

Generalized Entropy Stable Weighted Essentially Non- Oscillatory Finite Difference Scheme in Multi-Block Domains



PRESENTED BY

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Motivation

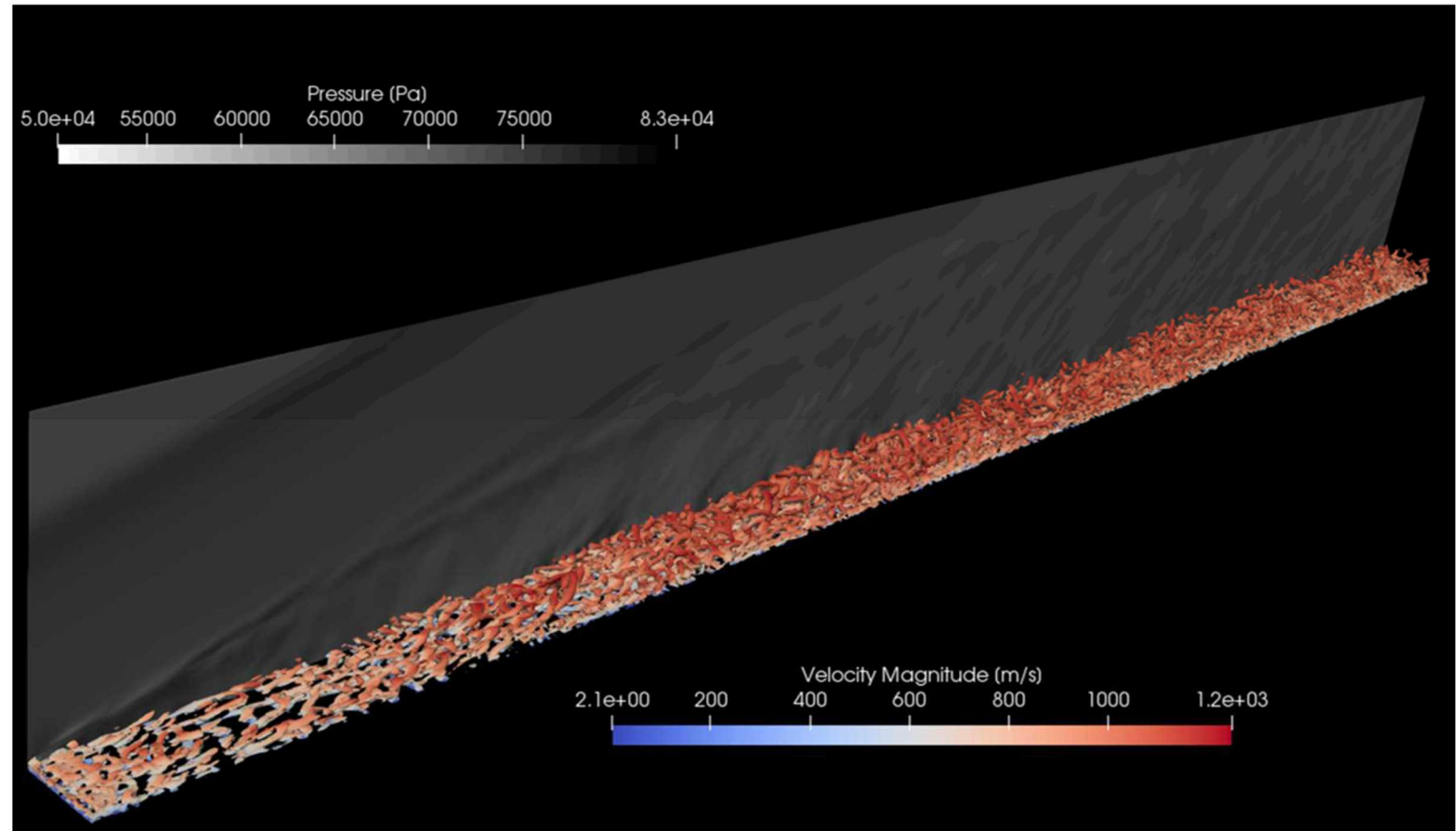
High-fidelity simulations of 3D turbulent compressible flows

Challenges

- Resolve various fluid scales
- Stability vs. accuracy
- Large-scale or complex geometry

Numerical requirements

- High-order
- Low dissipation
- Handle shocks
- Multi-block capable

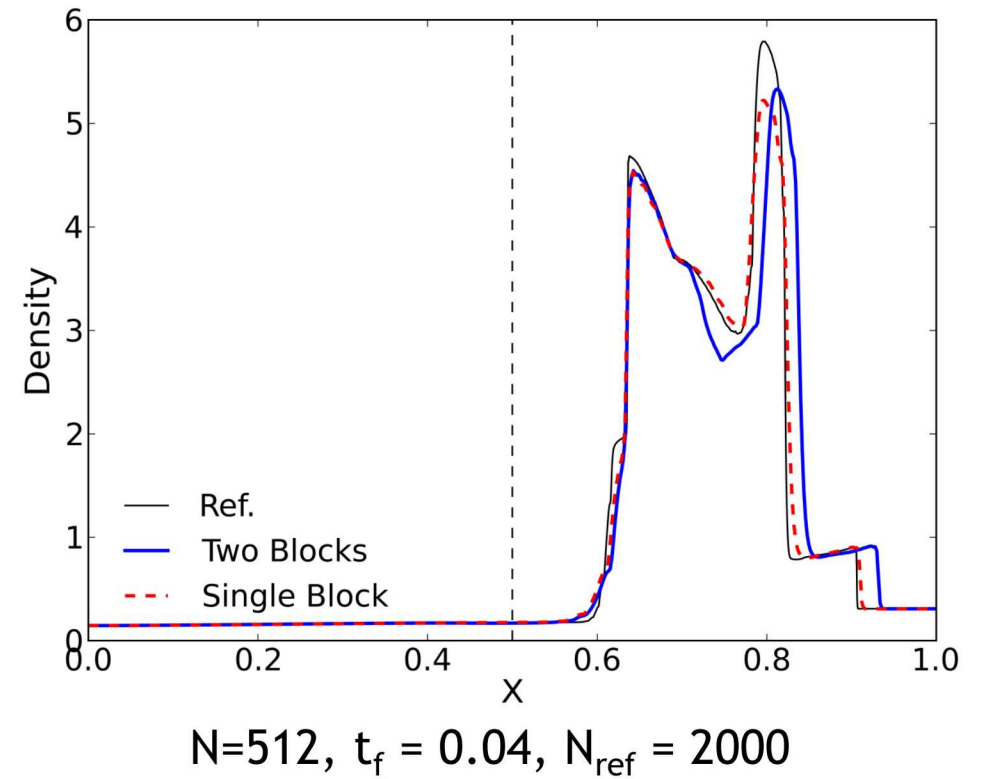
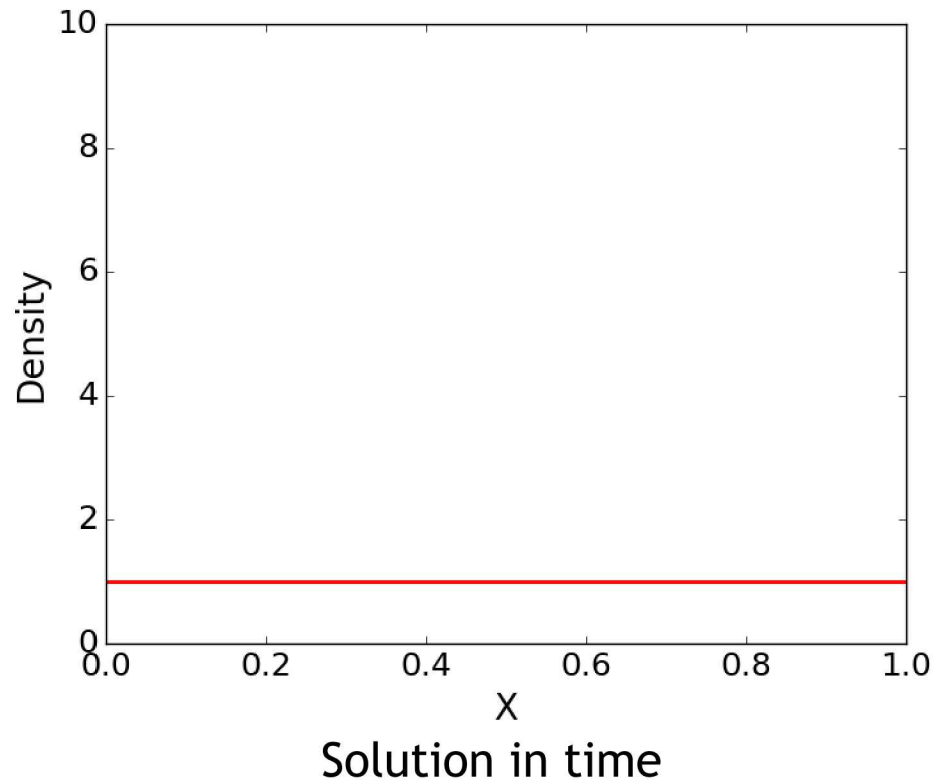


Mach 3.5 flat plate boundary layer DNS

Robust Shock Capturing Across Multi-Block Interface

Capturing shocks across multi-block interface is difficult!

Woodward Colella with **conventional** high-order finite difference



Numerical Basis: Generalized Summation-By-Parts

Nonlinear conservation laws

$$\mathbf{u}_t + (\mathbf{f}_k)_{x_k} = 0, \quad x_k \in \Omega, \quad t \in [0, \infty),$$

$$\mathbb{B}(\mathbf{u}) = \mathbf{g}^{bnd}, \quad x_k \in \partial\Omega, \quad t \in [0, \infty),$$

$$\mathbf{u}(x, 0) = \mathbf{g}_0(x_k), \quad x_k \in \Omega,$$

$$\mathbf{u}_t + \mathcal{D}_k \mathbf{f}_k = \mathcal{P}_k^{-1} \mathbf{g}_k^{int} + \mathcal{P}_k^{-1} \mathbf{g}_k^{bnd}, \quad k = 1, 2, 3$$

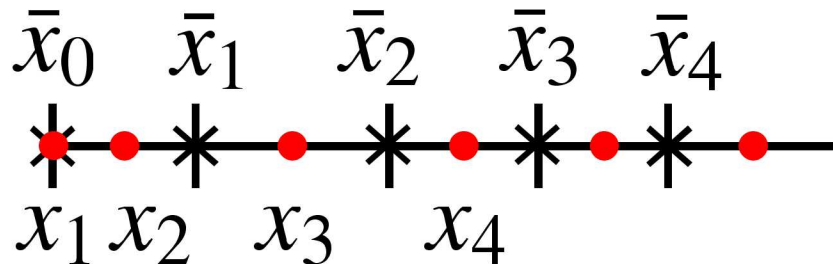
Generalized summation-by-parts (SBP) operator (Del Rey Fernandez, JCP 2014)

- Satisfies discrete analogue of **integration-by-parts**

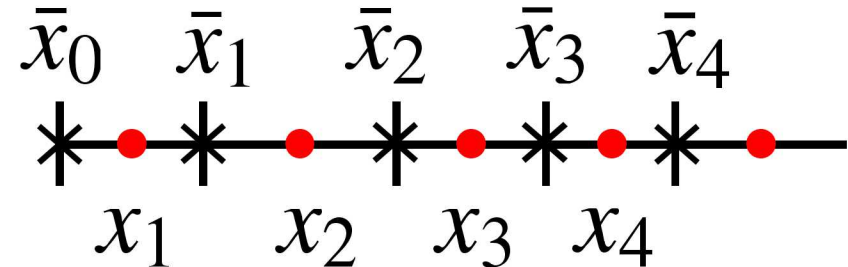
$$\begin{aligned} \mathcal{D} &= \mathcal{P}^{-1} Q, \quad \mathcal{P} = \mathcal{P}^T, \quad \boldsymbol{\xi}^T \mathcal{P} \boldsymbol{\xi} > 0, \quad \boldsymbol{\xi} \neq 0 \\ Q^T &= \mathcal{B} - Q, \quad \mathcal{B} = \mathbf{b}_1 \mathbf{b}_1^T - \mathbf{b}_{-1} \mathbf{b}_{-1}^T, \end{aligned}$$

- Generalization

$$\mathbf{b}_{-1} = (1, 0, 0, \dots, 0)^T$$



$$\mathbf{b}_{-1} = \left(\frac{35}{16}, -\frac{35}{16}, \frac{21}{16}, -\frac{5}{16}, 0, \dots, 0 \right)^T$$



Numerical Scheme Details

Entropy stable cell-centered high-order finite difference

$$\mathbf{u}_t + \mathcal{P}^{-1}[2Q \circ \mathcal{F}]\mathbf{1} = \mathcal{P}^{-1}\mathbf{g}^{int}$$

- **Complementary grid** enables us to recast gradient form to flux form
 - Important for entropy stable WENO flux (Fisher 2013)

$$\mathbf{f}(\mathbf{u})_x = \mathcal{P}^{-1}[2Q \circ \mathcal{F}]\mathbf{1} = \mathcal{P}^{-1}\Delta\bar{\mathbf{f}} \quad \longrightarrow \quad \mathbf{u}_t + \mathcal{P}^{-1}\Delta\bar{\mathbf{f}} = \mathcal{P}^{-1}\mathbf{g}^{int}$$

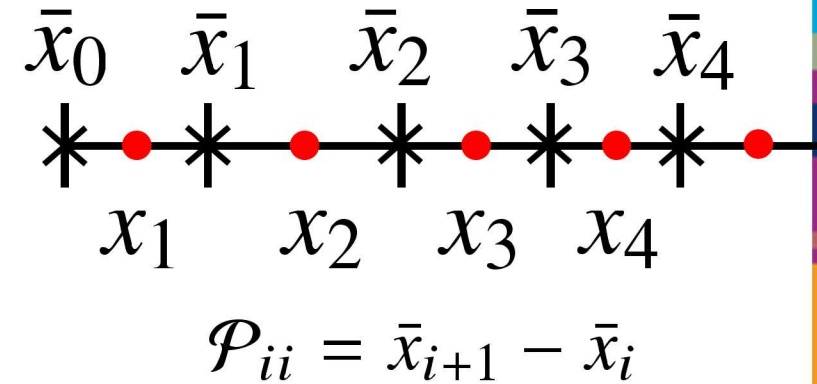
- Entropy stable two-point nonlinear flux

$$\frac{d}{dt}\mathbf{1}^T \mathcal{P}S + \sum_{k=1}^N \sum_{l=1}^N b_{1,k}b_{1,l}\bar{F}(u_l, u_k) - b_{-1,k}b_{-1,l}\bar{F}(u_l, u_k) = 0$$

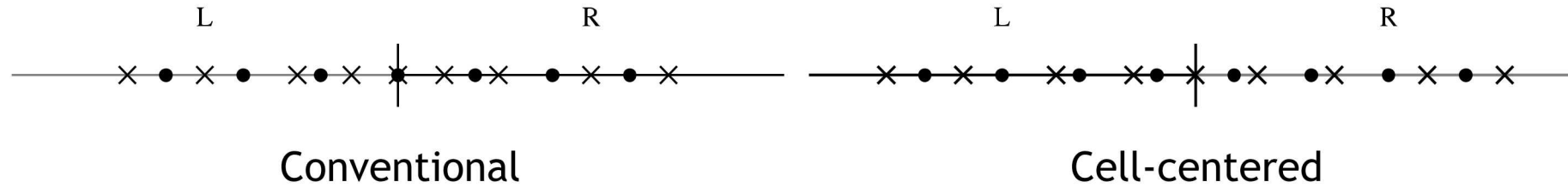
$$\frac{d}{dt}\mathbf{1}^T \mathcal{P}S + \bar{F}|_1 - \bar{F}|_{-1} = 0$$

$$\bar{f}_i^S = \sum_{k=i}^N \sum_{l=1}^i 2\hat{q}_{(l,k)}\bar{f}(u_l, u_k) + \sum_{k=i+1}^N \sum_{l=1}^N -b_{-1,l}b_{-1,k}\bar{f}(u_l, u_k) + \sum_{k=1}^i \sum_{l=1}^N b_{1,l}b_{1,k}\bar{f}(u_l, u_k), \quad 1 \leq i \leq N-1,$$

- Similar to finite volume and satisfies telescoping flux property
- Benefits of cell-centered approach
 - Stronger coupling across multi-block interface
 - Better shock capturing



Generalized Entropy Stable Interface Penalty



Two-domain finite difference in flux form

$$\mathbf{u}_t + \mathcal{P}^{-1} \Delta \bar{\mathbf{f}} = \mathcal{P}^{-1} \mathbf{g}^{int}$$

$$\Delta \bar{\mathbf{f}} = (Q + \mathcal{G}) \begin{bmatrix} \mathbf{f}_L \\ \mathbf{f}_R \end{bmatrix} \quad Q = \begin{bmatrix} Q_L & 0 \\ 0 & Q_R \end{bmatrix} \quad \mathcal{G} = \begin{bmatrix} -\frac{1}{2} \mathbf{b}_1^L \mathbf{b}_1^{L^T} & \frac{1}{2} \mathbf{b}_1^L \mathbf{b}_{-1}^{R^T} \\ -\frac{1}{2} \mathbf{b}_{-1}^R \mathbf{b}_1^{L^T} & \frac{1}{2} \mathbf{b}_{-1}^R \mathbf{b}_{-1}^{R^T} \end{bmatrix}$$

Generalized entropy stable interface penalty

$$\begin{aligned} \mathbf{g}^{int} = & \left\{ \left(\mathbf{b}_1^L \mathbf{b}_1^{L^T} \circ \mathbf{f}(\mathbf{u}, \mathbf{u}) - \mathbf{b}_1^L \mathbf{b}_{-1}^{R^T} \circ \mathbf{f}(\mathbf{u}, \mathbf{u}) \right) \right. \\ & \left. - \frac{1}{2} \mathbf{b}_1^L \mathbf{R} |\Lambda| \mathbf{R}^T \left(\mathbf{b}_1^{L^T} \mathbf{w} - \mathbf{b}_{-1}^{R^T} \mathbf{w} \right) \right\} \\ & - \left\{ \left(-\mathbf{b}_{-1}^R \mathbf{b}_1^{L^T} \circ \mathbf{f}(\mathbf{u}, \mathbf{u}) + \mathbf{b}_{-1}^R \mathbf{b}_{-1}^{R^T} \circ \mathbf{f}(\mathbf{u}, \mathbf{u}) \right) \right. \\ & \left. - \frac{1}{2} \mathbf{b}_{-1}^R \mathbf{R} |\Lambda| \mathbf{R}^T \left(\mathbf{b}_{-1}^{R^T} \mathbf{w} - \mathbf{b}_1^{L^T} \mathbf{w} \right) \right\} \end{aligned}$$

Shock Capturing Method: Weighted Essentially Non-Oscillatory

Entropy stable WENO (Fisher and Carpenter, JCP 2013)

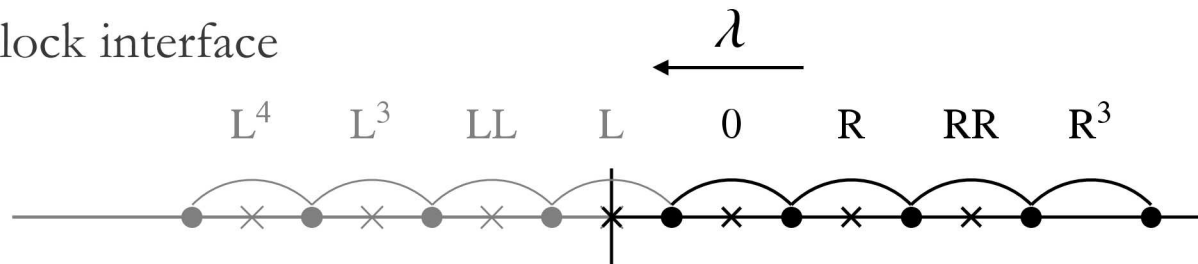
$$\bar{f}_i^W = \sum_{l=1}^{n_s} \bar{\omega}_i^l \bar{f}_i^{S_l}, \quad \bar{\omega}_i^l = \frac{\bar{\alpha}_i^l}{\sum_j \bar{\alpha}_i^j}, \quad \bar{\alpha}_i^l = \bar{d}_i^l \left(1 + \frac{\bar{\tau}_i^l}{\bar{\beta}_i^l + \bar{\epsilon}_i} \right), \quad l = 1, \dots, n_s$$

$$\bar{f}_i^{SSW} = \bar{f}_i^W + \delta(\bar{f}_i^S - \bar{f}_i^W), \quad \delta = \frac{\sqrt{b^2 + c^2} - b}{\sqrt{b^2 + c^2}}, \quad b = (w_{i+1} - w_i)^T (\bar{f}_i^S - \bar{f}_i^W), \quad c = 10^{-12}$$

- Entropy stability condition is satisfied with entropy stable WENO

$$(w_{i+1} - w_i)^T (\bar{f}_i^{SSW} - \bar{f}_i^S) \leq 0, \quad 0 \leq i \leq N - 1,$$

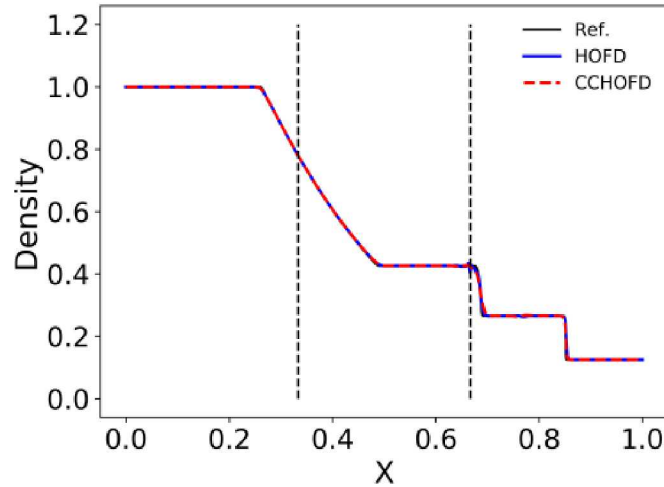
WENO across multi-block interface



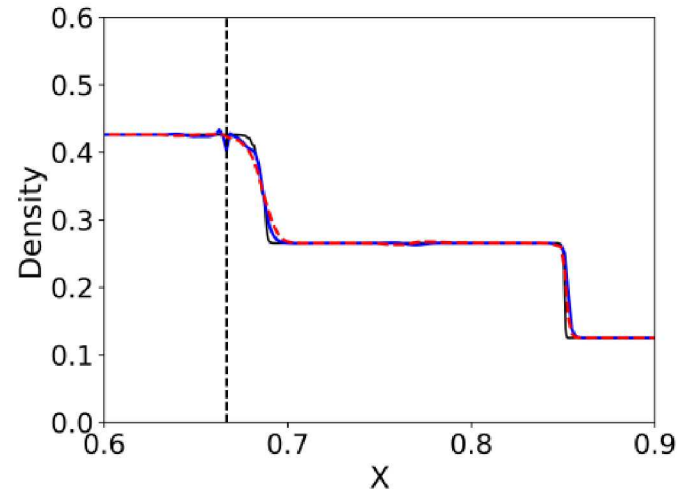
- Cell-centered SBP operator gives a strong coupling between blocks
- WENO target flux, weight, candidate stencil based on non-dissipative interface operator
- Need a different biasing due to larger stencil width

$$\bar{\alpha}_i^l = \begin{cases} \bar{d}_i^l \left(1 + \frac{\bar{\tau}_i^l}{\bar{\beta}_i^l + \bar{\epsilon}_i} \right), & \text{if } l \in [0, R, RR, R^3] \\ \gamma \bar{d}_i^l \left(1 + \frac{\bar{\tau}_i^l}{\bar{\beta}_i^l + \bar{\epsilon}_i} \right), & \text{if } l \in [L, LL, L^3, L^4] \end{cases}, \quad l = 1, \dots, n_s,$$

Multi-Block Shock Capturing WENO: 1D Shock Examples



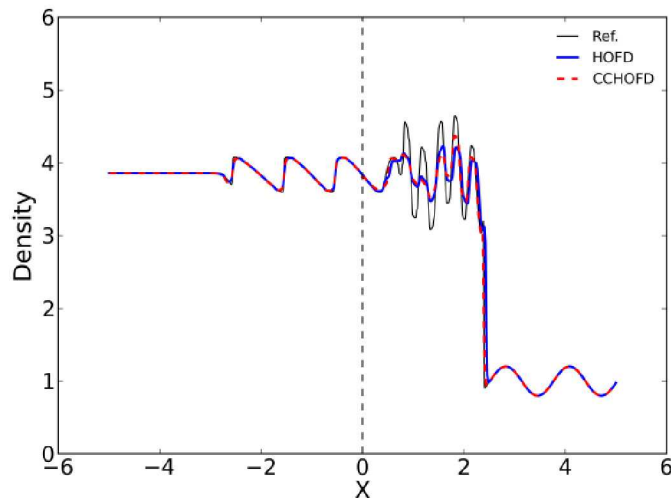
(a) Density



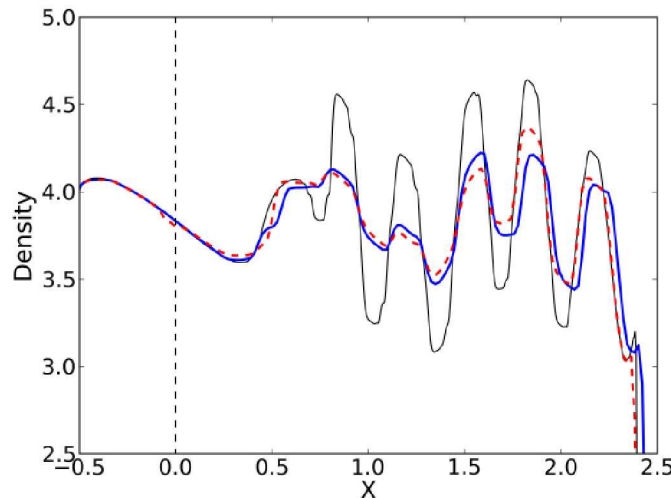
(b) Density close up near the contact discontinuity

$$(\rho, u, p) = \begin{cases} (1, 0, 1), & \text{if } x < 0.5 \\ (0.125, 0, 0.1), & \text{if } x \geq 0.5, \end{cases}$$

$N=512$, $t_f = 0.25$
Three-block



(a) Density

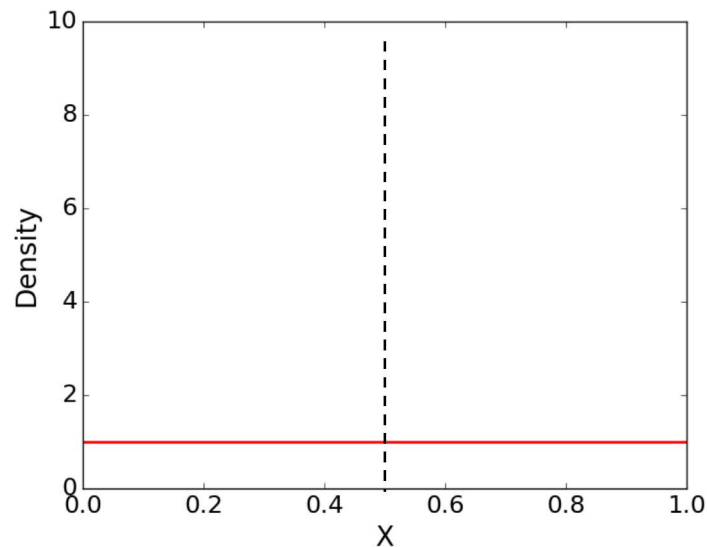


(b) Density close up

$$(\rho, u, p) = \begin{cases} (3.857143, 2.629369, 10.3333), & \text{if } x < -4.0 \\ (1 + 0.2 \sin(5x), 0, 1), & \text{if } x \geq -4.0. \end{cases}$$

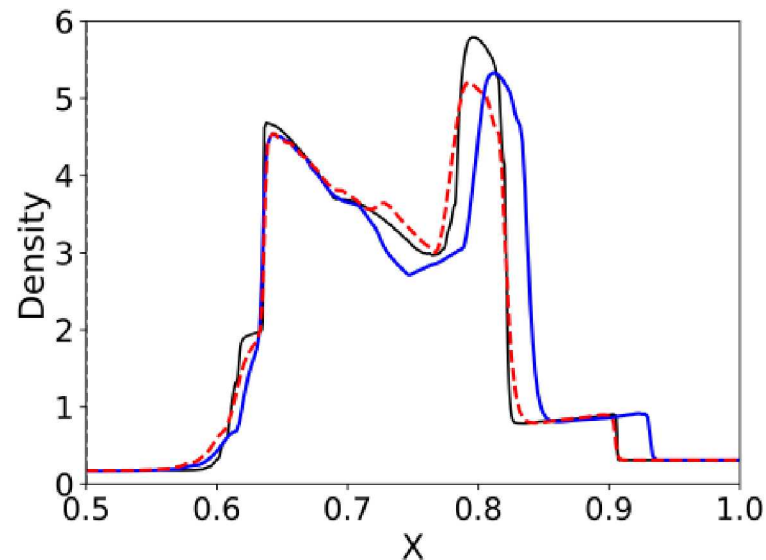
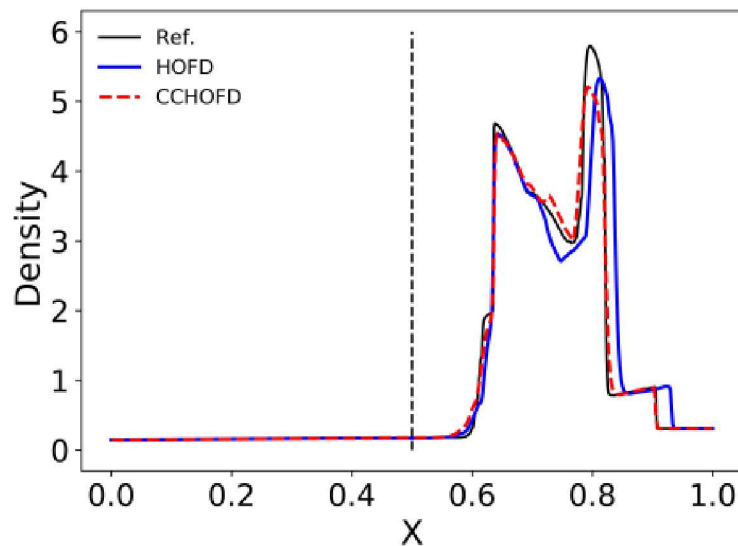
$N=512$, $t_f = 1.8$, $N_{\text{ref}} = 2000$
Two-block

Strong Shock Across Interface: Woodward Colella



Initial condition

$$(\rho, u, p) = \begin{cases} (1, 0, 1000), & \text{if } x < 0.1 \\ (1, 0, 100), & \text{if } 0.1 \leq x < 0.9 \\ (1, 0, 0.01), & \text{if } x \geq 0.9. \end{cases}$$



$N=512$, $t_f = 0.04$, $N_{\text{ref}} = 2000$
Two-block

Turbulence and Shocks Have Different Characteristics

Our observations so far...

Cell-centered high-order finite difference

- Non-dissipative
- Can't handle shocks

Shock capturing method: WENO

- Dissipative
- Destroys turbulent structures

Solution: Hybrid scheme

- Blend CCHOFD and shock capturing method
 - Non-dissipative + Handle shocks
- Shock sensor (Larsson, CTR 2011)
 - Activate shock capturing method when

$$-\nabla \cdot \mathbf{v} > \max \left(A \sqrt{\omega \cdot \omega}, B \frac{c}{h} \right) \quad \text{where } A > 1, B < 1$$

Turbulence Dissipation Characterization

Taylor Green vortex

- 3D periodic domain $[-\pi L \leq (x, y, z) \leq \pi L]$
- Initial condition

$$M = 0.1 \quad Re = 1600$$

$$\rho(x, y, z, 0) = 1.0$$

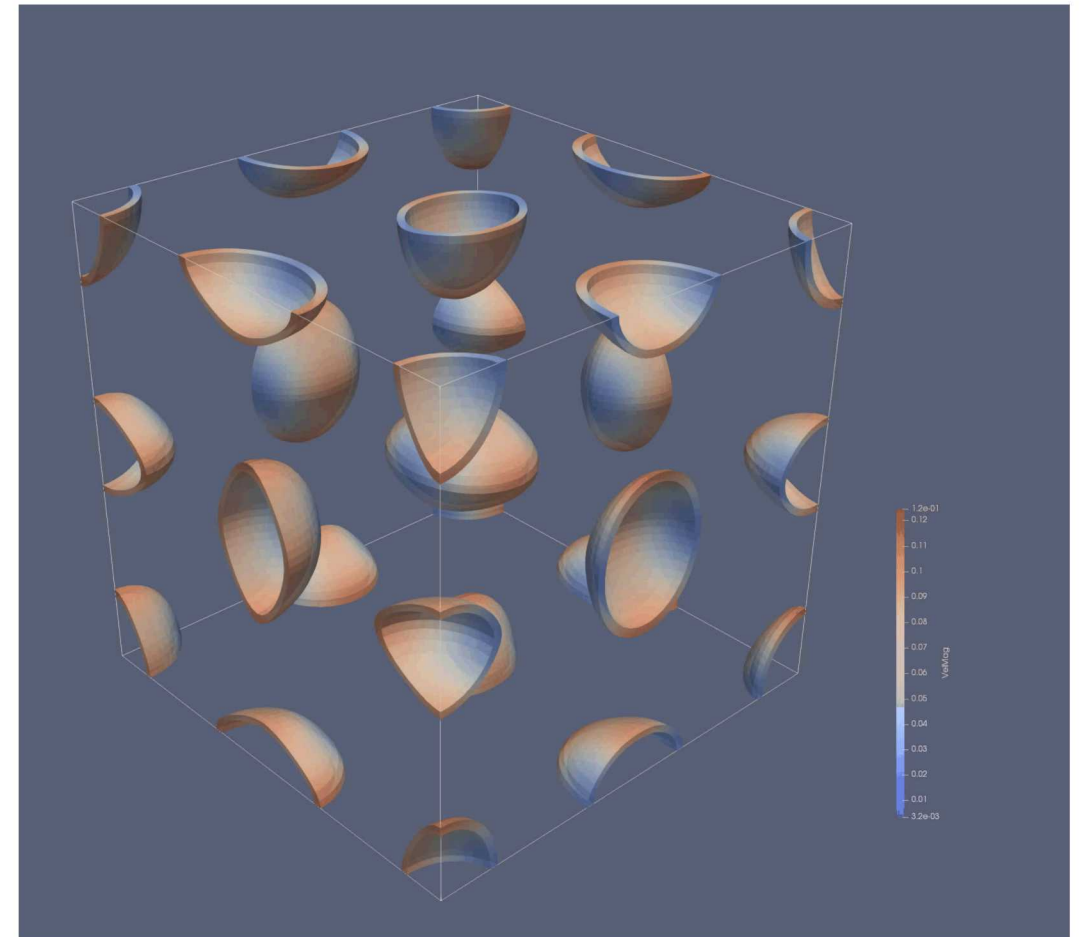
$$u(x, y, z, 0) = V_0 \sin \frac{x}{L} \cos \frac{y}{L} \cos \frac{z}{L}$$

$$v(x, y, z, 0) = -V_0 \cos \frac{x}{L} \sin \frac{y}{L} \cos \frac{z}{L}$$

$$w(x, y, z, 0) = 0$$

$$p(x, y, z, 0) = 1.0 + \frac{\gamma M_0^2}{16} \left(\cos \frac{2x}{L} + \cos \frac{2y}{L} + \cos \frac{2z}{L} \right) + 2$$

- Benchmark schemes
 - Cell-centered HOFD
 - Hybrid cell-centered HOFD
- Assess multi-block capability

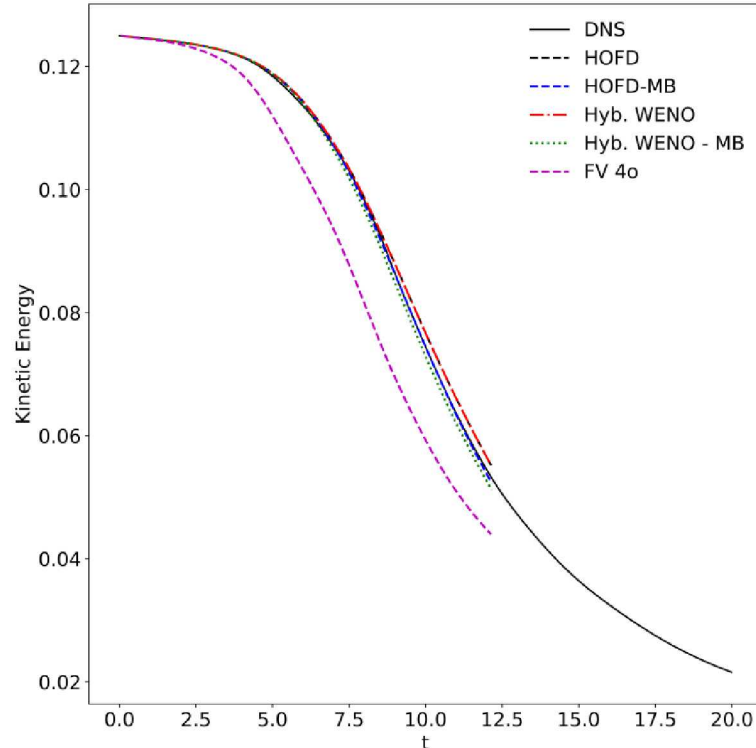


Vorticity visualization using cell-centered HOFD 64^3

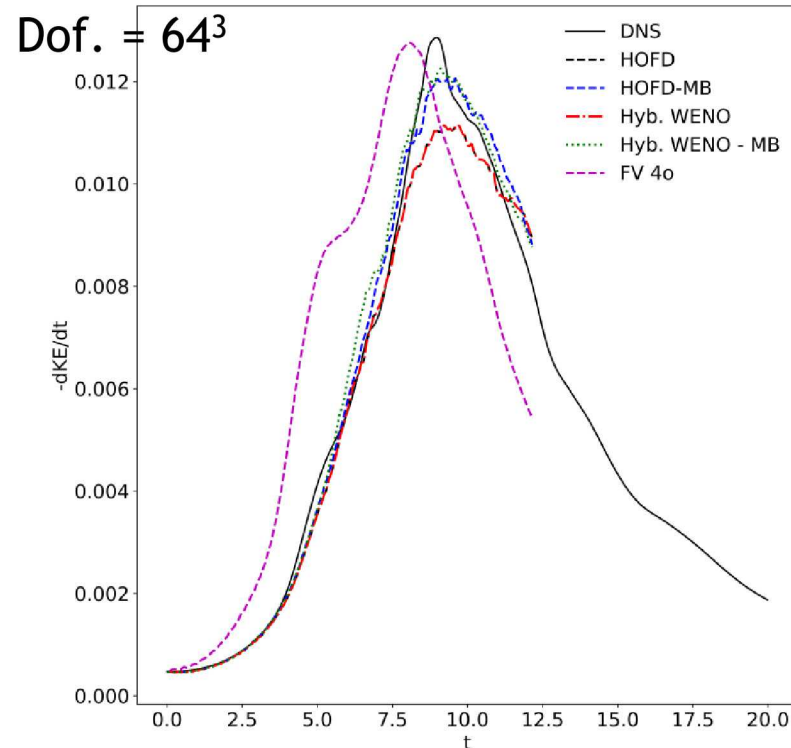
Quantitative Evaluation of Cell-Centered HOFD

Turbulence dissipation characteristics

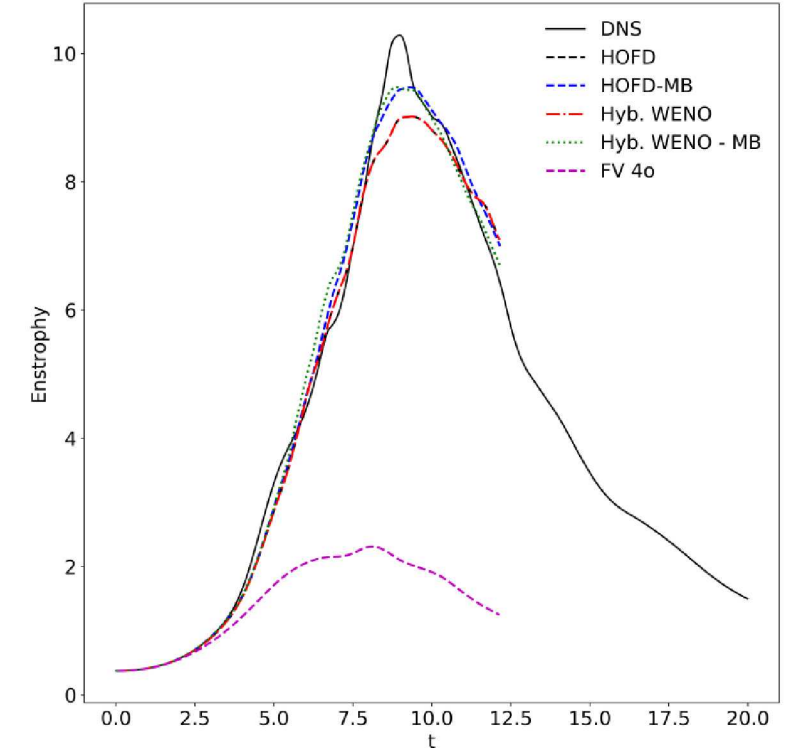
- DNS: 512^3 spectral method
- Cell-centered HOFD and Hybrid WENO: demonstrate multi-block structured mesh
- Popular CCFV : Subbareddy and Candler low dissipation method Mach sensor



Total kinetic energy



Total kinetic energy dissipation rate

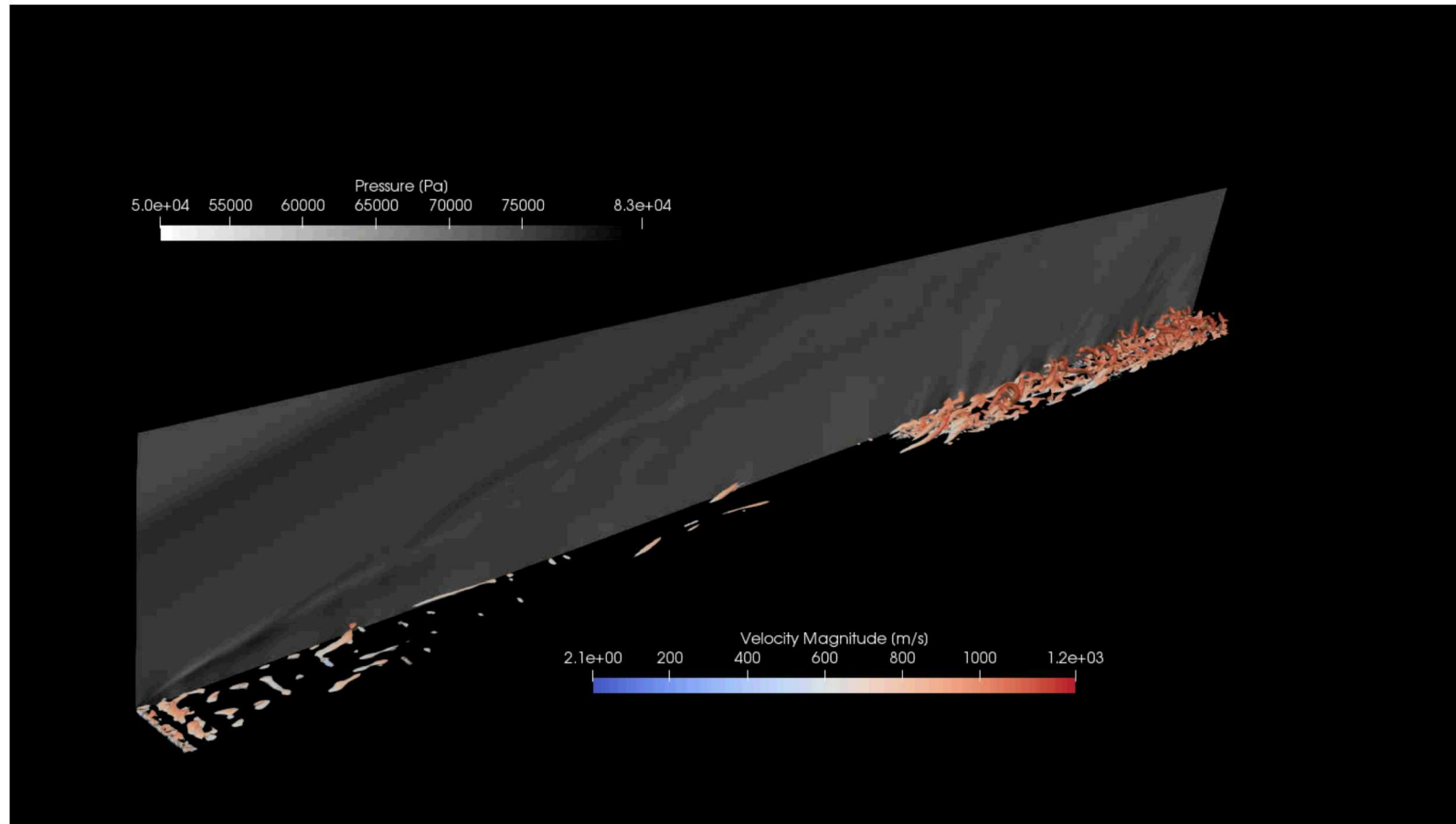


Enstrophy

Where We Are Headed

Toward full high-fidelity simulations of 3D turbulent compressible flows

- Low-dissipation
- Shock capturing
- Hybrid framework in multi-block structured configuration



Mach 3.5 flat plate boundary layer DNS w. ~Dof. 63 million (Barone, AIAA 2017)

Future Works

Demonstrate full scale multi-block LES/DNS with generalized interface penalty

Improve shock capturing scheme

- Interface WENO stencil biasing
- Develop a robust alternative shock capturing scheme, e.g. artificial viscosity

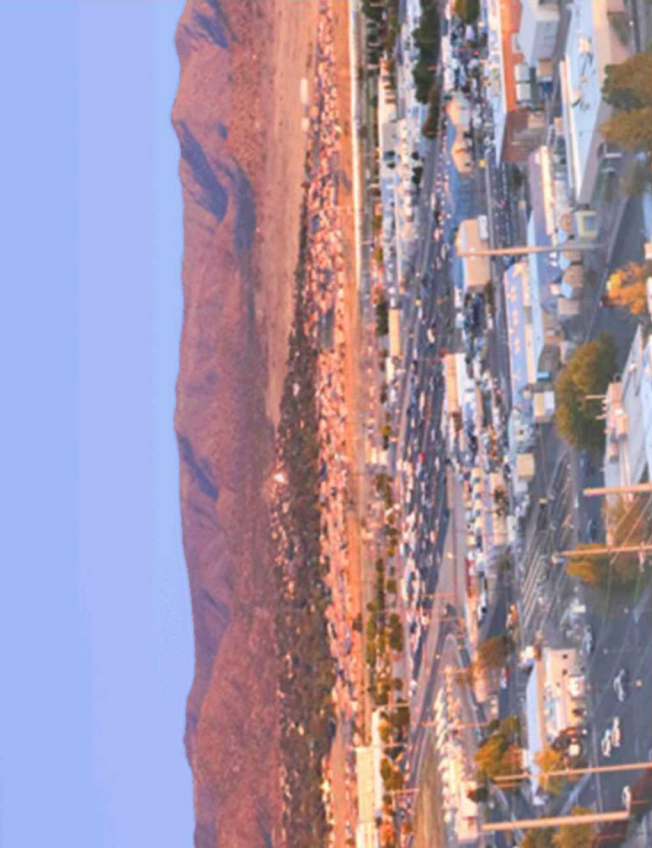
Improve hybrid scheme

- Improved shock sensor to detect shocks better
- Investigate an alternative hybrid scheme that treats under-resolved and well-resolved flow regions differently

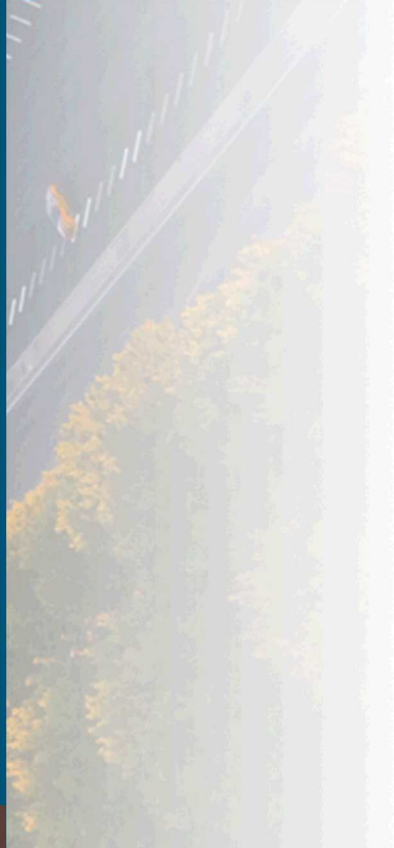
Acknowledgements

Colleagues and collaborators

- SNL: Travis Fisher, Jerry Watkins, Mike Hansen, Robert Knaus, Scott Miller
- NASA: Mark Carpenter

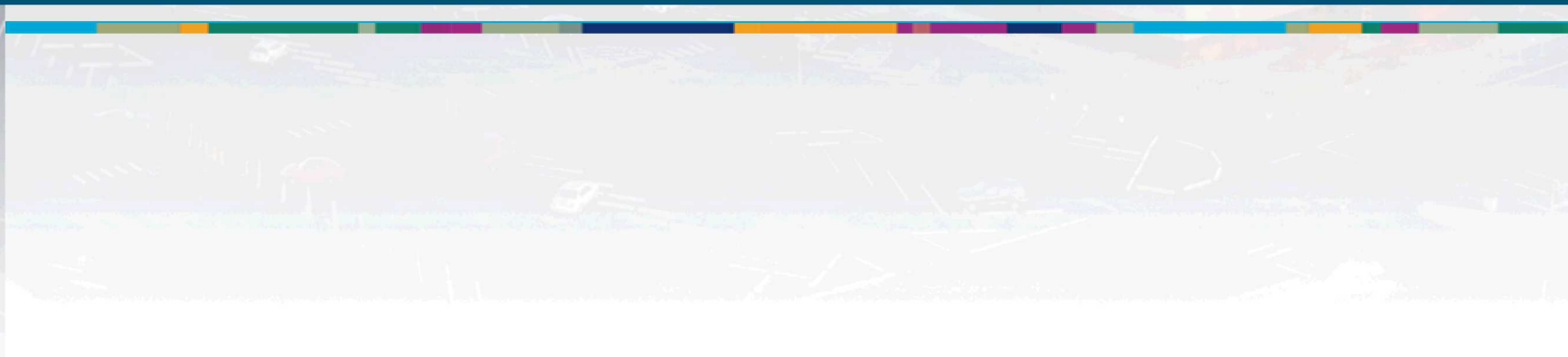
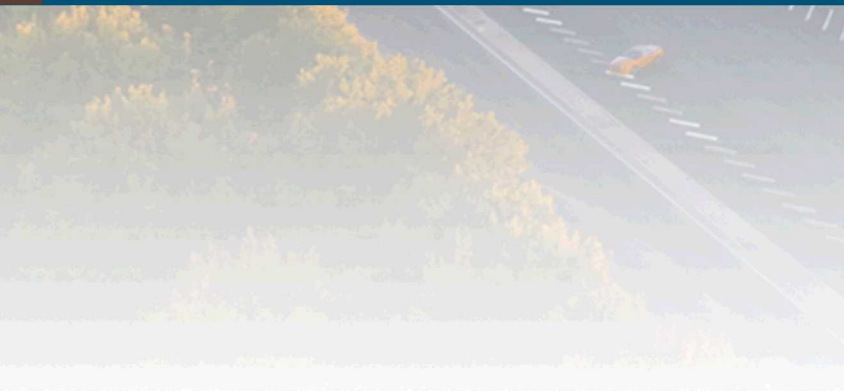


Questions?





Extra Slides



Method Verification

1D Euler MMS

$$\mathbf{u}_t + \mathbf{f}_x = \mathbf{s},$$

$$\mathbf{u} = (\rho, \rho u, \rho E)^T \quad \mathbf{f} = (\rho u, \rho u^2 + p, \rho u H)^T$$

- Initial condition

$$(\rho, u, T) = (2 + \cos(\pi x), 2 + \cos(\pi x), 2 + \cos(\pi x)),$$

$$\mathbf{s} = - \begin{pmatrix} 2\pi \sin(\pi x)(\cos(\pi x) + 2) \\ \pi \sin(\pi x)(\cos(\pi x) + 2)(2R + 3 \cos(\pi x) + 6) \\ \pi \sin(\pi x)((\cos(\pi x) + 2)^2(3c_p + 2 \cos(\pi x) + 4)) \end{pmatrix},$$

- Error convergence

Single block

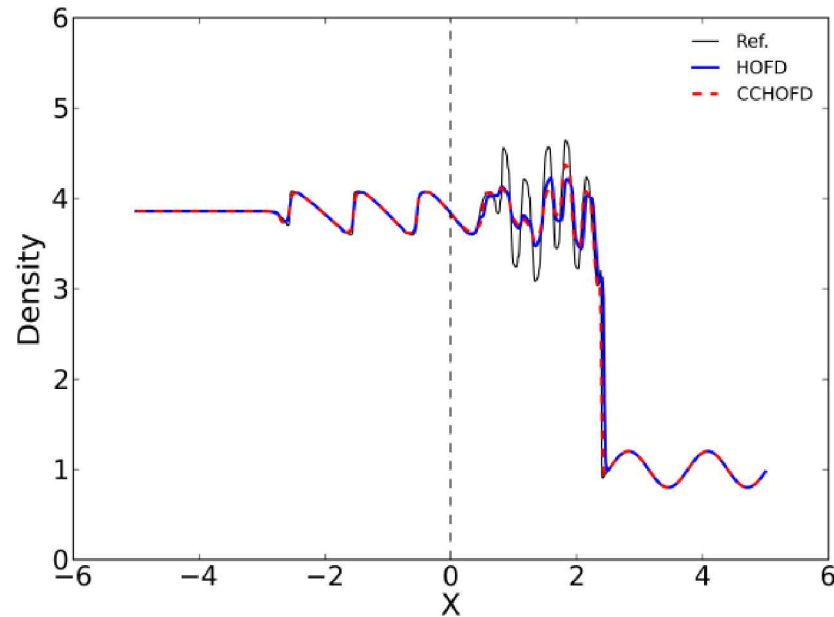
N	h	$\epsilon_2(\rho)$	Rate	$\epsilon_2(u)$	Rate	$\epsilon_2(T)$	Rate
32	6.06E-02	1.02E-03		5.96E-03		5.28E-01	
64	3.08E-02	5.67E-05	4.27	1.07E-03	2.53	5.66E-02	3.30
128	1.55E-02	4.09E-06	3.84	8.77E-05	3.66	4.29E-03	3.76
256	7.78E-03	2.80E-07	3.89	5.91E-06	3.91	2.87E-04	3.92
512	3.90E-03	1.81E-08	3.96	3.80E-07	3.97	1.83E-05	3.98

Two blocks

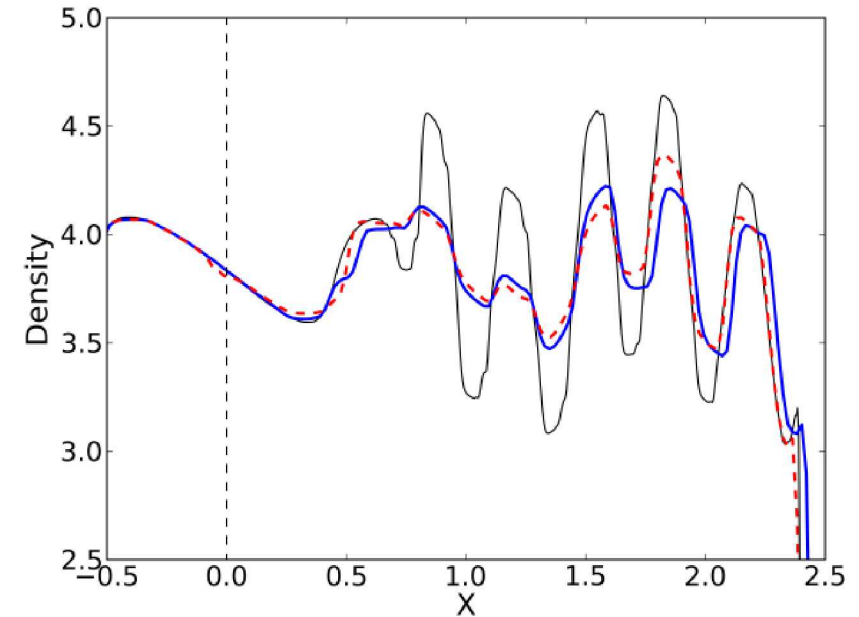
2N	h	$\epsilon_2(\rho)$	Rate	$\epsilon_2(u)$	Rate	$\epsilon_2(T)$	Rate
32	5.88E-02	9.03E-03		3.92E-02		3.51E+00	
64	3.03E-02	3.57E-04	4.87	1.22E-03	5.23	3.63E-01	3.42
128	1.54E-02	3.04E-05	3.64	9.73E-05	3.73	2.92E-02	3.72
256	7.75E-03	2.10E-06	3.90	6.89E-06	3.86	1.97E-03	3.93
512	3.89E-03	1.35E-07	3.98	4.36E-07	4.01	1.27E-04	3.98

Interaction of a strong shock and a standing entropy wave fluctuation

$$(\rho, u, p) = \begin{cases} (3.857143, 2.629369, 10.3333), & \text{if } x < -4.0 \\ (1 + 0.2 \sin(5x), 0, 1), & \text{if } x \geq -4.0. \end{cases}$$

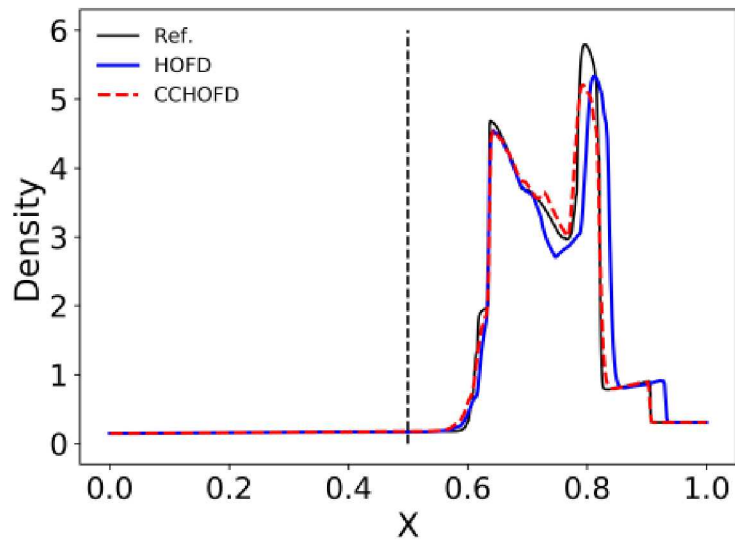


(a) Density

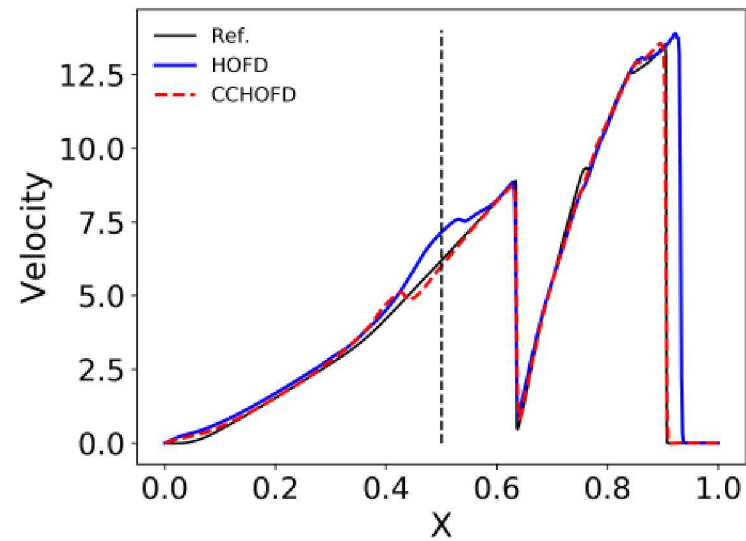


(b) Density close up

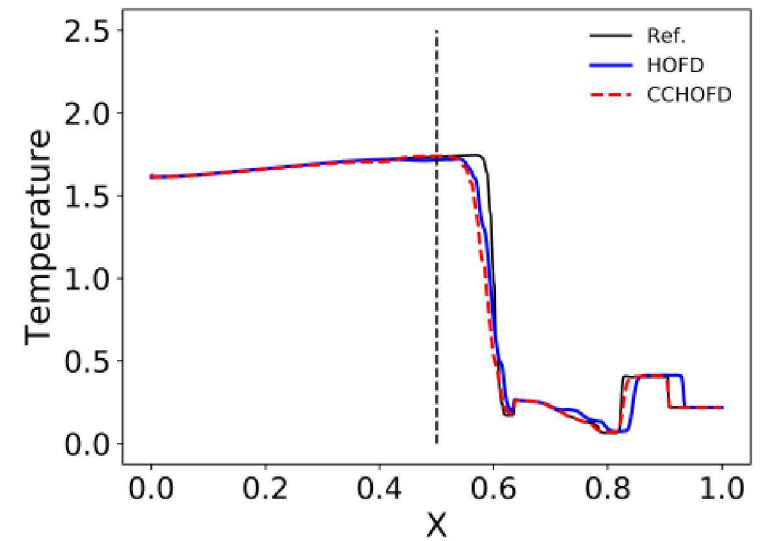
$$N=512, t_f = 1.8, N_{\text{ref}} = 2000$$



(a) Density



(b) Velocity



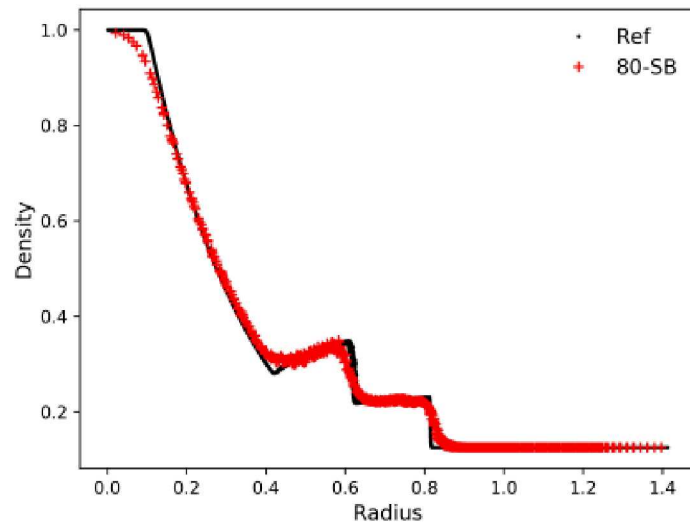
(c) Temperature

$N=512$, $t_f = 0.04$, $N_{\text{ref}} = 2000$

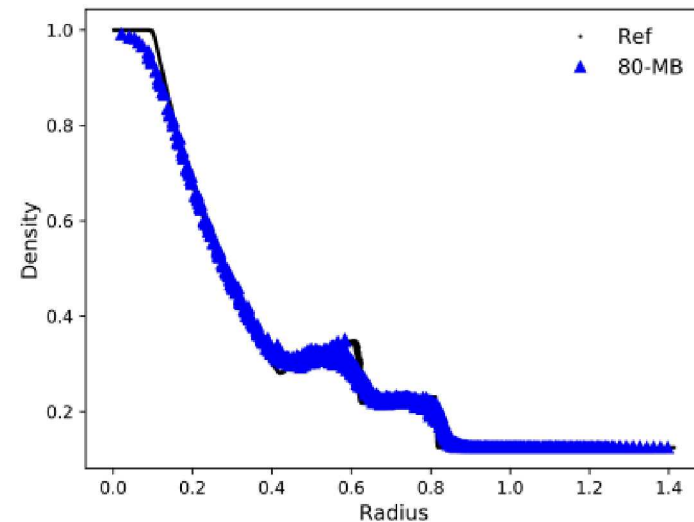
2D Sod Shock Tube

Single-block and multi-block comparison

- Domain $[-1 < (x, y) < 1]$
- Initial condition $(\rho, u, v, p) = \begin{cases} (1, 0, 0, 1), & \text{if } r \leq 0.4 \\ (0.125, 0, 0, 0.1), & \text{if } r > 0.4, \end{cases}$



(a) 80^2 mesh, a single block



(b) 80^2 mesh equally divided into 16 sub-blocks

$$T_f = 0.25, N_{\text{ref}} = 1280^2$$

Entropy Stable Cell-Centered High-Order Finite Difference

Entropy stability

- Entropy stability analysis

$$\begin{aligned}\mathbf{u}_t + \mathbf{f}_x &= 0 \\ \mathbf{w}^T \mathbf{u}_t + \mathbf{w}^T \mathbf{f}_u \mathbf{u}_x &= 0 \quad \text{where } \mathbf{w}^T = S_{\mathbf{u}} \\ S_t + F_x &= 0\end{aligned}$$

- Global entropy consistency

$$\begin{aligned}\frac{d}{dt} \mathbf{1}^T \mathcal{P}S + \sum_{k=1}^N \sum_{l=1}^N b_{1,k} b_{1,l} \bar{F}(u_l, u_k) - b_{-1,k} b_{-1,l} \bar{F}(u_l, u_k) &= 0 \\ \frac{d}{dt} \mathbf{1}^T \mathcal{P}S + \bar{F}|_1 - \bar{F}|_{-1} &= 0 \quad \text{SBP is satisfied}\end{aligned}$$

- Entropy stable condition (Fisher, JCP 2013)

$$\mathbf{w}^T \Delta \bar{\mathbf{f}} \geq \mathbf{w}^T \Delta \bar{\mathbf{f}}^S.$$

- Locally satisfy entropy stability condition

$$(\mathbf{w}_{i+1} - \mathbf{w}_i)^T \bar{\mathbf{f}}_i^S(\mathbf{u}_{i+1}, \mathbf{u}_i) = \psi_{i+1} - \psi_i$$

Generalized Entropy Stable Interface Penalty

Entropy stability

$$\frac{d}{dt} \mathbf{1}^T \mathcal{P} S = \bar{F}_L|_{-1} - \bar{F}_L|_{(-)} + \bar{F}_R|_{(+)} - \bar{F}_R|_{-1} + \mathbf{w}^T \mathbf{g}^{int}$$

$$\begin{aligned} & -\bar{F}_L|_{(-)} + \bar{F}_R|_{(R)} + \mathbf{w}^T \left(\mathbf{b}_1^L \mathbf{b}_1^{L^T} \circ \mathbf{f}(\mathbf{u}, \mathbf{u}) - \mathbf{b}_1^L \mathbf{b}_{-1}^{R^T} \circ \mathbf{f}(\mathbf{u}, \mathbf{u}) - \frac{1}{2} \mathbf{b}_1^L \mathbf{R} |\Lambda| \mathbf{R}^T \left(\mathbf{b}_1^{L^T} \mathbf{w} - \mathbf{b}_{-1}^{R^T} \mathbf{w} \right) \right) \\ & - \mathbf{w}^T \left(-\mathbf{b}_{-1}^R \mathbf{b}_1^{L^T} \circ \mathbf{f}(\mathbf{u}, \mathbf{u}) + \mathbf{b}_{-1}^R \mathbf{b}_{-1}^{R^T} \circ \mathbf{f}(\mathbf{u}, \mathbf{u}) - \frac{1}{2} \mathbf{b}_1^R \mathbf{R} |\Lambda| \mathbf{R}^T \left(\mathbf{b}_{-1}^{R^T} \mathbf{w} - \mathbf{b}_1^{L^T} \mathbf{w} \right) \right) = 0 \end{aligned}$$

$$\begin{aligned} & \psi|_{(-)} - \psi|_{(+)} + \mathbf{w}^T \left(-\mathbf{b}_1^L \mathbf{b}_{-1}^{R^T} \circ \mathbf{f}(\mathbf{u}, \mathbf{u}) - \frac{1}{2} \mathbf{b}_1^L \mathbf{R} |\Lambda| \mathbf{R}^T \left(\mathbf{b}_1^{L^T} \mathbf{w} - \mathbf{b}_{-1}^{R^T} \mathbf{w} \right) \right) \\ & + \mathbf{w}^T \left(\mathbf{b}_{-1}^R \mathbf{b}_1^{L^T} \circ \mathbf{f}(\mathbf{u}, \mathbf{u}) + \frac{1}{2} \mathbf{b}_1^R \mathbf{R} |\Lambda| \mathbf{R}^T \left(\mathbf{b}_{-1}^{R^T} \mathbf{w} - \mathbf{b}_1^{L^T} \mathbf{w} \right) \right) = 0 \end{aligned}$$

$$\begin{aligned} & \mathbf{w}^T \left(\mathbf{b}_{-1}^R \mathbf{b}_1^{L^T} \circ \mathbf{f}(\mathbf{u}, \mathbf{u}) + \frac{1}{2} \mathbf{b}_1^R \mathbf{R} |\Lambda| \mathbf{R}^T \left(\mathbf{b}_{-1}^{R^T} \mathbf{w} - \mathbf{b}_1^{L^T} \mathbf{w} \right) \right) - \mathbf{w}^T \left(\mathbf{b}_1^L \mathbf{b}_{-1}^{R^T} \circ \mathbf{f}(\mathbf{u}, \mathbf{u}) + \frac{1}{2} \mathbf{b}_1^L \mathbf{R} |\Lambda| \mathbf{R}^T \left(\mathbf{b}_1^{L^T} \mathbf{w} - \mathbf{b}_{-1}^{R^T} \mathbf{w} \right) \right) \\ & = \psi|_{(+)} - \psi|_{(-)} \end{aligned}$$

Shock Capturing Method: Artificial Viscosity

Artificial viscosity method (Shakib, 1991)

- Formulated using entropy variables

$$\hat{\mu} = \max \left[\left(\frac{(L\mathbf{u})^T \mathbf{w}_u(L\mathbf{u})}{\phi + (\mathbf{w}_{x_i})^T g_{ij} \mathbf{u}_w(\mathbf{w}_{x_j})} \right)^{1/2}, c_{ref} \frac{|u| + c}{h} \right], \quad L\mathbf{u} = \frac{\partial \mathbf{f}_k}{\partial \mathbf{v}} \frac{\partial \mathbf{v}}{\partial x_k} - \frac{\partial \mathbf{f}_k}{\partial x_k}$$

$$\mathcal{P}_k^{-1} \Delta_k \bar{\mathbf{f}}_k^{av} = \mathcal{D}_i g_{ij} \frac{\partial \mathbf{u}}{\partial \mathbf{w}} \hat{\mu} \mathcal{D}_j \mathbf{w}$$

$$\mathcal{P}_k^{-1} \Delta_k \bar{\mathbf{f}}_k^{ad} = \mathcal{D}_2 |\Lambda| \frac{\partial \mathbf{u}}{\partial \mathbf{w}} \mathcal{D}_2 \mathbf{w} \quad \text{where} \quad \mathcal{D}_2 = \Delta \Delta^T$$

- CCHOFD requires artificial dissipation due to lack of dissipation
- Design issues and concerns
 - Artificial dissipation introduces extra dissipation even in freestream regions
 - Shock sensor type adaptive mechanism is utilized to reduce dissipation

2D Curvilinear Problem

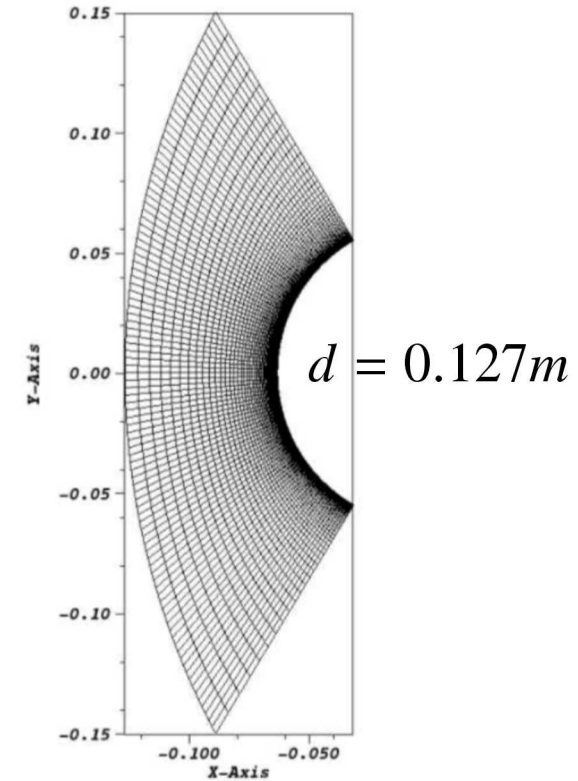
Blottner cylinder

- Perfect gas law
- Sutherland viscosity
- Inflow conditions

$$M = 5.0 \quad Re = 1.8875 \times 10^6$$

Property	Value
Density, $\rho [kg/m^3]$	8.788×10^{-2}
Velocity, $u[m/s]$	871.47
Temperature, $T[K]$	75.85

- Grids
 - 50x50
 - 100x100
 - 200x200

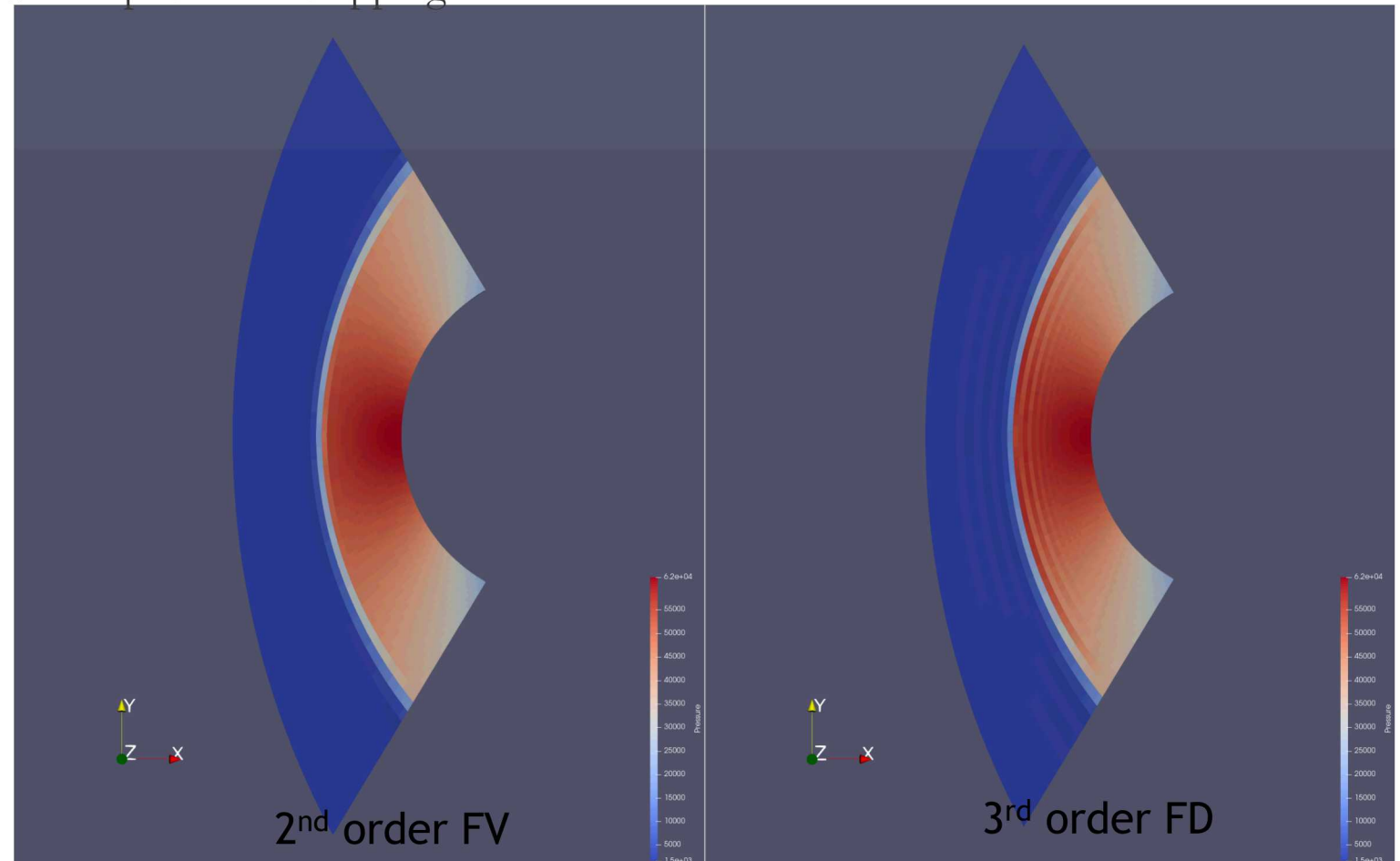


Curvilinear Multi-Block Shock Capturing : Artificial Viscosity

Qualitative comparison

- 2nd order cell-centered finite volume with implicit time stepping
 - Minmod limiter
- 3rd order cell-centered finite difference with explicit time stepping
 - Artificial viscosity

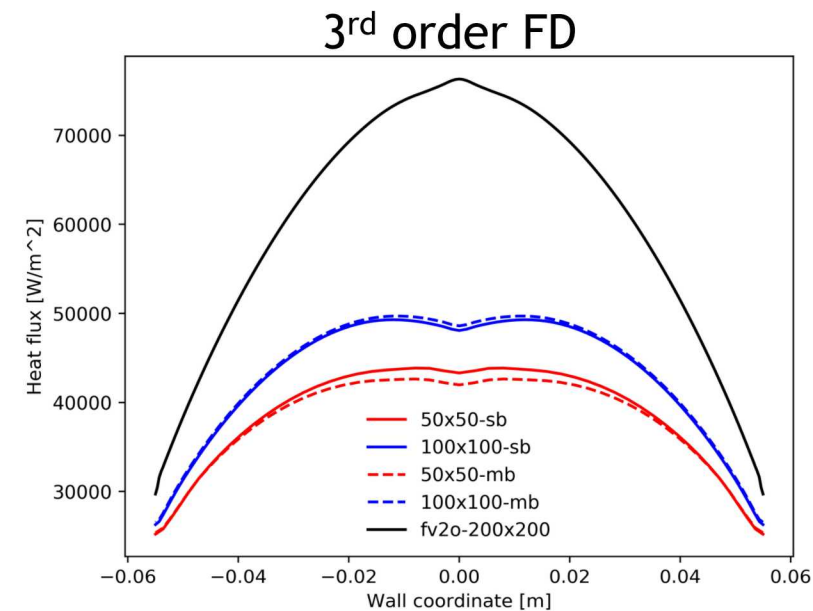
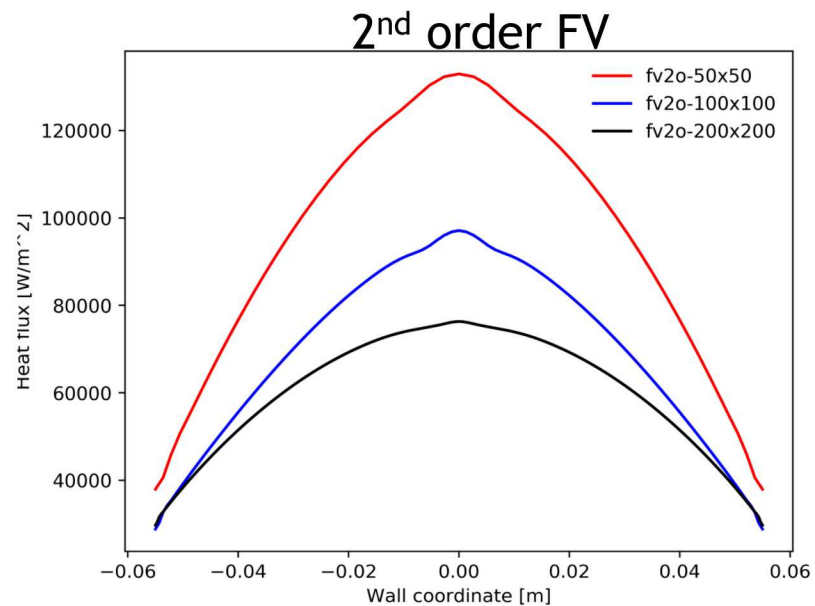
100x100 mesh in 2-block configuration
(longitudinal split)



Quantitative Comparison Multi-Block Capability

Wall heat flux prediction

- 2nd order cell-centered finite volume and 3rd order cell-centered finite difference
- FV 200x200 result close to reference
- FD 200x200 did not converge due to instability associated with AV near outflow boundary
- Multi-block and single block results converge



Entropy Stable Cell-Centered High-Order Finite Difference

Cell-centered summation-by-parts operator

- (2-4-2) operator: third-order

$$\mathbf{b}_{-1} = \left(\frac{35}{16}, -\frac{35}{16}, \frac{21}{16}, -\frac{5}{16}, 0, \dots, 0 \right)^T$$

$$\mathcal{P} = \text{diag} \left(\frac{433}{384}, \frac{95}{128}, \frac{451}{384}, \frac{367}{384}, 1, \dots, 1, \frac{367}{384}, \frac{451}{384}, \frac{95}{384}, \frac{433}{384} \right) \delta x,$$

$$Q = \begin{pmatrix} -\frac{1225}{512} & \frac{6695}{1536} & -\frac{4097}{1536} & \frac{359}{512} & 0 & 0 & 0 & \dots \\ \frac{655}{1536} & -\frac{1225}{512} & \frac{1415}{512} & -\frac{1225}{1536} & 0 & 0 & 0 & \dots \\ -\frac{313}{1536} & \frac{55}{512} & -\frac{441}{512} & \frac{533}{512} & -\frac{1}{12} & 0 & 0 & \dots \\ -\frac{9}{512} & \frac{175}{1536} & -\frac{323}{512} & -\frac{25}{512} & \frac{2}{3} & -\frac{1}{12} & 0 & \dots \\ 0 & 0 & \frac{1}{12} & -\frac{2}{3} & 0 & \frac{2}{3} & -\frac{1}{12} & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \end{pmatrix},$$

Numerical Approach

Cell-Centered Entropy Stable High-Order Finite Difference

- Extension to conventional/Node-centered scheme introduced by Fisher 2013
- Entropy stable non-dissipative base scheme
- High-order accurate: 3rd order or better
- Shock capturing method: Weighted Essentially Non-Oscillatory, artificial viscosity
- Multi-block capable

