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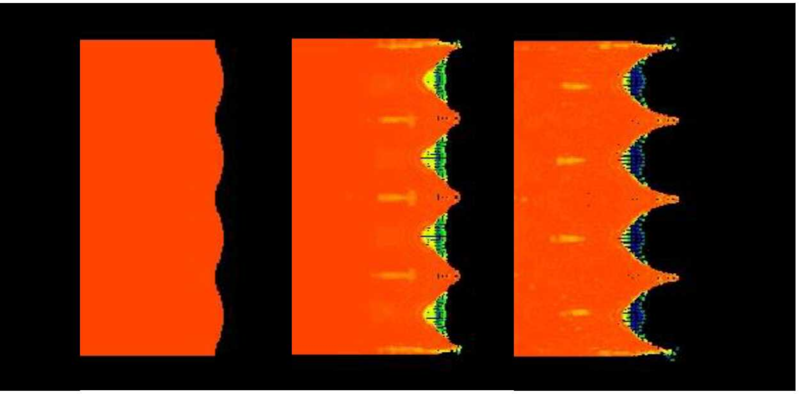

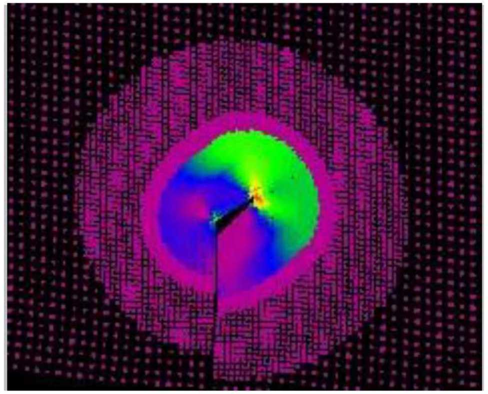

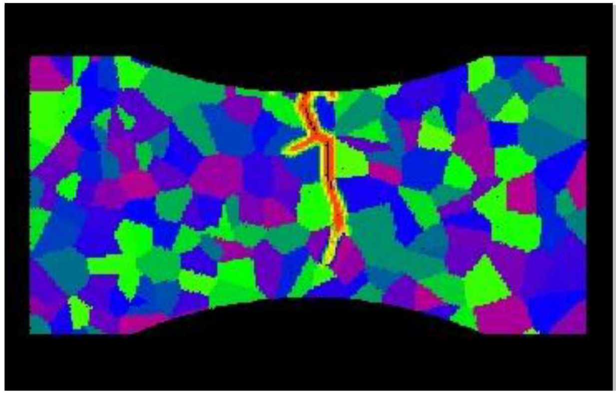

# The First Twenty Years of Peridynamics

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Albuquerque, New Mexico

USNCCM, Austin, TX, July 29, 2019

U.S. DEPARTMENT OF ENERGY

NNSA National Nuclear Security Administration

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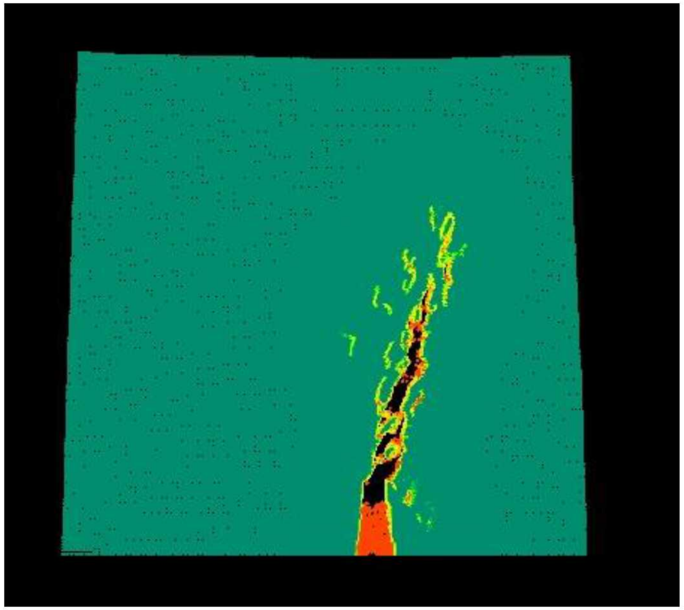
# Outline

- Peridynamics: What it is
- What it was originally intended to do
  - Fracture
- What people have discovered it can do
  - Lots of other stuff

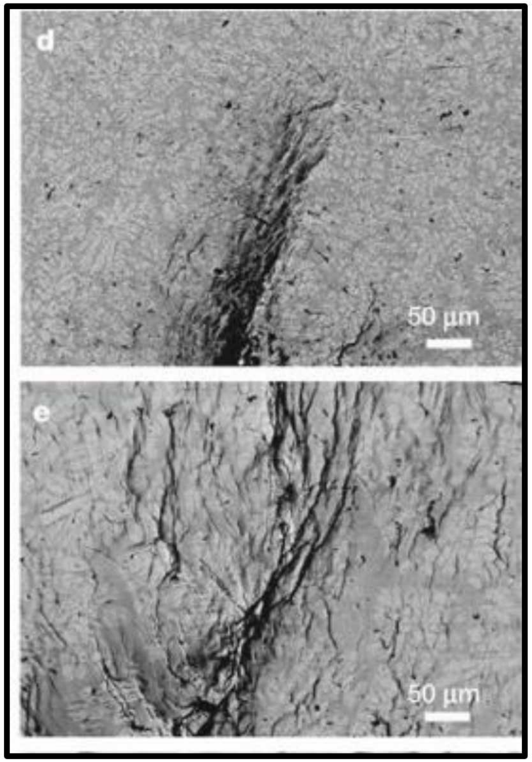
# How does a crack nucleate and grow?

How does a continuous deformation become discontinuous?

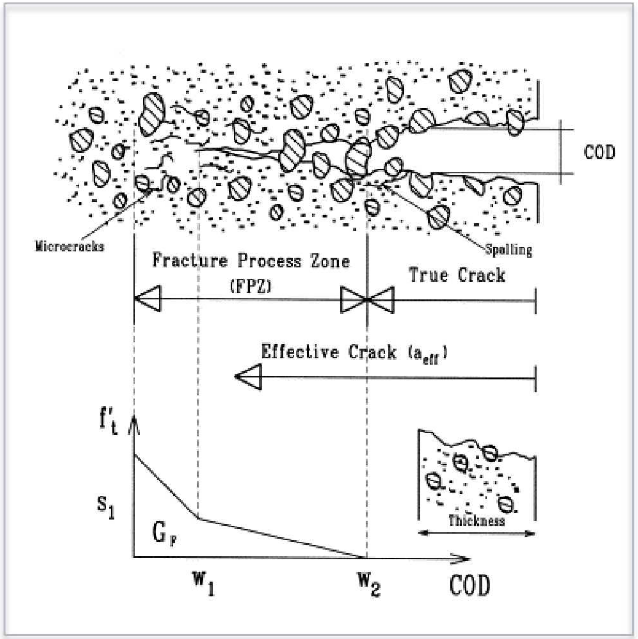
- To study this, we need a model that seamlessly transitions from one to the other within a consistent mathematical system.



Peridynamic simulation




Metallic glass crack tip*




Crack process zone idealization**

*Hofmann et al, Nature (2008)

**Abhimanew, https://commons.wikimedia.org/wiki/File:Fracture_Process_Zone.gif

# Peridynamic answers to some simple questions

- Why is fracture different from other kinds of deformation?
  - *It isn't.*
- Why are special modeling techniques needed for fracture?
  - *They aren't.*
- Why does nearly everybody think they are?
  - *Because nearly everybody uses partial differential equations (PDEs).*
- What might work better?
  - *Integral equations.*

Typical damage progression in a notched composite panel (photo courtesy Boeing)

# Peridynamic* momentum balance

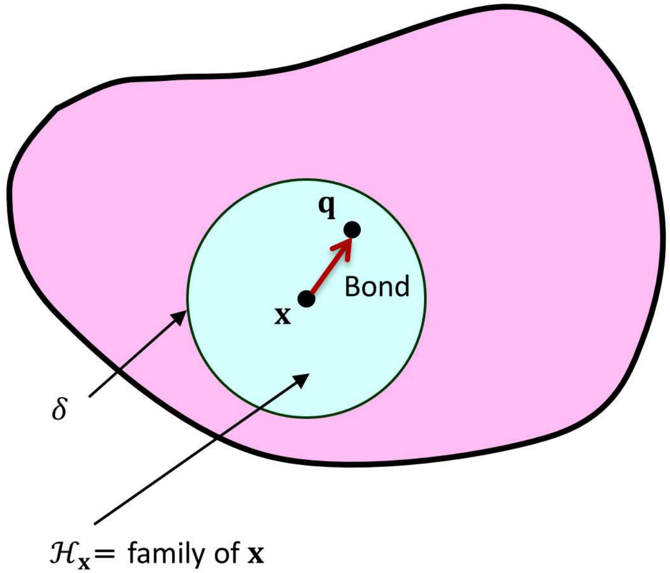
* Peri (near) + dyne (force)

- Any point $\mathbf{x}$ interacts directly with other points within a distance $\delta$ called the "horizon."
- The material within a distance $\delta$ of $\mathbf{x}$ is called the "family" of $\mathbf{x}$, $\mathcal{H}_{\mathbf{x}}$.

Peridynamic equilibrium equation

$$\int_{\mathcal{H}_{\mathbf{x}}} \mathbf{f}(\mathbf{q},\mathbf{x})\, dV_{\mathbf{q}} + \mathbf{b}(\mathbf{x}) = 0$$

**f** = bond force density (from the material model, which includes damage)



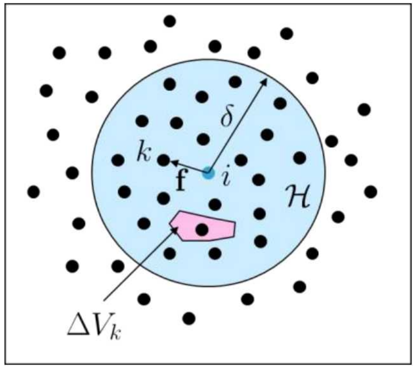

- If $\mathbf{f}$ satisfies $\mathbf{f}(\mathbf{x},\mathbf{q}) = -\mathbf{f}(\mathbf{q},\mathbf{x})$ for all $\mathbf{x},\mathbf{q}$ then linear momentum is conserved.

- SS, JMPS (2000)

# Simple particle discretization

- Integral is replaced by a finite sum: resulting method is meshless and Lagrangian.

$$\rho\ddot{\mathbf{y}}(\mathbf{x},t) = \int_{\mathcal{H}} \mathbf{f}(\mathbf{x}',\mathbf{x},t)\,dV_{\mathbf{x}'} + \mathbf{b}(\mathbf{x},t) \longrightarrow \rho\ddot{\mathbf{y}}_i^n = \sum_{k\in\mathcal{H}} \mathbf{f}(\mathbf{x}_k,\mathbf{x}_i,t)\,\Delta V_k + \mathbf{b}_i^n$$




- Good:
  - Simple.
  - Linear and angular momentum conserved exactly.
    - Why: the discretized system is itself a peridynamic body.
- Bad:
  - Dust!
  - If $\Delta x/\delta$ is held constant, fails to converge to PDEs as $\delta \to 0$.
  - Fails patch test for irregular grids.

- Discontinuous Galerkin is another viable method (used in LS-DYNA).

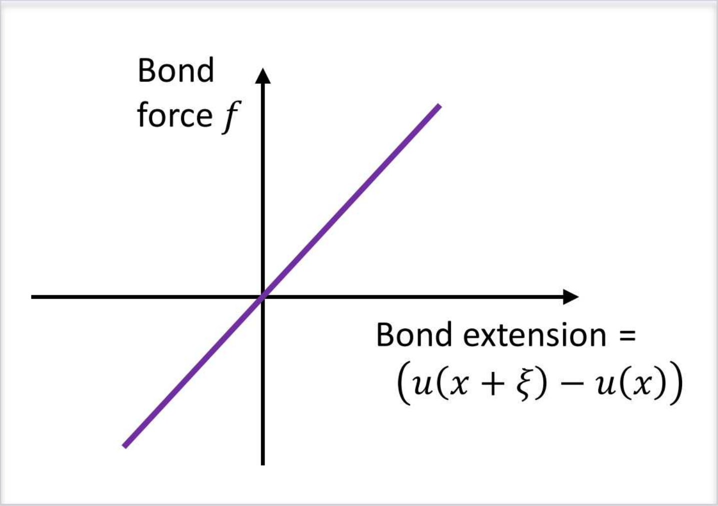

- SS & Askari, *Computers and Structures* (2005)
- Bobaru, Yang, Alves, SS, Askari, & Xu, *IJNME* (2009)
- Chen & Gunzburger, *CMAME* (2011)
- Du, Tian, & Zhao, *SIAM J Numerical Analysis* (2013)
- Tian & Du, *SIAM J Numerical Analysis*. (2014)
- Ganzenmüller, Hiermaier, May, in *Meshfree methods for partial differential equations VII,* Springer (2015)
- Seleson & Littlewood, *Computers & Mathematics with Applications* (2016)
- Du, in *Handbook of peridynamic modeling* (2016)

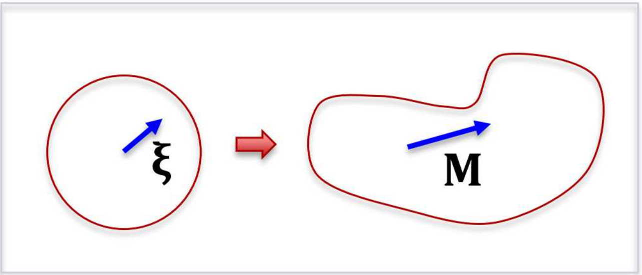
# Simplest material model: Microelastic

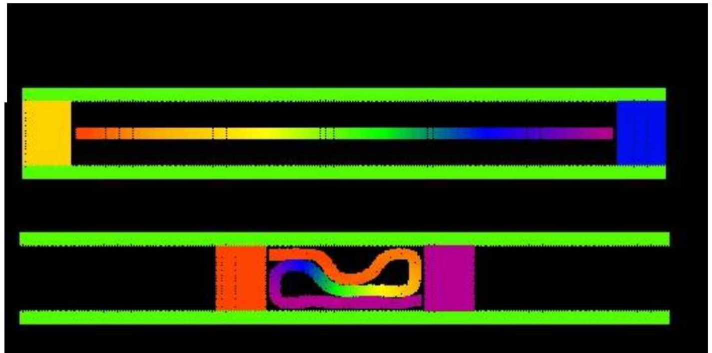
- Each bond acts like a linear spring.

  $$\mathbf{f} = \mathbf{C}(\boldsymbol{\xi})\big(\mathbf{u}(\mathbf{x}+\boldsymbol{\xi}) - \mathbf{u}(\mathbf{x})\big)\mathbf{M}$$

  - $\mathbf{u}$= displacement
  - $\mathbf{M}$= deformed bond direction
  - $\boldsymbol{\xi}$= bond vector
  - $\mathbf{f}$= bond force
  - $\mathbf{C}(\boldsymbol{\xi})$ = micromodulus (spring constant)
- Micromodulus and horizon determine the wave speeds.



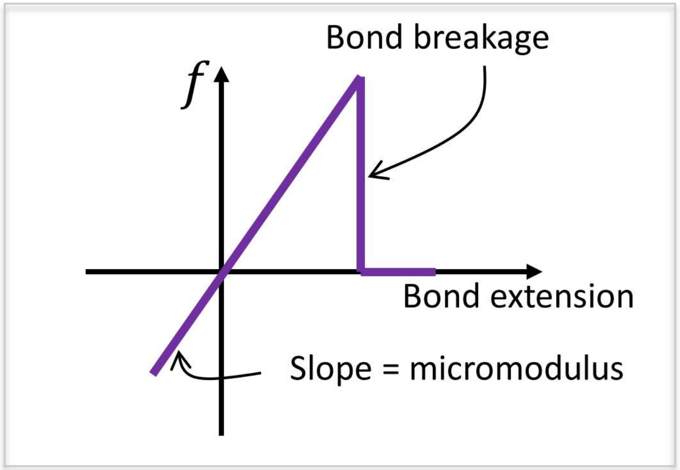



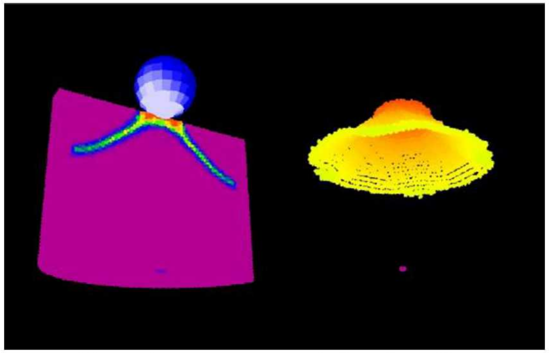

We get geometrical nonlinearity for free!
(Simulation includes contact forces.)

- SS, *JMPS* (2000)

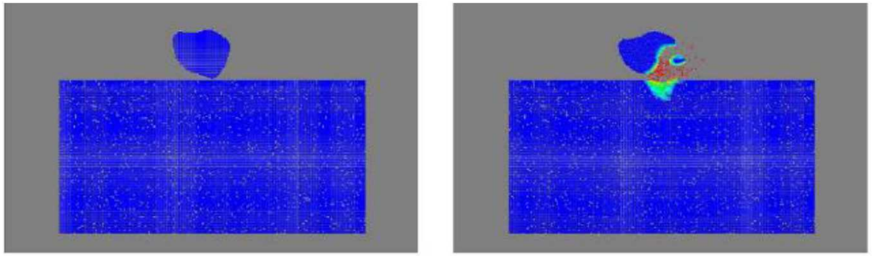
# Simplest material model for fracture: Brittle microelastic

- Damage can be modeled by bond breakage.
  - Breakage strain determines the energy release rate $G_{Ic}$.



Hertzian cone crack in glass*

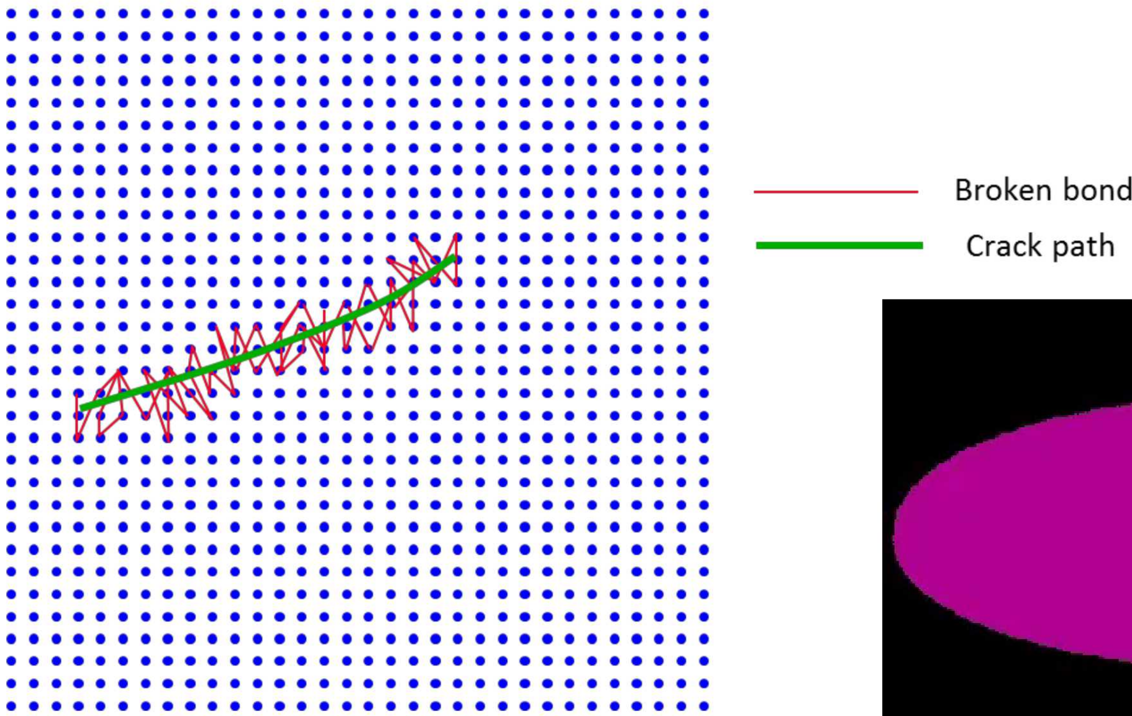


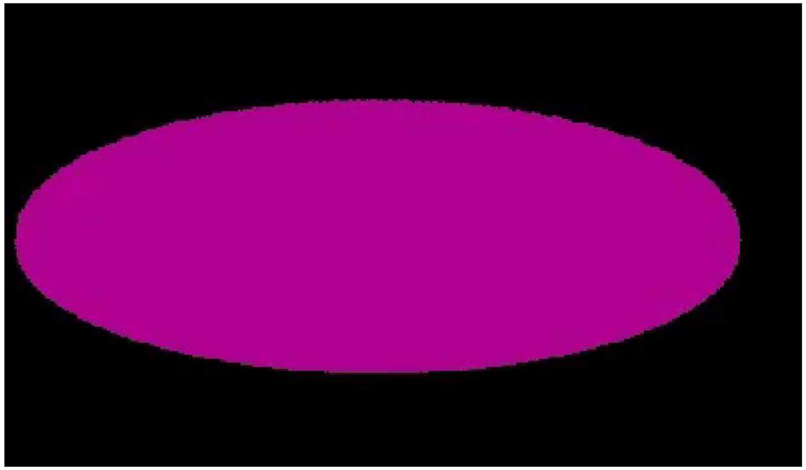
Sand particle impact erodes a glass target**

*SS & Askari, *Computers and Structures* (2005)
**Waxman & Guven, *Wear* (2019)

# Most commonly used capability: Autonomous crack growth

- Cracks do what they want (grow, arrest, branch, curve, oscillate, …)

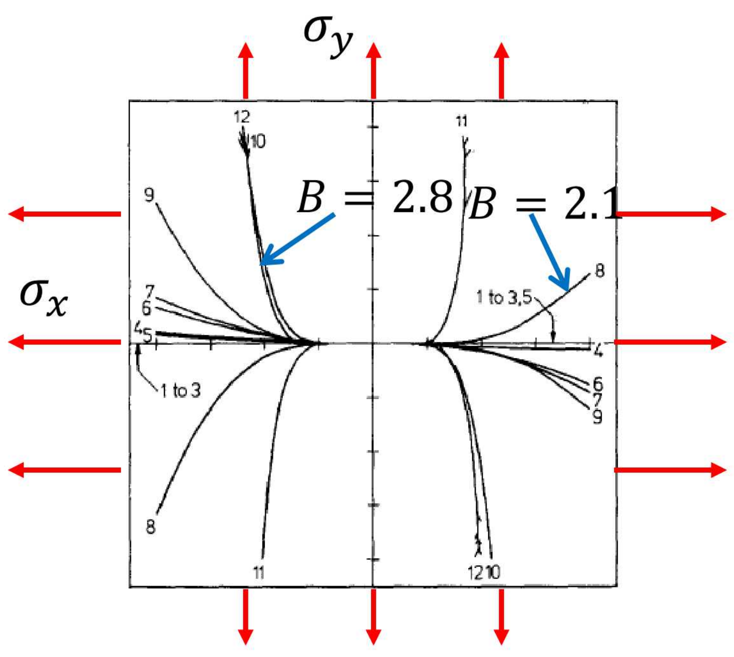


- SS & Askari, *Computers and Structures* (2005)

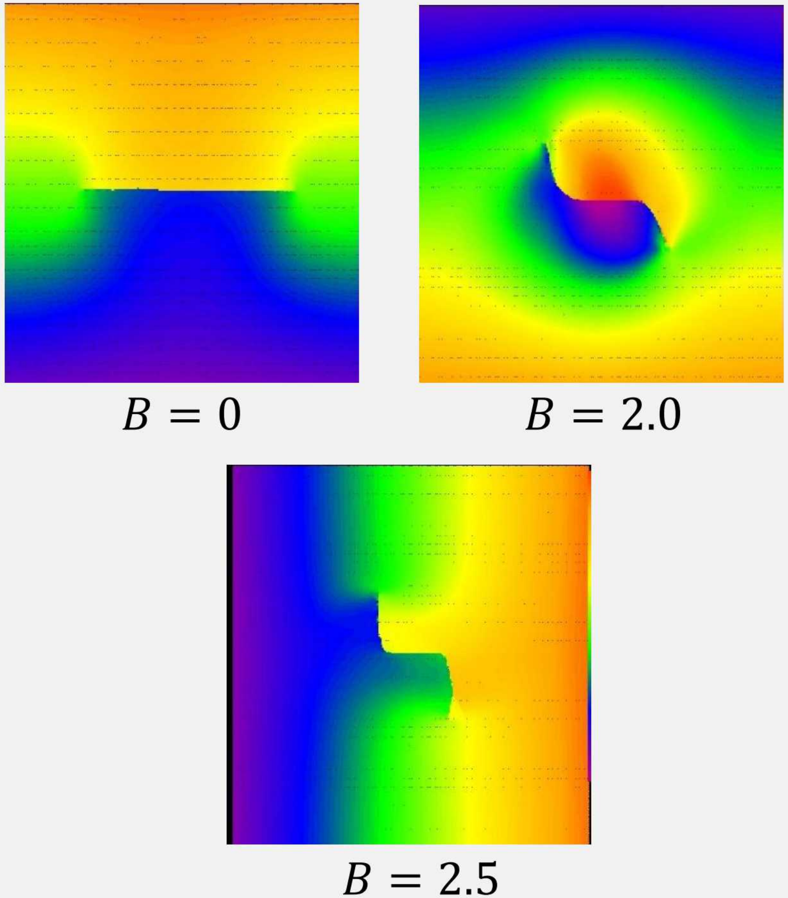
# Crack stability and mode transition

- Biaxial loading makes a crack turn.
- Center defect can grow in an S-shape.
- Biaxiality: $B = \sigma_x/\sigma_y$.



Observed crack paths in PMMA*

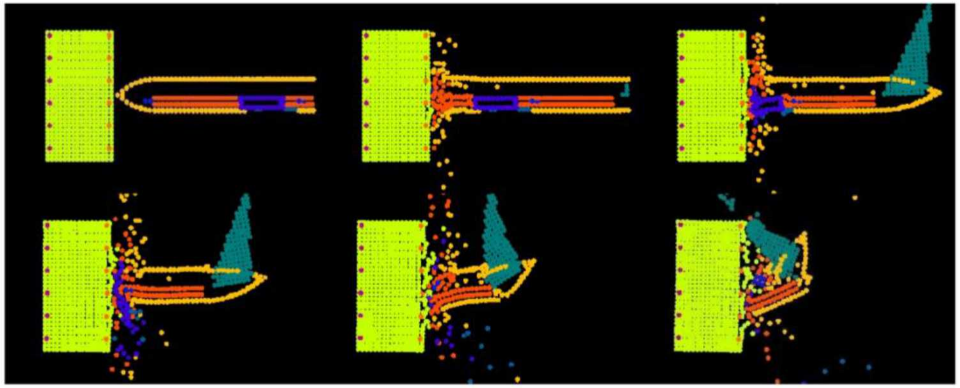
Simulated crack paths



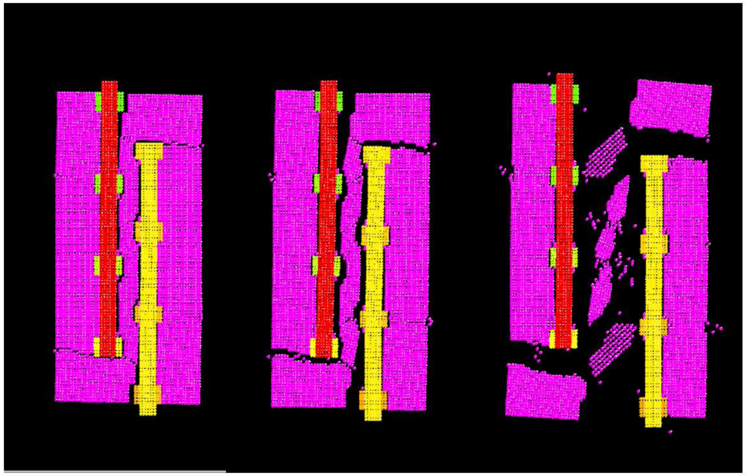

*Leevers, Radon, & Culver JMPS (1976)

# Early 2000's: Airplane crashes and reinforced concrete

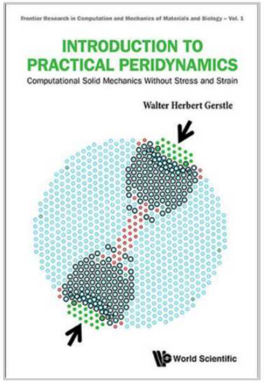
- Meshless discretization, simple contact algorithm, and geometrical nonlinearity are all important.



Emu simulation of F4 aircraft vs. concrete block*



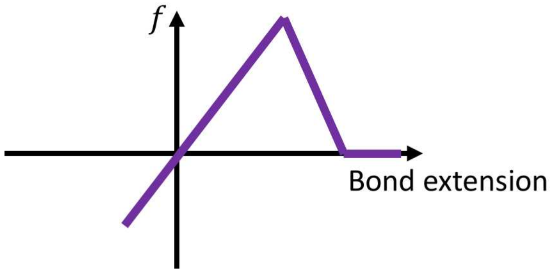
Model of rebar splice failure (courtesy W. Gerstle)



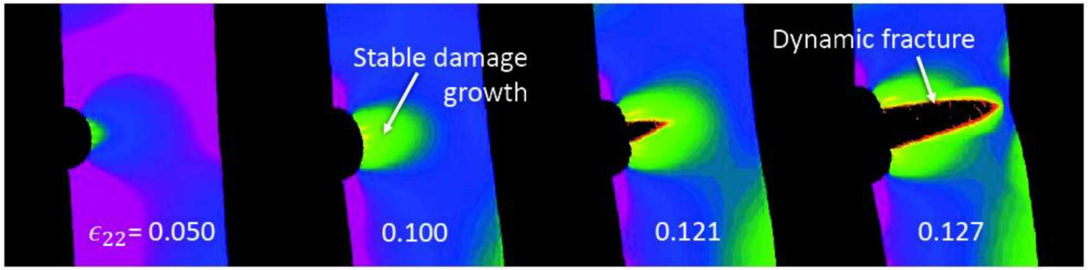

- Sugano et al, Nuclear Engineering and Design (1993)
- Gerstle, Sau, & SS, Nuclear Engineering and Design (2007)

# Damage laws beyond brittle microelastic

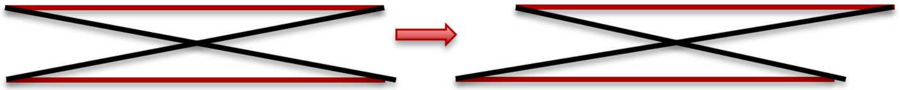
- The theory does not restrict what causes bonds to break. Examples:




Adding a tail to the simple bond breakage material decouples $G_{Ic}$ from the peak stress*




Peridynamic crack nucleation using nonlocal damage mechanics**

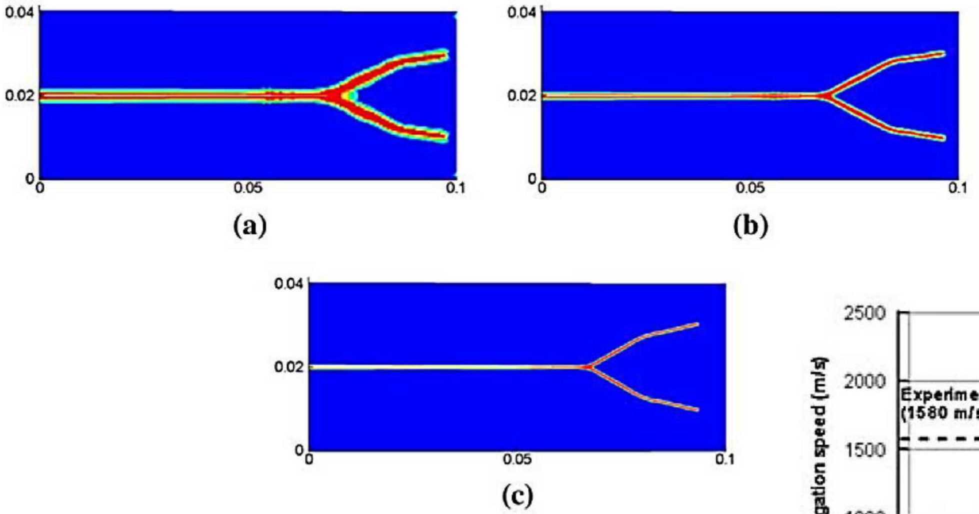


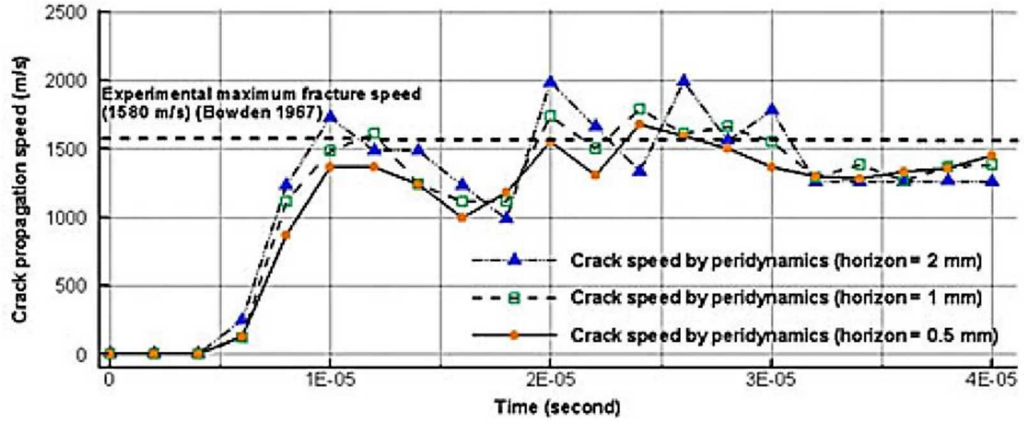
Damage criterion that depends on shear*** can be calibrated to $G_{IIc}$

*SS, in *Handbook of Peridynamic Modeling* (2016)
**SS, in *Handbook of Nonlocal Continuum Mechanics* (2019)
***E. Oterkus, thesis, Univ of Arizona (2010)

# 2010: Peridynamics reproduces measured dynamic crack velocity

- Fracture in soda-lime glass using 3 different grid spacings*.

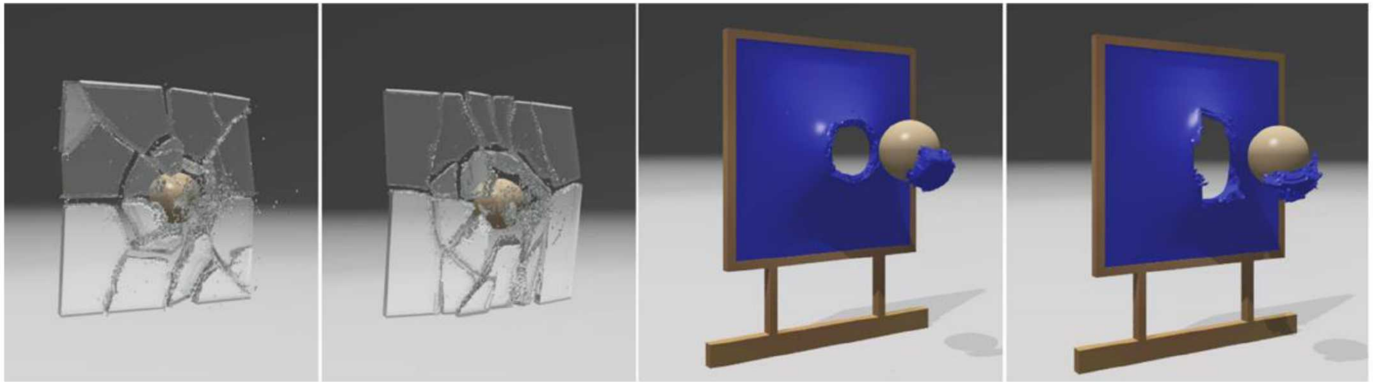



- Ha & Bobaru, *Int J Fracture* (2010)
- *Agwai, Guven, & Madenci, *Int J Fracture* (2011)
- Ha & Bobaru, *Engin Fracture Mech* (2011)
- Dipasquale, Zaccariotto, & Galvanetto, *Int J Fracture* (2014)
- Bobaru & Zhang, *Int. J Fracture* (2015)
- Zhou, Wang, & Qian, *European J Mechanics-A/Solids*. (2016)

# Fracture animation

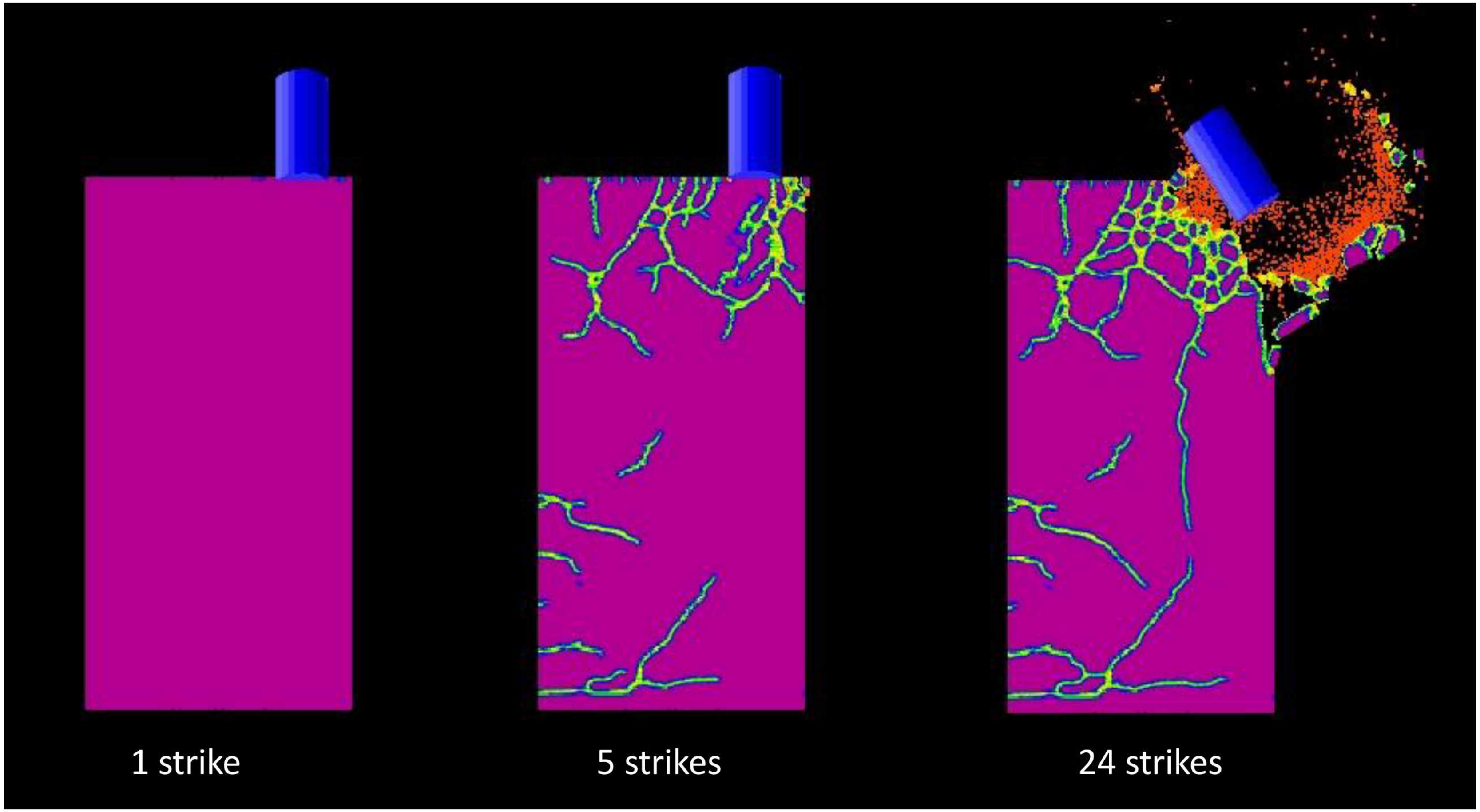
- Peridynamics is a viable method for making movies.



Chen, Zhu, Zhao, Li, & Wang*, Computer Graphics Forum* (2018)


Levine, Bargteil, Corsi, Tessendorf, & Geist, *ACM SIGGRAPH/Eurographics Symposium on Computer Animation* (2014)

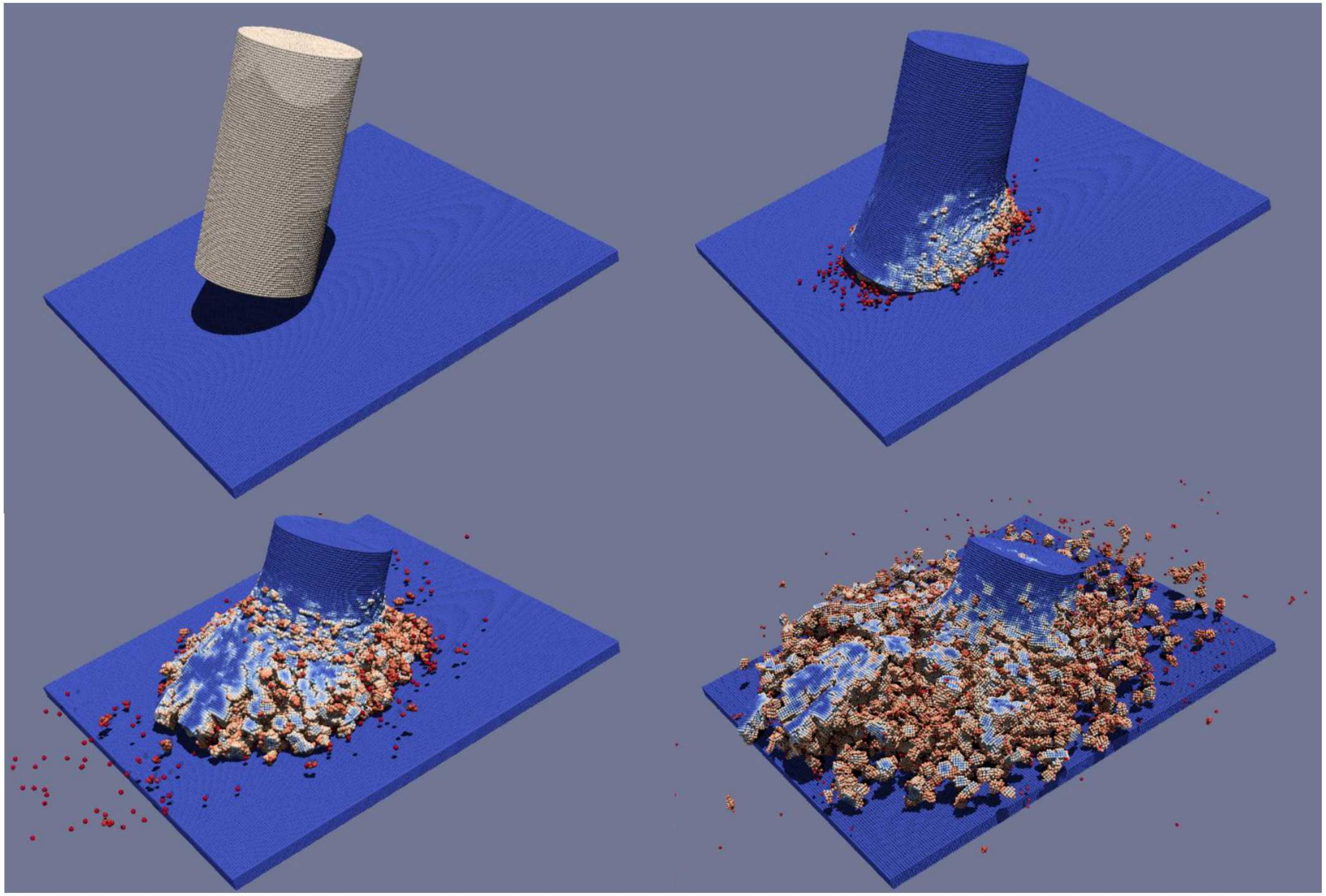
# Accumulation of damage: Hammering on a block




Colors show damage

# Fragmentation due to impact

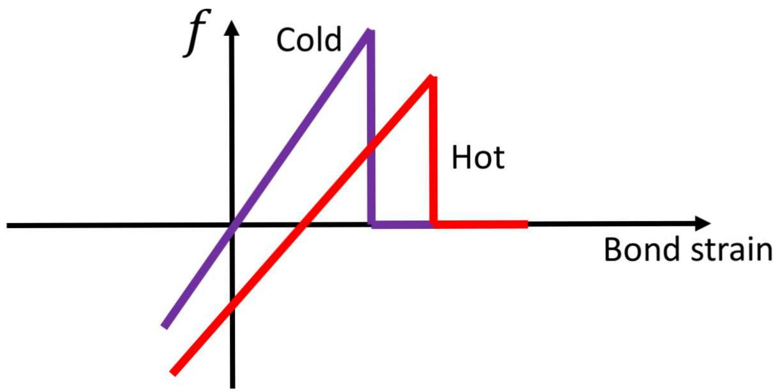
- Brittle cylinder vs. rigid plate at 1km/s.

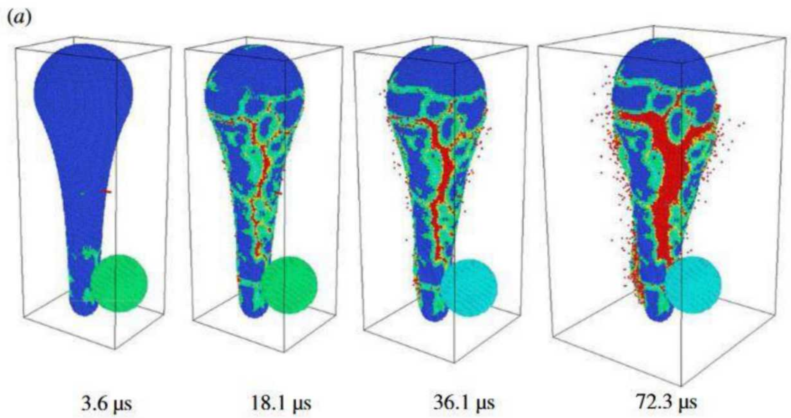


Colors show damage

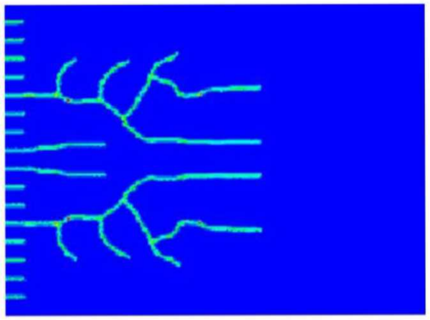
# Fracture driven by thermal stress

- A microelastic material model is easily modified to include thermal strain in bonds.
- A thermodynamically consistent state-based approach to thermoelasticity is available**.




Rupert test for thermal stress in glass**

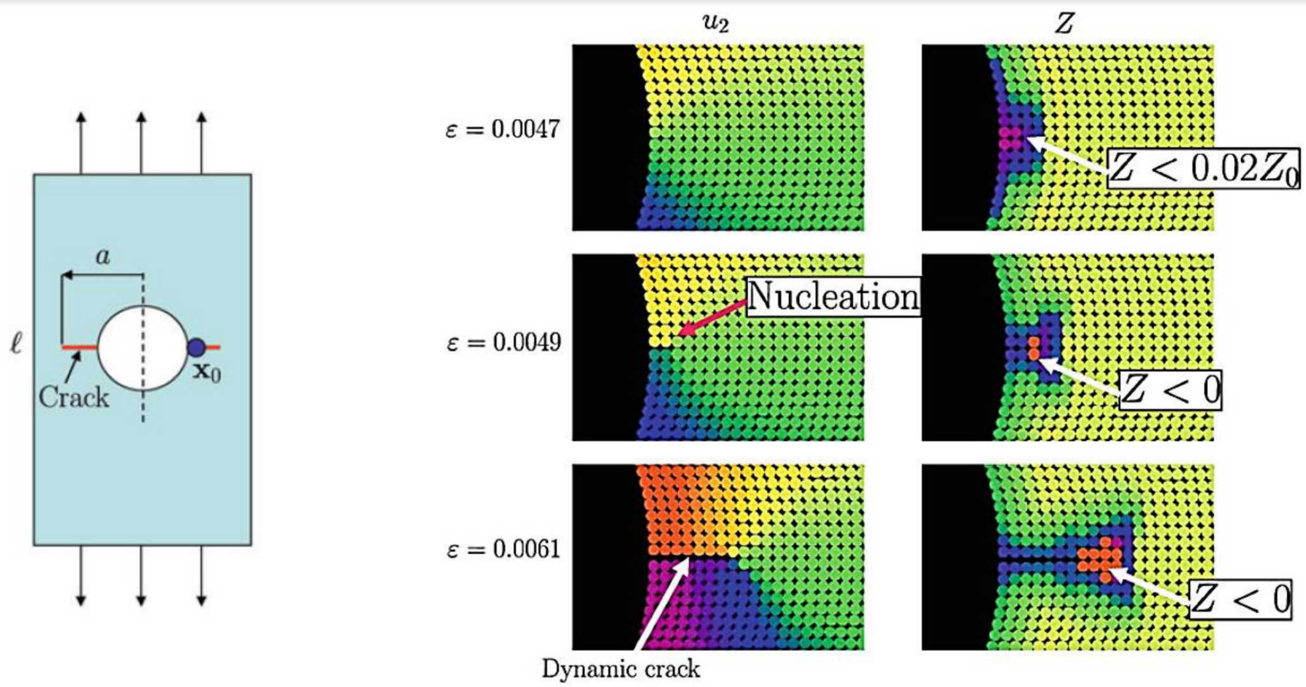


Crack branching driven by a temperature gradient*

- *Kilic & Madenci, *Int J Fracture* (2009)
- **S. Oterkus, Madenci, & Agwai, JMPS (2014)
- ***Jeon, Stewart, & Ahmed, *Proc Royal Society A* (2015)
- Xu, Zhang, Chen, & Bobaru, *Int J Fracture* (2018)

# Crack nucleation

- Crack nucleation happens due a material instability.
- Happens when a perturbation of any point fails to induce a force in the opposite direction.
- Does not in general coincide with peak in a stress-strain curve.

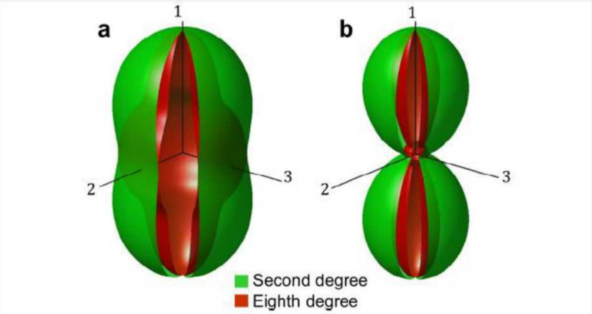



Crack nucleation near a hole in a stretchd brittle-microelastic plate.
Unstable material surrounds a crack tip.

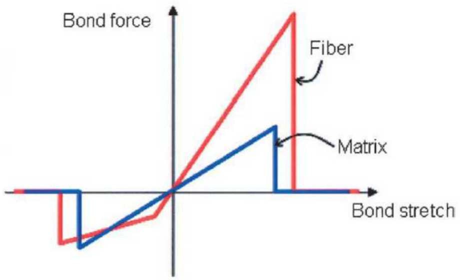
- SS, Weckner, Askari, & Bobaru*, Int J Fracture* (2010)
- Lipton, Lehoucq, & Jha, *J Peridynamics & Nonlocal Modeling* (2016)

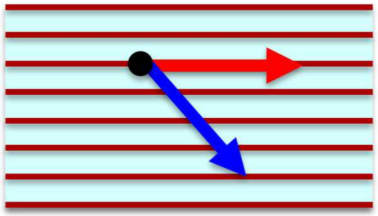
# 2009: Anisotropy and composites

- Apply different bond stiffness and breakage criteria in different directions.

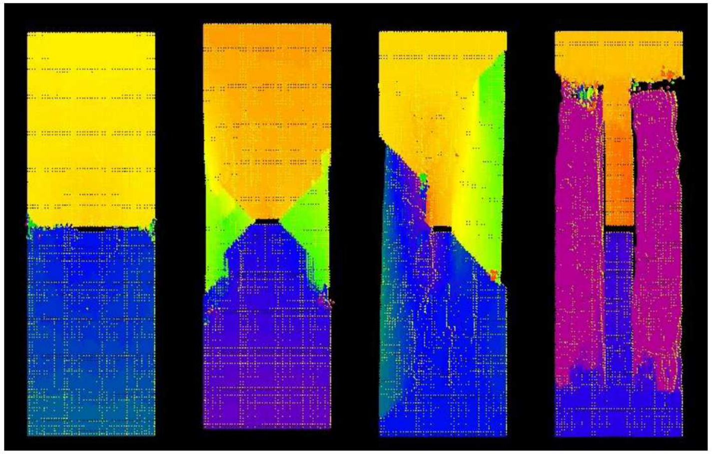



Bond stiffness as a function of direction using spherical harmonics*



Fiber and matrix bonds



EMU simulations of failure modes in a notched composite panel under tension for different layups

- Kilic, Agwai, & Madenci, *Composite Structures* (2009)
- E. Oterkus, thesis, Univ of Arizona (2010)
- Hu, Ha, & Bobaru*, Int J Multiscale Comp Engin* (2011)
- Hu, Ha, & Bobaru , *CMAME* (2012)
- *Ghajari, Iannucci, & Curtis, *CMAME* (2014)
- Yu & Wang, *Composite Structures* (2014)
- S. Oterkus & Madenci*, AIAA SciTech Forum* (2014)
- Hu, De Carvalho, & Madenci, *Composite Structures* (2015)
- Su & Huang, *Composite Structures* (2016)
- Bobaru, Mehrmashhadi, Chen, & Niazi, *US-Japan Conference on Composite Materials* (2018)
- Baber, Ranatunga, & Guven, *J Composite Materials* (2018)
- Trageser & Seleson, arXiv:1905.12761 (2019)
- Mikata, *IJSS* (2019)

# Many validation studies have been done

Sandia National Laboratories

- First issue of the new *Journal of Peridynamics and Nonlocal Modeling* has a review article by Diehl on published validation to date:

Journal of Peridynamics and Nonlocal Modeling
https://doi.org/10.1007/s42102-018-0004-x

REVIEWS

**A Review of Benchmark Experiments for the Validation of Peridynamics Models**

Patrick Diehl[1] · Serge Prudhomme[2] · Martin Lévesque[1]

Received: 2 November 2018 / Accepted: 25 December 2018

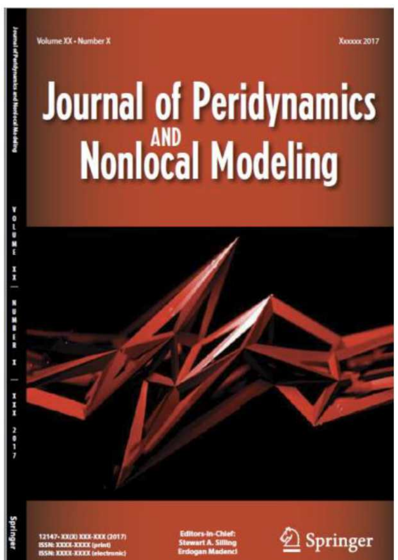




**Table 3** Applications of bond-based and state-based peridynamics for the comparison with experimental data

| Material | Mechanical test | B | S | Exp | Sim |
|---|---|---|---|---|---|
| Composite | Flexural test with an intial crack | ✓ | | [75] | [2] |
| Composite | Damage growth prediction (six-bolt specimen) | ✓ | | [120] | [96] |
| Composite | Damage prediction (center-cracked laminates) | ✓ | | [6, 12, 69, 134] | [70] |
| Composite | Dynamic tension test (prenoteched rectangular plate) | ✓ | | [12, 65] | [58] |
| Steel | Crack growth (Kalthoff-Winkler) | ✓ | ✓ | [66–68] | [3, 52, 114, 144] |
| Aluminum/Steel | Fracture (compact tension test) | ✓ | | [9, 77, 89, 91] | [135, 141, 142] |
| Aluminum | Taylor impact test | | ✓ | [4, 21] | [3, 43, 45] |
| Aluminum (6061-T6) | Ballistic impact test | | ✓ | [132] | [127] |
| Concrete | Lap-splice experiment | ✓ | | [48] | [48] |
| Concrete | 3-point bending beam | ✓ | ✓ | [19, 63] | [7, 51] |
| Concrete | Failure in a Barazilian disk under compression | | ✓ | [51] | [54] |
| Concrete | Anchor Bolt Pullout | ✓ | | [128] | [83] |
| Glass | Dynamic crack propagation (prenotched thin rectangular plate) | ✓ | | [15, 36, 100] | [2, 53, 144] |
| Glass | Impact damage with a thin polycarbonate backing | ✓ | | [8, 20, 40] | [59] |
| Glass | Single crack paths (quenched glass plate) | ✓ | | [13, 103, 136] | [71] |
| Glass | Multiple crack paths (quenched glass plate) | ✓ | | [102, 137] | [71] |
| Glass | Crack tip propagation speed | ✓ | | [15] | [52, 53, 144] |
| PMMA | Fast cracks in PMMA | ✓ | | [39] | [2] |
| PMMA | Tensile test | ✓ | | [124] | [32] |
| Soda-lime glass | Impact on a two-plate system | ✓ | | [16, 130] | [130] |

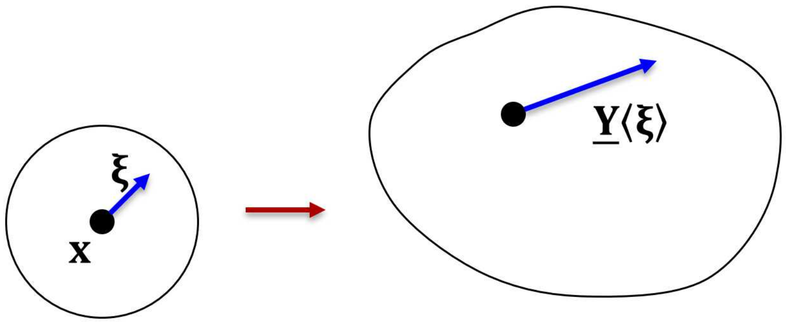
Legend: B refers to bond-based peridynamics, S refers to state-based peridynamics, Exp to experimental data, and Sim to simulation

# Books

Advances in Applied Mathematics
Handbook of
Peridynamic
Modeling
Edited by
Florin Bobaru
John T. Foster
Philippe H. Geubelle
Stewart A. Silling
CRC Press
A CHAPMAN & HALL BOOK
Erdogan Madenci · Erkan Oterkus
Peridynamic
Theory
and Its
Applications
Springer
George Z. Voyiadjis
Editor
Handbook of
Nonlocal
Continuum
Mechanics for
Materials and
Structures
Springer
Erdogan Madenci · Atila Barut
Mehmet Dorduncu
Peridynamic
Differential
Operator for
Numerical
Analysis
Springer
INTRODUCTION TO
PRACTICAL PERIDYNAMICS
Computational Solid Mechanics Without Stress and Strain
Walter Herbert Gerstle
World Scientific

# 2007: State-based material modeling

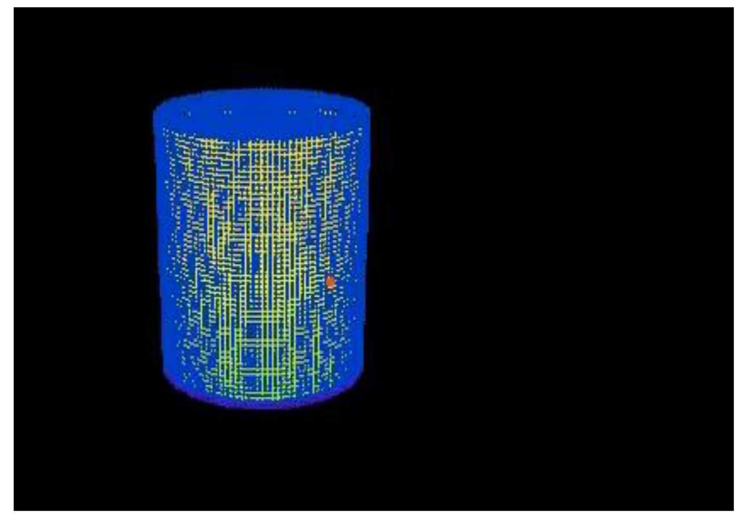
- Bond-based: each bond responds independently of the others.
- State-based: the deformation of the entire family collectively determines the bond forces.
- State-based momentum balance: $\rho \ddot{\mathbf{y}}(\mathbf{x},t) = \int_{\mathcal{H}} \underbrace{\{\underline{\mathbf{T}}[\mathbf{x}]\langle \mathbf{q}-\mathbf{x}\rangle - \underline{\mathbf{T}}[\mathbf{q}]\langle \mathbf{x}-\mathbf{q}\rangle\}}_{\mathbf{f}(\mathbf{q},\mathbf{x})} dV_{\mathbf{q}} + \mathbf{b}(\mathbf{x})$




- Example: compressible fluid

$$\underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle = k\vartheta \frac{\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle}{|\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle|}$$

$\vartheta$ = dilatation of family
= (deformed volume)/(initial volume)-1

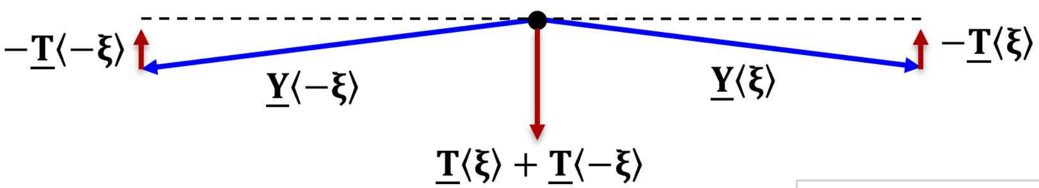


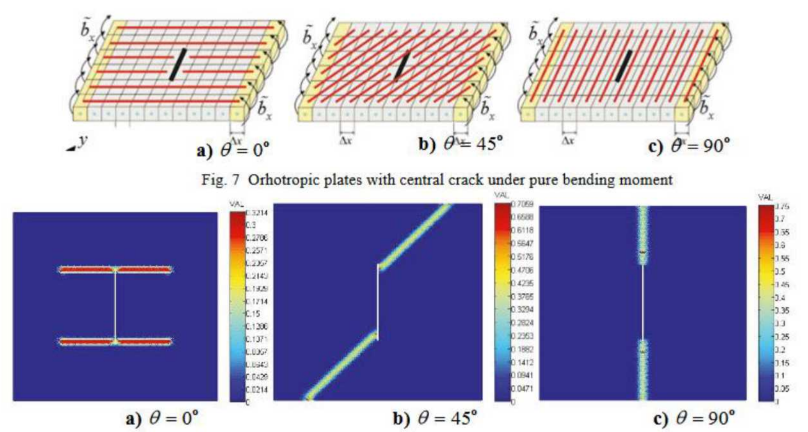
Hydraulic ram
VIDEO

- SS, Epton, Weckner, Xu, & Askari, *J Elasticity* (2007)
- Madenci & E. Oterkus, *Peridynamic Theory & Its Applications*, Springer (2014)

# 2014: State-based plates & shells

- A state-based model can resist the relative rotation of bonds in a family, i.e., bending.
- No need to create new field equations for beams, plates, & shells.




Bending and failure of orthotropic plates*

- O'Grady & Foster, *Int J Solids & Structures* (2014)
- Diyaroglu, E. Oterkus, S. Oterkus, & Madenci*, Int J Solids & Structures* (2015)
- Chowdhury, P. Roy, D. Roy, Reddy, *Int J Solids & Structures* (2016)
- *Taştan, Yolum, Güler, Zaccariotto, & Galvanetto, *Procedia Structural Integrity* (2016)

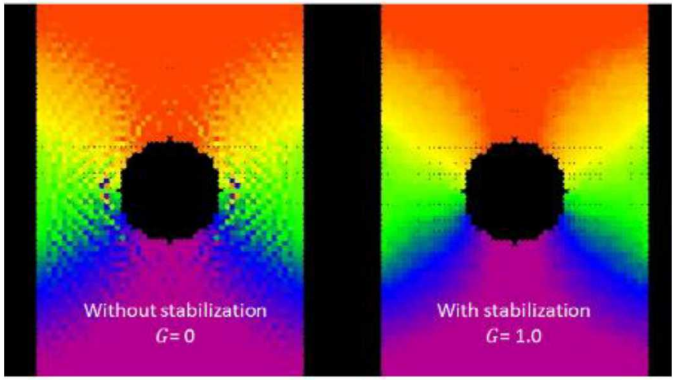
# Correspondence materials

Sandia National Laboratories

- This is a type of state-based material model that uses a stress tensor as an intermediate quantity in computing bond forces.

$$\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle \rightarrow \mathbf{F} \rightarrow \boldsymbol{\sigma} \rightarrow \underline{\mathbf{T}}\langle\boldsymbol{\xi}\rangle$$

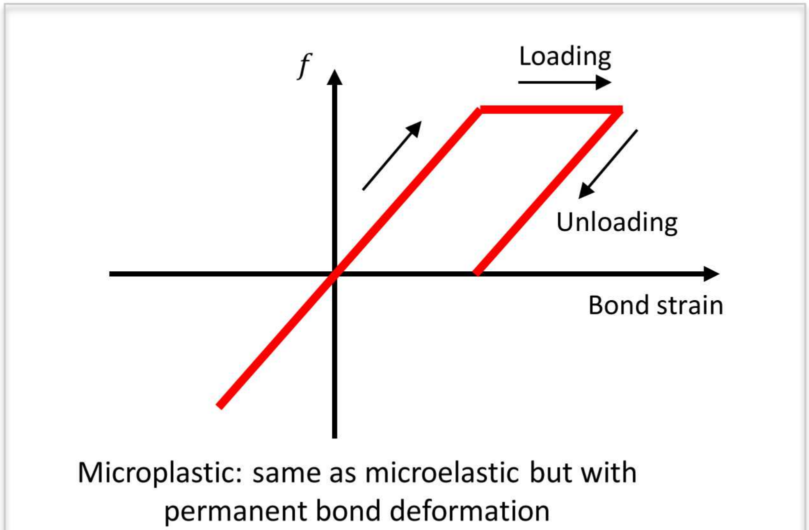
- Good:
  - Can use any material model from a finite element code.
- Bad:
  - Loose a lot of information about bonds.
  - Stability issues especially zero-energy modes.
- Some overlap with SPH except for damage.



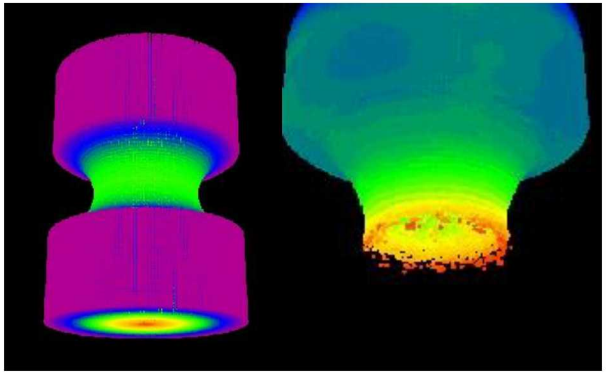

Correspondence material zero-energy mode instability and stabilization*

- SS, Epton, Weckner, Xu, & Askari, *J Elasticity* (2007)
- Warren, SS, Askari, Weckner, Epton, Xu, *Int J Solids & Structures* (2009)
- Tupek & Radovitzky, *JMPS* (2014)
- Bessa, Foster, Belytschko, & Liu, *Computational Mechanics* (2014)
- *SS, *CMAME* (2017)
- Du & Tian, *SIAM J Applied Math* (2018)
- Foster & Xu, *Int J Solids & Structures* (2018)
- Li, Hao, & Zhen, *CMAME* (2018)
- Nicely, Tang, & Qian, *CMAME* (2018)
- Chowdhury, P. Roy, D. Roy, & Reddy, *CMAME* (2019)
- Ganzenmüller, Hiermaier, May, *Computers & Structures* (2015)

# Ductile materials and failure

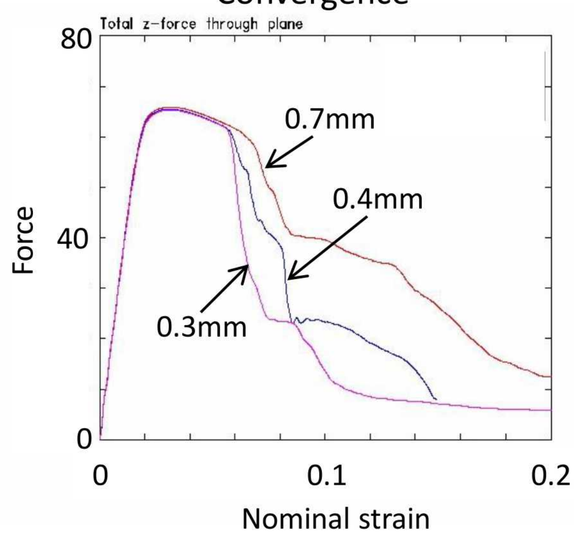



Microplastic: same as microelastic but with permanent bond deformation



Johnson-Cook plasticity with Wellman tearing model*

Convergence

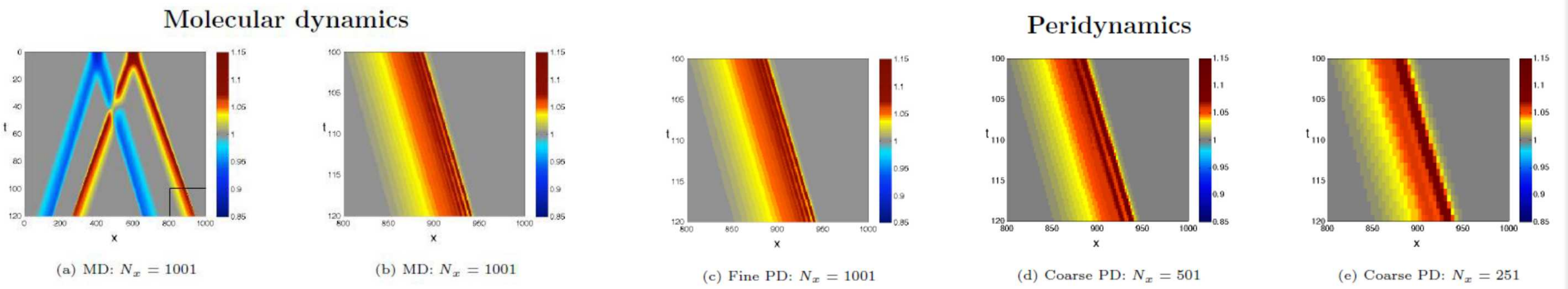



- Can use $J_2$ plasticity with correspondence material models.
- Crystal plasticity has been implemented**.
- Ordinary state based plasticity model appears possible***.


- Macek & SS, *Finite Elements in Analysis and Design* (2007)
- Foster, SS, & Chen, *IJNME* (2010)
- Foster, SS, & Chen, *Int J Multiscale Computational Engineering* (2011)
- ***Mitchell, Sandia tech report SAND2011-4974C (2011)
- *Wellman, Sandia tech report SAND2012-1343 (2012)
- **Sun & Sundararaghavan, *Int J Solids Structures* (2014)
- Amani, E. Oterkus, Areias, Zi, Nguyen-Thoi, & Rabczuk, *Int J Impact Engineering* (2016)
- Rahaman, P. Roy, D. Roy, & Reddy, *CMAME* (2017)

# Atomistics & peridynamics

- Intuitively, the nonlocal nature of PD should make it more compatible with molecular dynamics than the PDE theory is.
- A set of atoms can be represented as a PD body:
  - Each is a delta function in space.
  - Any interatomic potential can be included as a state-based PD material model.
- PD continuum equations can be derived from statistical mechanics using Irving-Kirkwood approach*.




Peridynamic model upscaled from the Embedded Atom Method**

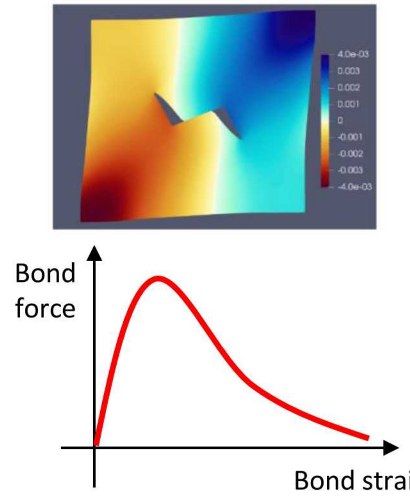

- Seleson, Parks, Gunzburger & Lehoucq, *Multiscale Modeling & Simulation*. (2009)
- **Seleson, thesis, Florida State Univ (2010)
- *Lehoucq & Sears, *Phys Rev E* (2011)
- Rahman & Haque, *Int J Computational Materials Science & Engineering*. (2012)
- Seleson & Parks, in *Handbook of Peridynamic Modeling* (2016)
- Tong & Li, *JMPS* (2016)

# Relation to the Cauchy theory

- In the limit of zero horizon:
  - If the material model is stable (monotonic for all bonds):
    - Peridynamic operator converges to Cauchy operator: $\int f \rightarrow \nabla \cdot \sigma$.
    - Displacements approach solutions in the Cauchy theory.
    - Peridynamic stress tensor approaches Cauchy stress.
    - Waves become non-dispersive.
    - Boundary conditions converge to expected forms.
  - Otherwise:
    - Converges to a deformation with Griffith cracks (!).


- Emmrich & Weckner, *Comm Math Sci* (2007)
- SS & Lehoucq, *J Elasticity* (2008)
- Zhou & Du, *SIAM J Numerical Analysis* (2010)
- Du & Zhou, *ESAIM* (2011)
- Hinds & Radu, *Appl Math & Computation* (2012)
- Erbay, Erkip, Muslu, *J Differential Equations* (2012)
- Du, Gunzburger, Lehoucq,  & Zhou, *J Elasticity* (2013)
- Emmrich & Puhst,  *Comm Math Sci* (2013)
- Mengesha & Du, *Discrete Contin. Dynam. Systems B* (2013)
- Bellido & Mora-Corral, *SIAM J Math Analysis* (2014)
- Lipton, *J Elasticity* (2014, 2015)
- Mangesha & Du, *J Elasticity* (2014)
- Mengesha & Du, *Proc Royal Society Edinburgh A: Math* (2014).
- Lipton, Lehoucq, & Jha, *J Peridynamics & Nonlocal Modeling* (2016)
- Du, in *Handbook of Peridynamic Modeling*, Taylor & Francis (2016)
- Coclite, Dipierro, Maddalena, & Valdinoci, *Nonlinearity* (2018)


Crack growth in a nonconvex (unstable) peridynamic material



# Usability issues

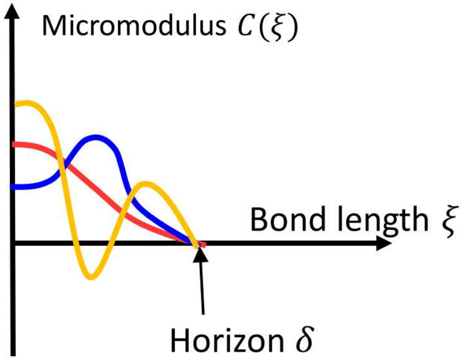
- Boundary conditions in peridynamics are different than with PDEs:
  - Integral equations are more compatible with "volume constraints".
- Surface effect:
  - Materials take on altered properties near free surfaces and interfaces.
- Dust:
  - In the simple particle discretization, nodes tend to break off completely once damage starts to occur.
- Many ideas have been proposed and demonstrated for mitigating these effects.
  - But software tools need to make peridynamics look more like just an option that works seamlessly.
  - LS-DYNA has made a lot of progress.
  - RKPM
  - Peridiynamic Differential Operator


- Macek & SS, *Finite Elements in Analysis & Design* (2007)
- Kilic, thesis, Univ of Arizona (2008)
- Mitchell, SS, Littlewood*, J Mechanics of Materials & Structures* (2015)
- Le & Bobaru*, Computational Mechanics* (2018)
- Madenci, Barut, & Dorduncu, *Peridynamic Differential Operator of Numerical Analysis*, Springer (2019)
- Bessa, Foster, Belytschko, & Liu, *Computational Mechanics* (2014).
- Pasetto, Leng, Chen, Foster, & Seleson, *CMAME*.(2018).
- Hillman, Pasetto, & Zhou. "Generalized Reproducing Kernel Peridynamics…", researchgate.net (2019).

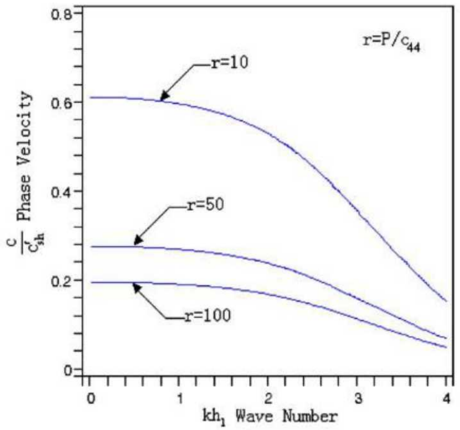
# Dispersive waves and the horizon

- Linear waves are dispersive in peridynamics.
  - Useful! (sometimes)
  - All real materials are dispersive to some extent.

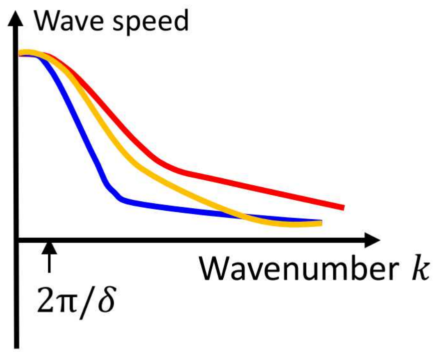
$$f = C(\xi)(u(x+\xi) - u(x))$$




Many choices of $C(\xi)$ can lead to the same bulk elastic modulus




Real dispersion curves in a composite*



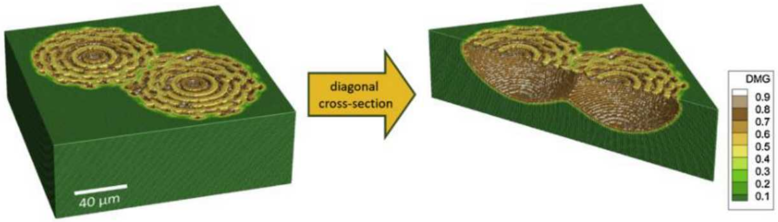

The choice of $C(\xi)$can be matched to dispersion data


- Weckner & SS, *Int J Multiscale Computational Engineering* (2011)
- Butt, Timothy, & Meschke, *Computational Mechanics* (2017)
- *Qian, Jin, Wang, & Kishimoto, *Int J Engineering Science* (2004)
- Bobaru & Hu*, Int J Fracture* (2012)

*What’s been learned about…*

# Quasi-static problems

- Large number of bonds results in dense matrices.
  - Great difficulty storing and manipulating these.
  - Solvers that are effective with FEM generally are not so good with peridynamics.
- Matrix can become ill-conditioned or singular if there is damage.
- Dynamic relaxation is often the most effective method.
  - But has known problems.
  - Better iterative solvers would be a good thing.




Peridynamic simulation of corrosion pit growth*


- Kilic & Madenci, Theoretical & Applied Fracture Mechanics (2010)
- Breitenfeld, Geubelle, Weckner, & SS, CMAME (2014)
- Zaccariotto, Luongo, & Galvanetto, Aeronautical Journal (2015)
- Du, Gunzburger, Lehoucq, Zhou, SIAM Review (2012)
- Katiyar, Foster, Ouchi, Sharma, J Computational Physics (2014)
- Jabakhanji & Mohtar, Adv Water Resources (2015)
- Ouchi, Katiyar, York, Foster, & Sharma, Computational Mechanics (2015)
- De Meo, Diyaroglu, Zhu, E. Oterkus, Siddiq. Int J Hydrogen Energy. (2016)
- S. Oterkus, Madenci, & E. Oterkus, Engineering Geology (2017)
- Chen & Bobaru, JMPS (2015)
- Chen, Zhang, & Bobaru, J Electrochemical Society. (2016)
- De Meo, Diyaroglu, Zhu, E. Oterkus, & Siddiq, Int J Hydrogen Energy (2016)
- De Meo & E. Oterkus, Ocean Engineering. (2017)
- Li, Chen, Tan, & Bobaru, Materials Science and Engineering: A. (2018)
- *Jafarzadeh, Chen, Zhao, & Bobaru, Corrosion Science (2019)

# Where peridynamics is now

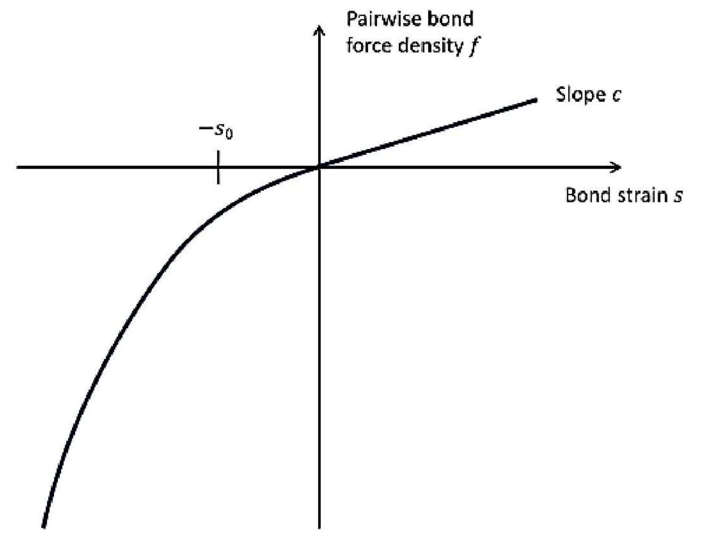
- Basic mechanical and mathematical principles have been worked out.
  - Sky is the limit in dreaming up new materials.
- People are continually discovering new applications.
- Well-posedness and relation to other theories is understood.
- Simple meshless discretization is still the workhorse but has some issues (DG is good too).
- Possible fertile areas for growth:
  - Atomistics: A2C coupling, coarse-graining.
  - Fundamentals of fracture.
  - Multiscale.
  - Fluids and fluid-structure coupling.
  - Post-failure behavior of solids.
  - Coupling of different physics.
  - FEM implementation.
  - Implicit solvers.
  - Material & structural stability*.
  - Nonlocal waves*.
  - Models from pixels.
  - Eulerian balance laws.
  - Fast integration methods.
  - Chemical reactions.
  - Geological media & flow.


It's in a lot of places!
(numbers indicate relative number of downloads)

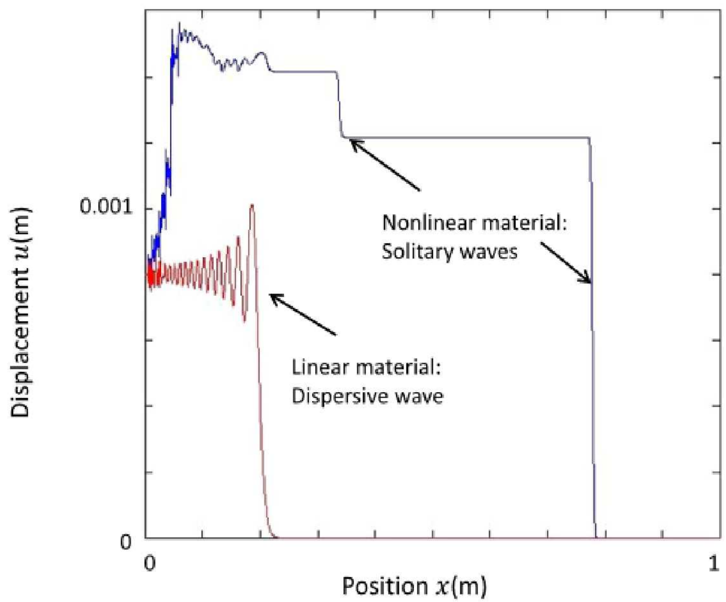
*Personal fave

# Nonlocal waves

- Linear waves are dispersive in peridynamics.
- Just by choosing the right nonlocal peridynamic material model, we can reproduce solitary waves.
  - Local theory cannot do this without replacing the governing 2nd order PDEs with 4th order.
  - E.g., KdV equation.




Nonlinear microelastic material model for a bond




Simulated dispersive waves and solitary waves. Impact on left edge of a bar.

- SS, *JMPS* (2016)
- Pego & Van, *J Elasticity* (2018)

# Extra slides

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# 2018: Ingeniously heterogeneous materials

- Peridynamic differential operator (PDDO):

$$\frac{d^p f(x)}{dx^p} = \int_{\mathcal{H}_{\mathbf{x}}} f(x+\xi) g_N^p(x,\xi)\, d\xi, \qquad g_N^p = \sum_{q=0}^{N} a_q^p w_q(\xi)\xi^q.$$

  where the $w_q$ are weighting functions and there is a procedure for determining the $a_q^p$ at each point.

- Replace derivatives in an ODE or PDE with a discretized form of the above, solve.
- Result is an accurate solution to the ODE or PDE using expressions that look just like peridynamics.
- Cost of this accuracy: the corresponding peridynamic material model is nonhomogeneous.
- Also there has been a lot of progress on RKPM for peridynamics.


- Madenci, Barut, & Dorduncu, Peridynamic *Differential Operator of Numerical Analysis*, Springer (2019)
- Bessa, Foster, Belytschko, & Liu, *Computational Mechanics* (2014).
- Pasetto, Leng, Chen, Foster, & Seleson, *CMAME*.(2018).
- Hillman, Pasetto, & Zhou. "Generalized Reproducing Kernel Peridynamics…", researchgate.net (2019).

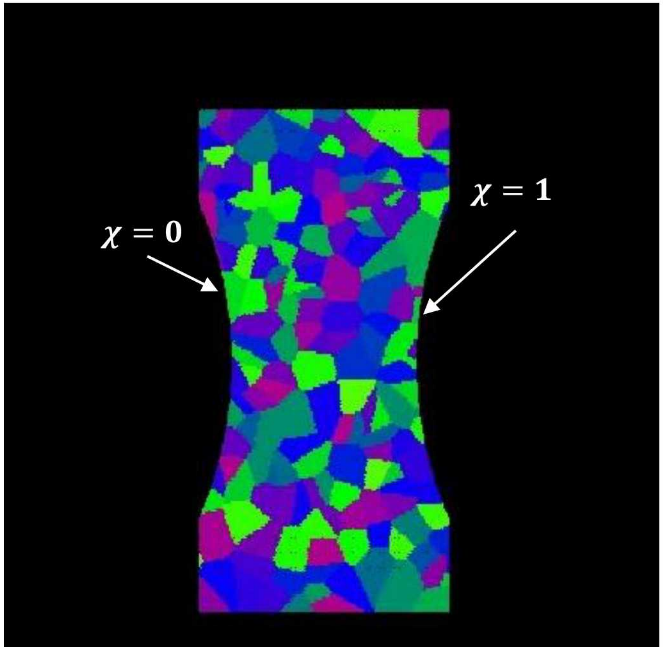
# 2012: Nonlocal diffusion

- A possible nonlocal mass diffusion law:

$$\dot{\chi}(\mathbf{x}) = \int_{\mathcal{H}_{\mathbf{x}}} D(\mathbf{q} - \mathbf{x})(\chi(\mathbf{q}) - \chi(\mathbf{x}))\, dV_{\mathbf{q}}$$

  where $\chi$ is concentration and $D$ is the microconductivity.

- $D$ can depend on bond breakage or other variables.
- Bond breakage parameters can depend on $\chi$.




Fatigue with contaminant transport


- Du, Gunzburger, Lehoucq, Zhou, SIAM Review (2012)
- Katiyar, Foster, Ouchi, Sharma, J Computational Physics (2014)
- Chen & Bobaru, JMPS (2015)
- Jabakhanji & Mohtar, Adv Water Resources (2015)
- Ouchi, Katiyar, York, Foster, & Sharma, Computational Mechanics (2015)
- De Meo, Diyaroglu, Zhu, E. Oterkus, Siddiq. Int J Hydrogen Energy. (2016)
- S. Oterkus, Madenci, & E. Oterkus, Engineering Geology (2017)

# 2010: State-based thermodynamics

- First law:

$$\dot{e} = \underline{\mathbf{T}} \bullet \underline{\dot{\mathbf{Y}}} + h + r$$

where $e$=internal energy density, $h$=rate of heat transport, and $r$=source rate.

- The integral in the first law only sums up the part of the bond forces due to its own material model.

- Second law:

$$\theta \dot{\eta} \geq h + r$$

where $\theta$=temperature and $\eta$=entropy density.

- Typical nonlocal heat transport model:

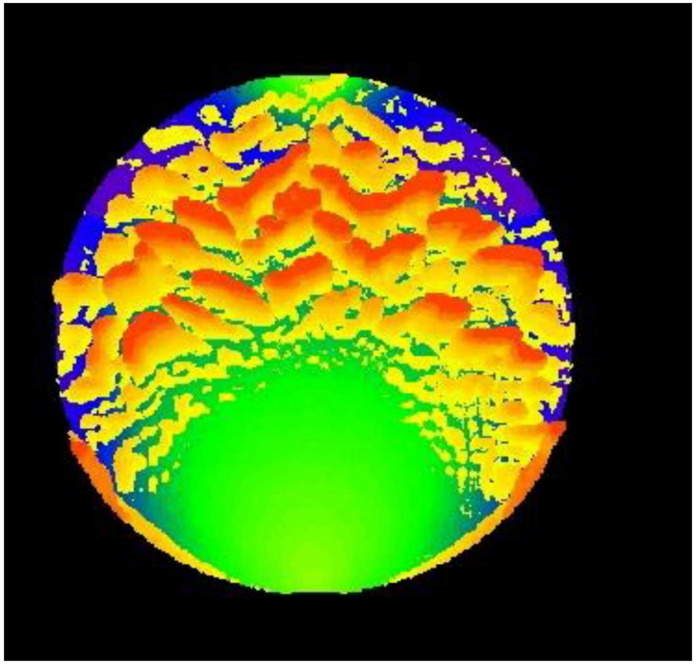
$$h(\mathbf{x}) = \int_{\mathcal{H}_\mathbf{x}} K(\mathbf{q} - \mathbf{x})(\theta(\mathbf{q}) - \theta(\mathbf{x}))\, dV_\mathbf{q}$$

where $K$ is the microconductivity.

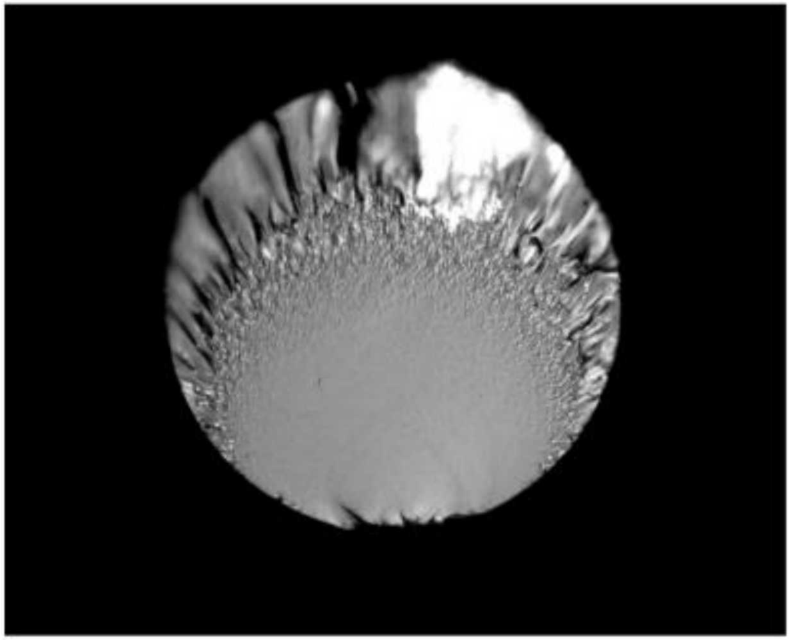
- SS & Lehoucq, *Advances in Applied Mechanics* (2010)
- Bobaru & Duangpanya, *Int J Heat & Mass Transfer* (2010)
- S. Oterkus, Madenci, & Agwai, *J Computational Physics* (2014)
- E. Oterkus & Madenci, *Peridynamic Theory & Its Applications*, Springer (2014)
- Chen & Bobaru, *Computer Physics Comm* (2015)

# Mirror-mist-hackle transition in glass

- Model predicts roughness and microbranches that increase in size as the crack grows.
- Transition radius decreases as initial stress increases – trend agrees with experiments.



3D peridynamic model

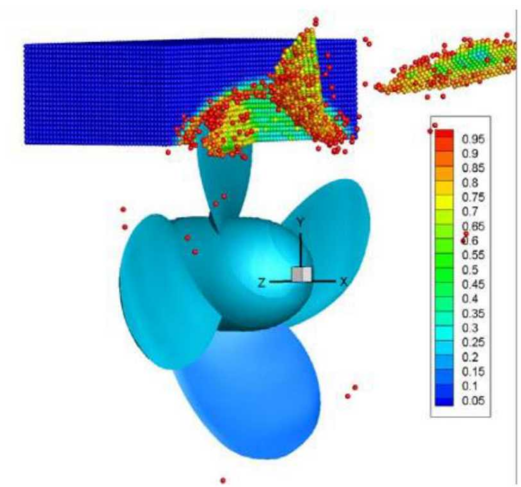


Fracture surface in a glass optical fiber*

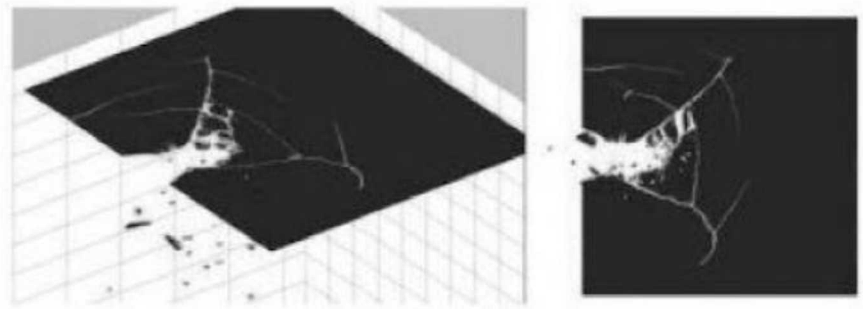
* Castilone, Glaesemann & Hanson, Proc. SPIE (2002)

# Fracture of cold things

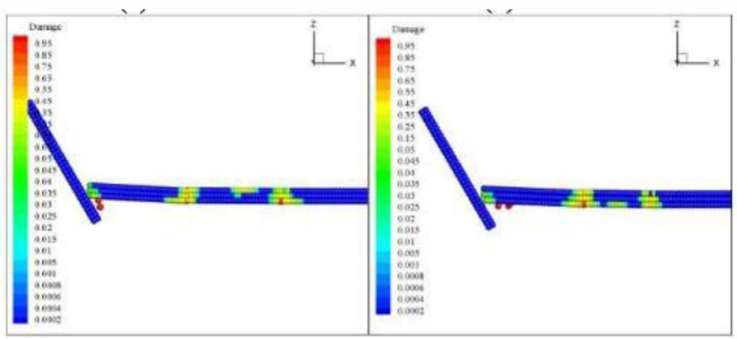
- Researchers are using peridynamics to model the behavior of floating ice.




Floating ice vs. propeller*



Floating ice vs. rigid cylindrical structure**

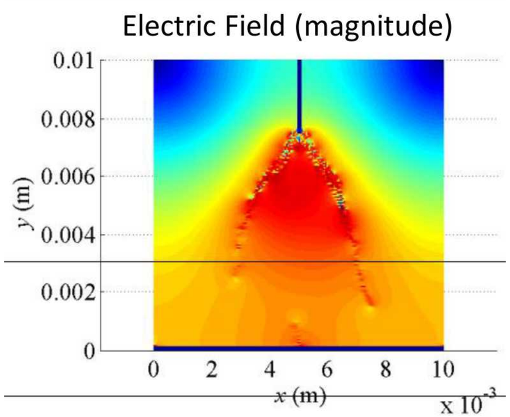


Floating ice vs. rigid slope***

*Ye, Wang, Chang, & Zhang, Ocean Engineering (2017)
**Vazic, E. Oterkus, & S. Oterkus, in Proc 7th International Conference on Marine Structures (2019)
***Lu, Wang, Jia, & Shi, in 28th International Ocean and Polar Engineering Conference (2018)

# Fracture of electrical things

- Damage and other peridynamic material properties can be coupled with electromagnetic fields and mass diffusion.




Damage due to high-voltage dielectric breakdown*

| Physics | Continuum | Peridynamic kernel | Peridynamic Parameters |
|---|---|---|---|
| Solid Mechanics | $\sigma_{xx} = E\frac{\partial u}{\partial x}$ | $f_F(\eta,\xi) = cg(\lvert\xi\rvert)s = cg(\lvert\xi\rvert)\left(\frac{\eta}{\lvert\xi\rvert}\right)$ | $c = \frac{2E}{(Area)\delta^2}$ |
| Thermal | $q_x = -k_T\frac{\partial T}{\partial x}$ | $f_T(\tau,\xi) = \kappa_T g(\lvert\xi\rvert)\tau$ | $\kappa_T = \frac{3k_T}{(Area)\delta^3}$ |
| Electrical | $j_x = -k_E\frac{\partial \Phi}{\partial x}$ | $f_E(\tau,\xi) = \kappa_E g(\lvert\xi\rvert)\phi$ | $\kappa_E = \frac{3k_E}{(Area)\delta^3}$ |
| Atomic Vacancy Diffusion | $J_x = -\frac{DC}{kT}\left[\Omega\frac{\partial\sigma_x}{\partial x} + Z^* q\frac{\partial\Phi}{\partial x}\right]$ | $f_C(\chi,\xi) = \frac{D_\xi C_\xi g(\lvert\xi\rvert)}{kT_\xi}\left[\kappa_{C\sigma}\frac{\Delta\sigma_x}{\lvert\xi\rvert} + \kappa_{C\Phi}\frac{\chi}{\lvert\xi\rvert}\right]$ | $\kappa_{C\sigma} = \frac{2\Omega}{(Area)\delta^2}$ $\kappa_{C\Phi} = \frac{2Z^* q}{(Area)\delta^2}$ |

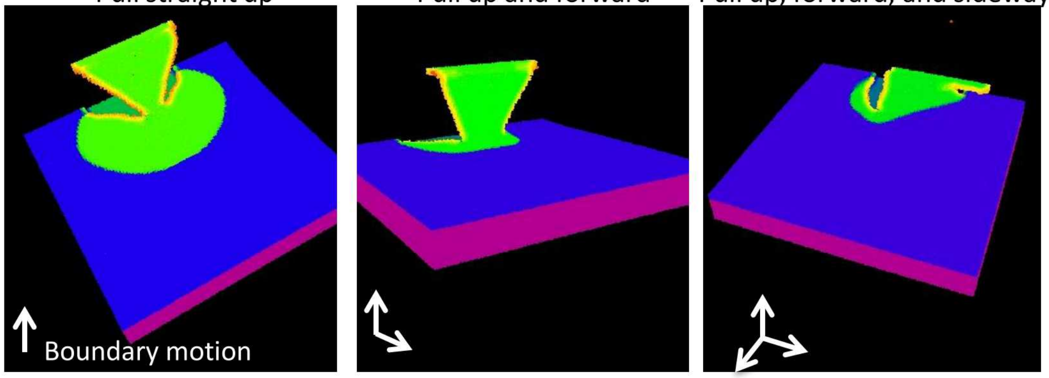
Peridynamic model for electromigration**

- *Wildman & Gazonas, *J Mechanics of Materials & Structures* (2015)
- **Gerstle, SS, Read, Tewary, Lehoucq, Comput Mater Continua (2008)
- S. Oterkus, Fox, & Madenci, in *IEEE 63rd Electronic Components and Technology Conference* (2013)

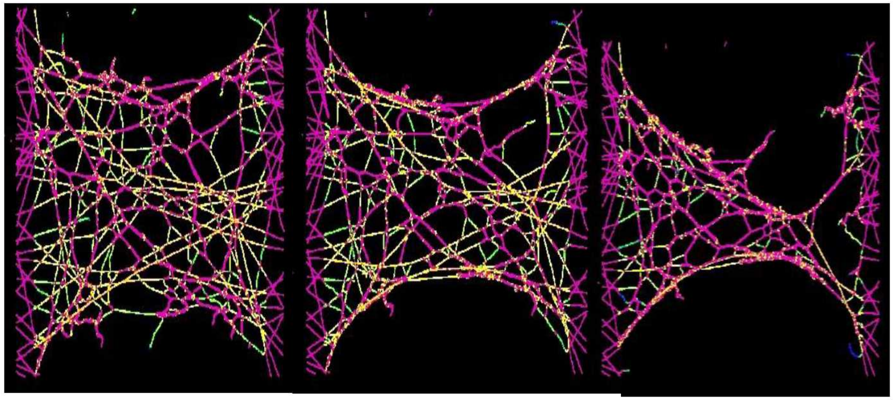
# Fracture of sticky things

- Peridynamic bonds between different materials can model adhesion.

Pull straight up | Pull up and forward | Pull up, forward, and sideways




Scotch tape

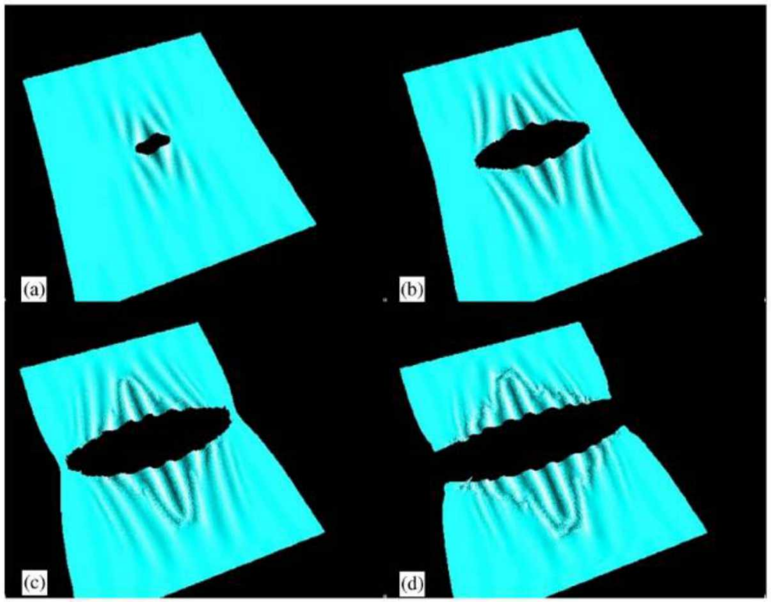


Nanofiber membrane*

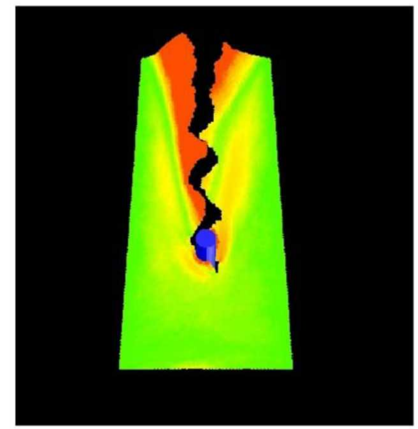
*Bobaru, *Modelling & Simulation in Materials Science and Engineering* (2007)

# Fracture of wrinkly things

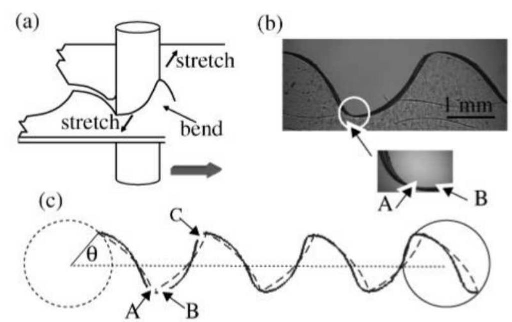
- Peridynamic bonds between different materials can model adhesion.




Center crack in a brittle-elastic membrane*



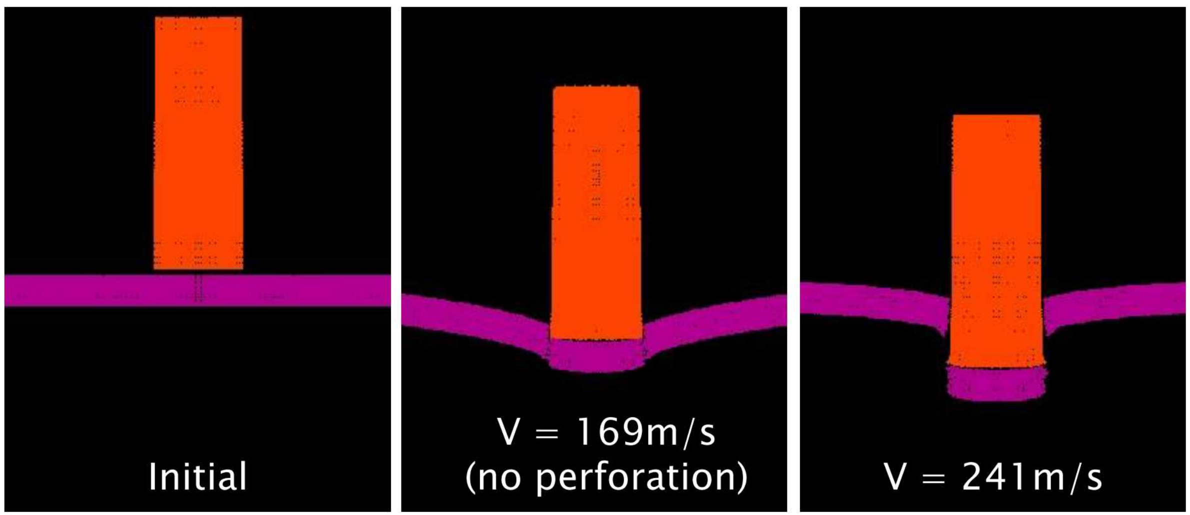
Oscillatory crack path in a film dragged over a rigid cylinder*,**

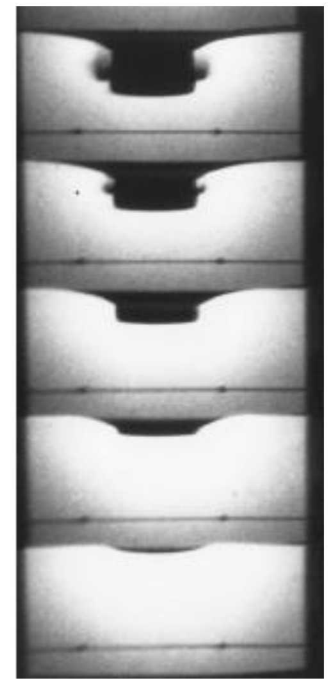



*Bobaru, *Modelling & Simulation in Materials Science and Engineering* (2007)
**Ghatak & Mahadevan, Physical review letters (2003)

# Blunt projectile vs. steel plate

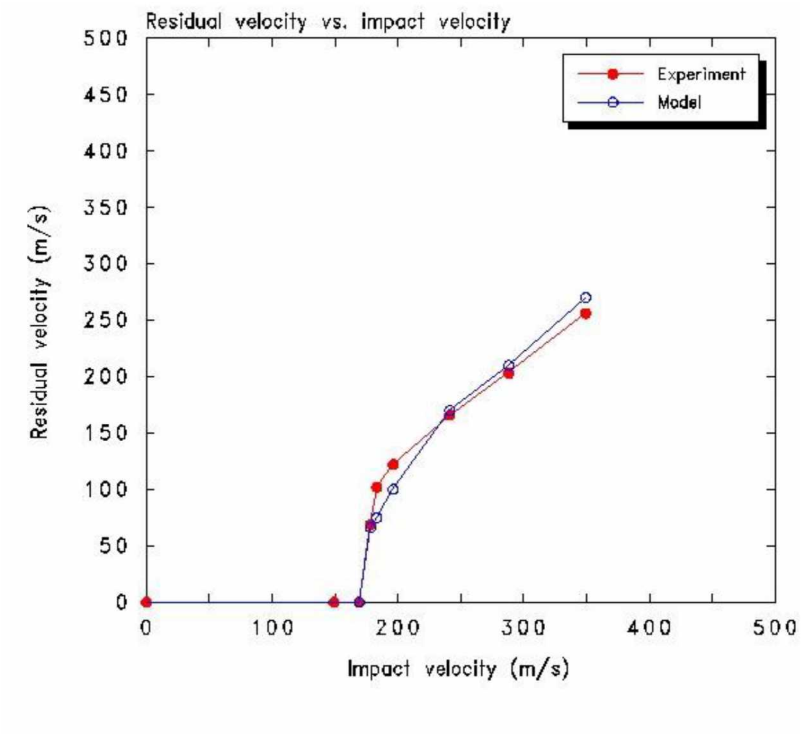
- Correspondence materials often work very well (requires stabilization).
  - 30mm diameter 4340 steel cylinder onto 10.5mm thick HY-100 steel plate.
  - Failure mode is plugging.
  - Both materials use Johnson-Cook plasticity within a peridynamic correspondence model.

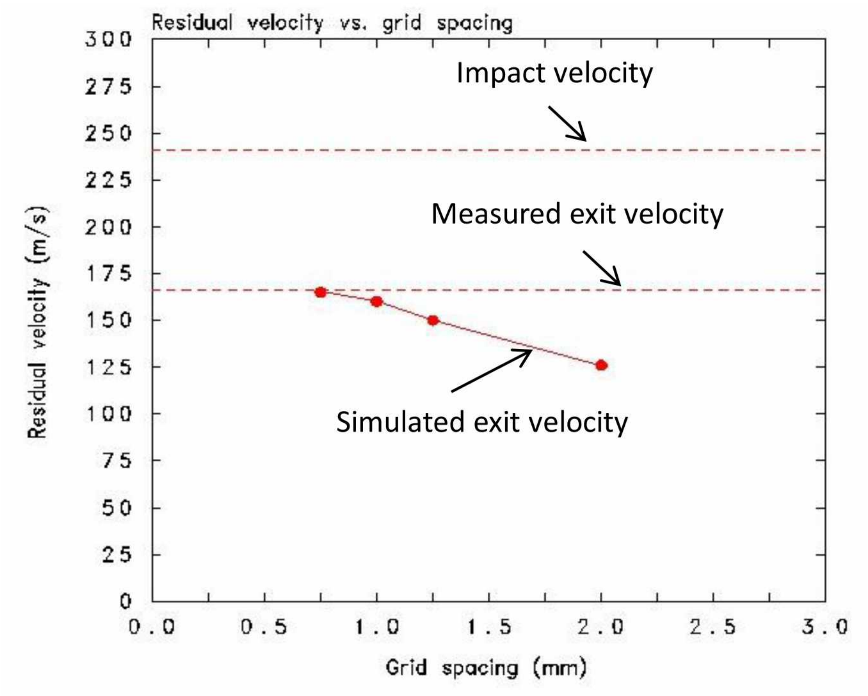


V = 246 m/s
Experiment
Forrestal & Hanchk, Int. J. Impact Eng. (1999)

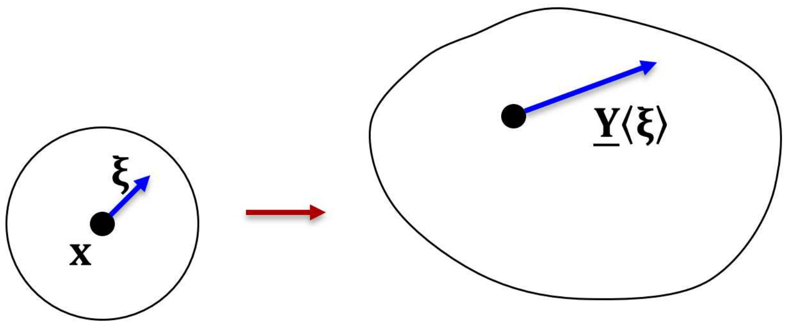
# Blunt projectile vs. steel plate, ctd: Exit velocity and convergence



# 2007: States

- In general, each bond force should depend on the deformation of the entire family.
- State: a function that associates some number or vector with each bond.




- Example: Deformation state

$$\underline{\mathbf{Y}}\langle\boldsymbol{\xi}\rangle = \mathbf{y}(\mathbf{x}+\boldsymbol{\xi}) - \mathbf{y}(\mathbf{x})$$

- This notation allows us to precisely define the bond force or other quantities for each bond in a body.
- Can do algebra with states. Example: Dot product:

$$\underline{\mathbf{A}} \bullet \underline{\mathbf{B}} = \int_{\mathcal{H}} \underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle \cdot \underline{\mathbf{B}}\langle\boldsymbol{\xi}\rangle \; \mathrm{d}V_{\boldsymbol{\xi}}$$


- SS, Epton, Weckner, Xu, & Askari, *J Elasticity* (2007)
- Madenci & E. Oterkus, *Peridynamic Theory & Its Applications*, Springer (2014)

# State-based elasticity

- State based material modeling is similar to conventional theory but with tensor gradients and other operators replaced by their peridynamic analogues.

- PDE theory:

$$\boldsymbol{\sigma} = \frac{\partial W}{\partial \mathbf{F}} \qquad \text{(tensor gradient)}$$

where $\boldsymbol{\sigma}$= stress tensor, $W$= strain energy, $\mathbf{F}$= deformation gradient tensor.

- Peridynamics:

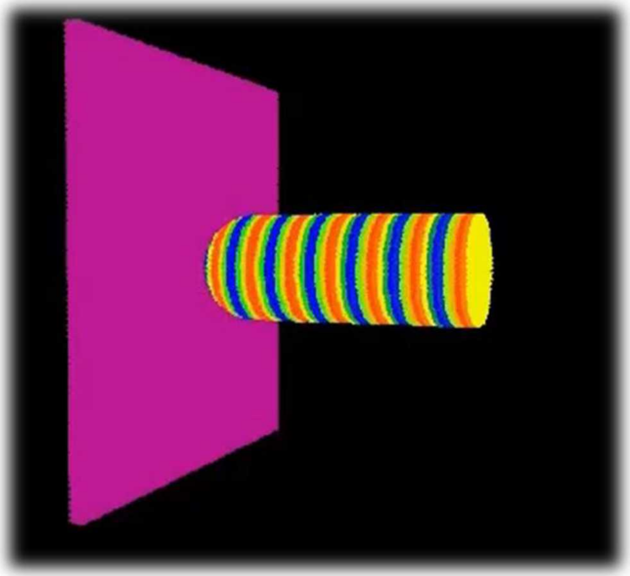
$$\underline{\mathbf{T}} = \frac{\partial W}{\partial \underline{\mathbf{Y}}} \qquad \text{(state gradient, i.e., Frechet derivative)}$$

where $\underline{\mathbf{T}}$= force state, $W$= strain energy, $\underline{\mathbf{Y}}$ = deformation state.

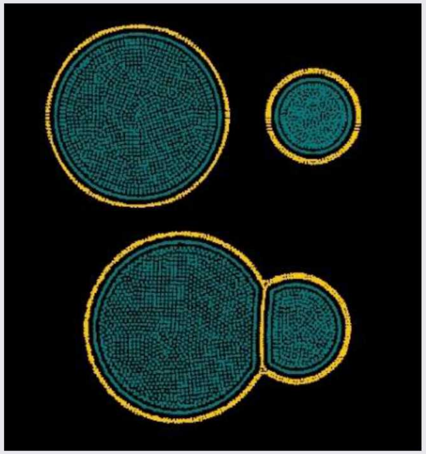

- SS, Epton, Weckner, Xu, & Askari, *J Elasticity* (2007)
- Madenci & E. Oterkus, *Peridynamic Theory & Its Applications*, Springer (2014)

# 2016: State-based fluids and soft materials

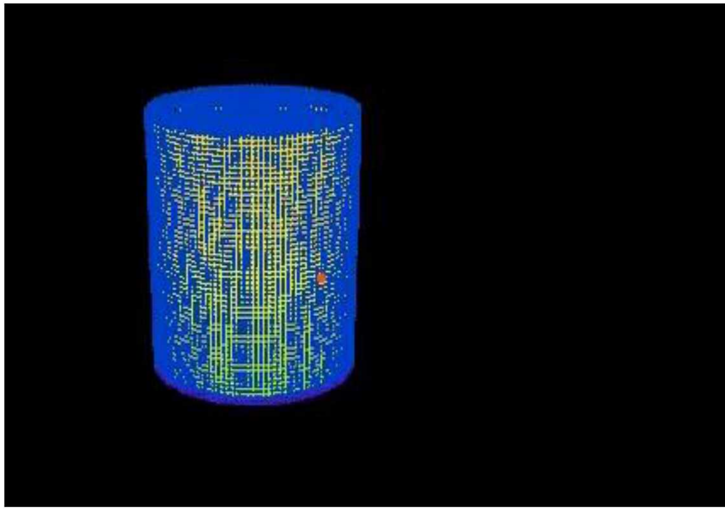
- Fluids can fracture.
- Long-range peridynamic forces can characterize surface and interface forces.



Gelatin bird strike simulant
VIDEO



Surface tension



Hydraulic ram
VIDEO

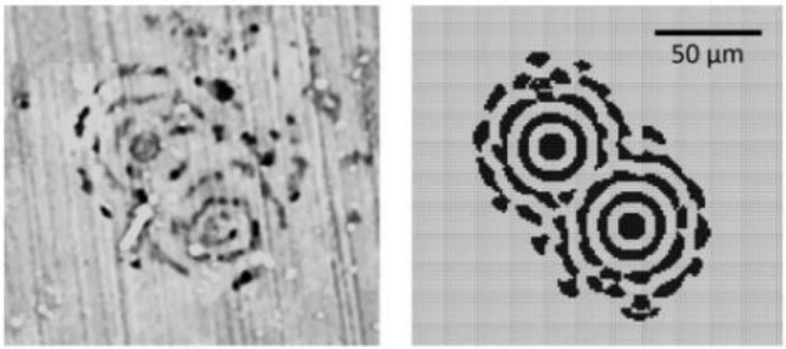
- SS, Parks, Kamm, Weckner, & Rassaian, *Int J Impact Engineering* (2017)

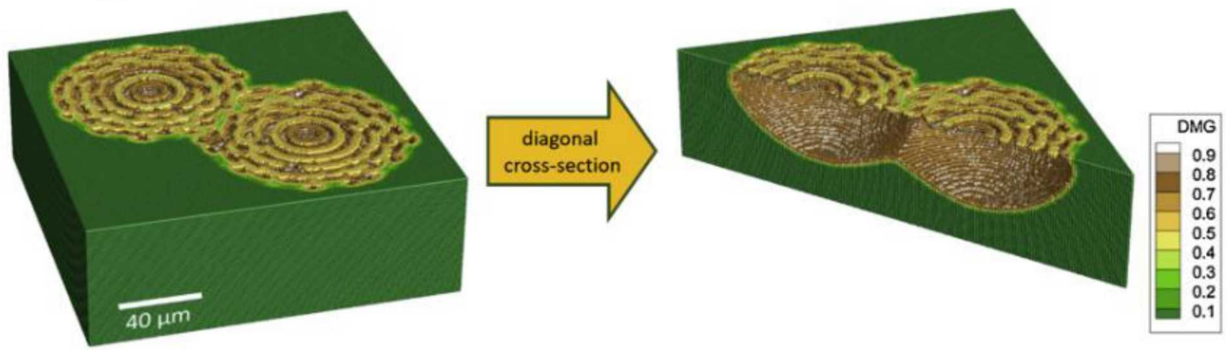
# 2015: Multiphysical problems: Corrosion



- Peridynamic simulation of corrosion pits and lacy covers*
- Can use a peridynamic diffusion model for transport.

Experiment Simulation



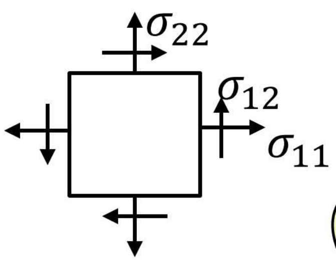



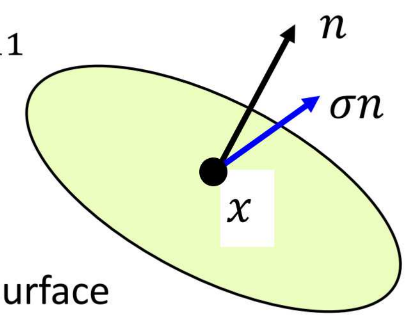

- Chen & Bobaru, JMPS (2015)
- Chen, Zhang, & Bobaru, J Electrochemical Society. (2016)
- De Meo, Diyaroglu, Zhu, E. Oterkus, & Siddiq, Int J Hydrogen Energy (2016)
- De Meo & E. Oterkus, Ocean Engineering. (2017)
- Li, Chen, Tan, & Bobaru, Materials Science and Engineering: A. (2018)
- *Jafarzadeh, Chen, Zhao, & Bobaru, Corrosion Science (2019)

# The nature of internal forces

## **Standard theory**

Stress tensor field
(assumes continuity of forces)



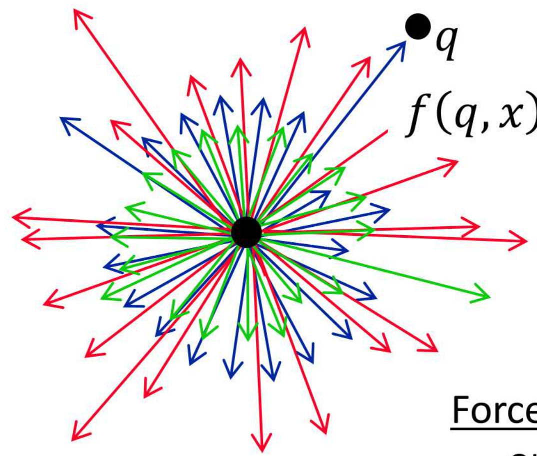

Stress tensor maps surface normal vectors onto surface forces

$$\rho \ddot{u}(x,t) = \nabla \cdot \sigma(x,t) + b(x,t)$$

Differentiation of surface forces

## **Peridynamics**

Bond forces between neighboring points
(allowing discontinuity)




Force state maps bonds onto bond forces

$$\rho \ddot{u}(x,t) = \int_{H_x} f(q,x) dV_q + b(x,t)$$

Summation over bond forces