SAND2019-7346C

Hyper-differential sensitivity analysis for PDE-constrained optimization: Methods and software

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ICIAM 2019
Valencia
July 19, 2019

[1] Sandia National Laboratories is a multimission laboratory managed and operated by National Technology and Engineering Solutions of Sandia, LLC., a wholly owned subsidiary of Honeywell International, Inc., for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-NA-0003525.

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1. Sensitivity Analysis Framework
   - Classical Framework
   - Hyper-Differential Sensitivity Analysis (HDSA) Framework
2. Method and Computation
   - Overview of the Method
   - Computational Considerations
3. Software Design
4. Applications
   - Parametric Sensitivity for Optimal Control
   - Parametric Sensitivity for Inverse Problems
   - Data Source Sensitivity for Inverse Problems
   - Distributional Sensitivity for Stochastic Optimization

Classical Framework

- model mapping uncertain parameters θ to a model output $\mathcal{M}(\theta)$

$$\mathcal{M} : \Theta \to \mathcal{Y}$$

- Θ and $\mathcal{Y}$ may be $\mathbb{R}^n$ or infinite dimensional function spaces
- common approaches to sensitivity analysis seek to perturb or vary θ and analyze the model output $\mathcal{M}(\theta)$
- a simple example is when θ corresponds to physical parameters (such as material properties) in a partial differential equation (PDE) and $\mathcal{M}(\theta)$ is a functional of the solution of the PDE
- some possible sensitivity analysis approaches are to
  - compute the derivative of $\mathcal{M}(\theta)$ with respect to θ
  - generate samples $\{\theta_i\}_{i=1}^N$ in Θ, evaluate $\mathcal{M}(\theta_i)$, $i = 1, 2, \ldots, N$, and estimate statistical properties of the model output

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HDSA Framework

Consider the PDE-constrained optimization problem

$$\min_{u,z} J(u,z,\theta) \qquad (1)$$
$$s.t. \quad c(u,z,\theta) = 0$$

where

- J is an objective function
- u is a (infinite dimensional) state variable
- z is a (possibly infinite dimensional) optimization variable
- c is a PDE
- θ corresponding to (possibly infinite dimensional) physical parameters, data, and/or probability distributions

Goal: Determine the sensitivity of the **solution** of (1) to changes in θ.

Comparison with the Classical Framework

$$
\begin{aligned}
&\min_{u,z} J(u,z,\theta) \\
&s.t. \quad c(u,z,\theta) = 0
\end{aligned}
\tag{1}
$$

where

- the model $\mathcal{M}(\theta)$ in the classical framework corresponds to the solution of $c(u,z,\theta) = 0$ (given a particular z)
- in many applications our ultimate goal is the optimal solution of (1) or the PDE state evaluated at the optimal solution
- HDSA is a goal-oriented sensitivity analysis
- HDSA has various interpretations and uses depending upon:
  - if (1) is a control, design, or inverse problem
  - if (1) is a deterministic or stochastic optimization problem
  - if θ corresponds to physical parameters, data, or probability distributions

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Optimization Problem

$$\min_{u,z} J(u,z,\theta) \quad (1)$$
$$s.t. \quad c(u,z,\theta) = 0$$

- denote the Lagrangian for (1) by

$$\mathcal{L}(u,z,\lambda,\theta) = J(u,z,\theta) + \langle c(u,z,\theta), \lambda \rangle$$

- assume that $(u^\star, z^\star, \lambda^\star)$ is a local minimum of (1) with $\theta = \theta_0$

Under mild assumptions [2], there exists a function

$$\mathcal{F} : \mathcal{N}(\theta_0) \to \mathcal{N}(u^\star, z^\star, \lambda^\star)$$

such that

$$\nabla \mathcal{L}(\mathcal{F}(\theta), \theta) = 0$$

for any θ in a neighborhood of θ_0.

---

[2] K. Brandes and R. Griesse, Quantitative stability analysis of optimal solutions in PDE-constrained optimization.

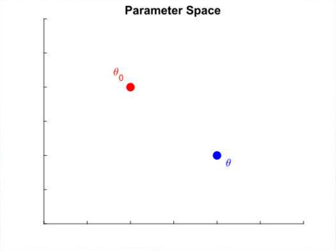
Parameter to Optimal Solution Mapping

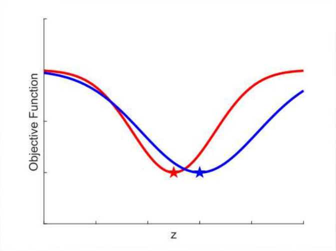
$$\mathcal{F} : \mathcal{N}(\theta_0) \rightarrow \mathcal{N}(u^\star, z^\star, \lambda^\star)$$

such that

$$\nabla \mathcal{L}(\mathcal{F}(\theta), \theta) = 0$$

$$\theta \qquad \mapsto \qquad \mathcal{F}(\theta)$$





Parameters $\mapsto$ Optimal Solution

Differentiating through the Optimization Problem

$$\min_{u,z} J(u,z) \qquad (1)$$
$$s.t. \quad c(u,z,\theta) = 0$$

$$\mathcal{F} : \mathcal{N}(\theta_0) \to \mathcal{N}(u^\star, z^\star, \lambda^\star)$$

- $\mathcal{F}(\theta)$ is the solution of (1) with parameters θ
  - it depends on the particular local minimum $(u^\star, z^\star, \lambda^\star)$
  - evaluating $\mathcal{F}(\theta)$ requires the solving (1) (expensive)
- the Fréchet derivative of $\mathcal{F}$ with respect to θ is given by [3]

$$\mathcal{D}\mathcal{F}(\theta) = -\mathcal{K}^{-1}\mathcal{B}$$

where $\mathcal{K}$ is the hessian of $\mathcal{L}$ and $\mathcal{B}$ is the Fréchet derivative of $\nabla\mathcal{L}$ with respect to θ (both evaluated at $(u^\star, z^\star, \lambda^\star)$)

[3] K. Brandes and R. Griesse, Quantitative stability analysis of optimal solutions in PDE-constrained optimization.

An Illustrative Example

$$\min_{u,z} \quad J(u,z) = (u-2)^2 + .0005z^2 \tag{2}$$
$$s.t. \quad u = \frac{1}{1+e^{-\theta_1 z}} + \theta_2$$

- $u_{opt}(\theta), z_{opt}(\theta)$ be the optimal solution of (2) (as a function of θ), then

$$\frac{\partial z_{opt}}{\partial \theta_1}(0.5, 0.5) = 9.99 \qquad \text{and} \qquad \frac{\partial z_{opt}}{\partial \theta_2}(0.5, 0.5) = 3.12.$$

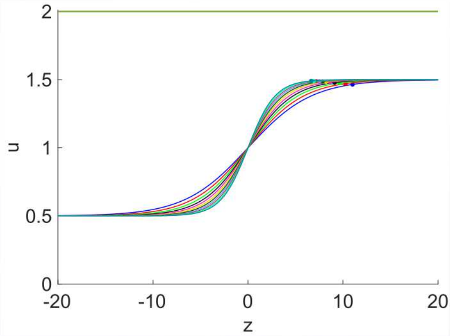
- If instead, (2) is solved with $\theta = (0.5, 0.5)$ and the derivative of

$$g(\theta_1, \theta_2) = \left(\frac{1}{1+e^{-\theta_1 z_{opt}(0.5,0.5)}} + \theta_2 - 2 \right)^2 + .0005 z_{opt}(0.5, 0.5)^2$$

is computed with respect to (θ_1, θ_2) we get

$$\frac{\partial g}{\partial \theta_1}(0.5, 0.5) = 0.135 \qquad \text{and} \qquad \frac{\partial g}{\partial \theta_2}(0.5, 0.5) = 1.03.$$

An Illustrative Example



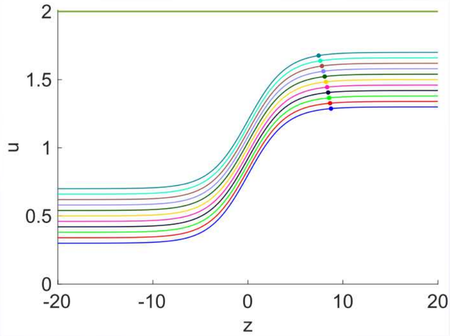


Figure: Plots of $u = \frac{1}{1+e^{-\theta_1 z}} + \theta_2$ and the optimal control strategies for different values of $\theta = (\theta_1, \theta_2)$. Left: varying θ_1 from 0.3 to 0.7 with $\theta_2 = 0.5$ fixed; right varying θ_2 from 0.3 to 0.7 with $\theta_1 = 0.5$ fixed.

Local Sensitivity Indices

$$\min_{u,z} J(u,z) \qquad (1)$$
$$s.t. \quad c(u,z,\theta) = 0$$

$$\mathcal{DF} : \mathcal{N}(\theta_0) \to \mathcal{N}(u^\star, z^\star, \lambda^\star)$$

- define $\Pi(u,z,\lambda) = z$ (or $\Pi(u,z,\lambda) = u$)
- define the local sensitivity function $\mathcal{S} : \Theta \to \mathbb{R}$ by

$$\mathcal{S}(\theta) = \frac{||\Pi \mathcal{K}^{-1} \mathcal{B} \theta||}{||\theta||} \qquad \theta \in \Theta$$

- $\mathcal{S}(\theta)$ measures the sensitivity of the solution of (1) to parameter perturbations in the direction θ
- if $\{\theta_1, \theta_2, \dots, \theta_n\}$ is a finite dimensional basis for θ, define the local sensitivity indices

$$S_i = \mathcal{S}(\theta_i), \qquad i = 1, 2, \dots, n$$

Global Sensitivity Indices

$$S_i = \frac{||\Pi \mathcal{K}^{-1} \mathcal{B} \theta_i||}{||\theta_i||} \qquad i = 1, 2, \ldots, n$$

- $\{S_i\}_{i=1}^n$ are local because the derivative $\mathcal{DF}$ is evaluated at an optimal solution $(u^\star, z^\star, \lambda^\star)$ corresponding to a nominal parameter θ_0
- we can define global sensitivity indices by sampling different nominal parameters
  - care must to taken to account for multiple local minima, we handle this by associating the local minima with the initial iterate of the optimization routine
  - extensive sampling of local sensitivities is intractable in most applications, but useful insight may be gained from sparse sampling

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Local Sensitivity: Computation

$$S_i = \frac{||\Pi \mathcal{K}^{-1} \mathcal{B} \theta_i||}{||\theta_i||} \qquad i = 1, 2, \dots, n$$

- requires solving the PDE-constrained optimization problem once, followed by n linear system solves with coefficient matrix $\mathcal{K}$
- n is large when, for instance, $\{\theta_1, \theta_2, \dots, \theta_n\}$ is a discretization of an infinite dimensional parameter field
- in many applications $\mathcal{B}$ possesses a low rank structure, leveraging the singular value decomposition (SVD) yields

$$||\Pi \mathcal{K}^{-1} \mathcal{B} \theta_i|| = \left|\left| \sum_{k=1}^{\infty} \sigma_k (v_k, \theta_i) u_k \right|\right| \approx \sqrt{\sum_{k=1}^{m} \sigma_k^2 (v_k, \theta_i)^2}$$

where (σ_k, u_k, v_k), $k = 1, 2, \dots$ are the singular triplets of $\Pi \mathcal{K}^{-1} \mathcal{B}$

- estimating the truncated SVD of $\Pi \mathcal{K}^{-1} \mathcal{B}$ allows efficient estimation of S_i, $i = 1, 2, \dots, n$

Singular Value Decomposition (SVD)

- the computational challenge is computing the truncated SVD (with appropriate inner products) of

$$\Pi\mathcal{K}^{-1}\mathcal{B} \tag{3}$$

- let M_θ and M_z denote the mass matrices (defining discretized functional space inner products using coordinate representations)
- the SVD of the discretized operator (3) may be extracted from the generalized eigenvalue problem

$$Ax = \lambda Mx$$

where

$$A = \begin{pmatrix} 0 & M_z\Pi\mathcal{K}^{-1}\mathcal{B} \\ \mathcal{B}^T\mathcal{K}^{-1}\Pi M_z & 0 \end{pmatrix}$$

and

$$M = \begin{pmatrix} M_z & 0 \\ 0 & M_\theta \end{pmatrix}$$

Generalized Eigenvalue Problem

$$\begin{pmatrix} 0 & M_z \Pi \mathcal{K}^{-1} \mathcal{B} \\ \mathcal{B}^T \mathcal{K}^{-1} \Pi M_z & 0 \end{pmatrix} \begin{pmatrix} \tilde{z} \\ \tilde{\theta} \end{pmatrix} = \lambda \begin{pmatrix} M_z & 0 \\ 0 & M_\theta \end{pmatrix} \begin{pmatrix} \tilde{z} \\ \tilde{\theta} \end{pmatrix}$$

- computational cost dominated by applying $\mathcal{K}^{-1}$ (a large linear system solve)
- use conjugate gradient since $\mathcal{K}$ is symmetric positive definite
- use randomized generalized eigenvalue solver from [4] to execute linear system solves in parallel on MPI subcommunicators
- postprocess dominant eigenvalues/vectors to estimate the truncated SVD (and hence local sensitivities)

[4] A.K. Saibaba, J. Lee,and P.K. Kitanidis, Randomized algorithms for generalized Hermitian eigenvalue problems with application to computing Karhunen-Loève expansion

Computational Cost Outline

$$\min_{u,z} J(u, z, \theta) \quad (1)$$
$$s.t. \quad c(u, z, \theta) = 0$$

Cost to compute one local sensitivity (assuming all eigenvalue solver matrix vector products are parallelized)

1. solve (1) once
2. solve 4 large linear systems (whose coefficient matrix is the hessian of the Lagrangian of (1) evaluated at local minimum) for each matrix vector product required by the eigenvalue solver
3. many other inexpensive matrix vector products

Software Infrastructure

- our codes are developed within the Rapid Optimization Library (ROL) in Trilinos [5]
- C++ implementation with
  - parallel linear algebra constructs
  - abstract interfaces
- abstract interfaces enable applications of HDSA in many different problem contexts
- multi-level parallelism facilitates using HDSA on large scale applications

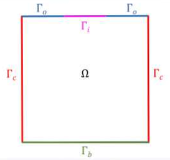
[5] Heroux, M., Bartlett, R., Hoekstra, V.H.R., Hu, J., Kolda, T., Lehoucq, R., Long, K., Pawlowski, R., Phipps, E., Salinger, A., Thornquist, H., Tuminaro, R., Willenbring, J., and Williams, A. An overview of Trilinos, Tech. Rep. SAND2003- 2927, 2003.

Rapid Optimization Library (ROL) [6]

- contains state-of-the-art matrix free derivative-based optimization algorithms
- enables large scale PDE-constrained optimization for control, design, and inverse problems
- full space and reduced space approaches
- deterministic and stochastic optimization algorithms

[6] Kouri, D.P., von Winckel, G., and Ridzal, D. ROL: Rapid Optimization Library.

Chemical Vapor Deposition Reactor



$$\min_{u,z} \frac{1}{2}\int_{\Omega} (\nabla \times v)^2 dx + \frac{\gamma}{2}\int_{\Gamma_c} z^2 dx$$

$s.t.$

$$u = (v_1, v_2, p, T)$$

$$-\epsilon(\theta)\nabla^2 v + (v \cdot \nabla)v + \nabla p + \eta(\theta) T g = 0 \qquad \text{in } \Omega$$

$$\nabla \cdot v = 0 \qquad \text{in } \Omega$$

$$-\kappa(\theta)\Delta T + v \cdot \nabla T = 0 \qquad \text{in } \Omega$$

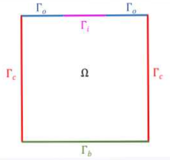
$$T = 0 \quad \text{and} \quad v = v_i \qquad \text{on } \Gamma_i$$

$$\kappa(\theta)\frac{\partial T}{\partial n} = 0 \quad \text{and} \quad v = v_o \qquad \text{on } \Gamma_o$$

$$T = T_b(\theta) \quad \text{and} \quad v = 0 \qquad \text{on } \Gamma_b$$

$$\kappa(\theta)\frac{\partial T}{\partial n} + \nu(\theta)(z - T) = 0 \quad \text{and} \quad v = 0 \qquad \text{on } \Gamma_c$$

Chemical Vapor Deposition Reactor



$$\min_{u,z} \frac{1}{2}\int_\Omega (\nabla \times v)^2 dx + \frac{\gamma}{2}\int_{\Gamma_c} z^2 dx$$

$s.t.$

$$u = (v_1, v_2, p, T)$$

$$-\epsilon(\theta)\nabla^2 v + (v\cdot\nabla)v + \nabla p + \eta(\theta)Tg = 0 \qquad \text{in } \Omega$$

$$\nabla \cdot v = 0 \qquad \text{in } \Omega$$

$$-\kappa(\theta)\Delta T + v\cdot\nabla T = 0 \qquad \text{in } \Omega$$

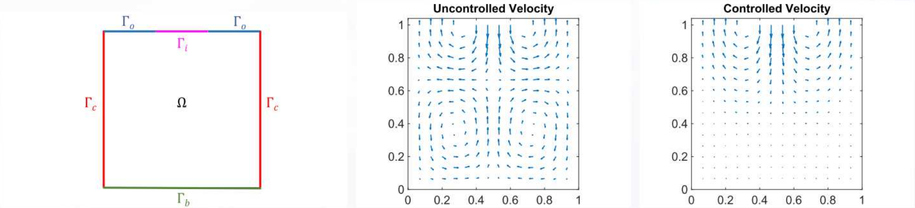
$$T = 0 \quad \text{and} \quad v = v_i \qquad \text{on } \Gamma_i$$

$$\kappa(\theta)\frac{\partial T}{\partial n} = 0 \quad \text{and} \quad v = v_o \qquad \text{on } \Gamma_o$$

$$T = T_b(\theta) \quad \text{and} \quad v = 0 \qquad \text{on } \Gamma_b$$

$$\kappa(\theta)\frac{\partial T}{\partial n} + \nu(\theta)(z - T) = 0 \quad \text{and} \quad v = 0 \qquad \text{on } \Gamma_c$$

Chemical Vapor Deposition Reactor



- particles are injected into the top of a container
- the temperature on side walls is controlled to minimize vorticities in the fluid
- uncertainties enter through properties of the fluid and spatially distributed thermal boundary conditions

Control Solutions

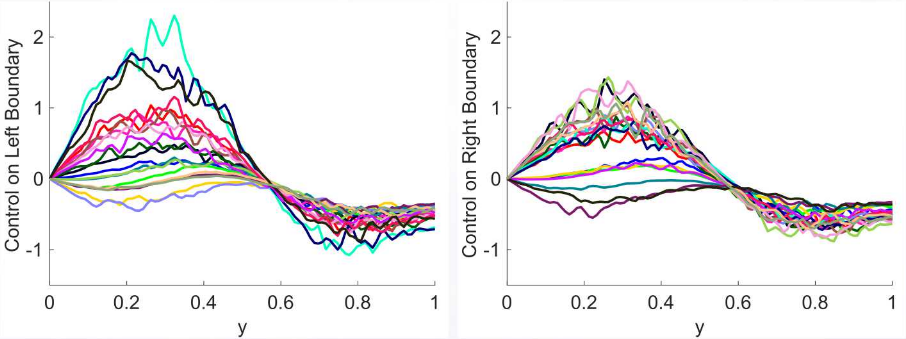


Figure: Control solutions corresponding to 20 different parameter samples. The left and right panels are the controllers on the left and right boundaries, respectively. Each curve is a control solution for a given parameter sample.

Singular Values

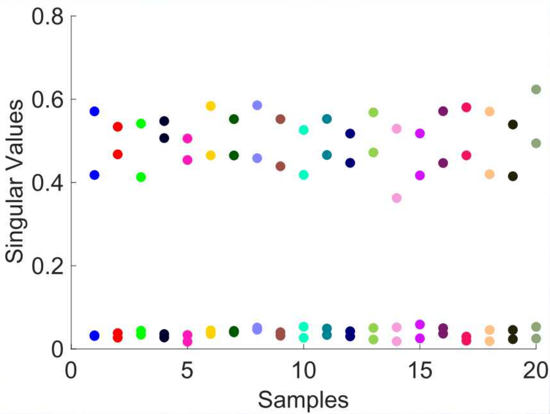


Figure: Leading 4 singular values at 20 different parameter samples. Each vertical slice corresponds to the leading 4 singular values for a fixed sample.

Local Sensitivities

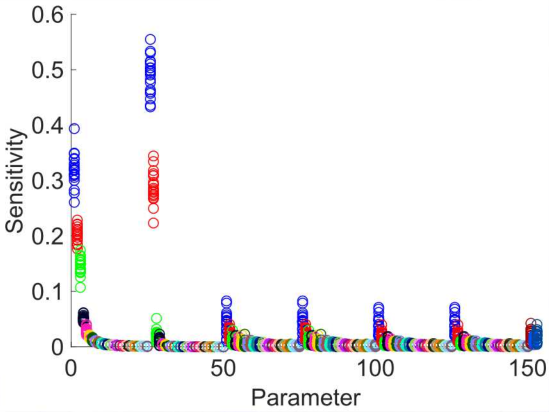


Figure: Local sensitivities for the 153 uncertain parameters. The 20 circles in each vertical slice indicates the sensitivity index for a fixed parameter as it varies over the 20 parameter samples.

Controller Singular Vectors

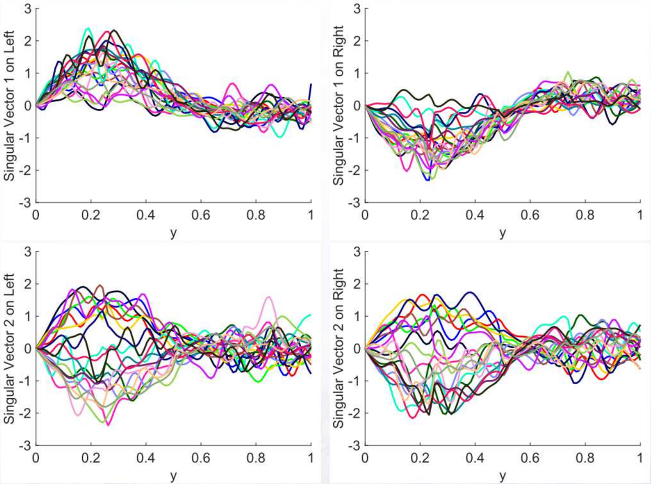


Figure: The top (bottom) row shows the first (second) singular vector on the left and right boundaries, respectively. Each curve corresponds to a different parameter sample.

Observations and Applications

Observations:

- local sensitivity analysis yields similar results for each parameter sample
- only around 10% of the uncertain parameters exhibit significant influence on the control strategy
- the bottom boundary condition, T_b, has the greatest influence on the control strategy

Applications:

- experimental efforts and/or modeling efforts should be invested toward characterizing the bottom thermal boundary condition

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Subsurface Contaminant Source Inversion

$$\min_{u,z} \frac{1}{2}\sum_{i=1}^{T}\sum_{j=1}^{M}(\mathcal{P}_{i,j}y - d_{i,j})^2 + \frac{\alpha}{2}\int_\Omega z(x,y)^2 dxdy$$

$s.t.$

$u = (p, y)$

$\nabla \cdot (-\kappa(\theta)\nabla p) = 0$

$$\frac{\partial y}{\partial t} + \nabla \cdot (-\epsilon(\theta)\nabla y) + (-\kappa(\theta)\nabla p)\cdot \nabla y = \chi_{[.01,.02]}(t)z \qquad \text{in } \Omega \text{ for}$$

$p = \psi(\theta)$ on Γ_L

$-\kappa(\theta)\nabla p \cdot n = 0$ on Γ_B

$\nabla u \cdot n = 0$ on $\Gamma_L \cup \Gamma_R \cup \Gamma_B \cup \Gamma_T$ for

$u = 0$ in Ω for

Subsurface Contaminant Source Inversion

$$\min_{u,z} \frac{1}{2} \sum_{i=1}^{T} \sum_{j=1}^{M} (\mathcal{P}_{i,j} y - d_{i,j})^2 + \frac{\alpha}{2} \int_\Omega z(x,y)^2 dxdy$$

$s.t.$

$$u = (p, y)$$

$$\nabla \cdot (-\kappa(\theta) \nabla p) = 0$$

$$\frac{\partial y}{\partial t} + \nabla \cdot (-\epsilon(\theta) \nabla y) + (-\kappa(\theta) \nabla p) \cdot \nabla y = \chi_{[.01,.02]}(t) z \qquad \text{in } \Omega \text{ for}$$

$$p = \psi(\theta) \qquad \text{on } \Gamma_L$$

$$-\kappa(\theta) \nabla p \cdot n = 0 \qquad \text{on } \Gamma_B$$

$$\nabla u \cdot n = 0 \qquad \text{on } \Gamma_L \cup \Gamma_R \cup \Gamma_B \cup \Gamma_T \text{ for}$$

$$u = 0 \qquad \text{in } \Omega \text{ for}$$

Pressure Equation

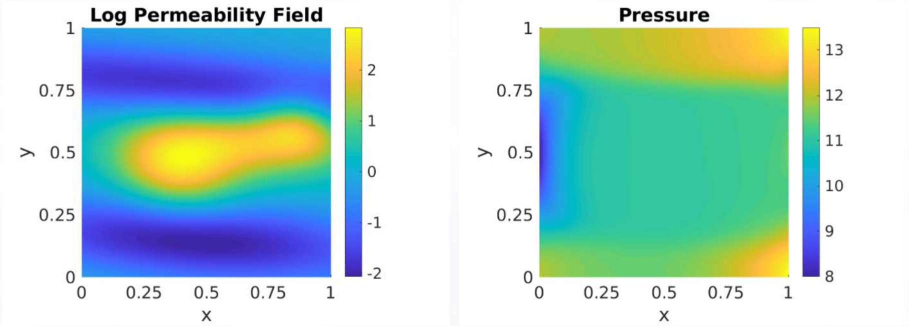

Figure: Left: the log (base 10) of the nominal permeability field; right: the solution of the pressure equation with nominal permeability field and boundary conditions.

- pressure field produces a right to left flow with high velocity around $y = 0.5$
- contaminant enters in the center of the domain and is advected to the left

Transport Equation

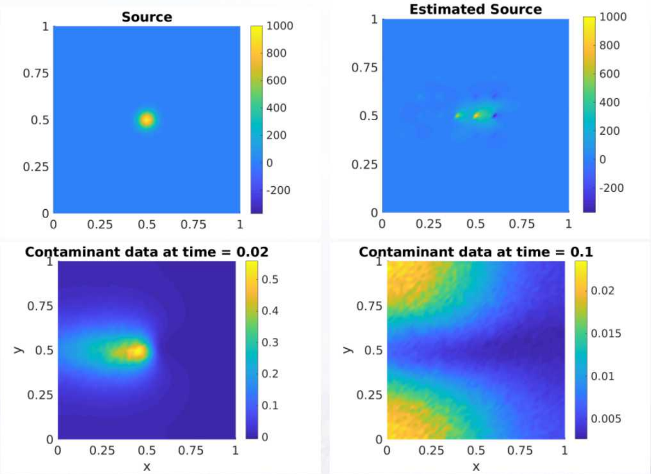

Figure: Top: the "true" source (left) used to generate synthetic data and the estimated source (right). Bottom: the noisy contaminant data at just after the contaminant injection (left) and after several time steps (right).

Singular Values

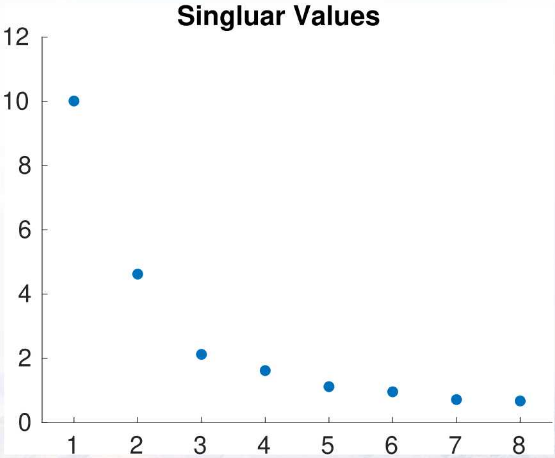

Figure: Singular values for local sensitivities computed at the nominal parameters.

Sensitivity Indices

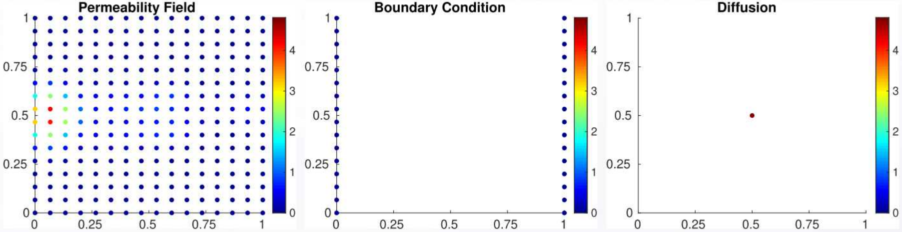

Figure: Sensitivity indices for the permeability field (left), pressure equation Dirichlet boundary conditions (center), and diffusion coefficient (right).

Observations and Applications

Observations:

- the estimated source is most sensitivity to the diffusion coefficient
- the permeability field is most important in the region near $x = 0.10$ and $y = 0.50$
- the pressure boundary condition ψ is less influential

Applications:

- the magnitude of the sensitivities relative to the magnitude of the source provides a level of confidence in our estimation
- the localized region of high permeability sensitivity indicates that we may focus data acquisition efforts in a small region
- the high sensitivity of the diffusion coefficient indicates that properties of the contaminant are more important (locally around the nominal estimates) than the subsurface properties (permeability and boundary conditions)

Tracer Test Permeability Inversion

$$\min_{u,z} \frac{1}{2} \sum_{j=1}^{M} \left(w_p (\mathcal{P}_j^p p - d_j^p)^2 + \sum_{i=1}^{T} w_y (\mathcal{P}_{i,j}^y y - d_{i,j}^y)^2 \right) + \frac{\alpha}{2} \int_\Omega ||\nabla z||^2 dxdy$$

$s.t.$

$u = (p, y)$

$\nabla \cdot (-e^z \nabla p) = 0$ in Ω

$\frac{\partial y}{\partial t} + \nabla \cdot (-\epsilon \nabla y) + (-e^z \nabla p) \cdot \nabla y = f$ in Ω for $t > 0$

$p = \psi$ on $\Gamma_L \cup \Gamma_R$

$-\kappa \nabla p \cdot n = 0$ on $\Gamma_B \cup \Gamma_T$

$\nabla u \cdot n = 0$ on $\Gamma_L \cup \Gamma_R \cup \Gamma_B \cup \Gamma_T$ for $t > 0$

$u = 0$ in Ω for $t = 0$

Tracer Test Permeability Inversion

$$\min_{u,z} \frac{1}{2}\sum_{j=1}^{M}\left(w_p(\mathcal{P}_j^p p - d_j^p - \theta_j^p)^2 + \sum_{i=1}^{T} w_y(\mathcal{P}_{i,j}^y y - d_{i,j}^y - \theta_{i,j}^y)^2 \right) + \frac{\alpha}{2}\int_\Omega ||\nabla z||^2 dxdy$$

$s.t.$

$$u = (p, y)$$

$$\nabla \cdot (-e^z \nabla p) = 0 \qquad \text{in } \Omega$$

$$\frac{\partial y}{\partial t} + \nabla \cdot (-\epsilon \nabla y) + (-e^z \nabla p) \cdot \nabla y = f \qquad \text{in } \Omega \text{ for } t > 0$$

$$p = \psi \qquad \text{on } \Gamma_L \cup \Gamma_R$$

$$-\kappa \nabla p \cdot n = 0 \qquad \text{on } \Gamma_B \cup \Gamma_T$$

$$\nabla u \cdot n = 0 \qquad \text{on } \Gamma_L \cup \Gamma_R \cup \Gamma_B \cup \Gamma_T \text{ for } t > 0$$

$$u = 0 \qquad \text{in } \Omega \text{ for } t = 0$$

Permeability Field and Tracer Injection

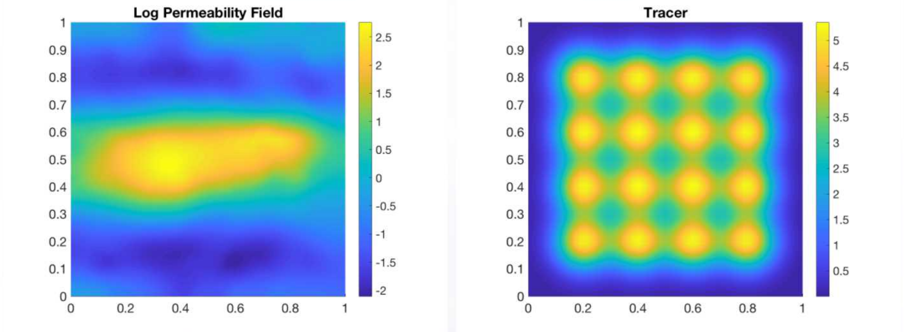

Figure: Left: the log (base 10) of the true permeability field; right: the tracer injection.

- pressure field produces a right to left flow with high velocity around $y = 0.5$
- tracer enters the domain through 16 injection points

Sensitivity Indices

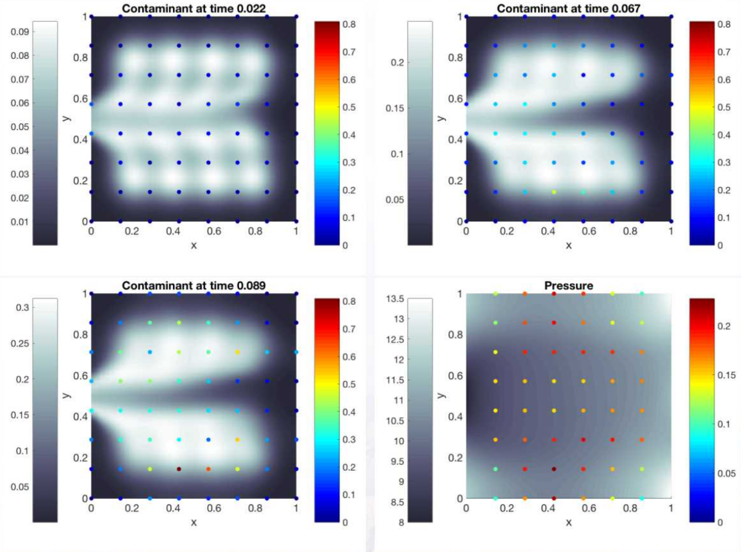

Figure: Sensitivity indices for contaminant and pressure data.

Observations and Applications

Observations:

- pressure data is more important in earlier times and tracer concentration data is more important in later times
- spatial structure in both pressure and tracer concentration sensitivities indicate that the most important sensors are
  - those lying in low permeability regions
  - those (moving in time) which measure the contaminant concentration on the back end of the high permeability flow region

Applications:

- provides insight into sensor design and deployment by distinguishing data types and spatiotemporal structure

Distributional Robustness for Darcy Flow Control

$$\min_{u,z} \mathbb{E}_\theta \left[\frac{1}{2} \sum_{j=1}^{M} (\mathcal{P}_j u(\theta) - d_j)^2 \right] + \frac{\alpha}{2} \int_\Omega ||z||^2 dxdy$$

$s.t.$

$$\nabla \cdot (-\kappa(\theta) \nabla u) = f(z) \qquad \text{in } \Omega$$

$$u = \psi \qquad \text{on } \Gamma_B \cup \Gamma_T$$

$$-\kappa(\theta) \nabla u \cdot n = 0 \qquad \text{on } \Gamma_L \cup \Gamma_R$$

Distributional Robustness for Darcy Flow Control

$$\min_{u,z} \sum_{i=1}^{N} w_i \left(\frac{1}{2} \sum_{j=1}^{M} (\mathcal{P}_j u_i - d_j)^2 \right) + \frac{\alpha}{2} \int_\Omega ||z||^2 dxdy$$

$s.t.$ for $i = 1, 2, \ldots, N$

$$\nabla \cdot (-\kappa(\theta_i) \nabla u_i) = f(z) \qquad \text{in } \Omega$$

$$u_i = \psi \qquad \text{on } \Gamma_B \cup \Gamma_T$$

$$-\kappa(\theta_i) \nabla u_i \cdot n = 0 \qquad \text{on } \Gamma_L \cup \Gamma_R$$

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State and Target

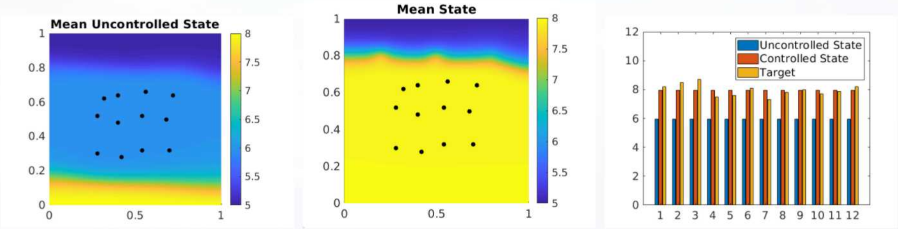

Figure: Left: uncontrolled mean state; center: controlled mean state; right: comparison of mean states with the target data. The black dots correspond to the locations of the target wells.

- control injection wells to get desired pressure at production wells
- uncertainty in the permeability field

Control and Sensitivity

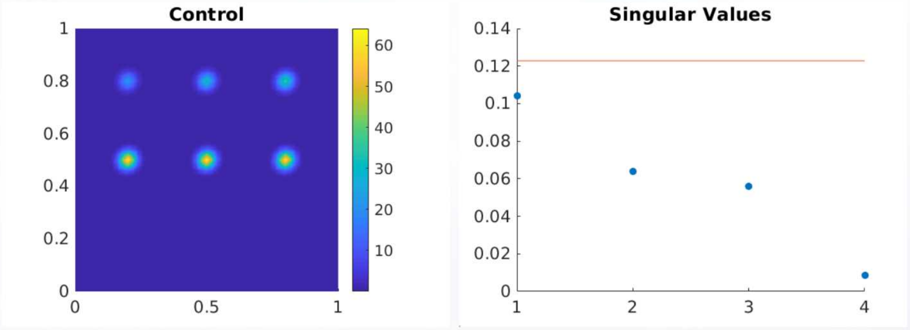

Figure: Left: optimal risk neutral control strategy; right: singular values of the sensitivity operator. The horizontal line above the singular values indicates the norm of the optimal controller.

- control strategy is parameterized by gaussian sources at 6 locations
- leading singular value is 85% of the magnitude of the optimal controller norm

Observations and Applications

Observations:

- a 10% relative perturbation of the weights produces approximately a 8.5% perturbation of the optimal controller

Applications:

- the control strategy is not robust to uncertainty in the distribution of the permeability field
- ongoing work explores how additional information may be leveraged from the singular vector or sensitivity indices to direct practitioners in such cases

Summary

- defined derivative-based local sensitivity indices for the solution of optimization problems with respect to auxiliary parameters
- identified computational challenges and proposed to use a randomized generalized eigenvalue solver to identify low rank structure and parallelize computation
- highlighted the software design and abstraction
- shown the versatility of the approach in various use cases
- presented numerical results and inferences made using HDSA for multi-physics problems

Questions?

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