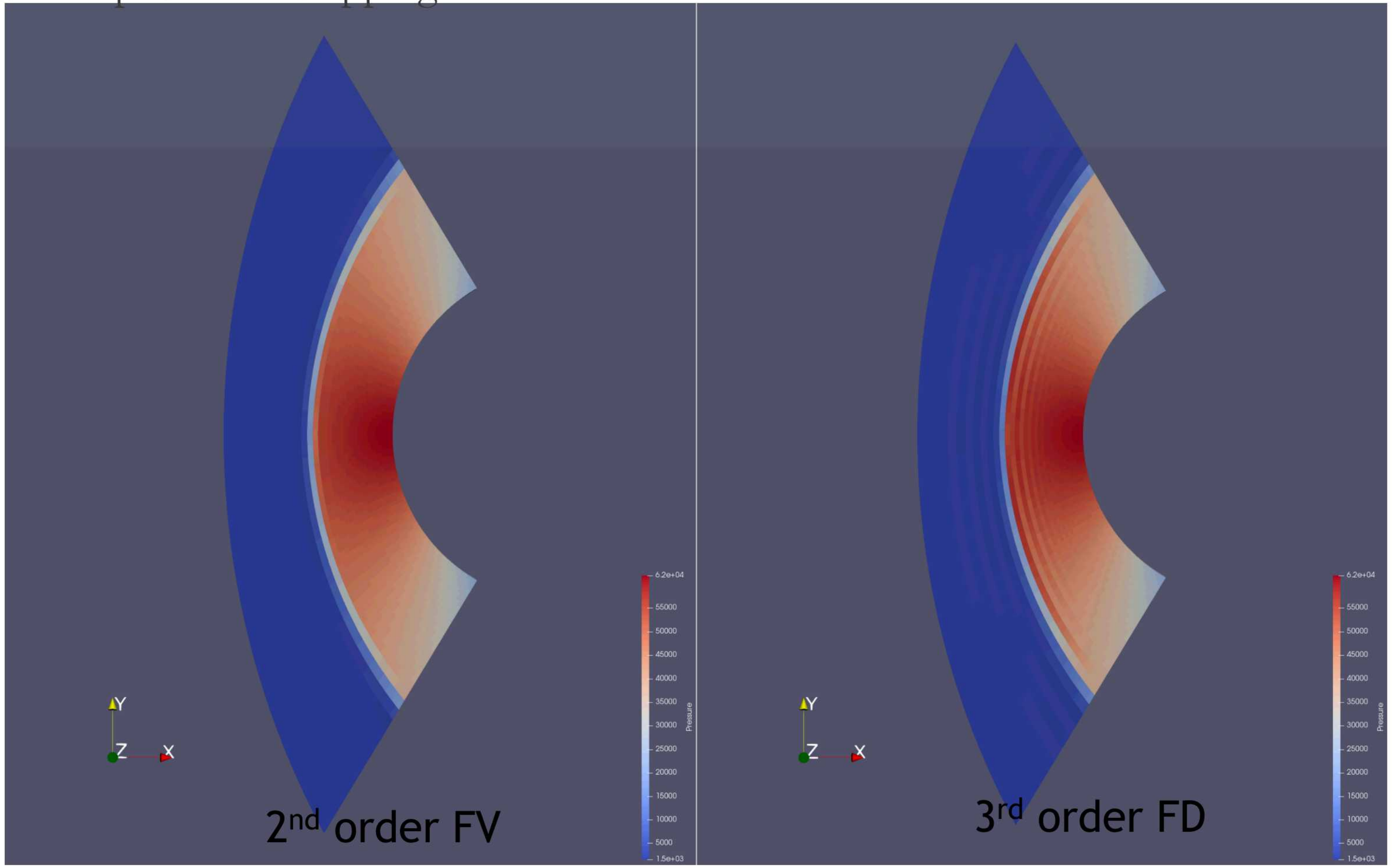
- Grids
  - 50x50
  - 100x100
  - 200x200



Curvilinear Multi-Block Shock Capturing : Artificial Viscosity

Qualitative comparison

- 2^{nd} order cell-centered finite volume with implicit time stepping
  - Minmod limiter
- 3^{rd} order cell-centered finite difference with explicit time stepping
  - Artificial viscosity

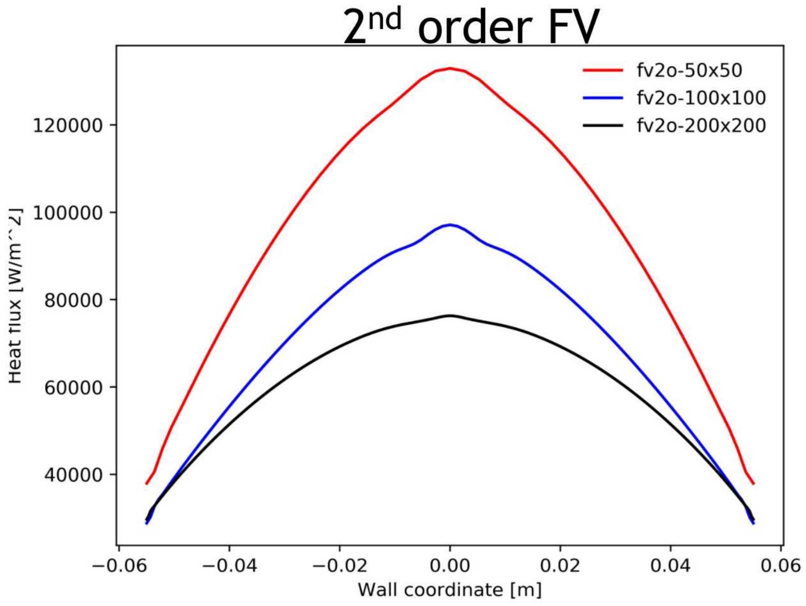


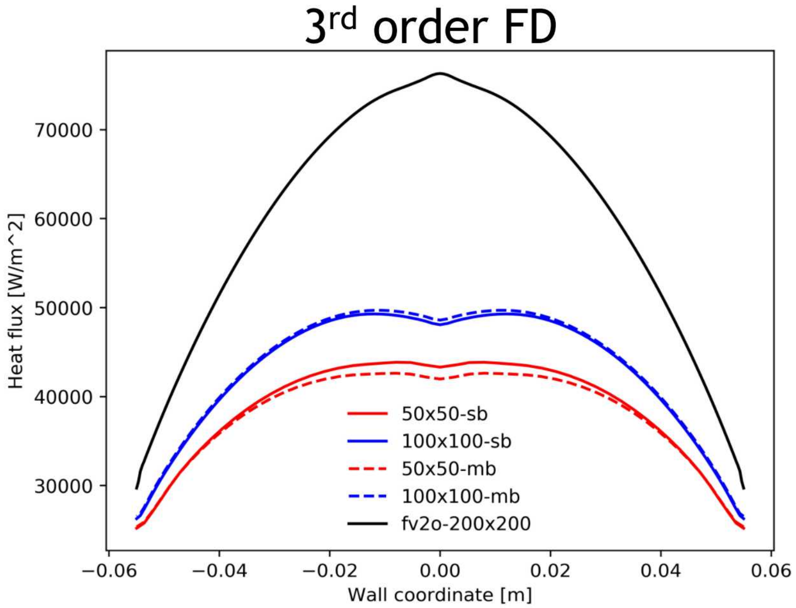
100x100 mesh in 2-block configuration (longitudinal split)

Quantitative Comparison Multi-Block Capability

Wall heat flux prediction

- 2^{nd} order cell-centered finite volume and 3^{rd} order cell-centered finite difference
  - FV 200x200 result close to reference
  - FD 200x200 did not converge due to instability associated with AV near outflow boundary
  - Multi-block and single block results converge





Method Verification

1D Euler MMS

$$\mathbf{u}_t + \mathbf{f}_x = \mathbf{s}, \qquad \mathbf{u} = (\rho, \rho u, \rho E)^T \quad \mathbf{f} = (\rho u, \rho u^2 + p, \rho u H)^T$$

- Initial condition

$$(\rho, u, T) = (2 + \cos(\pi x), 2 + \cos(\pi x), 2 + \cos(\pi x)),$$

$$\mathbf{s} = -\begin{pmatrix} 2\pi \sin(\pi x)(\cos(\pi x) + 2) \\ \pi \sin(\pi x)(\cos(\pi x) + 2)(2R + 3\cos(\pi x) + 6) \\ \pi \sin(\pi x)((\cos(\pi x) + 2)^2(3c_p + 2\cos(\pi x) + 4), \end{pmatrix},$$

- Error convergence

Single block

N	h	$\epsilon_2(\rho)$	Rate	$\epsilon_2(u)$	Rate	$\epsilon_2(T)$	Rate
32	6.06E-02	1.02E-03		5.96E-03		5.28E-01	
64	3.08E-02	5.67E-05	4.27	1.07E-03	2.53	5.66E-02	3.30
128	1.55E-02	4.09E-06	3.84	8.77E-05	3.66	4.29E-03	3.76
256	7.78E-03	2.80E-07	3.89	5.91E-06	3.91	2.87E-04	3.92
512	3.90E-03	1.81E-08	3.96	3.80E-07	3.97	1.83E-05	3.98

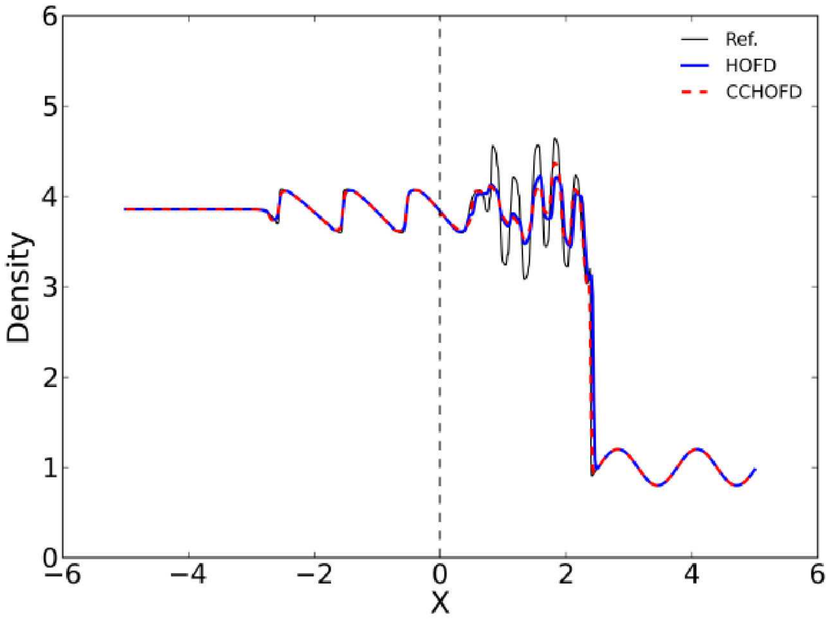
Two blocks

2N	h	$\epsilon_2(\rho)$	Rate	$\epsilon_2(u)$	Rate	$\epsilon_2(T)$	Rate
32	5.88E-02	9.03E-03		3.92E-02		3.51E+00	
64	3.03E-02	3.57E-04	4.87	1.22E-03	5.23	3.63E-01	3.42
128	1.54E-02	3.04E-05	3.64	9.73E-05	3.73	2.92E-02	3.72
256	7.75E-03	2.10E-06	3.90	6.89E-06	3.86	1.97E-03	3.93
512	3.89E-03	1.35E-07	3.98	4.36E-07	4.01	1.27E-04	3.98

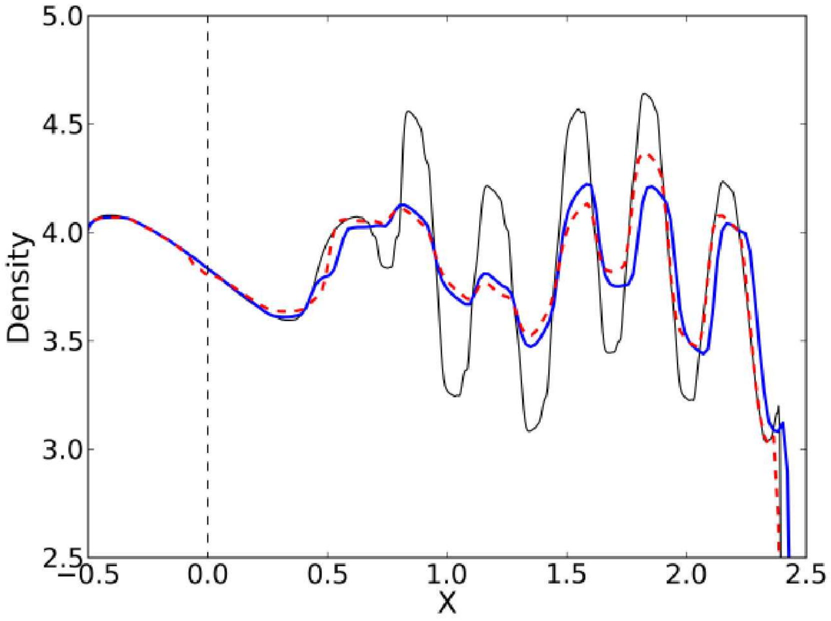
Shu Osher

Interaction of a strong shock and a standing entropy wave fluctuation

$$(\rho, u, p) = \begin{cases} (3.857143, 2.629369, 10.3333), & \text{if } x < -4.0 \\ (1 + 0.2\sin(5x), 0, 1), & \text{if } x \geq -4.0. \end{cases}$$



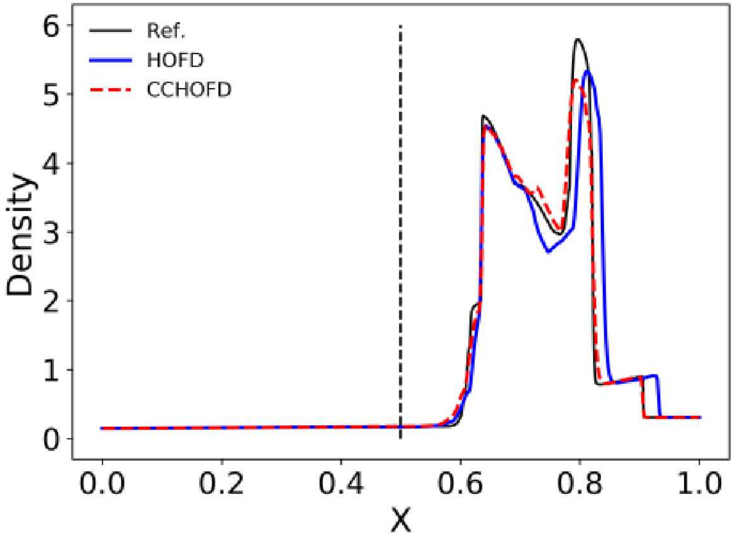
(a) Density



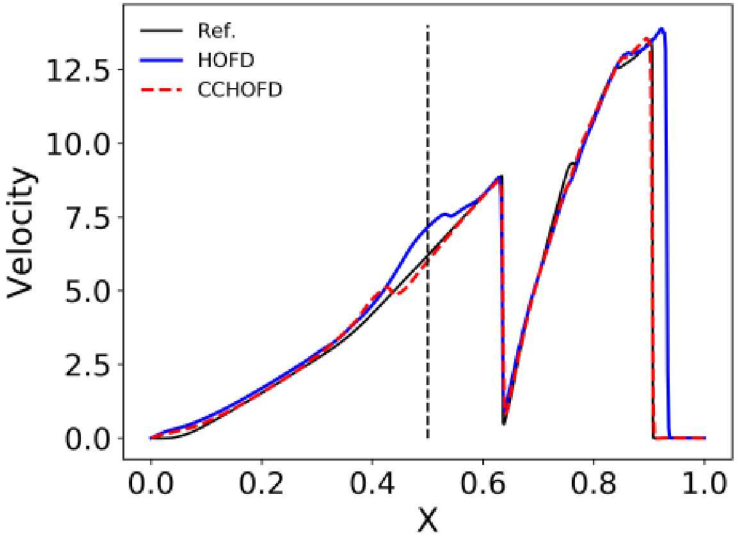
(b) Density close up

N=512, t_f = 1.8, N_{ref} = 2000

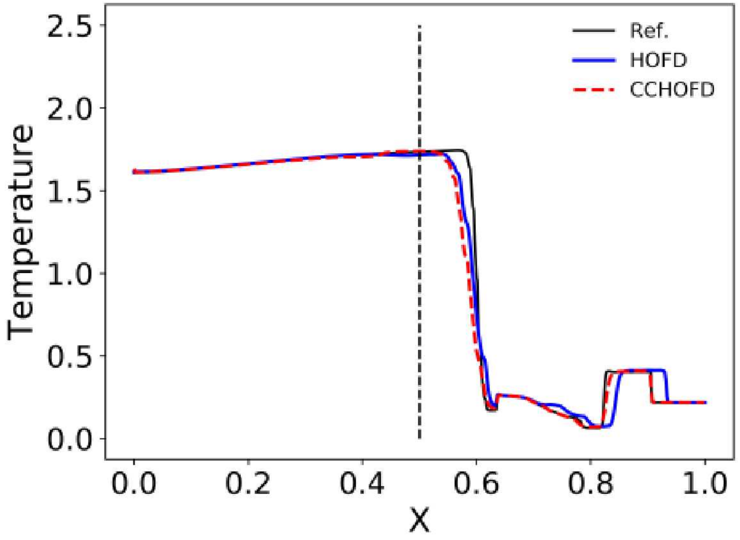
Woodward Colella



(a) Density



(b) Velocity



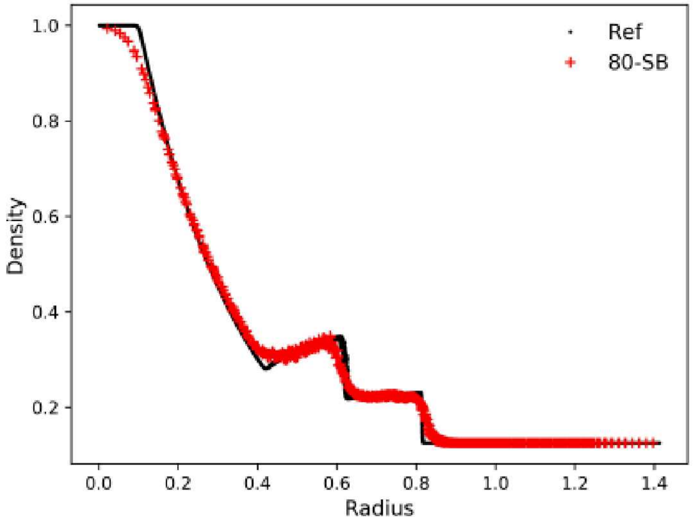
(c) Temperature

N=512, t_f = 0.04, N_{ref} = 2000

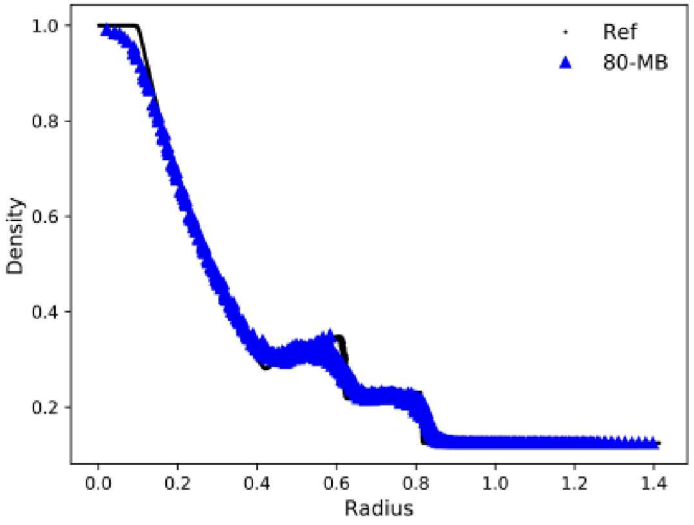
2D Sod Shock Tube

Single-block and multi-block comparison

- Domain $[-1 < (x, y) < 1]$
- Initial condition $(\rho, u, v, p) = \begin{cases} (1, 0, 0, 1), & \text{if } r \leq 0.4 \\ (0.125, 0, 0, 0.1), & \text{if } r > 0.4, \end{cases}$



(a) 80^2 **mesh, a single block**



(b) 80^2 **mesh equally divided into 16 sub-blocks**

$T_f = 0.25$, $N_{ref} = 1280^2$

Entropy Stable Cell-Centered High-Order Finite Difference

Cell-centered summation-by-parts operator

- (2-4-2) operator: third-order

$$\mathbf{b}_{-1} = \left(\frac{35}{16}, -\frac{35}{16}, \frac{21}{16}, -\frac{5}{16}, 0, \ldots, 0\right)^T$$

$$\mathcal{P} = \operatorname{diag}\left(\frac{433}{384}, \frac{95}{128}, \frac{451}{384}, \frac{367}{384}, 1, \ldots, 1, \frac{367}{384}, \frac{451}{384}, \frac{95}{384}, \frac{433}{384}\right)\delta x,$$

$$\mathcal{Q} = \begin{pmatrix} -\frac{1225}{512} & \frac{6695}{1536} & -\frac{4097}{1536} & \frac{359}{512} & 0 & 0 & 0 & \ldots \\ \frac{655}{1536} & -\frac{1225}{512} & \frac{1415}{512} & -\frac{1225}{1536} & 0 & 0 & 0 & \ldots \\ -\frac{313}{1536} & \frac{55}{512} & -\frac{441}{512} & \frac{533}{512} & -\frac{1}{12} & 0 & 0 & \ldots \\ -\frac{9}{512} & \frac{175}{1536} & -\frac{323}{512} & -\frac{25}{512} & \frac{2}{3} & -\frac{1}{12} & 0 & \ldots \\ 0 & 0 & \frac{1}{12} & -\frac{2}{3} & 0 & \frac{2}{3} & -\frac{1}{12} & \ldots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \end{pmatrix},$$